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The Schrijver system of the flow cone in series–parallel graphs

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ABSTRACT

We represent a *flow* of a graph $G = (V, E)$ as a couple (C, e) with C a circuit of G and e an edge of C, and its incidence vector is the $0/\pm 1$ vector $\chi^{C\setminus e} - \chi^{e}$. The *flow cone* of G is the cone generated by the flows of G and the unit vectors.

When G has no K_5-minor, this cone can be described by the system $x(M) \geq 0$ for all multicuts M of G. We prove that this system is box-totally dual integral if and only if G is series–parallel. Then, we refine this result to provide the Schrijver system describing the flow cone in series–parallel graphs.

This answers a question raised by Chervet et al., (2018).

1. Introduction

Totally dual integral systems were introduced in the late 70s and are strongly connected to min–max relations in combinatorial optimization [16]. A rational system of linear inequalities $Ax \leq b$ is *totally dual integral (TDI)* if the minimization problem in the linear programming duality:

$$\max\{cx : Ax \leq b\} = \min\{yb : y \geq \mathbf{0}, yA = c\}$$

admits an integer optimal solution for each integer vector c such that the maximum is finite. Such systems describe integer polyhedra when b is integer [13]. Schrijver [15] proved that every full-dimensional polyhedron is described by a unique minimal TDI system $Ax \leq b$ with A integer—its *Schrijver system* [6].

A stronger property is the box-total dual integrality, where a system $Ax \leq b$ is *box-totally dual integral (box-TDI)* if

$$Ax \leq b, \quad \ell \leq x \leq u$$

is TDI for all rational vectors ℓ and u (with possible infinite components). General properties of such systems can be found in Cook [5] and Chapter 22.4 of Schrijver [16]. Note that, although every rational polyhedron $\{x : Ax \leq b\}$ is described by a TDI system $\frac{1}{k}Ax \leq \frac{1}{k}b$, for some integer k, not every polyhedron is described by a box-TDI system. A polyhedron

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described by a box-TDI system is called a *box-TDI polyhedron*. As proved by Cook [5], every TDI system describing such a polyhedron is actually box-TDI.

In the last decade, several new box-TDI systems were exhibited. Chen, Ding, and Zang [1] characterized box-Mengerian matroid ports. In [2], they provided a box-TDI system describing the 2-edge-connected spanning subgraph polyhedron for series–parallel graphs. Ding, Tan, and Zang [10] characterized the graphs for which the TDI system of Cunningam and Marsh [9] describing the matching polytope is actually box-TDI. Ding, Zang, and Zhao [11] introduced new subclasses of box-perfect graphs. Cornaz, Grappe, and Lacroix [8] provided several box-TDI systems in series–parallel graphs. Recently, Chervet, Grappe, and Robert [3] gave new geometric characterizations of box-TDI polyhedra.

As mentioned by Pulleyblank [14], it is not uncommon that the minimal integer system and the Schrijver system of a polyhedron coincide. This is the case of the matching polytope and matroid polyhedra. However, this does not hold in general, as shown by Cook [4] and Pulleyblank [14] for the b-matching polyhedron, and by Sebő [18] for the T-join polyhedron.

In this paper, we are interested in TDI, box-TDI, and Schrijver systems for the flow cone of series–parallel graphs. Given a graph $G = (V, E)$, a *flow* of G is a couple (C, e) with C a circuit of G and e an edge of C. In a flow (C, e), the edge e represents a demand and $C \setminus e$ represents the path satisfying this demand. The *incidence vector* of a flow (C, e) is the $0/\pm 1$ vector $\chi^{C\setminus e} - \chi^e$. The *flow cone* of G is the cone generated by the flows of G and the unit vectors χ^e of $\mathbb{R}^E$.

The *cut* $\delta(W)$ is the set of edges having exactly one endpoint in a subset W of V. A *bond* is an inclusionwise minimal nonempty cut. Note that a nonempty cut is the disjoint union of bonds. Given a partition $\{V_1, \ldots, V_k\}$ of V, the set of edges having endpoints in two distinct V_i's is called *multicut* and is denoted by $\delta(V_1, \ldots, V_k)$. The *cut cone* of G is the cone generated by the incidence vectors of the cuts of G. Equivalently, it is the cone generated by the incidence vectors of the bonds of G, or by those of the multicuts of G.

When G has no K_5-minor, the flow cone of G is the polar of the cut cone and is described by $x(C) \geq 0$, for all cuts C of G [19]. Chervet, Grappe, and Robert [3] proved that the flow cone is a box-TDI polyhedron if and only if the graph is series–parallel. Moreover they provided the following box-TDI system:

$$\frac{1}{2}x(B) \geq 0 \quad \text{for all bonds } B \text{ of } G. \tag{1}$$

Quoting them, they "*leave open the question of finding a box-TDI system with integer coefficients, which exists by [16, Theorem 22.6(i)] and [5, Corollary 2.5]*".

Contribution. The goal of this paper is to answer the question of [3] mentioned above. Throughout, the main concept that we use is that of Hilbert basis, whose definition and connection with TDIness are given at the end of the introduction.

We first prove that

$$x(M) \geq 0 \quad \text{for all multicuts } M \text{ of } G, \tag{2}$$

is a TDI system describing the flow cone if and only if the graph is series–parallel. As the flow cone is a box-TDI polyhedron for such graphs, this implies that System (2) is a box-TDI system if and only if the graph is series–parallel. We then refine this result by providing the corresponding Schrijver system, which is composed of the so-called chordal multicuts—see Corollary 3.4.

This completely answers the question of [3].

Outline. In the next paragraph, we provide definitions and notation. In Section 2, we first characterize the graphs for which multicuts form a Hilbert basis. It follows that System (2) is box-TDI precisely for series–parallel graphs. In Section 3, we provide a minimal integer Hilbert basis for multicuts in series–parallel graphs. This gives the Schrijver system for the flow cone in series–parallel graphs.

Definitions. Given a finite set S and a subset T of S, we denote by $\chi^T \in \{0, 1\}^S$ the incidence vector of T, that is χ^T_s equals 1 if s belongs to T and 0 otherwise, for all $s \in S$. Since there is a bijection between sets and their incidence vectors, we will often use the same terminology for both.

Let $G = (V, E)$ be a loopless undirected graph. Given $U \subseteq V$, the graph $G[U]$ is obtained from G by removing all the vertices not in U. A set of edges M is a multicut if and only if $|M \cap C| \neq 1$ for all circuits C of G—see e.g. [7]. The *reduced graph* of a multicut M is the graph G_M obtained by contracting all the edges of $E \setminus M$. Note that a multicut of G_M is also a multicut of G. We denote respectively by $\mathcal{M}_G$ and $\mathcal{B}_G$ the set of multicuts and the set of bonds of G. A subset of edges of G is called a *circuit* if it induces a connected graph in which every vertex has degree 2. Given a circuit C, an edge of G is a *chord* of C if its endpoints are two nonadjacent vertices of C. A graph is 2-connected if it remains connected whenever a vertex is removed.

A graph is *series–parallel* if its 2-connected components either consist of a single edge or can be constructed from the circuit of length two C_2 by repeatedly adding edges parallel to an existing one, and subdividing edges, that is, replacing an edge by a path of length two. Series–parallel graphs are those having no K_4-minor [12]. A graph is *chordal* if every circuit of length 4 or more has a chord.

The cone $\mathcal{C}$ *generated* by a set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ of $\mathbb{R}^n$ is the set of nonnegative combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_k$, that is, $\mathcal{C} = \left\{\sum_{j=1}^k \lambda_j \mathbf{v}_j : \lambda_1, \ldots, \lambda_k \geq 0\right\}$. A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a *Hilbert basis* if each integer vector in their cone can

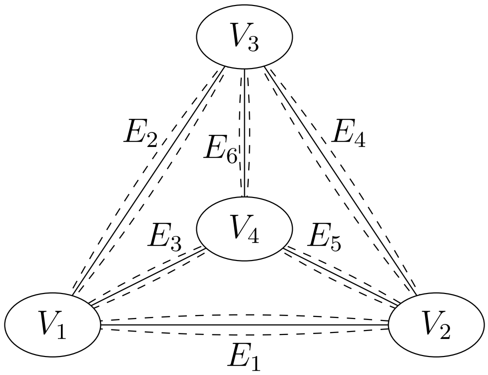

Fig. 1. Edges in the figure represent sets of edges of G having endpoints in distinct V_i's. Solid lines depict $e_1, \ldots, e_6$ given in the proof of Theorem 2.1.

be expressed as a nonnegative integer combination of $\mathbf{v}_1, \ldots, \mathbf{v}_k$. A Hilbert basis is *integer* if it is composed of integer vectors, and it is a *minimal integer Hilbert basis* if it has the smallest number of vectors among all integer Hilbert basis generating the same cone. Each pointed rational cone has a unique minimal integer Hilbert basis [15, Theorems 16.4]. The link between Hilbert basis and TDIness is in the following result.

Theorem 1.1 (*Corollary 22.5a of [16]*). *A system $Ax \geq \mathbf{0}$ is TDI if and only if the rows of A form a Hilbert basis.*

2. When do multicuts form a Hilbert basis?

2.1. Characterization

The following result characterizes the graphs for which the multicuts form a Hilbert basis.

Theorem 2.1. *The multicuts of a graph form a Hilbert basis if and only if the graph is series–parallel.*

Proof. First, let us show that the incidence vectors of the multicuts of a non series–parallel graph do not form a Hilbert basis. Suppose that $G = (V, E)$ has K_4 as a minor. Without loss of generality, we may assume G connected. Then, V can be partitioned into four sets $\{V_1, \ldots, V_4\}$ such that V_i induces a connected subgraph and at least one edge connects each pair V_i, V_j for $i, j = 1, \ldots, 4$. We subdivide $\delta(V_1, V_2, V_3, V_4)$ into $E_1, \ldots, E_6$ as in Fig. 1.

Let $\hat{E} = \{e_1, \ldots, e_6\}$ where $e_i \in E_i$ for all $i = 1, \ldots, 6$, and let $\mathbf{w} \in \mathbb{Z}^E$ be as follows:

$$\mathbf{w}_e = \begin{cases} 2 & \text{if } e \in E_1, \\ 1 & \text{if } e \in E_2, \ldots, E_6, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\mathbf{w} = \frac{1}{2}\chi^{\delta(V_1)} + \frac{1}{2}\chi^{\delta(V_2)} + \frac{1}{2}\chi^{\delta(V_1 \cup V_3)} + \frac{1}{2}\chi^{\delta(V_1 \cup V_4)}$, it belongs to the cut cone of G. Moreover, $\mathbf{w}^\top \chi^{\hat{E}} = 7$. Any conic combination of multicuts yielding $\mathbf{w}$ involves only multicuts contained in $\delta(V_1, \ldots, V_4)$. Each of these multicuts contains between 3 and 6 edges of $\hat{E}$. Hence, if $\mathbf{w}$ is an integer combination of such multicuts, it is the sum of two multicuts containing 3 and 4 edges of $\hat{E}$, respectively. This means that $\mathbf{w}$ is the sum of $\chi^{\delta(V_i)}$ and $\chi^{\delta(V_i, V_j)}$ for some $i \neq j$. Since $\mathbf{w}_{e_1} = 2$, we have $i \in \{1, 2\}$ and $j \in \{3, 4\}$. But then $\delta(V_i) \cap \delta(V_i, V_j)$ contains an edge among $e_2, \ldots, e_5$, a contradiction with $\mathbf{w}_{e_2} = \mathbf{w}_{e_3} = \mathbf{w}_{e_4} = \mathbf{w}_{e_5} = 1$.

Therefore, $\mathbf{w}$ is not an integer combination of multicuts, implying that the set of multicuts of G is not a Hilbert basis.

For the other direction, remark that each multicut of a series–parallel graph is the disjoint union of multicuts of its 2-connected components. Since they belong to disjoint spaces, if the set of multicuts of each 2-connected component forms a Hilbert basis, then so does their union. Hence, it is enough to prove that the multicuts of a 2-connected series–parallel graph form a Hilbert basis. From now on, assume the graph to be 2-connected.

We prove the result by induction on the number of edges of G. When $G = (\{u, v\}, \{e, f\})$ is the circuit of length two, the only nonempty multicut is $\{e, f\}$, and its incidence vector forms a Hilbert basis. Similarly, when G consists of a single edge, its incidence vector forms a Hilbert basis.

Now, let $\tilde{G} = (\tilde{V}, \tilde{E})$ be obtained from a 2-connected series–parallel graph $G = (V, E)$ by either adding a parallel edge or subdividing an edge. By the induction hypothesis, $\mathcal{M}_G$ is a Hilbert basis.

Suppose first that $\tilde{G}$ is obtained from G by adding an edge f parallel to an edge e of E. A subset of edges M of G containing (respectively not containing) e is a multicut if and only if $M \cup f$ (respectively M) is a multicut of $\tilde{G}$. Thus, the incidence vector of each multicut of $\tilde{G}$ is obtained by copying the component associated with e in the component of f. Since the incidence vectors of the multicuts of G are a Hilbert basis, so are the incidence vectors of the multicuts of $\tilde{G}$.

Suppose now that $\tilde{G}$ is obtained from G by subdividing an edge $\bar{e} \in E$. We denote by u the new vertex and by f and g the edges adjacent to it. A multicut M of $\tilde{G}$ can be expressed as the half-sum of the bonds of $\tilde{G}$. Moreover, as each bond is a multicut, bonds and multicuts of $\tilde{G}$ generate the same cone: the cut cone. Since System (1) is TDI in series–parallel graphs [3, end of Section 6.4], the set of vectors $\{\frac{1}{2}\chi^B : B \in \mathcal{B}_{\tilde{G}}\}$ forms a Hilbert basis.

Let $\mathbf{v}$ be an integer vector in the cut cone. There exist $\lambda_B \in \frac{1}{2}\mathbb{Z}_+$ for all $B \in \mathcal{B}_{\tilde{G}}$ such that $\mathbf{v} = \sum_{B \in \mathcal{B}_{\tilde{G}}} \lambda_B \chi^B$. The vector $\mathbf{v}$ is an integer combination of multicuts of $\tilde{G}$ if and only if $\mathbf{v} - \lfloor \lambda_{\delta(u)} \rfloor \chi^{\delta(u)}$ is, thus we may assume that $\lambda_{\delta(u)} \in \{0, \frac{1}{2}\}$. Define $\mathbf{w} \in \mathbb{Z}^E$ by:

$$\mathbf{w}_e = \begin{cases} \mathbf{v}_f + \mathbf{v}_g - 2\lambda_{\delta(u)} & \text{if } e = \bar{e}, \\ \mathbf{v}_e & \text{otherwise.} \end{cases}$$

Remark that $(B \setminus \bar{e}) \cup f$ and $(B \setminus \bar{e}) \cup g$ are bonds of $\tilde{G}$ whenever B is a bond of G containing $\bar{e}$. Moreover, a bond B of G which does not contain $\bar{e}$ is a bond of $\tilde{G}$. Since $\delta(u)$ is the unique bond of $\tilde{G}$ containing both f and g, we have:

$$\mathbf{w} = \sum_{B \in \mathcal{B}_G : \bar{e} \in B} (\lambda_{(B\setminus\bar{e})\cup f} + \lambda_{(B\setminus\bar{e})\cup g})\chi^B + \sum_{B \in \mathcal{B}_G : \bar{e} \notin B} \lambda_B \chi^B.$$

Thus, $\mathbf{w}$ belongs to the cut cone of G. Moreover, as $\lambda_{\delta(u)}$ is half-integer, $\mathbf{w}$ is integer. By the induction hypothesis, $\mathcal{M}_G$ is a Hilbert basis, hence there exist $\mu_M \in \mathbb{Z}_+$ for all $M \in \mathcal{M}_G$ such that $\mathbf{w} = \sum_{M \in \mathcal{M}_G} \mu_M \chi^M$. Consider the family $\mathcal{N}$ of multicuts of G where each multicut M of G appears μ_M times.

Suppose first that $\lambda_{\delta(u)} = 0$. Then, $\mathbf{v}_f + \mathbf{v}_g$ multicuts of $\mathcal{N}$ contain $\bar{e}$. Let $\mathcal{P}$ be a family of $\mathbf{v}_f$ multicuts of $\mathcal{N}$ containing $\bar{e}$ and $\mathcal{Q} = \{M \in \mathcal{N} : \bar{e} \in M\} \setminus \mathcal{P}$. Then, we have

$$\mathbf{v} = \sum_{M \in \mathcal{N} : \bar{e} \notin M} \chi^M + \sum_{M \in \mathcal{P}} \chi^{(M\setminus\bar{e})\cup f} + \sum_{M \in \mathcal{Q}} \chi^{(M\setminus\bar{e})\cup g},$$

hence $\mathbf{v}$ is a nonnegative integer combination of multicuts of $\tilde{G}$.

Suppose now that $\lambda_{\delta(u)} = \frac{1}{2}$. Then, $\mathbf{v}_f + \mathbf{v}_g - 1$ multicuts of $\mathcal{N}$ contain $\bar{e}$. Let $\mathcal{P}$ be a family of $\mathbf{v}_f - 1$ multicuts of $\mathcal{N}$ containing $\bar{e}$, let $\mathcal{Q}$ be a family of $\mathbf{v}_g - 1$ multicuts in $\{M \in \mathcal{N} : \bar{e} \in M\} \setminus \mathcal{P}$, and denote by N the unique multicut of $\mathcal{N}$ containing $\bar{e}$ which is not in $\mathcal{P} \cup \mathcal{Q}$. Then, we have

$$\mathbf{v} = \sum_{M \in \mathcal{N} : \bar{e} \notin M} \chi^M + \sum_{M \in \mathcal{P}} \chi^{(M\setminus\bar{e})\cup f} + \sum_{M \in \mathcal{Q}} \chi^{(M\setminus\bar{e})\cup g} + \chi^{N\setminus\bar{e}\cup\{f,g\}}.$$

Hence $\mathbf{v}$ is a nonnegative integer combination of multicuts of $\tilde{G}$. This proves that $\mathcal{M}_{\tilde{G}}$ is a Hilbert basis. □

2.2. An integer box-TDI system for the flow cone in series–parallel graphs

Combining the box-TDIness of the flow cone and Theorems 1.1 and 2.1 yields a box-TDI system for the flow cone of a series–parallel graph with only integer coefficients. This provides a first answer to the question of [3].

Corollary 2.2. *The following statements are equivalent:*

i. G is a series–parallel graph,
ii. System (2) is TDI,
iii. System (2) is box-TDI.

Proof (*i.*⇔*ii.*)**.** This equivalence follows by combining Theorems 1.1 and 2.1.

(*ii.*⇔*iii.*) If G is series–parallel, then System (1) is box-TDI [3, end of Section 6.4]. Hence, the flow cone of G is box-TDI. Since a TDI system describing a box-TDI polyhedron is a box-TDI system [5], point *ii.* implies point *iii.*. A box-TDI system being TDI by definition, point *iii.* implies point *ii.*. □

3. Which multicuts form Hilbert basis?

3.1. A minimal integer Hilbert basis

Theorem 2.1 provides the set of graphs whose multicuts form a Hilbert basis. The following theorem refines this result by characterizing the multicuts which form the minimal Hilbert basis.

A multicut is *chordal* when its reduced graph is 2-connected and chordal. Note that bonds are chordal multicuts.

Theorem 3.1. *The chordal multicuts of a series–parallel graph form a minimal integer Hilbert basis.*

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Proof. Let $G = (V, E)$ be a series–parallel graph. By Theorem 2.1, the multicuts of G form an integer Hilbert basis. Hence, the minimal integer Hilbert basis is composed of the multicuts which are not disjoint union of other multicuts. These multicuts are characterized in the following lemma, from which stems the desired theorem.

Lemma 3.2. *A multicut of a series–parallel graph G is chordal if and only if it cannot be expressed as the disjoint union of other nonempty multicuts.*

Proof. Let M be a multicut of G. Recall that every multicut of G_M is a multicut of G. Besides, since the disjoint union of multicuts is a multicut, a disjoint union of nonempty multicuts is actually the disjoint union of two nonempty multicuts.

We first prove that, if G_M is 2-connected and chordal, then M is not the disjoint union of two nonempty multicuts. By contradiction, suppose that G_M is 2-connected and chordal, and $M = M_1 \cup M_2$ where M_1, M_2 are disjoint multicuts of G_M. If C is a circuit of length at most three in G_M, then $C \subseteq M_i$ for some $i = 1, 2$. Indeed, the edges of C are partitioned by M_1 and M_2, and a multicut and a circuit intersect in either none or at least two edges.

Since G_M is 2-connected and M_i is nonempty for $i = 1, 2$, there exists at least a circuit containing edges of both M_1 and M_2. Let C be such a circuit, of smallest length. Then, C has length at least 4, as otherwise it would be contained in one of M_1 and M_2. Since G_M is chordal, there exists a chord c of C. Denote by P_1 and P_2 the two paths of C between the endpoints of c. For $i = 1, 2$, the circuit $P_i \cup \{c\}$ is strictly shorter than C. Since C is the shortest circuit intersecting both M_1 and M_2, we get that $P_i \cup \{c\} \subseteq M_i$ for $i = 1, 2$. But then $c \in M_1 \cap M_2$, a contradiction.

To prove the other direction, first suppose that G_M is not 2-connected. Then, the set of edges of each 2-connected component of G_M is a multicut of G, and M is the disjoint union of these multicuts. Now, suppose that G_M is not chordal, that is, G_M contains a chordless circuit C of length at least 4. We will apply the following.

Claim 3.3. *Let C be a circuit of length at least 4 in a series–parallel graph G. Then, there exists a pair of vertices nonadjacent in $G[V(C)]$ whose removal disconnects G.*

Proof. We can assume that there are two nonadjacent vertices u and v of $G[V(C)]$ such that there exists a path P between u and v that has no internal vertex in C. Indeed, otherwise, removing any two nonadjacent vertices of $G[V(C)]$ would disconnect G.

Let us show that removing u and v disconnects G. Denote by Q and R the two paths of C between u and v. By contradiction, suppose that $G\backslash\{u, v\}$ is connected. Then, there exists a path containing neither u nor v between an internal vertex of R and an internal vertex of either P or Q. Let S be a minimal path of this kind. Then, no internal vertex of S belongs to P, Q, or R, and the subgraph composed of P, Q, R and S is a subdivision of K_4. This contradicts the hypothesis that G is series–parallel. □

By Claim 3.3 there exist two vertices u and v of C, nonadjacent in $G[V(C)]$, whose removal disconnects G. Denote by $V_1, \dots, V_k$ the sets of vertices of the connected components of $G \setminus \{u, v\}$. Let $G_i = G[V_i \cup \{u, v\}]$ and denote by $E(G_i)$ the set of edges of G_i, for $i = 1, \dots, k$. Note that, since u and v are not adjacent, $E(G_i) \cap E(G_j) = \emptyset$ for all distinct i and j. Thus, M is the disjoint union of $E(G_1), \dots, E(G_k)$.

Let us prove that $E(G_i)$ is a multicut of G_M, for $i = 1, \dots, k$. Consider a circuit D of G_M. If D is contained in one of the G_i's, then $|D \cap E(G_j)| \neq 1$ for $j = 1, \dots, k$. Otherwise, D is the union of two paths from u to v, these paths being contained in two different G_i's. Without loss of generality, let these paths be $P_1 \in G_1$ and $P_2 \in G_2$. Then, we have $D \cap E(G_i) = P_i$ if $i = 1, 2$, and $\emptyset$ otherwise. Since u and v are not adjacent, the shortest path from u to v in each G_i is of length at least two, hence $|P_i| \geq 2$. Therefore $|D \cap E(G_i)| \neq 1$ for $i = 1, \dots, k$.

Therefore, $E(G_i)$ is a multicut of G_M, and hence of G, for $i = 1, \dots, k$. Hence, M is the disjoint union of multicuts of G. □

□

3.2. The Schrijver system of the flow cone in series–parallel graphs

Corollary 2.2 provides an integer box-TDI description of the flow cone in series–parallel graphs. However, this box-TDI description is not minimal: there are redundant inequalities whose removal preserves box-TDIness. Here, we provide the minimal integer box-TDI system for this cone. This completely answers the question of [3, end of Section 6.4].

Corollary 3.4. *The Schrijver system for the flow cone of a series–parallel graph G is the following:*

$$x(M) \geq 0 \quad \text{for all chordal multicuts } M \text{ of } G. \tag{3}$$

Moreover, this system is box-TDI.

Proof. By Theorems 1.1 and 3.1, System (3) is a minimal integer TDI system. Since every bond is a chordal multicut, this system describes the flow cone for series–parallel graphs. Therefore, by [5, Corollary 2.5] and by the flow cone being box-TDI for series–parallel graphs, System (3) is box-TDI. □

We mention that, by planar duality, Corollary 3.4 provides the Schrijver system for the cone of conservative functions [17, Corollary 29.2h] in series–parallel graphs.

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