Hard Problems On Box-Totally Dual Integral Polyhedra

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Abstract. In this paper, we study the complexity of some fundamental questions regarding box-totally dual integral (box-TDI) polyhedra. First, although box-TDI polyhedra have strong integrality properties, we prove that Integer Programming over box-TDI polyhedra is NP-complete, that is, finding an integer point optimizing a linear function over a box-TDI polyhedron is hard. Second, we complement the result of Ding et al. (The complexity of recognizing linear systems with certain integrality properties. Mathematical Programming 114(2), 321–334 (2008)) who proved that deciding whether a given system is box-TDI is co-NP-complete: we prove that recognizing whether a polyhedron is box-TDI is co-NP-complete.

To derive these complexity results, we exhibit new classes of totally equimodular matrices — a generalization of totally unimodular matrices — by characterizing the total equimodularity of incidence matrices of graphs.

Keywords: Box-TDI polyhedron · Totally equimodular matrix · Incidence matrix

1 Introduction

Totally dual integral systems were introduced in the late 70's and serve as a general framework for establishing various min-max relations in combinatorial optimization [22]. A rational system of linear inequalities $Ax \leq b$ is *totally dual integral (TDI)* if the minimization problem in the linear programming duality relation:

$$\max\{c^\top x \colon Ax \leq b\} = \min\{b^\top y \colon A^\top y = c, y \geq \mathbf{0}\}$$

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admits an integer optimal solution for each integer vector c such that the maximum is finite. As is well-known, such systems $Ax \leq b$ can be used to define every integer polyhedron, with b integral [13].

A stronger property is the box-total dual integrality, where a system $Ax \leq b$ is *box-totally dual integral (box-TDI)* if $Ax \leq b$, $\ell \leq x \leq u$ is TDI for all rational vectors ℓ and u (with possible infinite components). General properties of such systems can be found in Cook [6] and Section 22.4 of Schrijver [22].

Box-TDI systems are intimately related to totally unimodular matrices. A matrix is *totally unimodular (TU)* if every subset of linearly independent rows forms a unimodular matrix, a matrix being *unimodular* if it has full row rank and all its nonzero maximal minors have value ± 1. A matrix A is TU if and only if the system $Ax \leq b$ is box-TDI for each rational vector b [22, Page 318].

Until recently, the vast majority of known box-TDI systems were systems associated with TU matrices. For instance, König's Theorem [20] can be seen as a consequence of the fact that the vertex-edge incidence matrix of a graph is TU if and only if the graph is bipartite [16].

In the last two decades, several new box-TDI systems were exhibited. Chen, Ding, and Zang [9] characterized box-Mengerian matroid ports. In [3], they provided a box-TDI system describing the 2-edge-connected spanning subgraph polyhedron for series-parallel graphs. Ding, Tan, and Zang [10] characterized the graphs for which the Edmonds system for defining the matching polytope [12], which is always TDI as shown by Cunningam and Marsh [8], is box-TDI. Ding, Zang, and Zhao [11] introduced new subclasses of box-perfect graphs. Cornaz, Grappe, and Lacroix [7] provided several box-TDI systems in series-parallel graphs. More recently, these graphs have also been characterized by the box-TDIness of their flow cone [2] and that of their k-edge-connected polyhedron [1]. These last two results use characterizations of box-TDI polyhedra given by Chervet, Grappe, and Robert [4].

As stated before, every integer polyhedron can be defined by a TDI system. Yet, the statement no longer holds if we replace TDI by box-TDI. A polyhedron that can be described by a box-TDI system is a *box-TDI polyhedron*, and every TDI system describing it is actually box-TDI [6]. Box-TDI polyhedra characterize the following generalization of TU matrices. A matrix is *totally equimodular (TE)* if every subset of linearly independent rows forms an equimodular matrix, a matrix being *equimodular* if it has full row rank and all its nonzero maximal minors have the same absolute value. A matrix A is TE if and only if the polyhedron $\{x\colon Ax \leq b\}$ is box-TDI for each rational vector b [4].

Several complexity results relative to TDIness and box-TDIness are known. Deciding whether a system $Ax \leq b$ is TDI or whether it is box-TDI are two co-NP-complete problems [9]. The first problem remains co-NP-complete even for conic systems [21], that is, when $b = \mathbf{0}$. A tractable case for the recognition of box-TDI systems is when A is TU, since total unimodularity can be tested in polynomial time [23]. We continue along this line by providing two new hardness results.

Contributions. In this paper, we prove that the problem of deciding whether a given polyhedron is box-TDI is co-NP-complete. Our proof builds upon the hardness result of Ding et al. [9] about the recognition of box-TDI systems.

We also prove that the edge-vertex incidence matrix of any graph is TE. This implies that the edge relaxation of the stable set problem is a box-TDI polyhedron. From the NP-hardness of the maximum stable set problem, it follows that optimizing a linear function over $\{x \in \mathbb{Z}^n : Ax \leq \mathbf{1}\}$ is NP-hard when A is TE. Since the latter problem is polynomial when A is TU, this unveils a major difference between TE and TU matrices. Moreover, this hardness result also implies that integer optimization over box-TDI polyhedra is NP-hard.

Another difference between TE and TU matrices is that the transpose of a TE matrix is not always TE. We highlight this fact by characterizing the equimodularity and the total equimodularity of the vertex-edge incidence matrix of a graph.

Outline. In Section 2, we provide the definitions and some results needed throughout the paper. In Section 3, we characterize the equimodularity and the total equimodularity of the edge-vertex and of the vertex-edge incidence matrix of a graph. Based on these results, in Section 4, we characterize the box-TDIness of the stable set polytope and that of the edge cover dominant polyhedron of a graph. As a consequence, we prove that Integer Programming over box-TDI polyhedra is NP-complete and that recognizing whether a polyhedron is box-TDI is co-NP-complete.

2 Preliminaries

2.1 Matrices and polyhedra

In a given matrix, a *minor* is the determinant of any square submatrix. When the latter has maximal size, the associated minor is *maximal*.

Recall that an integer matrix is unimodular if it has full row rank and all its nonzero maximal minors are ± 1. More generally, a rational matrix is equimodular if it has full row rank and all its nonzero maximal minors have the same absolute value. As observed in [4], checking equimodularity can be done in polynomial time. Indeed, equimodular matrices are TU up to a basis change, and checking total unimodularity can be done in polynomial time [23].

A *face* of a polyhedron $P = \{x : Ax \leq b\}$ is the polyhedron obtained by imposing equality on some inequalities in the description of P. A matrix M is *face-defining* for a face F of P if it has full row rank and the affine space generated by F can be written as $\{x : Mx = d\}$ for some vector d of appropriate size. These matrices characterize box-TDI polyhedra as follows.

Theorem 1 (Chervet et al. [4]). *A polyhedron P is box-TDI if and only if every face-defining matrix of P is equimodular.*

In our proofs, we will use Theorem 1 combined with the following observation.

Observation 2 (see, for instance, Chervet et al. [4]). *A full row rank matrix M is face-defining for a face F of a polyhedron $P \subseteq \mathbb{R}^n$ if and only if there exist a vector d and a set $H \subseteq F \cap \{x \colon Mx = d\}$ of $\dim(F)+1$ affinely independent points such that $|H| + \mathrm{rank}(M) = n+1$.*

Recall that a matrix is TE if every subset of linearly independent rows forms an equimodular matrix. By Theorem 1, every polyhedron whose constraint matrix is TE is box-TDI. It turns out that this characterizes TE matrices.

Theorem 3 (Chervet et al. [4]). *A matrix A of $\mathbb{Q}^{m\times n}$ is totally equimodular if and only if the polyhedron $\{x \colon Ax \leq b\}$ is box-TDI for all $b \in \mathbb{Q}^m$.*

TE matrices are to box-TDI polyhedra what TU matrices are to box-TDI systems.

Theorem 4 (Hoffman et al. [16]). *A matrix A of $\mathbb{Z}^{m\times n}$ is totally unimodular if and only if the system $Ax \leq b$ is box-TDI for all $b \in \mathbb{Z}^m$.*

2.2 Matrices and graphs

In this paper, all graphs are undirected. Without loss of generality, we assume that they are simple, connected, and have at least one edge, as our results extend immediately to general undirected loopless graphs.

Let $G = (V, E)$ be a graph. Given $W \subseteq V$, let $\delta(W)$ (respectively $E(W)$) be the set of edges with exactly one extremity (respectively both extremities) in W. An edge uv is said to *cover* u and v. Given $F \subseteq E$, $V(F)$ is the union of the vertices covered by each edge of F. A graph $G' = (V', E')$ is a *subgraph of* G if $E' \subseteq E$ and $V' = V(E')$. A subgraph $G' \subseteq G$ is a *spanning subgraph* of G if $V' = V$. The *degree* of a vertex v of G is the number of edges of G covering v and is denoted by $d_G(v)$. A set of edges $C \subseteq E$ is a *circuit* if the subgraph $(V(C), C)$ is connected and all its vertices have degree 2. A *hole* is a circuit for which $E(V(C)) = C$ [4]. An *odd circuit* is a circuit with an odd number of edges, similarly, an *odd hole* is a hole with an odd number of edges. A graph is *bipartite* if it does not contain any odd circuit. A *perfect matching* of a graph is a set of pairwise nonadjacent edges covering all the vertices.

Let A_G denote the *edge-vertex incidence matrix* of G, that is the matrix whose rows are the characteristic vectors of the edges of G, where the *characteristic vector of an edge* $e = uv$ is the vector $\chi^e \in \{0,1\}^V$ with $\chi^e_w = 1$ if $w \in \{u, v\}$ and $\chi^u_w = 0$ otherwise. Similarly, $A_G^\top$ is the *vertex-edge incidence matrix*. When a result applies to both the edge-vertex and the vertex-edge incidence matrices, we simply write incidence matrix. For $F \subseteq E$, let A_F be the edge-vertex incidence matrix of the graph $(V(F), F)$. The *characteristic vector of a vertex* u is the vector $\chi^u \in \{0,1\}^V$ with $\chi^u_w = 1$ if $w = u$ and $\chi^u_w = 0$ otherwise.

Odd circuits are involved in the value of the determinants of incidence matrices.

[4] In this paper, triangles are considered holes.

Theorem 5 (Grossman et al. [14]). *For a connected graph G with n vertices and n edges, $|\det(A_G)|$ is equal to 0 if G is bipartite, and 2 otherwise.*

Theorem 5 comes from the fact that since G is connected, it has exactly one circuit, and then the value of the determinant of its incidence matrix depends on the parity of that circuit. Theorem 5 can be used to deduce a well-known result characterizing bipartite graphs, generally referred to as Hoffman and Kruskal's Theorem [16].

Theorem 6 (Hoffman et al. [16]). *The incidence matrix of a graph is totally unimodular if and only if the graph is bipartite.*

In our proofs, we will use the following lemma to show that a matrix is not equimodular.

Lemma 7. *For an odd circuit C, and for every $u \in V(C)$, the matrix $\left[A_C^\top, \chi^u\right]$ has full row rank but is not equimodular.*

Proof. Reordering the rows and the columns of $\left[A_C^\top, \chi^u\right]$, we may assume that the matrix is as follows.

$$\left[\begin{array}{ccccc|c} 1 & 1 & & & & \\ & 1 & 1 & & & \\ & & \ddots & \ddots & & \\ & & & 1 & 1 & \\ 1 & & & & 1 & 1 \end{array}\right]$$
$$\underbrace{\hphantom{1\quad 1\quad 1\quad 1\quad 1}}_{A_C^\top}\ \underbrace{\hphantom{1}}_{\chi^u}$$

Since C is an odd circuit, $|\det(A_C^\top)| = 2$, hence $\left[A^\top, \chi^u\right]$ has full row rank. Moreover, the last $|C|$ columns form a lower triangular matrix with 1s on the main diagonal, thus they have determinant 1. Therefore, the matrix is not equimodular. ☐

The definition of bipartite graphs can be generalized as follows. A graph G is *quasi-bipartite* if for each odd circuit C of G, the graph $G \setminus V(C)$ has at least one isolated vertex. These graphs characterize the box-TDIness of the system given in the following theorem, where K_4 denotes the complete graph with 4 vertices.

Theorem 8 (Ding et al. [9]). *Given a connected graph G, the system $A_G^\top x \geq \mathbf{1}, x \geq \mathbf{0}$ is box-TDI if and only if G is a quasi-bipartite graph different from K_4.*

3 Incidence matrices and total equimodularity

In this section, we characterize when the incidence matrix of a graph is TE. Since total equimodularity is not preserved under taking the transpose, this section is divided into two parts: edge-vertex incidence matrices and vertex-edge incidence matrices.

3.1 Edge-vertex incidence matrices

Recall that the edge-vertex incidence matrix of a graph is TU if and only if the graph is bipartite. This extends to all graphs as follows in the more general context of total equimodularity.

Theorem 9. *The edge-vertex incidence matrix of a graph is totally equimodular.*

Proof. Let $G = (V, E)$ be a graph and let M be a full row rank matrix formed by a subset of k rows of the edge-vertex incidence matrix of G. Let us prove that M is equimodular by induction on its number of rows: the base case is when M has one row, and then M is equimodular since a row has only values in $\{0, 1\}$. The matrix M encodes a subgraph $H = (V, F)$ of G with $k = |F|$ edges.

We have $|V(F)| \geq |F|$, as otherwise M would have too many columns of zeros to have full row rank. If $|V(F)| = |F|$, then M has exactly one $k \times k$ submatrix which is nonsingular, hence M is equimodular. If $|V(F)| > |F|$, then H has a vertex u of degree one. Indeed, if every vertex of $V(F)$ had degree at least two we would have $2|F| = \sum_{w \in V(F)} d_H(w) \geq 2|V(F)|$, a contradiction.

The column of u in M contains a single one, in uv's row, where v is the neighbor of u in H. Let M' be the matrix obtained from M by removing uv's row. Note that M' has full row rank since M has it. A nonsingular $k \times k$ submatrix N of M has to contain at least one of u and v, as otherwise it has only zeros in uv's row. When N contains exactly one of them, then develop, by using the cofactor expansion, with respect to uv's row. When N contains both of them, then develop, as before, with respect to u's column. In both cases, the determinant of N is equal to a maximal minor of M', up to the sign. By the induction hypothesis, M' is equimodular, so all these determinants are equal in absolute value. Therefore, so are the nonzero $k \times k$ determinants of M, and M is equimodular. □

In [14], the authors proved that the problem of determining the maximum absolute value of a minor of a given incidence matrix is NP-hard. Hence, Theorem 9 implies the following.

Corollary 10. *Determining the maximum absolute value of a minor of a totally equimodular matrix is NP-hard.*

3.2 Vertex-edge incidence matrices

In contrast to edge-vertex incidence matrices, vertex-edge incidence matrices of graphs are rarely TE. We characterize below the classes of graphs for which they are. We also characterize when these matrices are equimodular. Note that when the graph G is bipartite the incidence matrix of G does not have full row rank by Theorem 5. Otherwise, the determinant of a square incidence matrix is 2^k, where $k \geq 1$ is the number of vertex-disjoint odd circuits [14]. Therefore, to get an equimodular vertex-edge incidence matrix, one should forbid vertex-disjoint odd circuits. It turns out that it is an equivalence, as proved below.

Theorem 11. *The vertex-edge incidence matrix of a connected nonbipartite graph $G = (V, E)$ is equimodular if and only if G has no pair of vertex-disjoint odd circuits.*

Proof. Note that every maximal square submatrix of a vertex-edge incidence matrix induces a spanning subgraph of G having $|V|$ edges. Since a spanning tree of G has $|V| - 1$ edges, this subgraph contains a circuit.

($\Rightarrow$) Suppose that G has two vertex-disjoint odd circuits C_1 and C_2, and let e_1 and e_2 be edges of C_1 and C_2, respectively. Since G is connected, there exists a spanning tree T of G containing $C_1 \cup C_2 \setminus \{e_1, e_2\}$. Since C_1 and C_2 are vertex-disjoint, there exists an edge e of T whose removal splits T into two trees T_1 and T_2 with $C_1 \setminus \{e_1\} \subseteq T_1$ and $C_2 \setminus \{e_2\} \subseteq T_2$.

By Theorem 5, $|\det(A^\top_{T_i \cup \{e_i\}})| = |\det(A^\top_{T \cup \{e_i\}})| = 2$, for $i = 1, 2$. By construction, $|\det(A^\top_{T \cup \{e_1,e_2\} \setminus \{e\}})| = |\det(A^\top_{T_1 \cup \{e_1\}}) \det(A^\top_{T_2 \cup \{e_2\}})| = 4$. The determinants of the maximal nonsingular square submatrices $A^\top_{T \cup \{e_1,e_2\} \setminus \{e\}}$ and $A^\top_{T \cup \{e_1\}}$ of $A^\top_G$ differ in absolute value, thus $A^\top_G$ is not equimodular.

($\Leftarrow$) Suppose that G is not bipartite and has no two vertex-disjoint odd circuits. Note that since G is connected, it contains a nonbipartite connected spanning subgraph H with $|V|$ edges. By Theorem 5, we have $|\det(A^\top_H)| = 2$ and $A^\top_G$ has full row rank. This holds for every nonbipartite connected spanning subgraph with $|V|$ edges. The other spanning subgraphs of G with $|V|$ edges are either connected and bipartite or a product of smaller minors corresponding to connected subgraphs. In the first case, the associated minor is zero by Theorem 5. In the second case, by Theorem 5 and the fact that G has no two vertex-disjoint odd circuits, one of these smaller minors is zero. Therefore, every maximal minor of $A^\top_G$ belongs to $\{-2, 0, 2\}$, and $A^\top_G$ is equimodular. □

Theorem 11 gives a characterization of graphs having two vertex-disjoint odd circuits in terms of total equimodularity. A graph-theoretic characterization of these graphs was given by Lovász — see [24, page 546], or [18] for a proof without matroid decomposition. They also appear in the context of extended formulations [5] and unimodular covers [15]. In particular, since equimodularity can be tested in polynomial time [4, Section 4.1], Theorem 11 provides another polynomial-time algorithm for their recognition [19].

Theorem 12. *The vertex-edge incidence matrix of a connected graph $G = (V, E)$ is totally equimodular if and only if G is an odd hole or a bipartite graph.*

Proof. ($\Rightarrow$) Suppose that G is neither bipartite nor an odd hole. Then, G contains an odd hole C and two edges uv and uw in C and $\delta(V(C))$, respectively.

The submatrix of $A^\top_G$ restricted to the rows associated with $V(C)$ can be reordered such that the first $|C| + 1$ columns form the matrix $\left[A_C, \chi^u\right]$. By Lemma 7, it has full row rank but is not equimodular. This implies that $A^\top_G$ is not TE.

($\Leftarrow$) If G is bipartite, then $A_G^\top$ is TU by Theorem 6, and hence TE. Now, if G is an odd hole, then $A_G^\top$ is also the edge-vertex incidence matrix of an odd hole, and hence is TE by Theorem 9. $\square$

By Theorem 12, deciding whether a vertex-edge incidence matrix is TE can be done in polynomial time. This might be a first step towards the complexity of recognizing TE matrices, which is an open problem raised in [4].

4 Box-TDIness and complexity consequences

In this section, we provide several complexity results based on the characterization of total equimodularity of incidence matrices devised in the previous section.

4.1 Edge relaxation of the stable set polytope

Given a graph $G = (V, E)$, a *stable set* is a set of pairwise nonadjacent vertices. The polytope $\{x \in \mathbb{R}^V : A_G x \leq \mathbf{1}, x \geq \mathbf{0}\}$ is called the *edge relaxation of the stable set polytope* of G and its integer points are precisely the characteristic vectors of the stable sets of G.

By Theorems 3 and 9, every polyhedron of the form $\{x \in \mathbb{R}^V : A_G x \leq b\}$ with b rational is box-TDI. As adding $x \geq \mathbf{0}$ preserves box-TDIness, we have the following.

Corollary 13. *The edge relaxation of the stable set polytope is box-TDI.*

Since finding a maximum stable set in a given graph is NP-hard [17], Corollary 13 implies that integer programming over a box-TDI polyhedron is NP-hard.

Corollary 14. *Given a box-TDI polyhedron P and a cost vector c, finding an integer point x maximizing $c^\top x$ over P is NP-hard.*

4.2 Edge relaxation of the edge cover dominant

Since multiplying a row by -1 preserves total equimodularity, by Theorems 3 and 12, when G is an odd hole or a bipartite graph, the polyhedron $\{x \in \mathbb{R}^E : A_G^\top x \geq \mathbf{1}\}$ is box-TDI. It turns out that the converse holds.

Theorem 15. *Given a connected graph $G = (V, E)$, the polyhedron $\{x \in \mathbb{R}^E : A_G^\top x \geq \mathbf{1}\}$ is box-TDI if and only if G is an odd hole or a bipartite graph.*

Proof. To prove the reverse direction, suppose that G is neither an odd hole nor a bipartite graph. Let us build a subgraph $H = (V, F)$ of G for which the polytope is not box-TDI. Since $\{x \in \mathbb{R}^F : A_H^\top x \geq \mathbf{1}\}$ is the projection onto F of $\{x \in \mathbb{R}^E : A_G^\top x \geq \mathbf{1}\}$ intersected with $\{x \in \mathbb{R}^E : x_e = 0, \text{ for all } e \in E \setminus F\}$, this will imply that $\{x \in \mathbb{R}^E : A_G^\top x \geq \mathbf{1}\}$ is not box-TDI.

Since G is connected, nonbipartite, and different from an odd hole, it contains an odd hole C with $\delta(V(C))$ nonempty. Denote by U the set of vertices of $V \setminus V(C)$ whose neighbors are all in $V(C)$. Let S be a subset of $\delta(U)$ such that each vertex of U is covered by exactly one edge of S. Let $F = (E \setminus \delta(U)) \cup S$, and let $H = (V, F)$. This graph is obtained from G by deleting edges in $\delta(U)$ so that every vertex of U has exactly one neighbor in C.

Let M be the $|V(C)| \times |F|$ matrix formed by the rows of $A_H^\top$ associated with the vertices of $V(C)$. By considering the columns of M associated to C and an edge of $\delta(V(C))$, observe that M contains a matrix of the type $\left[A_C^\top, \chi^u\right]$ for some $u \in V(C)$. Therefore, by Lemma 7, M has full row rank but is not equimodular.

We now show that M is face-defining for $P = \{x \in \mathbb{R}^F : A_H^\top x \geq \mathbf{1}\}$. Since M has full row rank, by Observation 2 it is sufficient to exhibit $|F| - |V(C)| + 1$ affinely independent points of the face $Q = P \cap \{x : Mx = \mathbf{1}\}$ of P. Let $K = F \setminus (C \cup \delta(V(C))$, we define:

$$p^0 = \frac{1}{2}\chi^C + \chi^{S \cup K} + \frac{1}{2}\sum_{u \in V(C)} |\delta(u) \cap S|(\chi^{L_u} - \chi^{C \setminus L_u}),$$

where L_u is the unique perfect matching of the path $C \setminus \delta(u)$. Then, we define two types of points:

- $p^e = p^0 + \chi^e$, for each $e \in K$,
- $q^{uv} = p^0 + \chi^{uv} + \frac{1}{2}(\chi^{L_u} - \chi^{C \setminus L_u})$, for each $uv \in \delta(V(C))$ with $u \in V(C)$.

Together with p^0, the points p are affinely independent because $p^e - p^0 = \chi^e$, for each e in K. Adding the points q maintains affine independence since q^{uv} is the only point with uv's coordinate different from 1.

Moreover, all these points belong to Q since they satisfy $x(\delta(u)) = 1$ for all $u \in V(C)$ and $x(\delta(v)) \geq 1$ for all $v \in V \setminus V(C)$. To see this, note that for each u in $V(C)$, $\chi^{L_u} - \chi^{C \setminus L_u}$ satisfies $x(\delta(u)) = -2$ and $x(\delta(v)) = 0$ for all $v \neq u$. The number of points p is $|K| + 1 = |F| - |V(C)| - |\delta(V(C))| + 1$ and the number of points q is $|\delta(V(C))|$, hence M is face-defining for F.

The matrix M is nonequimodular and face-defining for $\{x \in \mathbb{R}^F : A_H^\top x \geq \mathbf{1}\}$. Therefore, the latter is not box-TDI by Theorem 1, and neither is $\{x \in \mathbb{R}^E : A_G^\top x \geq \mathbf{1}\}$. □

Given a graph $G = (V, E)$, an *edge cover* is a set of edges covering each vertex. The polyhedron $\{x \in \mathbb{R}^E : A_G^\top x \geq \mathbf{1}, x \geq \mathbf{0}\}$ is called the *edge relaxation of the edge cover dominant* of G and its binary points are precisely the characteristic vectors of the edge covers of G.

Since adding box constraints preserves box-TDIness, by Theorem 15, the edge relaxation of the edge cover dominant of an odd hole or a bipartite graph is box-TDI. The converse does not hold, because adding $x \geq \mathbf{0}$ might cut off faces defined by nonequimodular matrices, such as the one given in the proof of Theorem 15. The larger class of graphs to be considered to get the converse is given in Theorem 16 below.

The parallel between Theorem 8 and Theorem 16 highlights once more the subtle difference between the TDIness of a system and that of a polyhedron. In particular, for an odd hole C_n, the system $A_{C_n}^\top x \geq \mathbf{1}, x \geq \mathbf{0}$ is not box-TDI while the associated polyhedron is box-TDI. This means that this system is not TDI. This can be seen as the right-hand side is integer but, since n is odd, the point $\frac{1}{2}\mathbf{1}$ is a noninteger vertex of the associated polyhedron. A box-TDI system describing this polyhedron is obtained by adding the inequality $\mathbf{1}^\top x \geq \frac{|C|}{2}$, which is one half of the sum of every inequality in $A_{C_n}^\top x \geq \mathbf{1}$.

Theorem 16. *The edge relaxation of the edge cover dominant of a connected graph G is box-TDI if and only if G is an odd hole or a quasi-bipartite graph different from K_4.*

Proof. Let P_G denote the edge relaxation of the edge cover dominant of G.
($\Leftarrow$) By Theorem 8, if G is a quasi-bipartite graph different from K_4, then the system $A_G^\top x \geq \mathbf{1}, x \geq \mathbf{0}$ is box-TDI, hence P_G is box-TDI.

If G is an odd hole, P_G is the intersection of the polyhedron stated in Theorem 15 with the box $\{x \colon x \geq \mathbf{0}\}$. Theorem 15 and the definition of box-TDI polyhedra imply that P_G is box-TDI.

($\Rightarrow$) Let us show that P_{K_4} is not box-TDI. By definition, $P_{K_4} = \{x : A_{K_4}^\top x \geq \mathbf{1}, x \geq \mathbf{0}\}$, where

$$A_{K_4}^\top = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}.$$

The full row rank matrix formed by the last three rows of $A_{K_4}^\top$, say B, is not equimodular because the determinant of the first three columns is 1, whereas that of the last three is 2. Moreover, the four points $z_1 = (1,0,0,0,0,1)^\top$, $z_2 = (0,1,0,0,1,0)^\top$, $z_3 = (0,0,1,1,0,0)^\top$ and $z_4 = (1,1,1,0,0,0)^\top$ belong to P_G, satisfy $Bx = \mathbf{1}$, and are affinely independent. Therefore, by Observation 2, B is a face-defining matrix of P_{K_4}. This implies that P_{K_4} is not box-TDI by Theorem 1.

To complete the proof there remains to show that P_G is not box-TDI when G is neither quasi-bipartite nor an odd hole. In this case, there exists an odd circuit C such that $G \setminus V(C)$ is nonempty and has no isolated vertices. If C has a chord e, then $C \cup \{e\}$ contains a smaller odd circuit C'. Since $C \setminus C'$ is a path of length at least two, $G \setminus V(C')$ has no isolated vertices. Therefore, we may assume that C is an odd hole.

Let M be the submatrix of $A_G^\top$ formed by the rows associated with the vertices of $V(C)$. By construction, $\delta(V(C))$ is nonempty, hence M contains $\left[A_C^\top, \chi^u\right]$, for some $u \in V(C)$. By Lemma 7, M is not equimodular.

We show that M is face-defining for P_G. Since M has full row rank, by Observation 2 it is sufficient to exhibit $|E| - |V(C)| + 1$ affinely independent points of the face $F = P_G \cap \{x : Mx = \mathbf{1}\}$ of P_G. We exhibit the same points as in the proof of Theorem 15, the difference is that, here, the set U is empty since there are no isolated vertices when removing $V(C)$:

- $p^0 = \frac{1}{2}\chi^C + \chi^K$,
- $p^e = p^0 + \chi^e$, for each $e \in K$,
- $q^{uv} = \chi^{uv} + \chi^{L_u} + \chi^K$, for each $uv \in \delta(V(C))$ with $u \in V(C)$.

As shown in the proof of Theorem 15, these points are affinely independent and satisfy $A_G^\top x \geq \mathbf{1}, Mx = \mathbf{1}$. Since these points also satisfy $x \geq \mathbf{0}$, they belong to the face $P_G \cap \{x \colon Mx = \mathbf{1}\}$ for which M is a face-defining matrix. By Theorem 1, P_G is not box-TDI. $\square$

Theorem 16 implies that recognizing box-TDI polyhedra is co-NP-complete since recognizing quasi-bipartite graphs is [9].

Corollary 17. *Recognizing box-TDI polyhedra is co-NP-complete.*

Conclusion

In this paper, we provide two hardness results regarding box-TDI polyhedra, and their proofs are based on the exhibition of new classes of binary TE matrices. A natural subsequent question is the characterization of binary TE matrices, and further that of $\{0, 1, -1\}$ TE matrices.

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