Characterizations of Box-Totally Dual Integral Polyhedra

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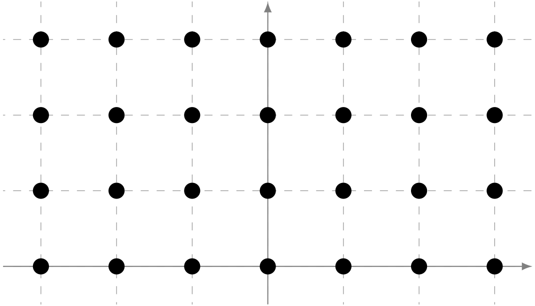
Université du Luxembourg

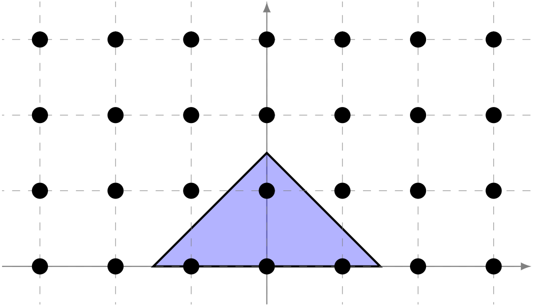
SPOC 22

Principally Box-Integer Polyhedra



Polyhedron: Intersection of a finite number of half-spaces $P = \{x : Ax \leq b\}$

Principally Box-Integer Polyhedra



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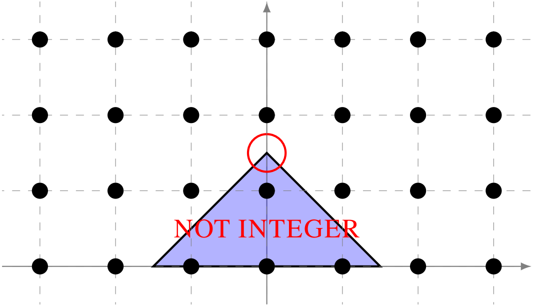

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- Integer: Each face of P contains an integer point

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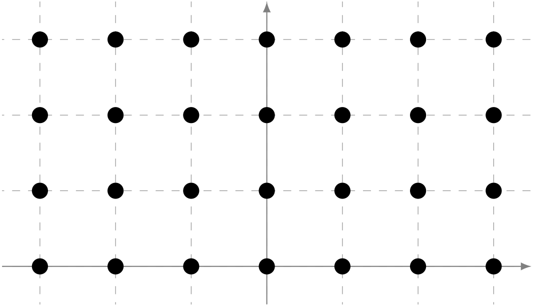

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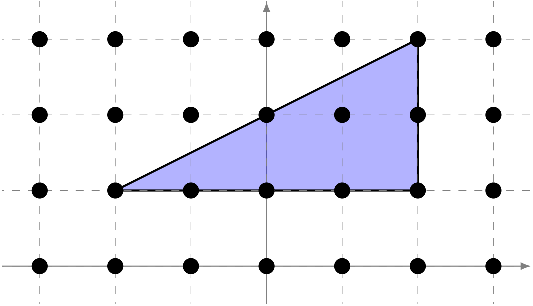

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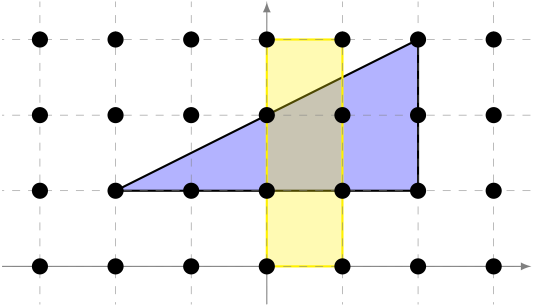

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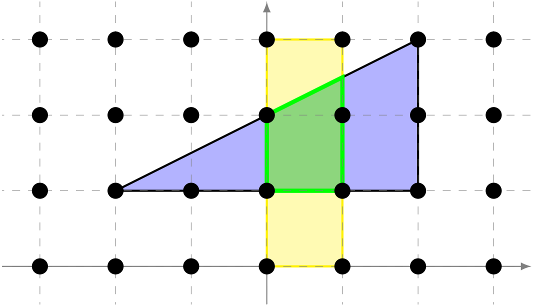

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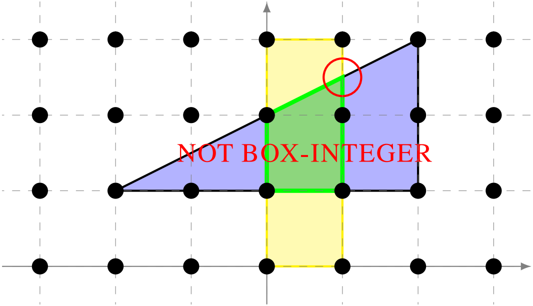

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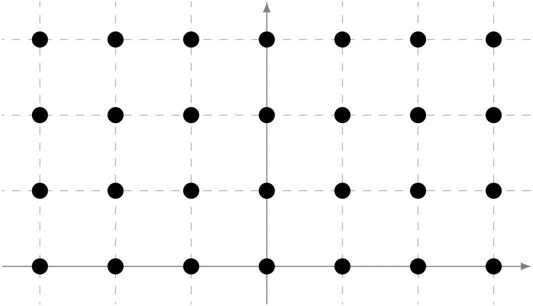

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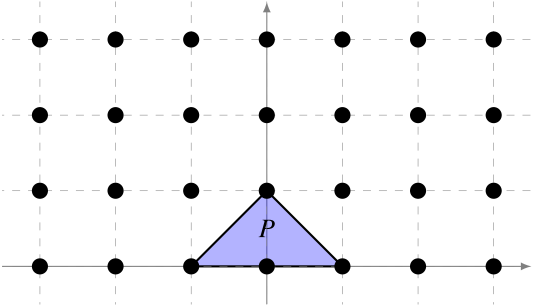

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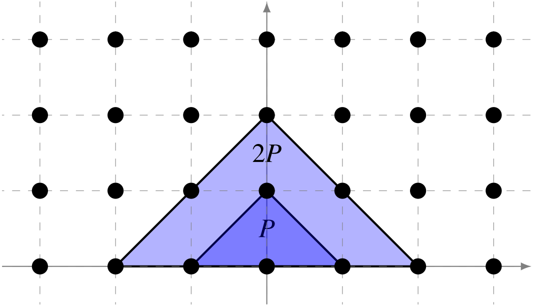

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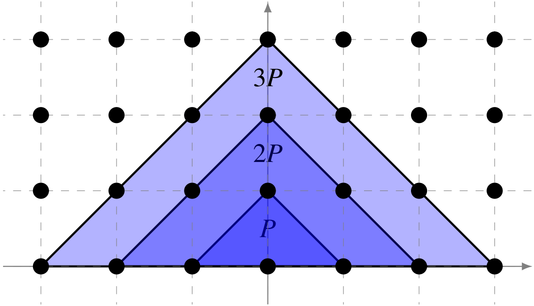

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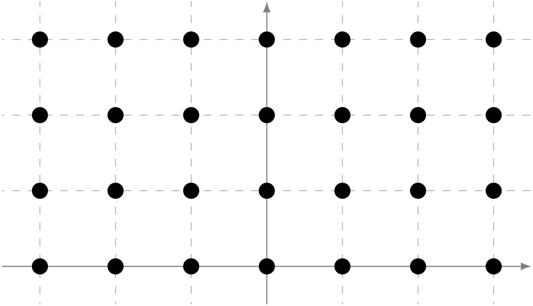

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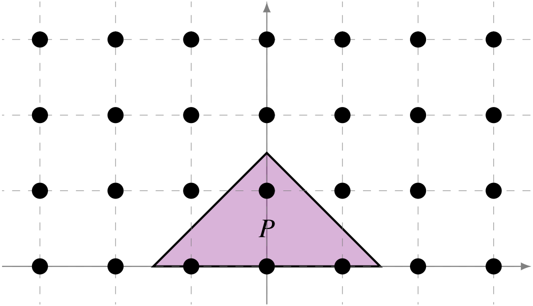

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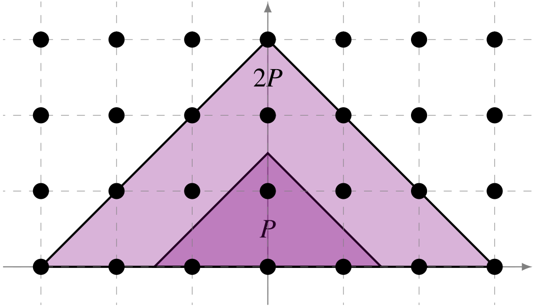

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Outline

Equimodular matrices

Characterizations of Principally Box-Integer Polyhedra

Box-Total Dual Integral Polyhedra

Box-Perfect Graphs

Equimodular Matrices

Full row rank $k \times n$ matrices

$$\begin{bmatrix} 1 & 1 & & -1 \\ 1 & -1 & 1 & -1 \\ & -1 & 1 & \end{bmatrix}$$

det $\square = +1$

Unimodular matrix (integer):

- all $k \times k$ nonzero determinants have absolute value $+1$

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Theorem (Veinott and Dantzig – 1967)

A is unimodular if and only if $\{x : Ax = b\}$ is fully box-integer for all $b \in \mathbb{Z}^k$.

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Proof:

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  - $\Rightarrow |det(D)| = |det(B^{-1}E)| = |\frac{det(E)}{det(B)}| = 1$ or 0
- Veinott and Dantzig's Theorem $\Rightarrow \{x : B^{-1}Ax = B^{-1}\ell b\}$ fully box-integer

Totally Equimodular Matrices

General $m \times n$ matrices

$$\begin{bmatrix} & & 1 & -1 \\ & 1 & & -1 \\ & 1 & -1 & \\ 1 & & -1 & \\ 1 & -1 & & \end{bmatrix}$$

Totally unimodular (TU) matrix:

- every full row rank submatrix is unimodular

Examples:

- network matrices
- incidence matrices of bipartite graphs

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Theorem (Hoffman and Kruskal – 1956)

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$$\begin{bmatrix} 2 & 2 & & -2 \\ 1 & & 1 & \\ & -1 & 1 & 1 \\ 1 & 1 & -1 & \\ & & -1 & -1 \end{bmatrix}$$

Totally equimodular (TE) matrix:

- every full row rank submatrix is equimodular

  $\Rightarrow A$ can be taken $0, \pm 1$

Totally Equimodular Matrices

General $m \times n$ matrices

$$\begin{bmatrix} & & 1 & -1 \\ & 1 & & -1 \\ & 1 & -1 & \\ 1 & & -1 & \\ 1 & -1 & & \end{bmatrix}$$

Totally unimodular (TU) matrix:

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Examples:

- network matrices
- incidence matrices of bipartite graphs

Theorem (Hoffman and Kruskal – 1956)

A is TU if and only if $\{x : Ax \leq b\}$ is fully box-integer for all $b \in \mathbb{Z}^m$.

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  $\Rightarrow A$ can be taken $0, \pm 1$

Theorem (Chervet, G., Robert – 2020)

A is TE if and only if $\{x : Ax \leq b\}$ is principally box-integer for all $b \in \mathbb{Z}^m$.

Outline

Equimodular matrices

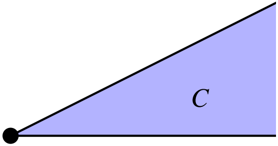
Characterizations of Principally Box-Integer Polyhedra

Box-Total Dual Integral Polyhedra

Box-Perfect Graphs

Principal Box-Integrality VS Equimodularity

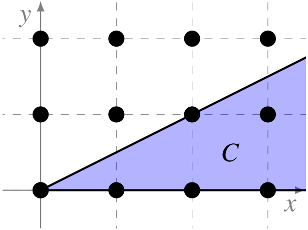
Non Box-Integer Cones



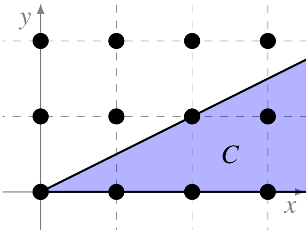
Principal Box-Integrality VS Equimodularity

Non Box-Integer Cones

For a cone $C = \{x : Ax \leq 0\}$:



For a cone $C = \{x : Ax \leq 0\}$: Any dilatation of C is C itself

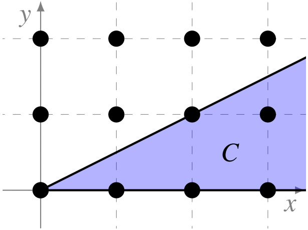


Principal Box-Integrality VS Equimodularity

Non Box-Integer Cones

For a cone $C = \{x : Ax \leq 0\}$: Any dilatation of C is C itself

- box-integrality $\Leftrightarrow$ full box-integrality $\Leftrightarrow$ principal box-integrality



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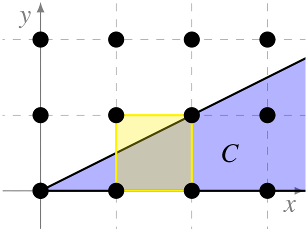

If the cone is NOT box-integer:

Principal Box-Integrality VS Equimodularity

Non Box-Integer Cones

For a cone $C = \{x : Ax \leq 0\}$: Any dilatation of C is C itself

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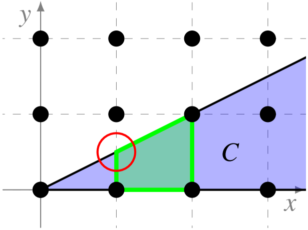

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Principal Box-Integrality VS Equimodularity

Non Box-Integer Cones

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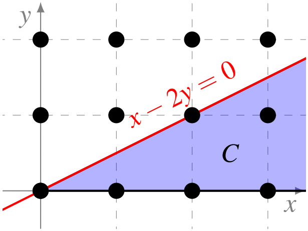


If the cone is NOT box-integer:

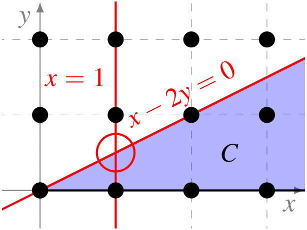
- Facet $H = \{x - 2y = 0\}$

PRINCIPAL BOX-INTEGRALITY VS EQUIMODULARITY

NON BOX-INTEGER CONES

For a cone $C = \{x : Ax \leq 0\}$: Any dilatation of C is C itself

- box-integrality ⇔ full box-integrality ⇔ principal box-integrality



If the cone is NOT box-integer:

- Facet $H = \{x - 2y = 0\}$
- $\left\{ 1 \times x - 2 \times y = 0 \right\} \cap \{x = 1\} = (1, \frac{1}{2})$

Principal Box-Integrality VS Equimodularity

Non Box-Integer Cones

For a cone $C = \{x : Ax \leq 0\}$: Any dilatation of C is C itself

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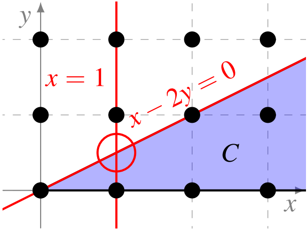

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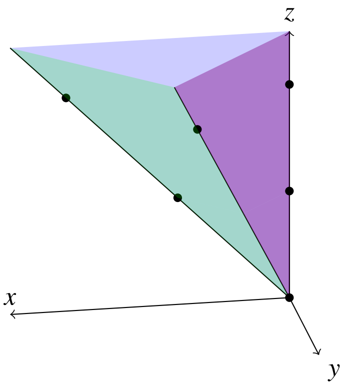
- $\begin{bmatrix} 1 & 2 \end{bmatrix}$ is NOT equimodular

Principal Box-Integrality VS Equimodularity

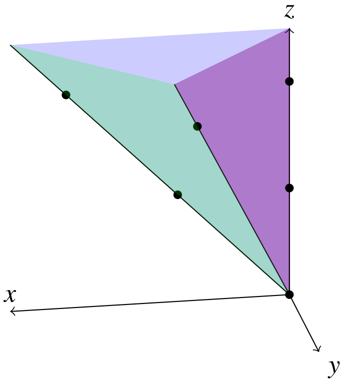
Non Box-Integer Cones



$$\text{Cone} \begin{cases} x - y + z \leq 0 \\ x - y \leq 0 \\ x, y, z \geq 0 \end{cases}$$

Principal Box-Integrality VS Equimodularity

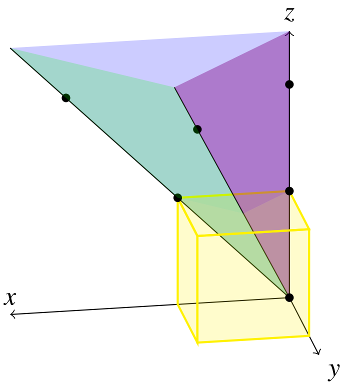
Non Box-Integer Cones



Cone $\begin{cases} x - y + z & \leq & 0 \\ x - y & \leq & 0 \\ x, y, z & \geq & 0 \end{cases}$

Principal Box-Integrality VS Equimodularity

Non Box-Integer Cones



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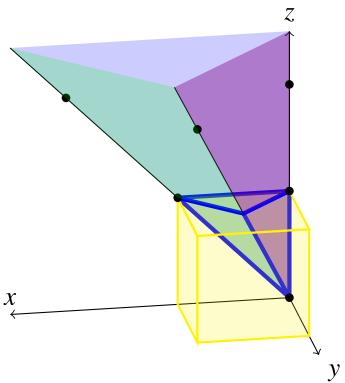

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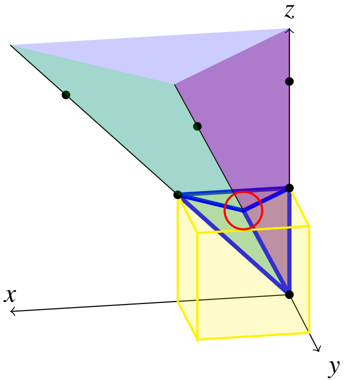

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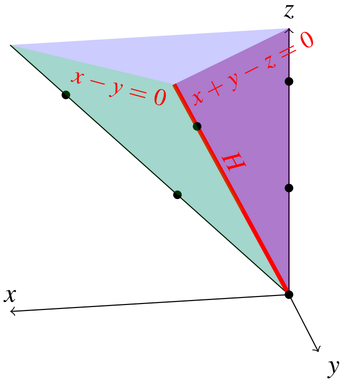
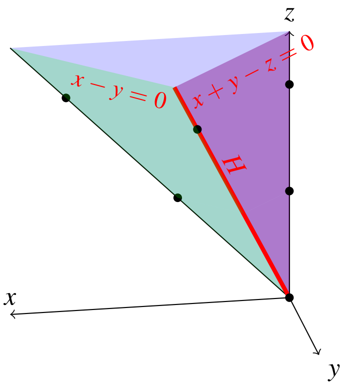


$$\text{Cone}\begin{cases} x - y + z \leq 0 \\ x - y \leq 0 \\ x, y, z \geq 0 \end{cases}$$

- Face $H = \{x + y - z = 0\} \cap \{x - y = 0\}$

PRINCIPAL BOX-INTEGRALITY VS EQUIMODULARITY

NON BOX-INTEGER CONES



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$$\text{Face } H : \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

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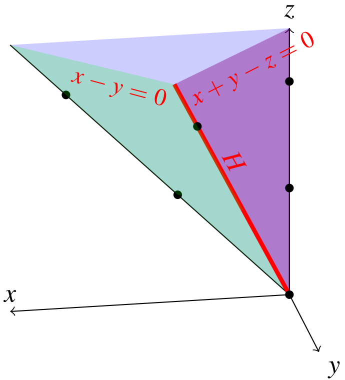

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det $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = -2$

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Principal Box-Integrality VS Equimodularity

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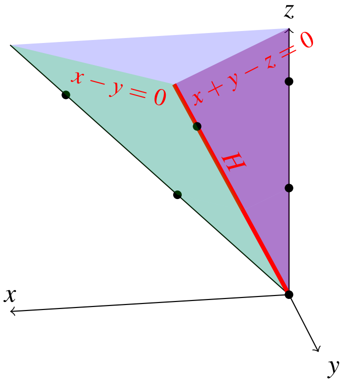

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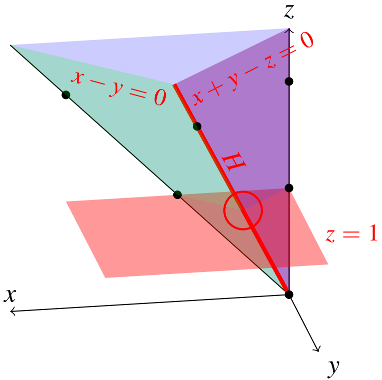

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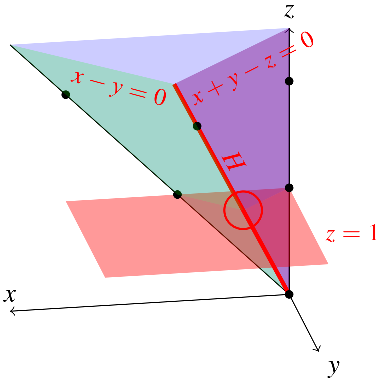

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PRINCIPAL BOX-INTEGRALITY VS EQUIMODULARITY

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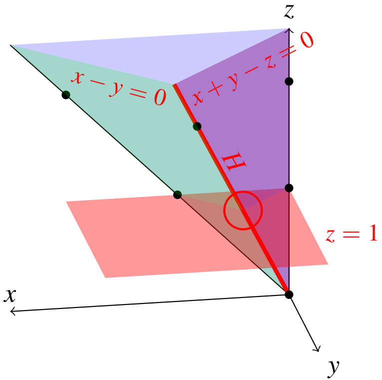

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Principal Box-Integrality VS Equimodularity

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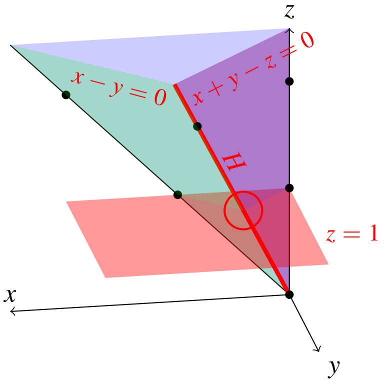

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Principal Box-Integrality VS Equimodularity

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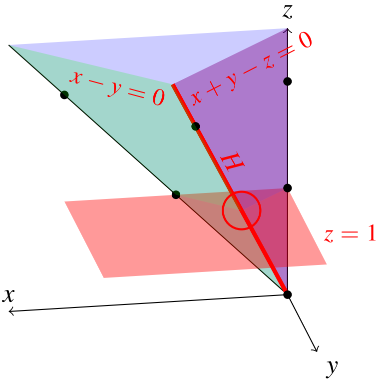

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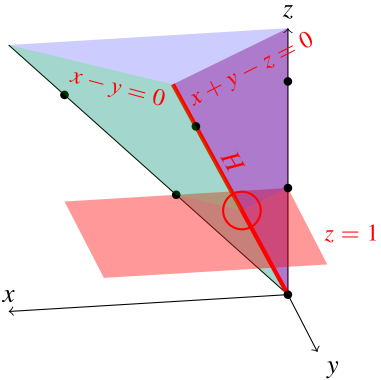
- Multilinearity of determinant $\Rightarrow$ NO equimodular face-defining matrix for H

Principal Box-Integrality VS Equimodularity

Non Box-Integer Cones



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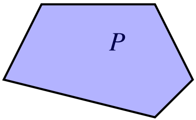
Lemma

A cone is box-integer if and only if all its face-defining matrices are equimodular

Principal Box-Integrality VS Equimodularity

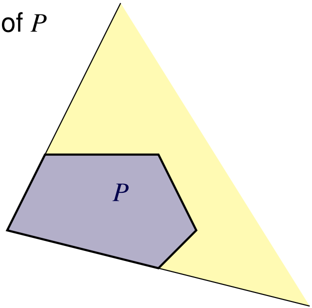
From Cones to Polyhedra

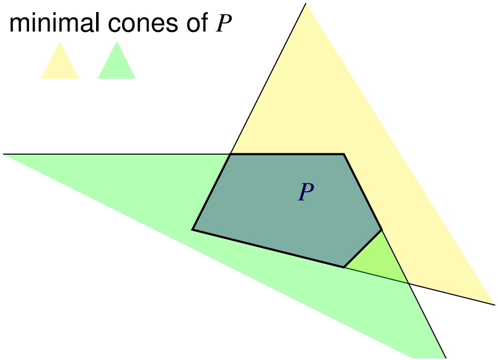


PRINCIPAL BOX-INTEGRALITY VS EQUIMODULARITY

FROM CONES TO POLYHEDRA

minimal cones of P

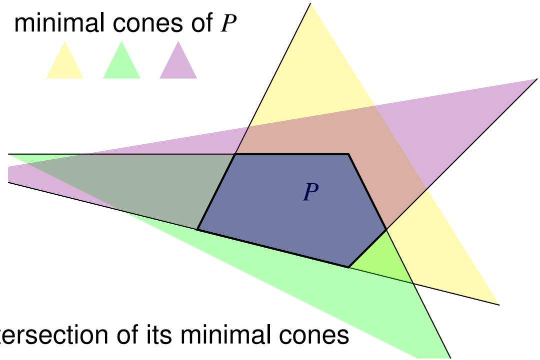




minimal cones of P
P

Principal Box-Integrality VS Equimodularity

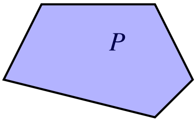
From Cones to Polyhedra



- $P =$ intersection of its minimal cones

Principal Box-Integrality VS Equimodularity

From Cones to Polyhedra



- $P =$ intersection of its minimal cones

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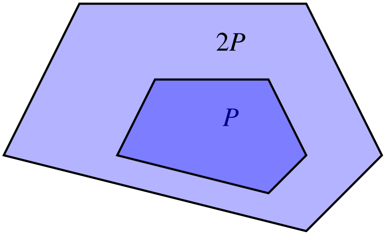

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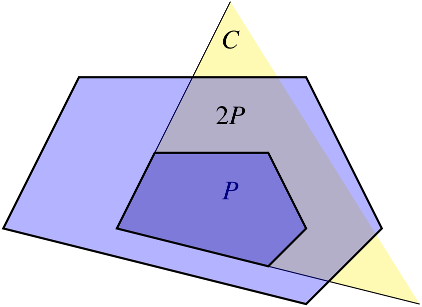

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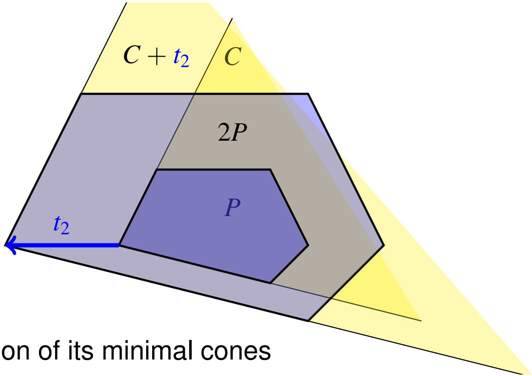

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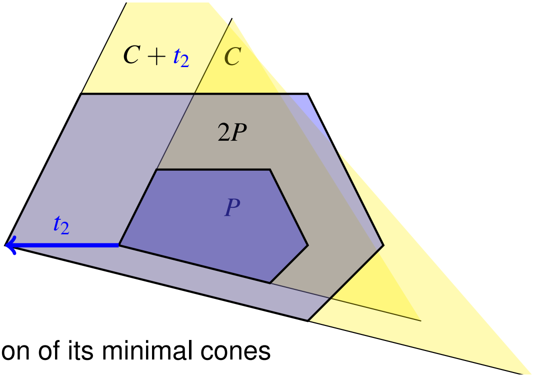

- $P =$ intersection of its minimal cones
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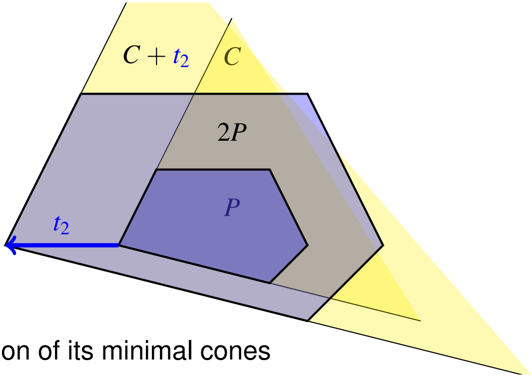

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Cor If a minimal cone of P is not box-integer, then P is not principally box-integer

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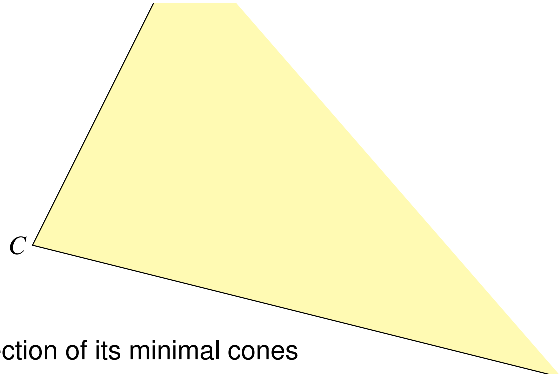

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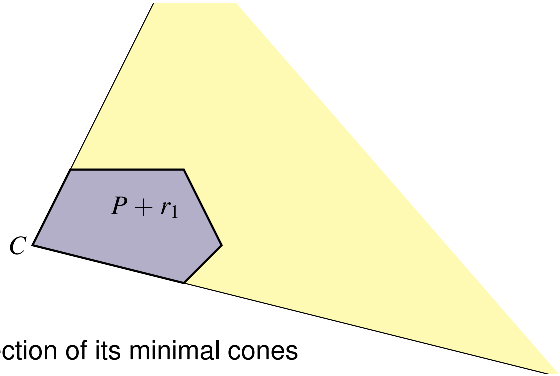

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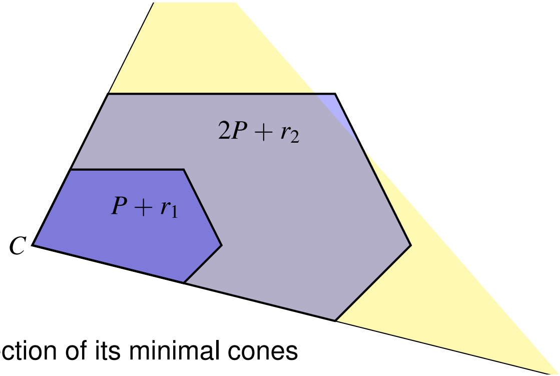

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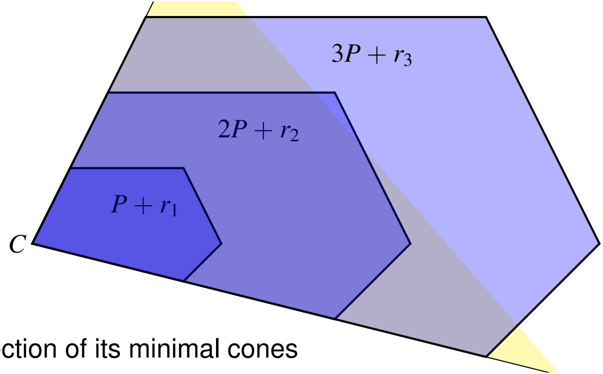

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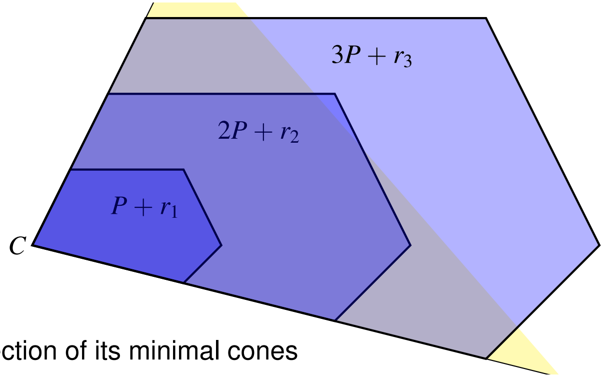

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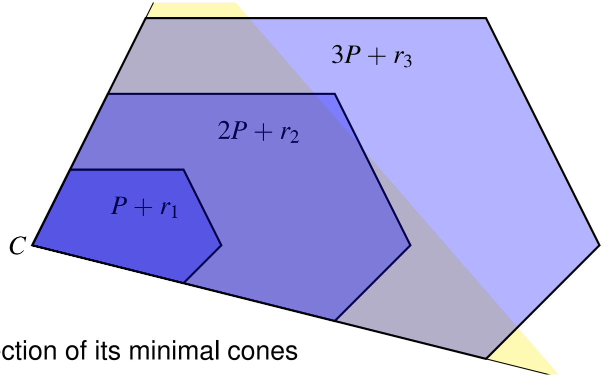

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Principal Box-Integrality VS Equimodularity

Theorem (Chervet, G., Robert – 2020)

Given a polyhedron P, the following statements are equivalent:

1. *P is principally box-integer*
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<u>Proof</u>:

$2 \Leftrightarrow 3$ earlier slide

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$3 \Leftrightarrow 4$ multilinearity of the determinant

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<u>PROOF</u>:

$$B^{-1}A = B^{-1}[B\ C] = \begin{bmatrix} 1 & & 0 & \\ & \ddots & & B^{-1}C \\ 0 & & 1 & \end{bmatrix}$$

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<u>Proof</u>:

$$B^{-1}A = B^{-1}[B\ C] = \left[\begin{array}{ccc|c} 1 & & 0 & \\ & \ddots & & B^{-1}C \\ 0 & & 1 & \end{array}\right]$$

where D denotes the submatrix formed by the last column of the identity block together with $B^{-1}C$.

- $|\det(D)| = 1$ or 0

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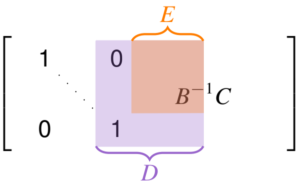
PROOF:

$B^{-1}A = B^{-1}\,[B\ C] =$



- $|\det(D)| = 1$ or 0
- $|\det(D)| = |\det(E)|$

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- $|\det(D)| = 1$ or 0
- $|\det(D)| = |\det(E)|$

$\Rightarrow |\det(E)| = 1$ or 0

Outline

Equimodular matrices

Characterizations of Principally Box-Integer Polyhedra

Box-Total Dual Integral Polyhedra

Box-Perfect Graphs

Total Dual Integral Systems

$$Ax \leq b$$

TOTAL DUAL INTEGRAL SYSTEMS

$$(P)\quad \begin{array}{ll} \text{min} & c^\top x \\ \text{s.t.} & Ax \leq \; b \end{array}$$

Total Dual Integral Systems

$$(P)\quad \begin{array}{ll} \min & c^\top x \\ \text{s.t.} & Ax \leq b \end{array} \qquad\qquad (D)\quad \begin{array}{ll} \max & b^\top y \\ \text{s.t.} & A^\top y = c \\ & y \geq 0 \end{array}$$

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$Ax \le b$ is Total Dual Integral (TDI) if (D) has an integer solution for all $c \in \mathbb{Z}^n$

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Examples of TDI systems:

- MaxFlow-MinCut theorem, Matchings, Mengerian clutters…

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Theorem (Edmonds and Giles – 1977)

If $Ax \leq b$ is TDI and b integer, then $P = \{x : Ax \leq b\}$ is an integer polyhedron.

Box-Total Dual Integral Polyhedra

A system $Ax \leq b$ is box-TDI if

- it is TDI

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A system $Ax \leq b$ is box-TDI if

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$$\begin{array}{lllll} \text{min} & c^\top x & \qquad & \text{max} & b^\top y + u^\top r - \ell^\top s \\ \text{s.t.} & Ax \leq b & & \text{s.t.} & A^\top y + r - s = c \\ & \ell \leq x \leq u & & & r, s, y \geq 0 \end{array}$$

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Obs: **Not** every polyhedron can be described by a box-TDI system

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\end{aligned}
$$

- $P = \{x : Ax \leq b\}$, $b \in \mathbb{Z}^m$
  $\Rightarrow kP = \{x : \frac{1}{k} Ax \leq b\}$
- $\ell, u \in \mathbb{Z}^n \Rightarrow$ P box-integer

Obs: Not every polyhedron can be described by a box-TDI system

- A polyhedron is box-TDI if it can be described by a box-TDI system

Box-Total Dual Integral Polyhedra

A system $Ax \leq b$ is box-TDI if

- it is TDI
- it remains TDI after the addition of $\ell \leq x \leq u$, for all $\ell, u \in \mathbb{Q}^n$

$$\begin{aligned} \text{min} \quad & c^\top x \\ \text{s.t.} \quad & \frac{1}{k} Ax \leq b \\ & \ell \leq x \leq u \end{aligned}$$

- $P = \{x : Ax \leq b\}$, $b \in \mathbb{Z}^m$
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Obs: If P is a box-TDI polyhedron, then P is principally box-integer

Theorem (Chervet, G., Robert – 2020)

A polyhedron P is box-TDI if and only if it is principally box-integer

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<u>Proof</u>:

- P is box-TDI if and only if all its minimal cones are box-TDI

Theorem (Chervet, G., Robert – 2020)

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Proof:

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Theorem (Chervet, G., Robert – 2020)

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Proof:

- P is box-TDI if and only if all its minimal cones are box-TDI
- A cone is box-TDI if and only if it is box-integer
- P is principally box-integer if and only if all its minimal cones are box-integer

Theorem (Chervet, G., Robert – 2020)

A polyhedron P is box-TDI if and only if it is principally box-integer

Corollary (Chervet, G., Robert – 2020)

Given a polyhedron P, the following statements are equivalent:

1. *P is box-TDI*
2. *kP is box-integer whenever kP is integer*
3. *all the face-defining matrices of P are equimodular*
4. *each face of P admits an equimodular face-defining matrix*
5. *each face of P admits a totally unimodular face-defining matrix*

Outline

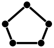
Equimodular matrices

Characterizations of Principally Box-Integer Polyhedra

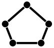
Box-Total Dual Integral Polyhedra

Box-Perfect Graphs

Box-perfect graphs

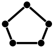
Perfect graphs: graphs with no induced odd  or 

BOX-PERFECT GRAPHS

Perfect graphs: graphs with no induced odd  or 

(STABLE)
$$\begin{aligned} x(K) &\leq 1, \text{ for all cliques } K \text{ of } G, \\ x &\geq 0 \end{aligned}$$

Box-perfect graphs

Perfect graphs: graphs with no induced odd  or 

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Theorem (Lovász – 1972, Chvátal – 1975)

Given a graph G, the following statements are equivalent:

1. *The graph G is perfect*
2. *The system* (STABLE) *is TDI*
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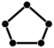
Perfect graphs: graphs with no induced odd  or 

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Box-perfect graph: graphs for which the system (STABLE) is box-TDI

OBS: A graph is box-perfect if and only if

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- its stable set polytope is box-TDI

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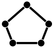
Perfect graphs: graphs with no induced odd  or 

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OPEN: Characterize box-perfect graphs (Cameron and Edmonds – 1982)

THANK YOU FOR YOUR ATTENTION!