Matrix Scaling: A New Heuristic for the Feedback Vertex Set Problem

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- Minimizing $\tau(G)$ is NP-Hard [Karp, 1972].

Motivations

- Finding feedback vertex sets in dependency digraphs can be used to resolve deadlock.

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- Selecting flip-flops in partial scan designs. It is a technique used in design for testing.

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1. Digraph reductions: Removing vertices and arcs without changing the problem.
2. Vertex selection: Choose a vertex to be in a FVS.
3. Removing redundant vertices: The FVS may not be minimal.

Strongly Connected Components

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- $O(|V| + |E|)$ running time

Levy and Low reductions

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- The operations can be done in any order [Levy and Low, 1988].

fvs_Max_Deg

Choosing a vertex based off of vertex degrees is quicker.

Algorithm 1: MaxDeg

Data: A Digraph $G = (X, U)$
Result: A FVS S

```
begin
    S ⟵ ∅
    LL_graph_reductions(G, S)
    L ⟵ get_SCC(G)
    while |L| ≠ 0 do
        remove g from L
        v ⟵ max(min(d⁺(v), d⁻(v))|v ∈ V(G))
        remove v from g
        S ⟵ S + {v}
        LL_reductions(g, S)
        L ⟵ get_SCC(g) + L
    end
    S ⟵ remove_redundant_nodes(G, S)
    return S
end
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- They selected the vertex with the smallest mean return time.
- Their method operates in about $O(|F|n^{2.376})$ time.

MFVSmean

Algorithm 2: MFVSmean

Data: A Digraph $G = (X, U)$
Result: A FVS S
begin
$S \longleftarrow \emptyset$
$LL\_graph\_reductions(G, S)$
$L \longleftarrow get\_SCC(G)$
while $|L| \neq 0$ **do**
remove g from L
$v \longleftarrow MFVSmean\_selection(g)$
remove v from g
$S \longleftarrow S + \{v\}$
$LL\_reductions(g, S)$
$L \longleftarrow get\_SCC(g) + L$
end
$S \longleftarrow remove\_redundant\_nodes(G, S)$
return S
end

MFVSmean

Algorithm 3: MFVSmean_selection

Data: A Digraph $G = (X, U)$
Result: A vertex v
begin
  $\mathbf{P} \longleftarrow \mathit{CreateTransitionMatrix}(G)$
  $\pi' \longleftarrow \mathit{ComputeStationaryDistributionVector}(\mathbf{P})$
  $\mathbf{P} \longleftarrow \mathit{CreateTransitionMatrix}(G^{-1})$
  $\pi'' \longleftarrow \mathit{ComputeStationaryDistributionVector}(\mathbf{P})$
  $\pi \longleftarrow \pi' + \pi''$
  determine $v \in V$ with $\pi_v = \|\pi\|_\infty$
  return v
end

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- DCUs are not a local property.
- It is reasonable to suspect that a vertex that is in many DCUs is in a minimum FVS.
- Finding all DCUs is hard.

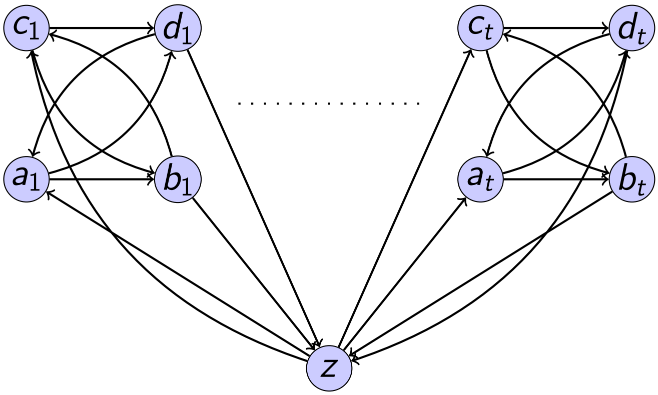

Figure: For $t \geq 2$ the vertex z is not in a minimum FVS, but is in nearly every DCU and most cycles.

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The permanent counts the number of spanning disjoint cycle unions. We can create an auxilary digraph H from G by adding loops to the vertices of G.

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$$perm(\mathbf{A}(H)) - 1 = |DCU(G)|$$

m_balance

Let H be the auxilary digraph created as before. From H we can create a matrix called the m_balance($\mathbf{A}$(H)) that gives the fraction of DCUs that every arc of H is in.

$$m\_bal(\mathbf{A}(H)) = \frac{a_{i,j} \times perm(\mathbf{A}(H)_{i,j})}{perm(\mathbf{A}(H))}. \tag{1}$$

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- The m_balance is doubly stochastic since every vertex is incident with every DCU.
- The loop with the smallest value in the m_balance corresponds to the vertex that is in the most DCUs.
- The m_balance is very hard to calculate.

Sinkhorn Balancing.

Algorithm 4: Sinkhorn_Selection

```
Data: A Digraph G = (V, U)
Result: A vertex v
begin
    A ⟵ adjacency matrix of G
    A ⟵ add ones to the diagonal of A
    for i ∈ {1, ..., ⌈log(n)⌉} do
        A ⟵ normalize the rows of A
        A ⟵ normalize the columns of A
    end
    v is the vertex corresponding to the lowest value on the
    diagonal of A
    return v
end
```

- Soules [1991] showed that Algorithm 4 converges quickly if $\mathbf{A}$ is totally supported. Reducing to strongly connected components guarantees this.

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Algorithm 5: Sinkhorn_Selection

Data: A Digraph $G = (V, U)$
Result: A vertex v
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  $\mathbf{A} \longleftarrow$ add ones to the diagonal of $\mathbf{A}$
  for $i \in \{1, \ldots, \lceil log(n) \rceil\}$ **do**
    $\mathbf{A} \longleftarrow$ normalize the rows of $\mathbf{A}$
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- Beichl and Sullivan [1999] showed that the limiting matrix of the Sinkhorn-Knopp algorithm can be used to estimate the permanent of **A**.

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Algorithm 6: Sinkhorn_Selection

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- Beichl and Sullivan [1999] showed that the limiting matrix of the Sinkhorn-Knopp algorithm can be used to estimate the permanent of $\mathbf{A}$.
- We observed that we only need to complete $log(n)$ iterations for the order to settle down.

fvs_sh_del

Algorithm 7: FVS_SH_Del

Data: A Digraph $G = (X, U)$
Result: A FVS S
begin
  $H \longleftarrow G$
  $S \longleftarrow \emptyset$
  $LL\_graph\_reductions(H, S)$
  $L \longleftarrow get\_SCC(H)$
  while $|L| \neq 0$ **do**
    remove g from L
    $v \longleftarrow Sinkhorn\_selection(g)$
    remove v from g
    $S \longleftarrow S + \{v\}$
    $LL\_reductions(g, S)$
    $L \longleftarrow get\_SCC(g) + L$
  end
  $S \longleftarrow remove\_redundant\_nodes(G, S)$
  return S
end

$$O(|S| log(n) n^2)$$

fvs_sh_del_mod

Algorithm 8: FVS_SH_DEL_MOD

Data: A Digraph $G = (X, U)$
Result: A FVS S
begin
  $H \longleftarrow G$
  $S \longleftarrow \emptyset$
  $LL\_graph\_reductions(H, S)$
  while $|V(H)| \neq 0$ **do**
    $v \longleftarrow Sinkhorn\_selection(H)$
    remove v from H
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  end
  $S \longleftarrow remove\_redundant\_nodes(G, S)$
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4. If it is a DAG, then v is redundant and we let $S_i = S_{i-1} - \{v\}$.

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- Let α, β be orderings of $V(G)$ and T respectively such that $c_X(\alpha_i) \geq c_X(\alpha_{i+1})$ and β and $c_X(\beta_i) \geq c_X(\beta_{i+1})$

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- $$\sum_{i=0}^{t} c_X(\alpha_i) \geq |X| \qquad \sum_{i=0}^{t'} c_X(\beta_i) \geq |X|.$$
- $$k\tau(G) \geq \sum_{i=0}^{t} c_X(\alpha_i) \geq |X| = \epsilon|S|.$$

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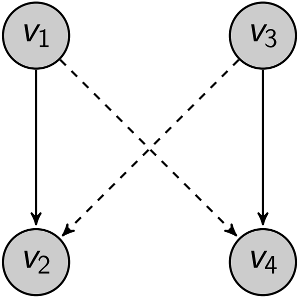
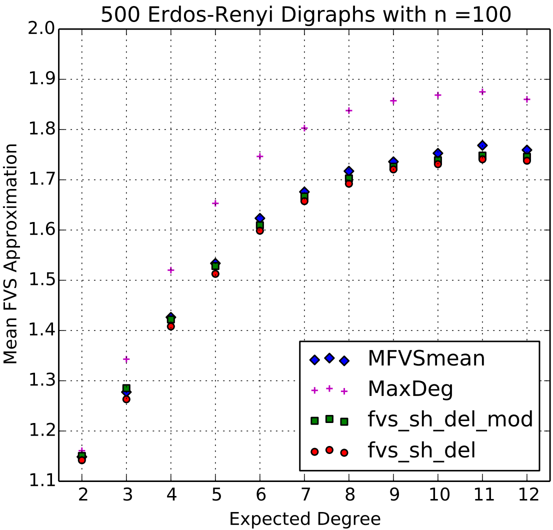
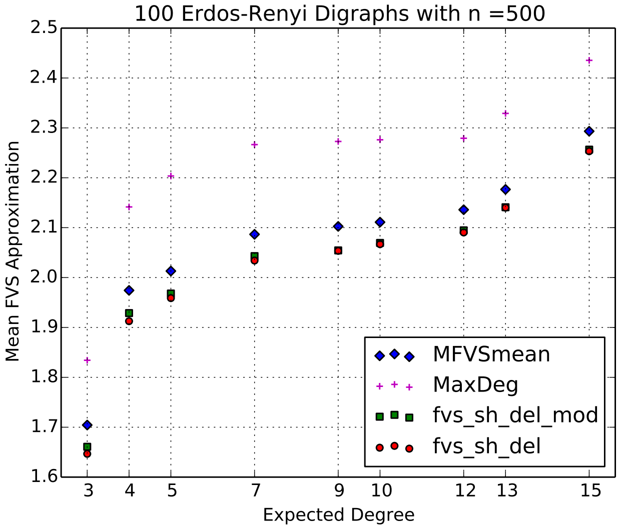
- We then perform $k^2 n$ arc switches to simulate a uniformly chosen one.


v_1
v_3
v_2
v_4

Erdos-Renyi Random Digraphs n = 100



Erdos-Renyi Random Digraphs n = 500



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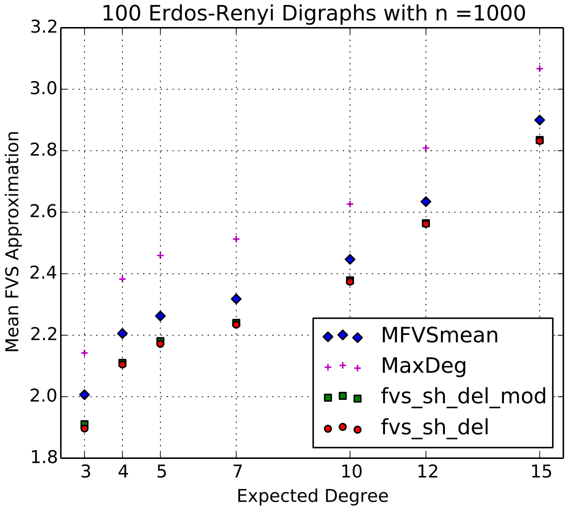

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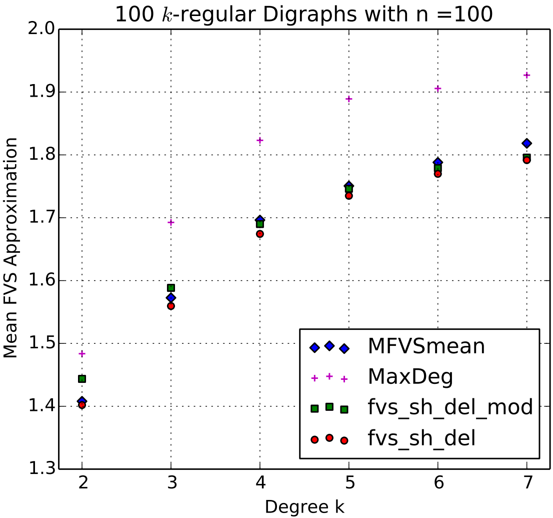

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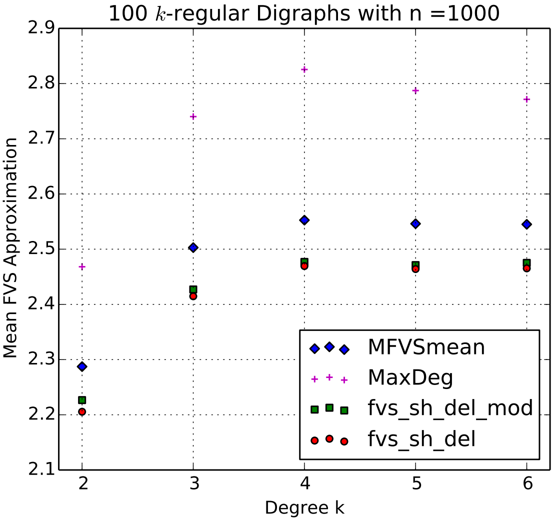

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$$entropy(\mathbf{A}) = -\sum_{i,j} a_{i,j} log(a_{i,j}).$$

Beichl and Sullivan showed the limiting matrix of the Sinkhorn-Knopp algorithm maximizes the entropy for all doubly-stochastic matrices with a given zero-one pattern.

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