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A weak box-perfect graph theorem

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ABSTRACT

A graph G is called *perfect* if $\omega(H) = \chi(H)$ for every induced subgraph H of G, where $\omega(H)$ is the clique number of H and $\chi(H)$ its chromatic number. The Weak Perfect Graph Theorem of Lovász states that a graph G is perfect if and only if its complement $\overline{G}$ is perfect. This does not hold for box-perfect graphs, which are the perfect graphs whose stable set polytope is box-totally dual integral.

We prove that both G and $\overline{G}$ are box-perfect if and only if $\overline{G}^+$ is box-perfect, where G^+ is obtained by adding a universal vertex to G. Consequently, G^+ is box-perfect if and only if $\overline{G}^+$ is box-perfect. As a corollary, we characterize when the complete join of two graphs is box-perfect.

In a graph, a *clique* is a set of pairwise adjacent vertices, and a *stable set* is a set of pairwise nonadjacent vertices. The *stable set polytope* $S(G)$ of a graph G is the convex hull of the incidence vectors of its stable sets. A graph G is called *perfect* if $\omega(H) = \chi(H)$ for every induced subgraph H of G, where $\omega(H)$ is the clique number and $\chi(H)$ the chromatic number of H. Lovász [9] proved the Weak Perfect Graph The-

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orem, which states that a graph G is perfect if and only if its complement $\overline{G}$ is perfect. It is also known [4,8] that perfect graphs are the graphs whose stable set polytope is described by the system composed of the clique inequalities and the nonnegativity constraints:

$$\begin{cases} \sum_{v \in C} x_v \leq 1 & \text{for each maximal clique } C \text{ of } G, \\ x \geq \mathbf{0}. \end{cases} \tag{1}$$

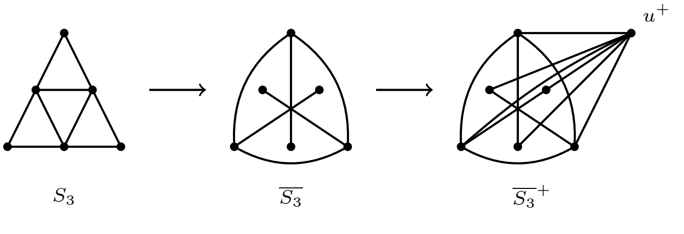
In fact, system (1) is totally dual integral if and only if G is perfect. A rational system of linear inequalities $Ax \leq b$ is *totally dual integral* (*TDI*) if the minimization problem in the linear programming duality:

$$\max\{c^\top x \colon Ax \leq b\} = \min\{b^\top y \colon A^\top y = c, y \geq \mathbf{0}\}$$

admits an integer optimal solution for each integer vector c such that the maximum is finite. A system $Ax \leq b$ is *box-totally dual integral* [7] (*box-TDI*) if $Ax \leq b$, $\ell \leq x \leq u$ is TDI for all rational vectors ℓ and u (with possible infinite components), and *box-TDI* polyhedra [5] are those that can be described by a box-TDI system. TDI and box-TDI systems were introduced in the late 1970's and serve as a general framework for establishing various min-max relations in combinatorial optimization [10].

A *box-perfect graph* is a graph for which system (1) is box-TDI. Equivalently, a graph is box-perfect if and only if it is perfect and its stable set polytope is box-TDI. The characterization of box-perfect graphs is a longstanding open question raised by Cameron and Edmonds in 1982 [1]. Mix-max relations about box-perfect graphs are discussed in [2]. Recent progress has been made on this topic by Ding, Zang, and Zhao [6]. They exhibit several new subclasses of perfect graphs and in particular prove the conjecture of Cameron and Edmonds [1] that parity graphs are box-perfect.

The Weak Perfect Graph Theorem does not hold for box-perfect graphs, as shown by S_3 below, which is not box-perfect (see e.g. [3, Section 6.2]) but whose complement $\overline{S_3}$ is. Adding a universal vertex u^+ to this complement destroys its box-perfection, that is, $\overline{S_3}^+$ is not box-perfect. Here, G^+ denotes the graph obtained from a graph G by adding a *universal* vertex, which is a new vertex connected to all the vertices of G.



We prove here that this holds in general. More precisely, we prove the following.

Theorem 1. *Given a graph $G = (V, E)$, the following statements are equivalent.*

1. *Both G and $\overline{G}$ are box-perfect,*
2. *$\overline{G}^+$ is box-perfect,*
3. *G^+ is box-perfect.*

Our starting point builds upon recent characterizations of box-TDI polyhedra [3]. In the context of box-perfect graphs, the combination of [3, Theorem 2] and [3, Observation 4] yields Theorem 2 below, for which a few definitions are required.

A subset U of V is also viewed as the row vector $(\chi^U)^\top$, where $\chi^U \in \{0,1\}^V$ denotes the incidence vector of U. A set of subsets of V is then viewed as a matrix whose rows correspond to those subsets. For a set C of columns and a set R of rows of a matrix M, we denote by M^C the submatrix of M formed by the columns in C, and by M_R and the submatrix of M formed by the rows in R.

A rational $r \times n$ matrix is *equimodular* if it has full row rank and its nonzero $r \times r$ determinants all have the same absolute value. A *face-defining pair* of a graph G is a pair $(\mathcal{K}, \mathcal{S})$, where $\mathcal{K}$ is a set of linearly independent cliques, $\mathcal{S}$ is a set of affinely independent stable sets, each clique of $\mathcal{K}$ intersects each stable set of $\mathcal{S}$, and[1] $|\mathcal{K}| + |\mathcal{S}| = |V| + 1$. Such a pair is *equimodular* when the matrix whose rows are the cliques of $\mathcal{K}$ is equimodular. Equivalently, as explained below, the matrix whose rows are $(\chi^T - \chi^S)^\top$, $T \in \mathcal{S} \setminus \{S\}$ is equimodular for each[2] $S \in \mathcal{S}$.

Theorem 2. *A perfect graph is box-perfect if and only if all its face-defining pairs are equimodular.*

A *face-defining matrix*[3] of a polyhedron $P = \{x : Ax \le b\}$ is a linearly independent set A_R of rows of A such that the affine hull of some face F of P can be written $\{x : A_R x = b_R\}$. [3, Theorem 2] asserts that a polyhedron is box-TDI if and only if all its face-defining matrices are equimodular. By [3, Observation 4], the cardinality and independence conditions on a face-defining pair $(\mathcal{K}, \mathcal{S})$ ensure that $\mathcal{K}$ is face-defining for the stable set polytope. When the graph is perfect, the face-defining pairs encode all the face-defining matrices of system (1) without nonnegativity constraints. The fact that nonnegativity constraints need not be considered in Theorem 2 relies on the following: if $F \cap \{x \ge \mathbf{0}\}$ is not box-TDI for some face F of P, then neither is F.

We will use that a face-defining pair $(\mathcal{K}, \mathcal{S})$ is equimodular if and only if the matrix whose rows are $(\chi^T - \chi^S)^\top$, $T \in \mathcal{S} \setminus \{S\}$ is equimodular for each $S \in \mathcal{S}$. Indeed,

[1] At this point, $|\mathcal{K}| + |\mathcal{S}| \le |V| + 1$ always holds by geometric arguments.
[2] Here, *each* can be replaced by *some*, see [3, Corollary 6].
[3] Compared to [3], face-defining matrices here are “from the system”. It is implicit therein that [3, Theorem 2] also holds under these settings.

when $(\mathcal{K}, \mathcal{S})$ is a face-defining pair, $\mathcal{K}$ is a face-defining matrix of the affine hull of $\mathcal{S}$. By [3, Theorem 2], since the latter has only itself as a face, it is box-TDI if and only if $\mathcal{K}$ is equimodular. Statements 2 and 3 of [3, Corollary 6] imply the announced equivalence as the vectors $(\chi^T - \chi^S)$ for $T \in \mathcal{S} \setminus \{S\}$ form a basis of the associated linear space, for each $S \in \mathcal{S}$.

Note that box-perfection is preserved under taking induced subgraphs [2]. Besides, each clique in a face-defining pair can be assumed maximal, because it can be assumed maximal in system (1). We can now prove Theorem 1.

Proof of Theorem 1. Replacing G by $\overline{G}$ shows that it is enough to prove ($2 \Rightarrow 1$) and ($1 \Rightarrow 3$). Moreover, G, $\overline{G}$, G^+, and $\overline{G}^+$ are all perfect as long as one of them is, hence we just have to deal with the box-TDIness of their stable set polytopes. Let u^+ denote the universal vertex of G^+ and $\overline{u}^+$ that of $\overline{G}^+$.

($2 \Rightarrow 1$) Suppose that $\overline{G}^+$ is box-perfect. Then, so is $\overline{G} = \overline{G}^+ \setminus \{\overline{u}^+\}$. To prove that G is box-perfect, by Theorem 2, we show that every face-defining pair $(\mathcal{K}, \mathcal{S})$ of G is equimodular.

Each element of $\overline{\mathcal{K}} = \{S \cup \{\overline{u}^+\} : S \in \mathcal{S}\}$ is a clique of $\overline{G}^+$ and each element of $\overline{\mathcal{S}} = \mathcal{K} \cup \{\{\overline{u}^+\}\}$ is a stable set of $\overline{G}^+$. Let us prove that $(\overline{\mathcal{K}}, \overline{\mathcal{S}})$ forms a face-defining pair of $\overline{G}^+$. Firstly, $\overline{\mathcal{K}}$ is linearly independent because $\mathcal{S}$ is affinely independent and $\overline{\mathcal{K}}$ is obtained from $\mathcal{S}$ by adding a $\mathbf{1}$ column. Secondly, $\overline{\mathcal{S}}$ is affinely independent because it is linearly independent. Thirdly, each stable set of $\overline{\mathcal{S}}$ intersects each clique of $\overline{\mathcal{K}}$. Finally, $|\overline{\mathcal{K}}| + |\overline{\mathcal{S}}| = |V \cup \{\overline{u}^+\}| + 1$, thus $(\overline{\mathcal{K}}, \overline{\mathcal{S}})$ forms a face-defining pair of $\overline{G}^+$.

By Theorem 2, $(\overline{\mathcal{K}}, \overline{\mathcal{S}})$ is equimodular, and so is the matrix whose rows are $(\chi^K - \chi^{\{\overline{u}^+\}})^\top$, for all $K \in \overline{\mathcal{S}} \setminus \{\{\overline{u}^+\}\}$. Removing $\overline{u}^+$'s column from this matrix yields $\mathcal{K}$, hence $(\mathcal{K}, \mathcal{S})$ is equimodular.

($1 \Rightarrow 3$) Suppose that G and $\overline{G}$ are both box-perfect, and let $(\mathcal{K}^+, \mathcal{S}^+)$ be a face-defining pair of G^+ with r cliques in $\mathcal{K}^+$. We may assume that each clique of $\mathcal{K}^+$ is maximal, hence each of them contains u^+. In particular, $(\mathcal{K}^+)^{\{u^+\}} = \mathbf{1}$ and we may assume that $\{\{u^+\}\} \subsetneq \mathcal{S}^+$. Let $\mathcal{K} = \{K \setminus \{u^+\} : K \in \mathcal{K}^+\}$ and $\mathcal{S} = \mathcal{S}^+ \setminus \{\{u^+\}\}$.

Let us prove that $(\mathcal{K}, \mathcal{S})$ forms a face-defining pair of G. In $\mathcal{K}^+$, column u^+ is a linear combination of the columns of $\mathcal{K}$, because $\mathcal{K}\chi^S = \mathbf{1}$ for $S \in \mathcal{S}$. Thus, the linear independence of $\mathcal{K}^+$ implies that of $\mathcal{K}$. The affine independence of $\mathcal{S}$ comes from that of $\mathcal{S}^+$. Since no stable set of $\mathcal{S}$ contains u^+, each $S \in \mathcal{S}$ intersects each $K \in \mathcal{K}$. Finally, $|\mathcal{K}| + |\mathcal{S}| = |V| + 1$. Since G is box-perfect, $(\mathcal{K}, \mathcal{S})$ is equimodular by Theorem 2. Therefore, all the $r \times r$ nonzero determinants of $\mathcal{K}^+$ not containing column u^+ have the same absolute value.

Recall that cliques and stable sets of G are respectively stable sets and cliques of $\overline{G}$, and let us prove that $(\mathcal{S}, \mathcal{K})$ forms a face-defining pair of $\overline{G}$. From the last paragraph, all that remains to show is the linear independence of $\mathcal{S}$, which holds because $\mathcal{S}$ is affinely independent and $\mathcal{S}\chi^K = \mathbf{1}$ for $K \in \mathcal{K}$. Since $\overline{G}$ is box-perfect, $(\mathcal{S}, \mathcal{K})$ is equimodular by Theorem 2. Therefore, for some $K \in \mathcal{K}$, so is the matrix whose rows are $(\chi^L - \chi^K)^\top$

for all $L \in \mathcal{K} \setminus \{K\}$. This matrix is obtained by pivoting in $\mathcal{K}^+$ using L's row in u^+'s column, hence all the $r \times r$ nonzero determinants of $\mathcal{K}^+$ containing column u^+ have the same absolute value.

To prove that $(\mathcal{K}^+, \mathcal{S}^+)$ is equimodular, all that remains to show is that $|\det(B)| = |\det(C)|$ for some nonsingular $r \times r$ submatrices B and C of $\mathcal{K}^+$ with column u^+ in exactly one of them. For $S \in \mathcal{S}$, the columns of $\mathcal{K}^S$ are linearly independent. Since $\mathcal{K}$ has full row rank, $\mathcal{K}^S$ can be completed into a nonsingular $r \times r$ submatrix B of $\mathcal{K}$. The sum of the columns of B associated with S is $\mathbf{1}$, hence replacing one of them by $\mathbf{1}$ does not change the determinant. Reordering the columns provides the desired matrix C. ■

The *complete join* of two graphs G and H is the graph obtained by connecting each vertex of G to each vertex of H. This operation preserves perfection, but not box-perfection. Indeed, $\overline{S_3}^+$ is not box-perfect and is the complete join of two box-perfect graphs, namely $\overline{S_3}$ and a single vertex $\{u\}$.

Corollary 3. *The complete join of G and H is box-perfect if and only if both G^+ and H^+ are box-perfect.*

Proof. Let J be the complete join of G and H. If J is box-perfect, then G^+ and H^+, as induced subgraphs of J, are also box-perfect. Conversely, suppose that G^+ and H^+ are box-perfect. Equivalently, by Theorem 1, $\overline{G}^+$ and $\overline{H}^+$ are box-perfect. Note that $\overline{J}^+$ is the graph obtained from $\overline{G}^+$ and $\overline{H}^+$ by identifying their universal vertex u. Let us prove that $\overline{J}^+$ is box-perfect. Then so is J by Theorem 1, and the proof is done.

By contradiction, suppose that $\overline{J}^+$ is not box-perfect, and let $(\mathcal{K}, \mathcal{S})$ be a nonequimodular face-defining pair of $\overline{J}^+$ given by Theorem 2. We may assume that each clique of $\mathcal{K}$ is maximal, and then u belongs to each of them. Given the structure of the graph, $\mathcal{K}$ is composed of cliques $\mathcal{K}_G$ and $\mathcal{K}_H$ of respectively $\overline{G}^+$ and $\overline{H}^+$, with $K_G \cap K_H = \{u\}$ for all $K_G \in \mathcal{K}_G$ and $K_H \in \mathcal{K}_H$. The latter implies that each nonzero $|\mathcal{K}| \times |\mathcal{K}|$ determinant of $\mathcal{K}$ is the product of a nonzero $|\mathcal{K}_G| \times |\mathcal{K}_G|$ determinant of $\mathcal{K}_G$ by a nonzero $|\mathcal{K}_H| \times |\mathcal{K}_H|$ determinant of $\mathcal{K}_H$. Since $\mathcal{K}$ is not equimodular, at least one of $\mathcal{K}_G$ and $\mathcal{K}_H$ is not equimodular.

Let $\mathcal{S}_G$ be a maximal family of affinely independent stables sets of $\overline{G}$ distinct from $\{u\}$ and intersecting each clique of $\mathcal{K}_G$. Define $\mathcal{S}_H$ similarly. Take $S_G \in \mathcal{S}_G$ and $S_H \in \mathcal{S}_H$ and let $\mathcal{S}' = \{\{u\}\} \cup \{S \cup S_H \text{ for all } S \in \mathcal{S}_G\} \cup \{S_G \cup S \text{ for all } S \in \mathcal{S}_H\}$. Given the structure of the graph, a dimensional analysis shows that $(\mathcal{K}, \mathcal{S}')$, $(\mathcal{K}_G, \{u\} \cup \mathcal{S}_G)$, and $(\mathcal{K}_H, \{u\} \cup \mathcal{S}_H)$ are respectively face-defining pairs of $\overline{J}^+$, $\overline{G}^+$, and $\overline{H}^+$. This contradicts the fact that $\overline{G}^+$ and $\overline{H}^+$ are box-perfect. ■

We mention that the arguments of the last two paragraphs can be adapted to prove that a graph is box-perfect if and only if all its 2-connected components are box-perfect.

Data availability

No data was used for the research described in the article.

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