POLYHEDRA:
SOME MATRICIAL PERSPECTIVES

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Introduction

This manuscript surveys some of my research of the past decade by exploring specific aspects of polyhedra. Two different properties of polyhedra are discussed, principally under a matricial point of view, and this work is organized in two parts that can be read independently. Each part consists of two chapters: a first chapter that overviews theoretical results, and a second chapter in which a few applications are put forward.

It is intended as an introduction to the addressed topics. Only basic algebra should be needed, yet some familiarity with polyhedra, integer programming, and combinatorial optimization might be helpful for the reader. Mainly, intuitive arguments will be given, and there will be few formal proofs. For the complete proofs, the reader is referred to the corresponding papers, which can be found in the appendix.

A *polyhedron* is the intersection of a finite number of half-spaces, or, equivalently, the set of points satisfying a finite number of linear inequalities. When an affine space intersects only the boundary of a polyhedron, the intersection is called a non trivial *face* of the polyhedron[1]. Faces of dimension zero are *vertices* and faces of dimension one less than the dimension of the polyhedron are *facets*. Particular families of polyhedra will play an important role here: *polytopes*, which are bounded polyhedra; *cones*, which are polyhedra containing a point that lies on all its nonempty faces; and *integer* polyhedra, which are polyhedra having an integer point in each nonnempty face.

Many polyhedra arise from combinatorial optimization problems. Imagine being interested in the sets of pairwise nonadjacent nodes of a given graph, called *stable sets*. To each stable set is associated its *incidence vector*, which is the vector indexed by the nodes of the graph and whose coordinate equals one when the node is in the stable set and zero otherwise. The convex hull of these incidence vectors forms the *stable set polytope* of the graph. Now, if each node has a cost, finding a stable set of maximum total cost reduces to optimizing a linear function over this polytope. This is how a combinatorial problem can be modeled with a polyhedron.

This polyhedral point of view allows the use of geometric and algebraic tools from polyhedral theory to derive new insights towards the essential properties of the underlying combinatorial problem.

There may be various alternatives to model a given combinatorial problem as a polyhedron. The first part of this document is devoted to the systematic study of these models. Roughly speaking, we shall introduce a measure of how good a model is, with the following question in mind: when does a problem admit a good model?

The second part studies a specific family of polyhedra. In a few words, we are interested in the geometric properties of these polyhedra, especially with respect to the integer points of the

[1]Trivial faces are the emptyset and the polyhedron itself.

space. The question behind it is: are there alternative expressions of these properties, and how can they be used in practice?

Essentially, in both parts, we discuss the matricial counterparts implied in and by the polyhedral representation.

Extended Formulations

The first part of this document is devoted to extended formulations. An *extended formulation* of a given polyhedron is a polyhedron in a higher dimensional space that can be projected onto the original polyhedron.

This is an important notion regarding complexity questions in combinatorial optimization. When the number of inequalities describing a polyhedron is polynomial, one can find an optimal solution in polynomial time, whichever objective function is chosen. It turns out that adding extra variables might reduce the number of inequalities needed to describe the polyhedron. For instance, the hexagon in Figure 1 is described by 6 inequalities, and it can be seen as the projection of a prism, for which 5 inequalities suffice.

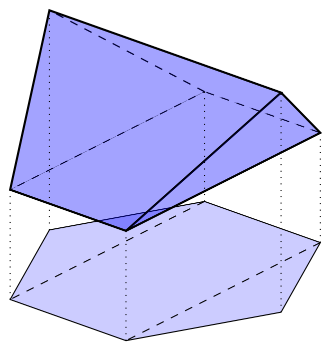


Figure 1: The hexagon as the projection of a prism.

The minimum number of inequalities in an extended formulation of a given polyhedron is called its *extension complexity*. Thus, if a polyhedron has polynomial extension complexity, then the associated optimization problem is solvable in polynomial time, even if the original polyhedron is described by an exponential number of inequalities. This is indeed the case for several well studied polytopes, such as the *spanning tree polytope*, which is the convex hull of incidence vectors of the spanning trees of a graph, or the *permutohedron*, which is the convex hull of all permutations of $(1, 2, \dots, n)$.

Beside these complexity aspects, extended formulations can be a powerful tool to find the inequalities describing the convex hull of a finite set of points. It is not always obvious how to describe such a set by means of linear inequalities: sometimes it is easier to obtain a description of the points by adding extra variables. Then, one can use projection techniques to get rid of these extra variables and thereby find a description in the original space.

The notion of extension complexity is intertwined with the factorization properties of a particular family of matrices called *slack matrices*. In essence, the slack matrix of a polytope encodes the distances from each vertex to each facet of the polytope. Such a matrix is always nonnegative. The *nonnegative rank*, which is an analogue of the rank for nonnegative matrices, plays an important role in our context. It is the smallest number of nonnegative rank one matrices whose sum is the starting matrix. Yannakakis [63] proved that the extension complexity of a polytope equals the nonnegative rank of its slack matrix.

The first chapter of Part I of this manuscript is devoted to the study of slack matrices. We first characterize them in geometric and combinatorial ways, and discuss their recognition problem. One geometric characterization involves the cone and the space generated by the columns of the matrix. Another one, of more combinatorial nature, relies on the structure of the zeros of the matrix. Cones play an significant role here because a polyhedron is a section of its homogeneization cone.

Finally, we explain how the nonnegative rank can be computed by using randomized communication protocols.

These results come from joint works with J. Gouveia, V. Kaibel, K. Pashkovich, R. Z. Robinson, and R. R. Thomas [38] and with Y. Faenza, S. Fiorini, and H. R. Tiwary [29].

The second chapter of Part I presents examples and techniques to derive extended formulations. These techniques are used to obtain the description of the lexicographical polytope in its original space. We also explain how to combine induction and a theorem of Balas [6] to obtain an implicit extended formulation of the circuit polytope of a series-parallel graph.

These are joint works with M. Barbato, M. Lacroix, and C. Pira [5] and S. Borne, P. Fouilhoux, M. Lacroix, and P. Pesneau [8].

Box-Totally Dual Integral Polyhedra

The second part of this manuscript studies a certain class of polyhedra called *box-totally dual integral polyhedra*. They play an important role in combinatorial optimization because they are associated to linear systems that have strong integrality properties.

In combinatorial optimization, many important results come from min-max relations. Let us mention two famous examples. The MaxFlow-MinCut theorem of Ford and Fulkerson [31] states that, in a directed graph having a source s, a sink t, and flow capacities on its arcs, the maximum amount of flow from the source to the sink without exceeding the capacities of the arcs equals the minimum capacity of a set of arcs to be removed to disconnect s from t. Another example is König's theorem [56, Theorem 16.2], which asserts that, in a bipartite graph, the maximum number of disjoint edges equals the minimum number of nodes that cover every edge of the graph. Often, and it is the case for these two examples, min-max relations come from integrality properties of some linear systems.

A *linear problem* is a problem where one has to optimize a linear objective function over a linear system. To each linear problem is associated another linear problem called its *dual*. The original problem is then called the *primal*. The strong duality in linear programming states that when the optima of these problems are finite, they have the same value. Moreover, when the inequalities of the primal problem describe an integer polyhedron, there always exist an integer optimal solution. Such an integer solution can be interpreted as a combinatorial object, like a flow in a graph. It is similar when the dual has an integer optimal solution, for instance these solutions are associated to st-cuts in the case of the dual of the maximum flow problem. Linear systems with the property that there always exists an integer optimal solution in their dual[1] are

[1]When the optimum is finite.

called *totally dual integral* systems. The linear system behind the MaxFlow-MinCut theorem is an example of totally dual integral system.

Among totally dual integral systems, there are some systems that yield stronger min-max relations: *box-totally dual integral* systems. These systems are those that remain totally dual integral when lower and upper bounds are imposed on some variables. The effect of being box-totally dual integral in the dual is that one can modify the objective function of the primal problem by an integer amount and at a certain cost before solving the resulting dual problem. Depending on what the primal variables represent, this can have various combinatorial interpretations.

These systems were introduced around the eighties, yet not much was known about them before the last twenty years. One of our recent referees describes the situation as follows: "Box-totally dual integrality is not that well-studied as totally dual integral systems, but the list of illustrious results from the last two decades shows that it is indeed an interesting notion that is worth considering". For instance, until recently, the vast majority of known box-totally dual integral systems were defined by a *totally unimodular* matrix, which is a matrix whose subdeterminants are all 0, 1, or -1. This is the case for the MaxFlow-MinCut theorem and König's theorem mentioned above. In particular, the linear systems behind these results are box-totally dual integral.

The polyhedra described by such systems are called *box-totally dual integral* polyhedra and they are the main subject of my research since 2016. Incidentally, in this part, we will meet several open problems along the way[1].

In the first chapter of Part II, we review several recent characterizations of these polyhedra, essentially geometric and matricial ones. A geometric characterization involves the integrality of the polyhedron and its dilations when being intersected with integer boxes. Other characterizations involve matrices defining the faces of the polyhedron. We will see how these characterizations allow us to easily recover several previously known results, and what new insights they provide. A new class of matrices, that we call *totally equimodular* and that generalize totally unimodular matrices, will come up naturally. They are the matrices for which all the associated polyhedra are box-totally dual integral, and we will briefly discuss some of their properties.

These characterizations and their consequences mostly come from joint work with P. Chervet and L.-H. Robert [14]. Results on totally equimodular matrices come from an ongoing work with and P. Chervet, M. Lacroix, F. Pisanu, L.-H. Robert, and R. Wolfler Calvo [13].

In the second chapter of Part II, we provide concrete examples of box-totally dual integral systems and polyhedra. Most of these examples show how the matricial characterization gives an easy way to disprove box-total dual integrality. For instance, we use it to refute a conjecture on box-perfect graphs. Then, we discuss possible connections between box-total dual integrality and the integer decomposition property. We also explain the relation between two results about Mengerian clutters. Finally, we provide several box-totally dual integral systems and polyhedra in series-parallel graphs.

The results mentioned in this chapter come from joint works with D. Cornaz and M. Lacroix [21], P. Chervet and L.-H. Robert [14], M. Barbato M. Lacroix, E. Lancini, and R. Wolfler Calvo [4] and M. Barbato, M. Lacroix, and E. Lancini [3].

[1]The hungry reader can find them directly Page 61.

Contents

Part I

EXTENDED FORMULATIONS

Chapter 1

Extended Formulations, Slack Matrices, and Communication Protocols

This chapter is devoted to the interplays between extended formulations, slack matrices, and communication protocols. The first section explains the link between the extension complexity of a polytope and nonnegative factorizations of its slack matrices, a result due to Yannakakis [63]. The next two section build on this result to present my work on this topic.

In Section 1.2, we overview geometric and combinatorial characterizations of slack matrices. Cones play an important role because any polyhedron is a section of its homogeneization cone. We will also discuss the complexity of recognizing whether a given matrix is a slack matrix. These results come from joint work with J. Gouveia, V. Kaibel, K. Pashkovich, R. Z. Robinson, and R. R. Thomas [38]

In Section 1.3, we show another way to compute the nonnegative rank of a nonnegative matrix by adapting tools from communication complexity theory. Namely, we prove that the nonnegative rank of a matrix equals the minimum complexity of a randomized protocol computing the matrix. The results of this section are joint work with Y. Faenza, S. Fiorini, and H. R. Tiwary [29].

1.1 Extended Formulations and Slack Matrices

In this section, we define extended formulations and slack matrices of a given polyhedron. Then, we explain the connection, due to Yannakakis, between these two notions. First, let us introduce the necessary notation.

1.1.1 Notation

An element M of $\mathbb{R}^{m\times n}$ will be thought of as a matrix with m rows and n columns, and an element b of $\mathbb{R}^m$ as a column vector. Given a set of vectors M, we denote by vect(M) the linear space they generate, by aff(M) their affine hull, by cone(M) the set of their nonnegative combinations, that is, the *cone* generated by M, and by conv(M) the convex combinations of these vectors. The set vect(M) is also called the *column span* of M, and conv(M) its *convex hull*. The transpose of a matrix M will be denoted by $M^\top$. Thus, vect$(M^\top)$ is the linear space generated by the rows of M, also called the *row span* of M. We will often consider a matrix M as the set of its column vectors.

A *polyhedron* is the intersection of a finite number of half spaces $a_i^\top x \geq b_i$, for $i = 1, \dots, m$. We write this $P = \{x : Ax \geq b\}$. By Minkowski-Weyl's theorem [18, Theorem 3.13], there exists a set of points V and a set of vectors R such that $P = \text{conv}(V) + \text{cone}(R)$. Vectors in R are called *rays*. The dimension $\dim(P)$ of P is the dimension of its affine hull. The *faces* of P are the polyhedra obtained by imposing some equalities among $Ax \geq b$. A face of dimension zero is a *vertex*, and a face of dimension $\dim(P) - 1$ is a *facet*. The set cone(R) is the set of directions in which we can move towards $+\infty$ while staying inside P, and is the *recession cone* of P, denoted by rec(P). The *lineality space* of P is lin$(P) = \text{rec}(P) \cap \text{rec}(-P)$. It is the set of directions in which we can move towards both $+\infty$ and $-\infty$ within P. The polyhedron P has vertices if and only if lin(P) is empty, that is, rec(P) contains no line. Polyhedra having vertices are *pointed*, and we can choose V to be their set of vertices in the above description.

A *polytope* is a bounded polyhedron, or equivalently a polyhedron whose recession cone is empty. A polytope of $\mathbb{R}^2$ is a *polygon*. We mention that a cone $C = \text{cone}(R)$ is equivalently a polyhedron which contains the origin in all its nonempty faces, that is, the cone C is a polyhedron of the form $C = \{x : Ax \geq \mathbb{0}\}$. The *polar* of C is the cone $\{x : R^\top x \leq \mathbb{0}\}$. A closely related cone is the *dual cone* of C, which is the cone $C^\circ = \{x : R^\top x \geq \mathbb{0}\} = \text{cone}(A^\top)$. For λ in $\mathbb{R}$, the *dilation* λP of P is the polyhedron $\lambda P = \{x : Ax \geq \lambda b\}$.

To ease the definitions introduced, we assume our polyhedra to be full-dimensional and pointed. Thanks to this assumption, we may assume that the descriptions of our polyhedra are irredundant, that is, there are no unnecessary inequalities, points in V, or rays. We explain in Remark 1.3 how the results mentioned here extend to the case where P is not full-dimensional or P has no vertices.

1.1.2 Extended Formulations

In combinatorial optimization, the set of solutions to a given problem is often described as a polytope. Surprisingly, many reasonable combinatorial problems, solvable in polynomial time, admit unreasonable descriptions in their natural space, involving an exponential number of inequalities. This is the case for instance for the *matching polytope*, which is, given a graph, the convex hull of the incidence vectors of subsets of edges covering each vertex at most once.

The question of the number of inequalities needed to describe a polytope is usually related to the computational difficulty of the underlying combinatorial problem. To try to obtain a smaller description, an idea is to add extra variables and find another formulation in an extended space.

An *extension* of a polyhedron P of $\mathbb{R}^n$ is a polyhedron Q in a higher dimension p such that P is the image of Q under a linear projection $\pi : \mathbb{R}^p \to \mathbb{R}^n$. More precisely, $P = \pi(Q)$ is the set of x such that there exists y in Q with $x = \pi(y)$. An *extended formulation* of P is a linear description of Q by means of linear inequalities and equations. The minimum number of inequalities in an extended formulation of P is called the *extension complexity* of P, and is denoted by $\mathrm{xc}(P)$. The polyhedron P can be though of as the shadow of Q.

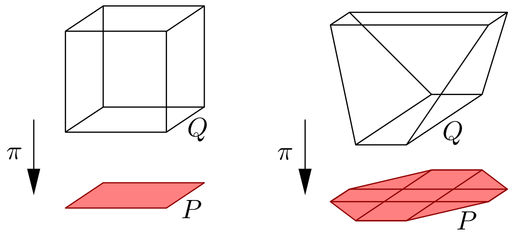


Figure 1.1: Examples of extended formulations.[1]

For instance, the octahedron in Figure 1.1, which is described by 8 inequalities, has an extended formulation described by 6 inequalities. Thus, its extension complexity is at most 6: adding extra variables might decrease the number of inequalities required.

In 1991, Yannakakis [63] developped tools to derive bounds on the extension complexity of a polytope. Among other things, he proved that there are no *symmetric*[2] extended formulations for the traveling salesman polytope. The *traveling salesman polytope* is the convex hull of the incidence vectors of the tours of the complete graph, a *tour* being a cycle going through each vertex exactly once. Finding a tour of minimum cost is a well-known NP-hard problem, thus if there were a polynomial size extended formulation of this polytope, it would imply that P = NP. He concluded his paper with the following impression about the symmetry assumption: “We do not think that asymmetry helps much”. Afterwards, for almost two decades, extended formulations were rather used as a tool than they were the object of a systematic study.

Extended formulations returned to the front stage approximately ten years ago, mainly thanks to three papers. The first one answers negatively Yannakakis’ conjecture about the impact of symmetry: Kaibel, Pashkovich, and Theis exhibit in [44] specific matching polytopes for which there are non-symmetric extended formulations of polynomial size, while no symmetric extended formulation of polynomial size exists. In the same period, for the first time[3], the exact extension complexity of a polyhedron was determined, by Goemans in [37]: therein, an upper bound for the extension complexity of the permutohedron is provided, together with

[1]This nice picture and many others in this section are to be credited to Samuel Fiorini.

[2]Roughly speaking, symmetric means that the formulation remains invariant under all permutations of the nodes of the graph.

[3]To the best of my knowledge.

an extended formulation achieving this bound. Almost simultaneously, Conforti, Cornuéjols, and Zambelli wrote a thorough survey [17] on extended formulations in combinatorial optimization that made the topic accessible to a broader audience.

Since then, many deep results were published about extended formulations. An important one is due to Rothvoß [52] and proves that there exists $0/1$ polytopes having exponential extension complexity. More concrete results followed, such as the proof that the traveling salesman polytope has no extended formulation of polynomial size, symmetric or not[4], due to Fiorini, Massar, Pokutta, Tiwary, and de Wolf [30]. Surprisingly, it was also shown by Rothvoß [53] that the matching polytope has no polynomial size extended formulation. Most of these results use the relation between the extension complexity of a polytope and the nonnegative factorizations of its slack matrices.

1.1.3 Slack Matrices

The *slack* of a point v with respect to an inequality $a^\top x \geq b$ is $a^\top v - b$. Slack matrices were originally introduced for polytopes. The slack matrix of a polytope contains the slack of each vertex with respect to each facet of the polytope. Essentially, this matrix encodes the distances from each vertex to each facet.

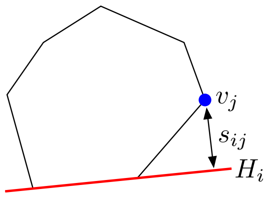


Figure 1.2: Slack s_{ij} of vertex v_j with respect to facet H_i.

The notion of slack matrices can be extended to general polyhedra, even if its meaning somewhat loses this visual interpretation. Slack matrices for cones were introduced in [38], and an extension to polyhedra is provided below.

More precisely, for a polytope $P = \{x : Ax \geq b\} = \text{conv}(V)$ of $\mathbb{R}^n$, the slack of vertex v_j with respect to facet $a_i^\top x \geq b_i$ is $a_i^\top v_j - b_i$. Up to to normalizing $a_i^\top x \geq b_i$, the slack represents the distance from the point v_j to the facet $H_i = \{x : a_i^\top x = b_i\}$. In particular, if v_j belong to H_i, then the slack is zero. The *slack matrix* of P with respect to these descriptions is the $m \times n$ matrix S_P whose entry (i, j) is $a_i^\top v_j - b_i$. In other words,

$$S_P = [A, -b] \cdot \begin{bmatrix} V \\ \mathbb{1}^\top \end{bmatrix} . \tag{1.1}$$

Let $C = \{x : Ax \geq \mathbb{0}\} = \text{cone}(R)$ be a pointed cone. The *slack matrix* of C with respect to these representations is the matrix $S_C = AR$. Let $P = \{x : Ax \geq b\} = \text{conv}(V) + \text{cone}(R)$ be a polyhedron. The *slack matrix of* P with respect to these representations is the matrix $S_P = [W, T]$, where $W_{ij} = a_i^\top v_j - b_i$ is the slack of vertex v_j with respect to inequality

[4]This problem was posed 20 years before by Yannakakis [63].

$a_i^\top x \geq b_i$ and $T_{ij} = a_i^\top r_j$ is the slack of r_j with respect to inequality $a_i^\top x \geq 0$. In matricial form, it is written as follows:

$$S_P = [A, -b] \cdot \begin{bmatrix} V & R \\ \mathbb{1}^\top & \mathbb{0}^\top \end{bmatrix}. \tag{1.2}$$

In other words, S_P is composed of the slacks of the vertices with respect to the facets and of the slack matrix of the recession cone of P. This definition contains the two previous ones: when the polyhedron is a cone, then V is empty; and when P is a polytope, the set R is empty.

These three definitions can be unified with another point of view. The *homogenization cone* C_P of a polyhedron $P = \{x : Ax \geq b\}$ is the set of (x, λ) with $x \in \lambda P$.

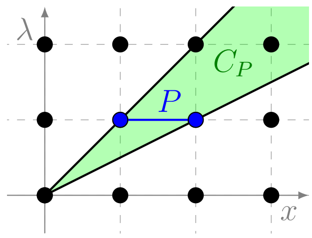


Figure 1.3: A one-dimensional polytope $P = \text{conv}((1), (2))$ and its homogenization cone $C_P = \text{cone}((1,1), (2,1))$.

When P is a cone, P is its own homogenization cone (up to removing λ). When $P = \text{conv}(V)$ is a polytope, C_P is described as follows:

$$C_P = \{(x, \lambda) \in \mathbb{R}^n \times \mathbb{R} : Ax \geq \lambda b\}.$$

Note that the slack matrix of the polytope P is also the slack matrix of C_P with respect to the representations

$$C_P = \{(x, \lambda) : [A, -b] \begin{bmatrix} x \\ \lambda \end{bmatrix} \geq \mathbb{0}\} = \text{cone}\left(\begin{bmatrix} V \\ \mathbb{1}^\top \end{bmatrix}\right).$$

This remark straightforwardly extends to polyhedra. To sum up, the slack matrix of a polyhedron is a slack matrix of its homogenization cone.

By definition, slack matrices are *nonnegative matrices*, that is, all their entries are nonnegative. Moreover, the zeros of the slack matrix of a polytope P record the face lattice of P, and hence the combinatorial structure of P.

To derive connections between extensions and slack matrices, we need to consider the nonnegative rank of a matrix. This notion adds nonnegativity constraints to the classical notion of rank for general matrices. Recall that $\text{rank}(M)$, the *rank* of a matrix M of size $m \times n$, is the smallest integer r such that $M = AB$, where A and B are matrices of size $m \times r$ and $r \times n$, respectively. Equivalently, the rank of M is the smallest r such that M can be written as the sum of r matrices of rank one.

The *nonnegative rank* of a nonnegative matrix M of size $m \times n$ is the smallest integer r such that $M = AB$, where A and B are nonnegative matrices of size respectively $m \times r$ and $r \times n$. It is denoted by $\text{rank}_+(M)$. Here, AB is called a *nonnegative factorization* of M. Note that, when $M = AB$, we have $M = \sum_{i=1}^r A_i B$ where the i^{th} column of A_i is that of A and A_i

has zeros everywhere else. Thus, similarly to the rank, the nonnegative rank of M is also the minimum integer r such that M is the sum of r nonnegative matrices of rank one.

For a given (full-dimensional) polyhedron, the facet-defining inequalities can be defined up to any positive scaling factor. Moreover, the rays of the recession cone can be chosen up to a positive factor. Thus different slack matrices are possible, depending on the chosen description. They are obtained from one another by multiplying some rows and some columns by a positive factor. As these scaling can be integrated in either matrices of a nonnegative factorization, such operations do not change the nonnegative rank. Therefore, we have the following.

Observation 1.1. *If R and S are two slack matrices of the same polyhedron, then their nonnegative ranks are equal.*

1.1.4 Yannakakis' Theorem

For polytopes, Yannakakis [63] proved the following connection between slack matrices and extensions. We mention that his proof and his result extend to polyhedra with the definition of slack matrices of polyhedra given in Section 1.1.3. His result is generalized to convex sets by J. Gouveia, P. A. Parrilo, and R. R. Thomas in [39].

Theorem 1.2 (Yannakakis [63]). *The extension complexity of a polytope of dimension at least one is equal to the nonnegative rank of any of its slack matrices.*

Let us explain why the extension complexity of a polytope is at most the nonnegative rank of its slack matrices. The other direction is slightly more technical and we refer the interested reader to [63] for more details.

Let $P = \{x \in \mathbb{R}^n : Ax \geq b\} = \text{conv}(V)$ be a polytope with m facets and let S be a slack matrix of P. Actually, Yannakakis [63] proves the stronger statement that any nonnegative factorization $S = TU$ with T of size $m \times r$ yields an extended formulation of P with at most r facets.

Let $S = TU$ be such a factorization. Then, $Q = \{(x, y) \in \mathbb{R}^{n+r} : Ax - Ty = b, y \geq \mathbb{0}\}$ is an extended formulation of P. Indeed, since $y \geq 0$ and T is nonnegative, each x in the projection of Q onto the x variables satisfies $Ax \geq b$. That is, $\text{proj}_x(Q)$ is contained in P. Moreover, by definition of the slack matrix, for each vertex v of P the vector $S^v = TU^v$ contains the slacks of v with respect to each facet of P. Hence $Av - TU^v = b$, and then, for each vertex v of P, the point (v, U^v) is in Q because U is nonnegative. We thus get the reverse inclusion, hence Q is an extension of P.

Notice that all the facets of Q are among $y \geq \mathbb{0}$, thus Q has at most r facets. Therefore, we have $\text{xc}(P) \leq r$, and taking $r = \text{rank}_+(S)$ implies $\text{xc}(P) \leq \text{rank}_+(S)$. Recall that, by Observation 1.1, all the slack matrices of a given polytope have the same nonnegative rank.

We mention that this result is existential: it does not provide explicitly an extended formulation of appropriate size of the polytope, even if the nonnegative factorization of the slack matrix is explicit. Indeed, to get an explicit formulation one should "clean" the system $Ax - Ty = b$, that is, remove redundant equalities to get $n + r - \dim(Q)$ equalities.

We conclude this section by explaining briefly how these results extend when the polyhedron is not full-dimensional or has no vertices.

Remark 1.3. *When the polytope is not full-dimensional, it can be written* $P = \{x : Ax \leq b, Cx = d\}$ *where* $\{x : Cx = d\}$ *is the smallest affine space containing* P*. Since the slack of every vertex of* P *with respect to the equalities is zero, the above results are straightforwardly extended to this case by integrating* $Cx = d$ *into any extended formulation of* P *and removing the rows of zeros added to the slack matrix of* P*.*

When the polyhedron P *has no vertices, intersect it with a hyperplane orthogonal to* $\text{lin}(P)$*, then take the slack matrix of the resulting polyhedron. Nonnegative factorizations of this matrix correspond to extensions of the starting polyhedron.*

Theorem 1.2 is the cornerstone in the study of extended formulations. It is the reason why we study slack matrices in the next section, where we characterize them in several ways.

1.2 Characterizations of Slack Matrices

Determining the nonnegative rank of a nonnegative matrix is NP-complete in general, as shown in [62]. Yet the complexity of this problem remains open for slack matrices, which form a strict subclass of nonnegative matrices.

In this section, we characterize these matrices in geometric and combinatorial ways. The provided characterizations are restricted to nonnegative matrices of rank at least two, as it is easy to see that no matrix of rank at most one is a slack matrix of a nontrivial polytope.

1.2.1 Geometric Characterizations

The first characterizations involve properties of the cones, vector spaces, and affine spaces generated by either the rows or the columns of the matrix.

Slack Matrices of Cones

A necessary and sufficient condition for a nonnegative matrix S to be the slack matrix of some cone is that the cone generated by the columns of S coincides with the nonnegative part of column span of S. Let us explain why this is necessary, and we refer the reader to [38] for sufficiency.

Let $S = AR$ be the slack matrix of $C = \{x : Ax \geq 0\} = \text{cone}(R)$. Then, $\text{cone}(S)$ is the set $K = \{y \geq 0 : y = Sx \text{ for some } x \in \mathbb{R}^n\}$. Indeed, since S is nonnegative, we have $\text{cone}(S) \subseteq K$. To see the converse, take y in K. Since y is nonnegative and $y = Sx = ARx$, we have $ARx \geq 0$, that is, $z = Rx$ belongs to C. Therefore, z is a nonnegative combination of the generators R of C, namely $z = Rx'$ for some $x' \geq \mathbb{0}$. Hence, $y = Az = ARx' = Sx'$ is in $\text{cone}(S)$.

This, which is Statement 2 of Theorem 1.4 below, characterizes slack matrices of cones.

Theorem 1.4 ([38])**.** *For a matrix M of $\mathbb{R}_+^{p\times q}$ with $\text{rank}(M) \geq 2$, the following statements are equivalent.*

1. *M is the slack matrix of a cone.*
2. $\text{cone}(M) = \text{vect}(M) \cap \mathbb{R}_+^p$.
3. $\text{cone}(M^\top) = \text{vect}(M^\top) \cap \mathbb{R}_+^q$.

Statement 3 of Theorem 1.4 is the polar version of Statement 2. Indeed, if $M = AR$ is the slack matrix of a cone $C = \{x : Ax \geq 0\} = \text{cone}(R)$, then $M^\top = R^\top A^\top$ is the slack matrix of its dual cone $C^\circ = \{x : R^\top x \geq 0\} = \text{cone}(A^\top)$. Applying Statement 2 of Theorem 1.4 to $M^\top$ gives Statement 3 of Theorem 1.4.

In other words, Statement 3 of Theorem 1.4 means that the cone generated by the rows of M coincides with the nonnegative part of the space generated by the rows of M.

Slack Matrices of Polytopes

As we saw in Section 1.1.3, the slack matrix of a polytope is the slack matrix of its homogenization cone. Thus, for a matrix to be the slack matrix of some polytope, it has to be the slack matrix of some cone.

There is another condition for that to happen. Recall that a cone is a polyhedron which contains the origin in every facet. This is not true for polytopes[1]. It means that if $P = \{x : Ax \geq b\}$ is a polytope, then $Ax = b$ has no solution. When a system of linear equalities has no solution, then one can write any absurdity such as "zero equals one" using linear combinations of these equalities. For instance, there exists y such that $b^\top y = -1$ and $A^\top y = \mathbb{0}$. Note that, for such a y and for a slack matrix S_P of P, we have

$$S_P^\top \cdot y = [V^\top, \mathbb{1}] \cdot \begin{bmatrix} A^\top \\ -b^\top \end{bmatrix} \cdot y = [V^\top, \mathbb{1}] \cdot \begin{bmatrix} \mathbb{0} \\ 1 \end{bmatrix} = \mathbb{1}.$$

In other words, the all one vector has to be in the linear space generated by the rows of S_P.

These two conditions characterize slack matrices of polytopes.

Theorem 1.5 ([38])**.** *For a matrix M of $\mathbb{R}_+^{p\times q}$ with* $\operatorname{rank}(M) \geq 2$*, the following statements are equivalent.*

1. *M is the slack matrix of a polytope.*
2. $\operatorname{cone}(M) = \operatorname{vect}(M) \cap \mathbb{R}_+^p$ *and* $\mathbb{1}$ *belongs to* $\operatorname{vect}(M^\top)$.
3. $\operatorname{cone}(M^\top) = \operatorname{vect}(M^\top) \cap \mathbb{R}_+^q$ *and* $\mathbb{1}$ *belongs to* $\operatorname{vect}(M^\top)$.

The polarity result that holds for cones breaks down for polytopes, and we refer to [38, Example 15] for an example where it fails. Nevertheless, a polarity result holds for polytopes if one is allowed to chose which slack matrix one prefers. The *polar* of a polytope P of $\mathbb{R}^n$ is

$$P^* = \{y \in \mathbb{R}^n : x^\top y \leq 1 \text{ for all } x \in P\}.$$

Since translating P does not change its slack matrices, we may assume that the origin is in the interior of P, and it is well-known [55, Theorem 9.1] that in this case P^* is also a polytope. Then, after scaling its inequalities, P can be described by $P = \{x \in \mathbb{R}^n : Ax \leq \mathbb{1}\}$ and then $P^* = \operatorname{conv}(A^\top)$. Similarly, if $P = \operatorname{conv}(V)$, then $P^* = \{x \in \mathbb{R}^n : V^\top x \leq \mathbb{1}\}$. This implies that, with respect to these representations, the slack matrix of P is the transpose of the slack matrix of P^*.

Proposition 1.6 ([38])**.** *For any polytope P, there exists a slack matrix M of P such that $M^\top$ is also a slack matrix of a polytope.*

In the light of Theorem 1.5, this says that slack matrices of polytopes (which already have $\mathbb{1}$ in their row span) allow positive scalings of their rows that puts $\mathbb{1}$ into their column span as well.

Slack Matrices of Polyhedra

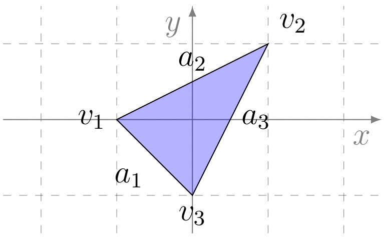
Recall that the slack matrix of a polyhedron is also the slack matrix of its homogeneization cone. Since a cone is a polyhedron, a matrix is the slack matrix of a polyhedron if and only if it is the slack matrix of a cone. Thus, Theorem 1.4 also characterizes slack matrices of polyhedra.

[1]Provided that the polytope is of dimension at least one, which we assume for the rest of this paragraph (note that a polytope of dimension zero is a cone).

1.2.2 A Combinatorial Characterization of Slack Matrices

When trying to build a nonnegative matrix M such that $\text{cone}(M) = \text{vect}(M) \cap \mathbb{R}^p_+$, one quickly gets the feeling that the zero coordinates of its column vectors have to be structured in some combinatorial sense. For instance, any $p \times p$ nonsingular matrix M satisfies $\text{vect}(M) \cap \mathbb{R}^p_+ = \mathbb{R}^p_+$. Therefore, for such an M to be a slack matrix of some cone, that is by Theorem 1.4 to satisfy $\text{cone}(M) = \mathbb{R}^p_+$, the matrix M has to contain a positive scaling of each unit vector of $\mathbb{R}^p_+$. That is, up to positive scalings, M is the identity matrix.

To interpret this phenomenon combinatorially, let us introduce incidence matrices. For a matrix M, we denote by M_{inc} the $0/1$ matrix with $(M_{\text{inc}})_{ij} = 1$ if and only if $M_{ij} = 0$. A matrix M_{inc} arising from a slack matrix M of a polytope P is called an *incidence matrix* of P. In this case, $(M_{\text{inc}})_{ij}$ is 1 if and only if vertex i belongs to facet j.



$$\begin{array}{c@{\;}c} & \begin{matrix} v_1 & v_2 & v_3 \end{matrix} \\ \begin{matrix} a_1 \\ a_2 \\ a_3 \end{matrix} & \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

Figure 1.4: A triangle and its incidence matrix.

Clearly, if M is the slack matrix of a polytope P, then M_{inc} is an incidence matrix of P. Moreover, by the definition of a slack matrix of a polytope, if P is full-dimensional, then $\text{rank}(M) = \dim(P) + 1$. These two conditions yield the characterization below.

We give the characterization of slack matrices of polytopes, since the corresponding statement for cones is immediately deduced by removing the condition that $\mathbb{1}$ is in the row span of the matrix.

Theorem 1.7 ([38]). *A nonnegative matrix M with* $\text{rank}(M) \geq 2$ *is a slack matrix of some polytope if and only if M_{inc} is an incidence matrix of some* $(\text{rank}(M)-1)$*-dimensional polytope and $\mathbb{1}$ is contained in the row span of M.*

In dimension two, since polygons have a very simple combinatorial structure, Theorem 1.7 readily yields a characterization of their slack matrices. Here, a *facet-vertex slack matrix* of a polygon P is a slack matrix of P whose rows and columns are in one-to-one correspondence with the vertices and facets of P, respectively. In particular, the following characterizes slack matrices of rank three.

Corollary 1.8 ([38]). *A matrix $M \in \mathbb{R}^{n \times n}_+$ ($n \geq 3$) is a facet-vertex slack matrix of a polygon with n vertices if and only if* $\text{rank}(M) = 3$ *and its rows and columns can be permuted such that the non-zero entries appear exactly at the positions (i,i) for $1 \leq i \leq n$, and $(i, i-1)$ for $2 \leq i \leq n$, and $(1, n)$.*

The graph associated to a polytope has the vertices of the polytope as nodes and an edge connects two nodes when the corresponding vertices lie on a same face of dimension 1. Steinitz' theorem [61] says that a graph G is associated to a three-dimensional polytope if and only if G is planar and three-connected. Using this, it can be checked in polynomial time whether a given $0/1$ matrix is an incidence matrix of a three-dimensional polytope. For every fixed $d \geq 4$, it is however NP-hard to decide whether a given $0/1$ matrix is an incidence matrix of a d-dimensional polytope [51].

1.2.3 Recognition of Slack Matrices

In this section, we discuss complexity aspects of the *slack matrix recognition problem*, which is the problem of deciding whether a given nonnegative matrix is the slack matrix of some polytope. By Theorems 1.4 and 1.5, since deciding whether the all one vector is in the linear space generated by the rows of a matrix can be done in polynomial time (by solving a linear system) the complexity is the same if we ask whether a matrix is the slack matrix of a cone, a polytope, or a polyhedron.

First, the slack matrix recognition problem is in co-NP because of Theorem 1.5. For a decision problem, to be in *co-NP* means that there is a polynomial certificate proving that the answer is no. Here, it means that when the matrix is not a slack matrix, then there exists a proof for it, and that this proof has polynomial size.

For any nonnegative matrix M, we have $\text{cone}(M) \subseteq \text{vect}(M) \cap \mathbb{R}^p_+$. By Theorem 1.4, M is the slack matrix of some cone if and only if equality holds. Therefore, to prove that M is not a slack matrix, one has to verify that the inclusion is strict. This can be done by exhibiting a point in $\text{vect}(M) \cap \mathbb{R}^p_+$ that does not belong to $\text{cone}(M)$ and a hyperplane separating this point from $\text{cone}(M)$. The coordinates of the point and the hyperplane can be chosen with encoding lengths bounded polynomially in the encoding length of M, hence they form the desired polynomial certificate.

Theorem 1.9 ([38]). *The slack matrix recognition problem is in co-NP.*

It is unkown whether the slack matrix recognition problem is NP-hard or solvable in polynomial time. It seems unlikely to determine its complexity easily as it is equivalent to the polyhedral verification problem [45], whose complexity is unknown. The *polyhedral verification problem* is the problem of deciding, given A, b, and V, whether $\{x : Ax \geq b\} = \text{conv}(V)$.

An intuition of why these two problems are connected lies in the definition of a slack matrix. Recall that a slack matrix of a polytope $P = \{x : Ax \geq b\} = \text{conv}(V)$ can be expressed as:

$$S_P = [A, -b] \cdot \begin{bmatrix} V \\ \mathbb{1}^\top \end{bmatrix}. \tag{1.3}$$

Thus, a nonnegative matrix M is the slack matrix of some polytope if and only if M can be decomposed as in (1.3), where A, b, and V are such that the polytopes $\{x : Ax \geq b\}$ and $\text{conv}(V)$ coincide. Proving that the slack matrix recognition problem and the polyhedral verification problem are indeed polynomially equivalent requires a bit more work, and we refer to [38] for a proof of the following.

Theorem 1.10 ([38]). *The slack matrix recognition problem can be reduced in polynomial time to the polyhedral verification problem, and conversely.*

1.3 Slack Matrices and Communication Protocols

Along with his result connecting extended formulation of a polytope and the nonnegative rank of its slack matrices, Yannakakis [63] provides a method to derive upper bounds on the nonnegative rank of a $0/1$ matrix. His method relies on tools from communication complexity and involves deterministic communication protocols.

Here, we first explain how his idea works. Then, we extend the notion of communication protocols by involving randomization. This extension allows to capture exactly the nonnegative rank of any nonnegative matrix, hence the extension complexity of a polytope when applied to one of its slack matrices.

1.3.1 Deterministic Protocols

Let X and Y be two finite sets and $f : X \times Y \to \mathbb{R}_+$ be a function. Two players, Alice and Bob, wish to compute $f(x, y)$ for some inputs x in X and y in Y. Alice knows only x and Bob knows only y. They must therefore exchange information to be able to compute $f(x, y)$. (We assume that each player possesses unlimited computational power.)

The communication is carried out as a protocol that is agreed upon beforehand by Alice and Bob, on the sole basis of the function f. At each step of the protocol, one of the players has the token. Whoever has the token sends a bit to the other player, that depends only on their input and on previously exchanged bits. This is repeated until the value of f on (x, y) is known to both players. The *complexity* of a protocol is the maximum number of bits exchanged over all possible inputs x and y. The minimum number of bits exchanged between the players in the worst case to be able to evaluate f by any protocol is called the *communication complexity of f*. In other words, it is the minimum complexity of a protocol cumputing f.

A protocol can be viewed as a rooted binary tree where each node is associated to either Alice or Bob. The leaves contain the values of the function. An execution of the protocol on a particular input is a path in the tree starting at the root and ending at a leaf. At a node owned by Alice, following the path to the left subtree corresponds to Alice sending a zero to Bob and taking the right subtree corresponds to Alice sending a one to Bob; and similarly for nodes owned by Bob.

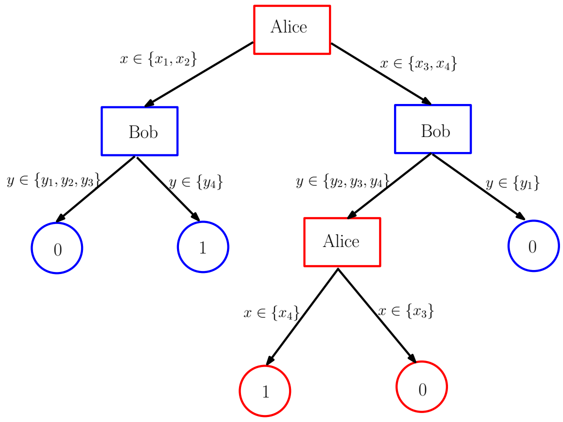
When presenting a protocol, we will often say that one of the two players sends an integer k rather than a binary value. This should be interpreted as the player sending the binary encoding of k, or as a (sub)tree of height $\lceil \log(k) \rceil$.

The function $f : X \times Y \to \mathbb{R}_+$ can be represented as a nonnegative matrix of size $|X| \times |Y|$ whose entry (x, y) is $f(x, y)$ for all $(x, y) \in X \times Y$. We will also denote this matrix by f. In the matrix setting, this means that Alice is given a row index x and Bob a column index y of the matrix, and that they want to compute the entry (x, y) of the matrix.

Let us consider an example. The following matrix represents the function f for which Alice and Bob want to develop a communication protocol.

	y_1	y_2	y_3	y_4
x_1	0	0	0	1
x_2	0	0	0	1
x_3	0	0	0	0
x_4	0	1	1	1

For a given function f, numerous communication protocols are possible. Here is an example of such a protocol computing the function f above.



Note that, in such a protocol, each leaf corresponds to a rectangle of the matrix containing a unique value. The indices of the entries of this rectangle are given by the couples (x, y) of inputs of Alice and Bob leading to the leaf. For instance, the leaf on the left of the above protocol corresponds to the rectangle $\{x_1, x_2\} \times \{y_1, y_2, y_3\}$. The deterministic property of the protocol implies that these rectangles are disjoint, and hence this induces a partition of the matrix into rectangles containning a unique value, as follows.

	y_1	y_2	y_3	y_4
x_1	0	0	0	1
x_2	0	0	0	1
x_3	0	0	0	0
x_4	0	1	1	1

Any communication protocol computing f yields such a partition. Conversely, one can check that a partition of the matrix into rectangles containing a unique value provides a communication protocol.

Moreover, each of these rectangles is a nonnegative matrix of rank one, and hence a partition into such rectangles allows us to write the matrix f as a sum of nonegative rank one matrices. Therefore, it provides a nonnegative factorization of f. Since a binary tree of height c has at most 2^c leaves, this gives the following observation.

Observation 1.11. *If a communication protocol of complexity c computes f, then* $\operatorname{rank}_+(f) \leq 2^c$.

Let us apply this on a concrete example: the stable set polytope of a claw-free perfect graph. This example builds on some ideas of Yannakakis and comes from joint work Y. Faenza, S. Fiorini et H. R. Tiwary [29]. A graph is *claw-free* when it does not contain the graph of Figure 1.5 as an induced subgraph.

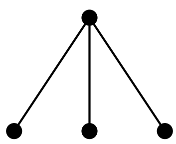


Figure 1.5: A claw.

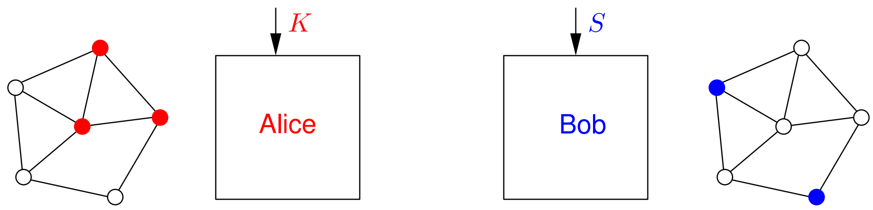
Recall from the introduction that the stable set polytope $\text{STAB}(G)$ of a graph G is the convex hull of the incidence vectors of the stable sets of G, and that the latter are sets of pairwise nonadjacent nodes. A *clique* is a set of pairwise adjacent nodes. As a clique and a stable set have at most one node in common, the system in the theorem below always contains the stable set polytope of the graph. It describes exactly the stable set polytope of G when G is a perfect graph, as shown by Chvátal [16].

Theorem 1.12 (Chvátal [16]). *A graph G is perfect if and only if* $\text{STAB}(G) = \{x \in \mathbb{R}^V_+ : x(K) \leq 1,$ *for each clique K of G*$\}$.

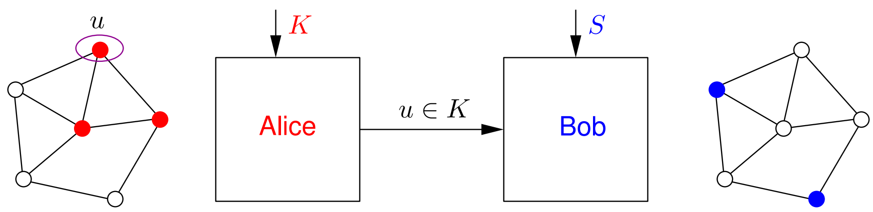
By exhibiting a well chosen communication protocol computing the slack matrix S_G arising from the description given in Theorem 1.12 when G is claw-free and perfect, we will obtain the following.

Theorem 1.13 ([29]). *For a claw-free perfect graph G with n nodes, we have* $xc(\text{STAB}(G)) = O(n^3)$.

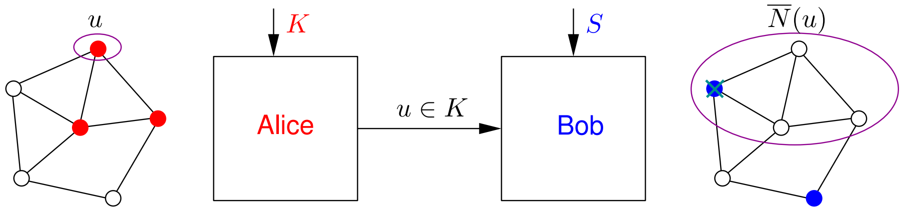
The protocol relies on the following properties of the slack matrix of the stable set polytope of a perfect graph. First, for a perfect graph G, note that S_G has only $0/1$ entries. Indeed, a row of S_G correspond to a clique K, a column to a stable S of G, and the slack of S with respect to K is $1 - |S \cap K|$. This is either 0 or 1 since a clique and a stable set have at most one node in common. Therefore, we want to devise a protocol in which, Alice is given a clique K and Bob a stable set S and they want to determine whether K and S intersect each other, by exchanging a minimum number of bits in the worst case.



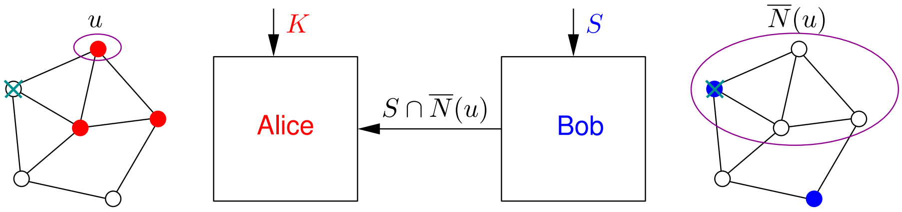
Alice picks a node u of her clique and sends this node to Bob. Since there are n nodes, sending the name of this node u costs $\log_2(n)$ bits.



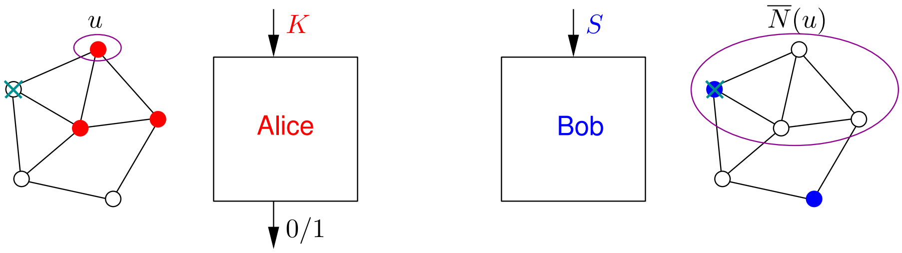
Now, Bob knows a node u of Alice's clique. In particuler, he knows that every node that Alice has is a neighbor of u: Alice's clique is contained in the closed neighborhood $\overline{N}(u)$ of u.



Since the graph is claw-free, the stable set of Bob has at most two nodes in the closed neighborhood of u. Bob sends the name of these possible nodes to Alice.



Alice now has sufficient information to decide whether K and S intersect: they intersect if and only if one of the nodes that Bob sent her is in her clique. Therefore, Alice can send the slack of S with respect to K.



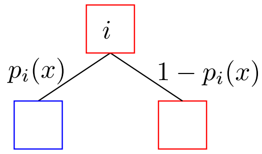
In the worst case, by communicating following this protocol, Alice sends the name of a node and Bob the name of two nodes, which gives $3\log_2(n)$ to be exchanged. By Observation 1.11 and Theorem 1.2, this protocol gives the bound announced in Theorem 1.13.

1.3.2 Randomized Protocols

The previous section revealed the potential of the idea of Yannakakis. Recall that the protocols introduced therein give a partition of the matrix into disjoint rectangles containing a unique value. Therefore, it only gives an upper bound on the nonnegative rank of the matrix. How can the protocols be modified in order to capture better the nonnegative rank of the considered matrix?

With Y. Faenza, S. Fiorini et H. R. Tiwary [29], we introduce randomized communication protocols and it allows to capture exactly the nonnegative rank of any nonnegative matrix. The version of the randomized protocols presented here will be less technical than that of [29] as some technical aspects are left aside to ease the exposition of the concept.

Let us make more formal what we mean by *randomized communication protocol*. The process is similar to the deterministic case: Alice and Bob agree on a decision tree. The random aspects come into play at each node of the tree, where Alice and Bob can send their bit randomly. More precisely, if node i of the tree representing the protocol concerns Alice, who knows $x \in X$, then she will send 0 with probability $p_i(x)$ and 1 with probability $1 - p_i(x)$, where $p_i(x)$ is a value between 0 and 1.

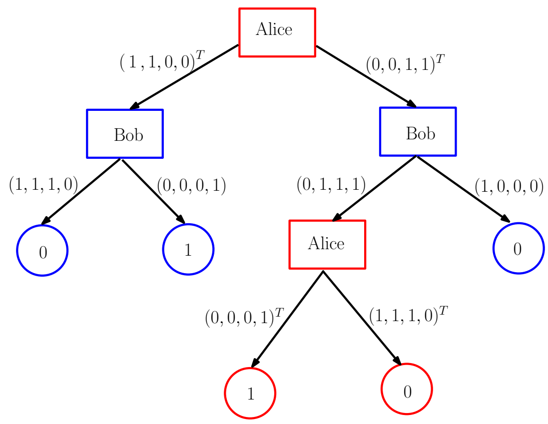


Consequently, given an entry (x, y), the value output by the protocol is random. We say that the protocol *computes f on average* if for every (x, y) in $X \times Y$, the mean of the values output by the protocol is equal to $f(x, y)$:

$$E[\text{value output by the protocol for the entry } (x, y)] = f(x, y).$$

The *complexity* of such a protocol is the height of the associated binary tree.

Note that randomized communication protocols generalize the deterministic protocols seen in the previous section. Indeed, the latter can be seen as randomized protocols where all the probabilities are either 0 or 1. For instance, below, the vector $(1, 1, 0, 0)^\top$ on the left of the top node contains the values $(p_1(x_1), p_1(x_2), p_1(x_3), p_1(x_4))$. The present values mean that Alice will send a 0 with probability 1 if her input is in $\{x_1, x_2\}$ and with probability 0 if it is in $\{x_3, x_4\}$.



It turns out that these randomized communication protocols capture exactly the nonnegative rank of any nonnegative matrix.

Theorem 1.14 ([29])**.** *If* $\mathrm{c_{min}}(S)$ *is the minimum complexity of a randomized protocol computing a nonnegative matrix* S *on average, then* $\mathrm{c_{min}}(S) = \lceil \log_2(\mathrm{rank}_+(S)) \rceil$.

For the first direction of this theorem, that is, if there exists a randomized communication protocol computing S on average, then $\mathrm{rank}_+(S) \leq 2^c$, we refer the reader to [29] to see how randomness allows to cover the matrix with not necessarily disjoint rectangles, each one containing a unique (nonnegative) value, hence to write the matrix as a sum of nonnegative rank one matrices. Each of these rectangles corresponds to a leaf of the tree associated to the protocol, and the latter, of complexity c, has at most 2^c leaves, which gives the announced upper bound.

For the other direction, we provide a randomized communication protocol computing the matrix S on average, of complexity $\log_2(\text{rank}_+(S))$.

First, we chose a normalized version of the matrix S. Let $r = \text{rank}_+(S)$ and $S = TU$ be a nonnegative factorization of S, where T is in $\mathbb{R}_+^{m\times r}$ and U in $\mathbb{R}_+^{r\times n}$. Recall that if we divide row i of S by a strictly positive λ, then the matrix S' we obtain satisfies $\text{rank}_+(S') = \text{rank}_+(S)$. Indeed, $S' = T'U$ where T' is obtained from T by dividing the i^{th} row of T by λ. That way, by dividing each row of S by the sum of the coefficients of the corresponding row of T, we may assume that $S = TU$ where T is a matrix in which the sum of the coefficients of each row is equal to 1. Then, each row of T corresponds to a probability distribution on its columns.

Then, here is a randomized communication protocol computing S on average:

1. Alice is given a row index i, Bob is given a column index j.
2. Alice picks randomly a column index k of T, with probability T_{ik}, and sends it to Bob.
3. Bob outputs the value U_{kj}.

On average, given the entry (i, j), this protocol outputs the value $\sum_{k=1}^{r} T_{ik}U_{kj} = S_{ij}$, hence indeed computes S on average. The complexity of this protocol is the complexity of sending the index of a column of T, which has r columns. Therefore, its complexity is $\log_2(r) = \log_2(\text{rank}_+(S))$, which gives the desired bound for Theorem 1.14.

The combination of Theorem 1.14 and Theorem 1.2 gives the following characterization of the extension complexity of a polytope.

Note that the trick of the previous proof of replacing the matrix S by its normalized version also makes sense in the context of slack matrices, as different equivalent descriptions of a given polytope can be obtained by dividing inequalities by a strictly positive value.

Corollary 1.15 ([29])**.** *If $c_{\text{min}}(S)$ is the minimum complexity of a randomized communication protocol computing a slack matrix S of a polytope P on average, then, $c_{\text{min}}(S) = \lceil \log_2(xc(P) \rceil$.*

Chapter 2

Examples of Extended Formulations

In this chapter, we overview two different techniques to derive extended formulations. In Section 2.1, we obtain the description of lexicographical polytopes by projecting an extended formulation based on paths in a digraph. In Section 2.2, we will combine induction and Balas' union theorem [6] to obtain an implicit extended formulation for the circuit polytope of series-parallel graphs. These results are joint works with M. Barbato, M. Lacroix, and C. Pira [5] and S. Borne, P. Fouilhoux, M. Lacroix, and P. Pesneau [8], respectively.

2.1 Lexicographical Polytopes

In this section, ℓ, u, s will denote integer points of $\mathbb{Z}^n$ satisfying $\ell \leq s \leq u$, that is, s is within $[\ell, u]$[1]. An integer point x of $\mathbb{Z}^n$ is *lexicographically smaller than* s, denoted by $x \preccurlyeq s$, if $x = s$ or the first nonzero coordinate of $s - x$ is positive. In other words, the first coordinate in which x differs from s is smaller than that of s, and the following coordinates are unconstrained. For instance, $(2,1)$, $(2,0)$, and $(1,3)$ are lexicographically smaller than $(2,1)$, but $(2,2)$ and $(3,1)$ are not.

The *top-lexicographical polytope* $L_{\ell,u}^{\preccurlyeq s} = \text{conv}(x \in \mathbb{Z}^n : \ell \leq x \leq u, x \preccurlyeq s)$ is the convex hull of the integer points within $[\ell, u]$ that are lexicographically smaller than s. The goal of this section is to provide the description of a top-lexicographical polytope by means of linear inequalities. First, we study the polytope formed by the componenwise maximal points of a top-lexicographical polytope.

Here is an example in dimension 2.

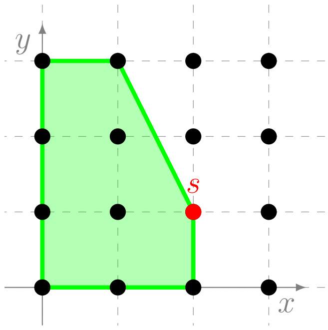


Figure 2.1: The top-lexicographical polytope $L_{\ell,u}^{\preccurlyeq s}$ with $\ell = (0,0)$, $u = (3,3)$, and $s = (2,1)$.

The *submissive* of a set of points Y of $\mathbb{R}^n$ is the set $\{x : x \leq y \text{ for some } y \text{ in } Y\}$ and can be written $Y + \mathbb{R}^n_-$. Observe that $L_{\ell,u}^{\preccurlyeq s}$ is the intersection of the submissive of the blue polytope $\text{conv}(s, p^1, p^2)$ with the nonnegative orthant. That is, $L_{\ell,u}^{\preccurlyeq s} = (\text{conv}(s, p^1, p^2) + \mathbb{R}^2_-) \cap \{x \geq \mathbb{0}\}$[2]. The blue polytope is the convex hull of the componentwise maximal integer points of the top-lexicographical together with p^2.

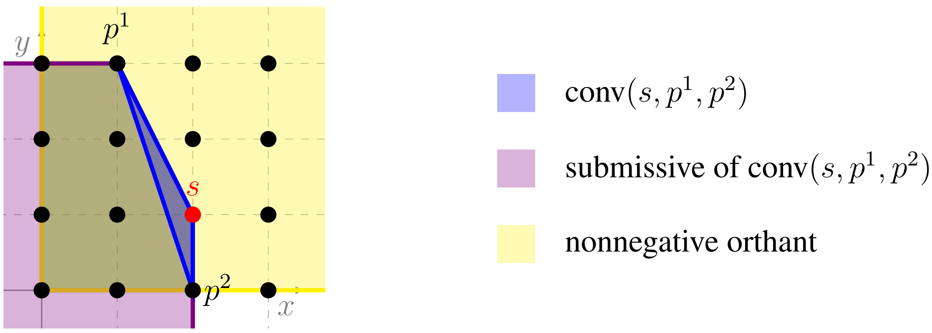


Figure 2.2: $L_{\ell,u}^{\preccurlyeq s} = (\text{conv}(s, p^1, p^2) + \mathbb{R}^2_-) \cap \{x \geq \mathbb{0}\}$.

[1] For sake of simplicity, we assume that $s_i > \ell_i$ for $i = 1, \ldots, n$, and we refer to [5] for the extra details needed when this does not hold.

[2] This equality holds also without the point p^2 in the blue polytope. Adding this point simplifies the projection of the extended formulation given later.

It turns out that this observation holds in general. More precisely, let $X^{\preccurlyeq s}_{\ell,u}$ be the set of the componentwise maximal integer points of $L^{\preccurlyeq s}_{\ell,u}$ together with $p^n = (s_1, \dots, s_{n-1}, s_n - 1)$. In other words, since we assumed $s_i > \ell_i$ for $i = 1, \dots, n$, the set $X^{\preccurlyeq s}_{\ell,u}$ is composed of the points $p^k = (s_1, \dots, s_{k-1}, s_k - 1, u_{k+1}, \dots, u_n)$, for $k = 1, \dots, n+1$, where $p^{n+1} = s$ by definition.

By definition of the submissive and since $x \leq y$ implies $x \preccurlyeq y$, the set $(\text{conv}(X^{\preccurlyeq s}_{\ell,u}) + \mathbb{R}^n_-) \cap \{x \geq \ell\}$ is contained in $L^{\preccurlyeq s}_{\ell,u}$. Moreover, since $\text{conv}(X^{\preccurlyeq s}_{\ell,u})$ is integer and contained in $\{x \geq \ell\}$, the polyhedron $(\text{conv}(X^{\preccurlyeq s}_{\ell,u}) + \mathbb{R}^n_-) \cap \{x \geq \ell\}$ is integer. Therefore, we have the following.

Observation 2.1. $L^{\preccurlyeq s}_{\ell,u} = (\text{conv}(X^{\preccurlyeq s}_{\ell,u}) + \mathbb{R}^n_-) \cap \{x \geq \ell\}$.

Let us model the integer points in $X^{\preccurlyeq s}_{\ell,u}$ as paths in the digraph given in Figure 2.3, and this will give an extended formulation of their convex hull. This digraph is composed of $n+1$ layers, each containing two nodes except the first and the last ones. There are three arcs connecting the layer k to the layer $k+1$, an upper arc y_k, a diagonal arc t_k, and a lower arc z_k. The only exception concerns the first level, which does not have the upper arc.

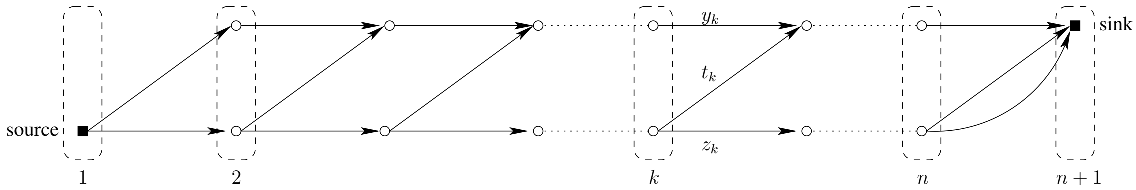


Figure 2.3: Path representation of the points of $X^{\preccurlyeq s}_{\ell,u}$.

Each path P from the source to the sink corresponds to a point x in $X^{\preccurlyeq s}_{\ell,u}$ as follows. Given the structure of the digraph, the path P contains exactly one diagonal arc. This diagonal arc connects a horizontal path composed of lower arcs starting from the source to a horizontal path composed of upper arcs ending at the sink. One of these horizontal paths might be empty. The lower arcs represent the coordinates in which the point x of $X^{\preccurlyeq s}_{\ell,u}$ meets s, the diagonal arc is the first coordinate of x different from that of s, and the upper arcs are the coordinates where x meets the upper bound. More precisely:

$$x_k = \begin{cases} u_k & \text{if } y_k \in P, \\ s_k - 1 & \text{if } t_k \in P, \\ s_k & \text{if } z_k \in P, \end{cases}$$

There are $n+1$ different paths from the source to the sink, and each of them yields a point of the type $(s_1, \dots, s_{k-1}, s_k - 1, u_{k+1}, \dots, u_n)$, that is, a point of $X^{\preccurlyeq s}_{\ell,u}$. This correspondence between a path P from the source to the sink and a point x of $X^{\preccurlyeq s}_{\ell,u}$ can be expressed linearly

$$x_i = u_i y_i + (s_i - 1)t_i + s_i z_i \qquad \text{for } i = 1, \dots, n, \tag{2.1}$$

where y_k, t_k, z_k are respectively associated to each upper arc, diagonal arc, and lower arc and have value 1 when they belong to the path P, and 0 otherwise. Therefore, if Q denotes the convex hull, in the (y, z, t) variables, of the incidence vectors of the paths from the source to the sink in our digraph, then we have the following extension of $\text{conv}(X^{\preccurlyeq s}_{\ell,u})$:

$$\text{conv}(X^{\preccurlyeq s}_{\ell,u}) = \text{proj}_x\{(x, y, z, t) : x \text{ satisfies (2.1) for some } (y, z, t) \text{ in } Q\}.$$

By [56, Theorem 13.10], since the digraph is acyclic, Q is described by:

- the sum of the values of the arcs entering the sink has to be 1,
- the sum of the values of the arcs entering each node but the source and the sink equals the sum of the values of the arcs leaving the node,
- nonnegativity of the variables.

Given the structure of the digraph in Figure 2.3, the projection of the above extended formulation onto the x variables (hence, the description of conv$(X^{\preccurlyeq s}_{\ell,u})$) is not difficult to obtain by induction, see [5]. Then, no inequality $ax \leq b$ with a negative coefficient in a is valid for a submissive. Therefore, thanks to Observation 2.1, to retrieve from the description of conv$(X^{\preccurlyeq s}_{\ell,u})$ that of the associated top-lexicographical polytope, one first discards from the resulting inequalities the ones having negative coefficients. Let us skip the details, the inequalities obtained are:

$$(x_k - s_k) + (u_k - s_k) \sum_{i=1}^{k-1} \left(\prod_{j=i+1}^{k-1} (u_j - s_j + 1) \right) (x_i - s_i) \leq 0, \text{ for } k = 1, \dots, n. \tag{2.2}$$

To conclude, a small geometric argument shows that no other inequalities are needed. We refer to [5] for a detailed proof of the following.

Theorem 2.2 ([5]). $L^{\preccurlyeq s}_{\ell,u} = \{x \in \mathbb{R}^n$ *satisfying* (2.2) *and* $\ell \leq x \leq u\}$.

2.2 The Circuit Polytope in Series-Parallel Graphs

In this section, we will study the circuit polytope in a specific class of graphs. A *circuit* in an undirected graph is a subset of edges inducing a connected subgraph in which every node has degree two[1]. To each circuit C of a graph $G = (V, E)$ is associated its incidence vector χ^C in $\{0, 1\}^E$, whose e^{th} coordinate equals 1 if and only if e is in C. The convex hull of these incidence vectors of circuits forms the *circuit polytope of* G, denoted by $C(G)$. Since finding a circuit of maximum cost in a planar graph is NP-complete [32], it is unlikely to obtain a complete description of the circuit polytope associated to such graphs. We consider a subclass of planar graphs: series-parallel graphs.

We explain how induction combined with Balas' Theorem [6] implicitely gives an extended formulation for the circuit polytope of a series-parallel graph. This idea can be applied to various combinatorial optimization problems.

To be able to retrieve the description of the polytope into the original space[2], one has to have a good guess of the result so that induction can be used confortably in the projection. Such a guess might come from the study of a bunch of examples. This is what we did with S. Borne, P. Fouilhoux, M. Lacroix, and P. Pesneau in [8] for the circuit polytope of a series-parallel graph.

Here, we shall only explain how to obtain an implicit extended formulation, and we will not explain how to use it to obtain the description in the original space, the reason being that the resulting description is rather technical. For these details, we refer the interested reader to [8].

Note that an edge whose removal disconnects the graph belongs to no circuit, hence we will assume that there are no such edges. A graph is 2-*connected* if it remains connected after the removal of any node. A circuit of a graph G is a circuit of one its 2-connected components, where a 2-*connected component of* G is a maximal 2-connected subgraph of G. Thus, we will restrict ourselves to 2-connected graphs.

There is a constructive characterization of series-parallel graphs, and we shall use it here as a definition. 2-*connected series-parallel graphs* are built as follows: starting from the circuit of length two, one repeatedly performs one of the following operations:

- *parallelization*: add an edge parallel to an existing one;
- *subdivision*: replace an edge by a path of length two.

These operations allows induction to be used because the set of circuit of a 2-connected series-parallel graphs is deduced straightforwardly from the circuits of the smaller graph it was obtained from.

More precisely, let $G = (V, E)$ be a 2-connected series-parallel graph. When G is obtained from a graph H by replacing an edge e of H by a path of length two $\{e, f\}$, the circuits of G are either circuits of H not containing e, or circuits $C \cup \{f\}$ for each circuit C of H containing e.

Therefore, the circuits of G contain either both e and f or none of them, and their incidence vectors can be described as follows according to a description of the circuits of H.

Observation 2.3. *Suppose G is obtained from H by replacing an edge e of H by a path of length two $\{e, f\}$. Then, adding a variable x_f to any extended formulation of $C(H)$ and imposing $x_e = x_f$ provides an extended formulation for $C(G)$.*

[1] We also consider that the emptyset is a circuit.

[2] That is, to project the implicit formulation.

When G is obtained from a graph H by adding an edge f parallel to an edge e of H, then the circuits of G are each circuit of H, the circuits obtained from those of H containing e by replacing e by f, and the circuit of length two $\{e, f\}$. Then, here is how to derive an extended formulation for the circuit polytope of G if we have one for the circuit polytope of H.

Lemma 2.4 ([8]). *Suppose G is obtained from H by adding a parallel edge f to an edge e of H and let $Q(H)$ be an extended formulation of $C(H)$. Then,*

1. *The polytope $S(G)$ obtained by replacing x_e by $x_e + x_f$ in $Q(H)$ and setting $0 \leq x_e$ and $0 \leq x_f$ is an extended formulation of the convex hull of the incidence vectors of all the circuits of G different from $\chi^{e,f}$.*

2. *The convex hull of $S(G)$ union $\chi^{e,f}$ is an extended formulation of $C(G)$.*

Lemma 2.3 and Statement 1 of Lemma 2.4 can be applied immediately to any extended formulation of the circuit polytope of H. To obtain an extended formulation of the circuit polytope of G, one has to handle Statement 2 of Lemma 2.4, and this is done using the following theorem of Balas [6]. His result gives an extended formulation for any finite union of polyhedra, yet we only state what we need here, the union of two polytopes.

Theorem 2.5 (Balas [6]). *Given two polytopes $P_1 = \{x \in \mathbb{R}^n : A^1x \leq b^1\}$ and $P_2 = \{x \in \mathbb{R}^n : A^2x \leq b^2\}$, we have* $\text{conv}(P_1 \cup P_2) = \text{proj}_x(Q)$*, where $Q = \{x = y^1 + y^2, A^1y^1 \leq (1-\lambda)b^1, A^2y^2 \leq \lambda b^2, 0 \leq \lambda \leq 1\}$.*

The idea in Balas' result is to express in a linear way that x is a convex combination $x = \lambda x^1 + (1-\lambda)x^2$ of x^1 in P_1 and x^2 in P_2 with $0 \leq \lambda \leq 1$. Since λx^1 is not a linear expression, this is done by replacing λx^1 by y^1 and saying that y^1 belongs to λP_1. Similarly, $(1-\lambda)x^2$ is replaced by y^2 which belongs to $(1-\lambda)P_2$, and x is simply $y^1 + y^2$.

An immediate consequence of Theorem 2.5 is the following.

Corollary 2.6. *Given two polytopes P_1 and P_2, there exists an extended formulation of* $\text{conv}(P_1 \cup P_2)$ *whose size is two plus the sizes of P_1 and P_2.*

Note that the extended formulation given by Statement 1 of Lemma 2.4 involves two new inequalities, and that applying Corollary 2.6 in Statement 2 of Lemma 2.4 provides an extended formulation with two more inequalities. Thus, if G is obtained from H by adding a parallel edge, then an extended formulation for the circuit polytope of G has four more inequalities than an extended formulation for $C(H)$. Moreover, if G is obtained from H by subdividing an edge, then an extended formulation for $C(G)$ has the size of an extended formulation for $C(H)$.

By construction of 2-connected series-parallel graphs, and since the circuit polytope of the circuit of length two $\{e, f\}$ is described by the inequalities $x_e = x_f, x_e \geq 0, x_f \geq 0, x_e + x_f \leq 2$, repeatedly applying the above observations gives the following: there exists an extended formulation for the circuit polytope of G of size $O(|E|)$.

The extended formulation is implicit. We refer to [8] for how to use the above remarks to explicitly obtain the description of the circuit polytope of a series-parallel graph.

Part II

BOX-TOTALLY DUAL INTEGRAL POLYHEDRA

Chapter 3

Box-Total Dual Integrality, Box-Integrality, and Equimodular Matrices

Box-total dual integral (box-TDI) systems are the linear systems that yield the strongest min-max relations. To introduce this concept, we will overview the MaxFlow-MinCut theorem of Ford and Fulkerson [31]: the linear system behind it is a box-TDI one.

Then, we will get to the main topic of this chapter: box-TDI polyhedra, the polyhedra that can be described by such systems. The goal here is to provide and explain several characterizations of these polyhedra. Cones play an important role towards that again, but not in the way they did in Part I. Indeed, a crucial observation here is that every polyhedron is the intersection of its minimal tangent cones.

Thus, we will first characterize box-TDI cones, in two ways. The first is about their integrality properties; and the second involves matrices defining their faces. Thanks to the connection between a polyhedron and its minimal tangent cones, these characterizations of box-TDI cones will offer characterizations of box-TDI polyhedra. We will use these characterizations to retrieve easily well-know properties of box-TDI polyhedra. To conclude, we introduce a class of matrices generalizing totally unimodular matrices: they are the matrices for which all the associated polyhedra are box-TDI, and they are called totally equimodular matrices. We will briefly discuss some of their properties.

This chapter is mainly based on joint work with P. Chervet and L.-H. Robert [14], and also on a ongoing work with P. Chervet, M. Lacroix, F. Pisanu, L.-H. Robert, and R. Wolfler Calvo [13].

3.1 MaxFlow–MinCut: A Celebrated Box-TDI System

Consider a directed graph having a source s and a sink t, in which each arc has an integer capacity. A flow in this graph is composed of a nonnegative value on each arc not exceeding its capacity so that the amount of flow entering each node other than the source and the sink equals the amount of flow leaving the node. It can be tought of as water running trough a set of pipes. The value of the flow is the amount of flow leaving the source. The *maximum flow problem* asks to send a maximum amount of flow from the source to the sink.

Ford and Fulkerson [31] proved that the value of a maximum flow is always equal to the minimum capacity of an st-cut, an st*-cut* being a set of arcs leaving a set of nodes containing the source but not the sink. Their result, called the MaxFlow-MinCut theorem, is probably the most famous result in combinatorial optimization. They used it as a stopping criterion to devise a combinatorial algorithm that finds a maximum flow in polynomial time.

Beside the immediate applications of water transit or running fuel through pipelines, the maximum flow problem can also model assignment problems for instance. It is also a powerful theoretical tool to model new problems. We saw an example in Section 2.2 where we modeled the componentwise maximal points of the lexicographical polytope as paths from the source to the sink of a particular digraph, and actually such a path is nothing but a flow of value 1. The problem of finding an st-cut of minimum capacity also has applications in network design and image segmentation.

The maximum flow problem starts this chapter because of the integrality properties of its formulation as a linear problem, thus let us write down this linear problem. Let $D = (V, A)$ be a directed graph, with a source s in V and a sink t in V, and a capacity $c_a \in \mathbb{Z}_+$ for each arc a in A. Associate to each arc a in A a variable $x_a \in \mathbb{R}$ that will represent the amount of flow going through the arc. The maximum flow problem is expressed as follows by means of linear equalities and inequalities:

$$
\begin{aligned}
\max \quad & x(\delta^+(s)) \\
& x(\delta^+(v)) = x(\delta^-(v)), \text{ for all } v \in V \setminus \{s,t\}, \\
& x_a \le c_a, \text{ for all } a \in A, \\
& x_a \ge 0, \text{ for all } a \in A,
\end{aligned}
\tag{3.1}
$$

where $\delta^+(v)$ is the set of arcs leaving node v, $\delta^-(v)$ is the set of arcs entering node v, and for $B \subseteq A$ we denote $x(B) = \sum_{b \in B} x_b$. The function $x(\delta^+(s))$ is called the *objective*.

In System (3.1), the family of equalities ensures the flow conservation at each node but the source and the sink, and the two families of inequalities impose nonnegativity of the values and respecting the capacities. This system has strong integrality properties that we explain below, and so does the associated polyhedron, the one described by the equalities and inequalities in System (3.1).

Indeed, Ford and Fulkerson [31] proved that there always exists a maximum flow with only integer values. Actually, this also holds whichever objective function is chosen in System (3.1). This implies that the associated polyhedron is integer, which means here that all its vertices are integer. Moreover, it is also the case in the dual problem — recall that to every linear problem is associated a dual that has the same value. Here, for all integer objective functions in System (3.1), there exists an integer optimal solution in the dual. Systems with this property are called *totally dual integral (TDI)* systems — thus, System (3.1) is TDI. We mention that its dual problem amounts to finding an st-cut of minimum capacity.

TDI systems are systems that yield min-max relation between combinatorial objects, and numerous graph theoretical results can be seen as the TDIness of a specific linear system. We refer the reader to [56] for many such examples. A more formal definition will be given in the next section.

Not only that, but all the integrality properties of the dual hold for any choise of capacities, and still hold if lower bounds are imposed on the variables x_a. Such bounds are sometimes called box-constraints — they mean intersecting the polyhedron with a box — and TDI systems that remain TDI after the addition of any box-constraints are called *box-totally dual integral (box-TDI)* systems. Not all TDI systems are box-TDI, hence the latter systems yield stronger min-max relations.

There is a somewhat combinatorial interpretation of box-TDIness compared to TDIness, further than the possibility of adding bounds to the variables while preserving the existence of integer solutions in the dual. Each addition of bound on the primal variables makes a new variable appear in the dual. Its cost is given by the value of the bound. These variables represent the following variant of the original dual problem: one is allowed to decrease or increase at a certain cost, by an integer amount, the primal objective function before solving the resulting dual problem. Depending on what the primal variables represent, this can have various combinatorial interpretations.

The scope of this chapter is the study of the polyhedra associated to box-TDI systems. These polyhedra are called *box-TDI polyhedra*, as they are the ones that can be described by box-TDI systems — which is not the case of all polyhedra. We will characterize them in several ways, matricially and geometrically. To get a hint of where matrices come into play, we mention that the box-TDIness of System (3.1) comes from the total unimodularity of the incidence matrix of a directed graph. We shall briefly review this in the next sections, and strengthen the link between box-TDI polyhedra and totally unimodular matrices. An interesting generalization of these matrices will appear along the way, and we will make a small detour to talk about some of their properties.

Box-TDI systems and polyhedra received a lot of attention from the combinatorial optimization community around the eighties. Originally, box-TDI systems were closely related to totally unimodular matrices. Indeed, any system with a totally unimodular matrix of constraints is box-TDI. Actually, until recently, the vast majority of known box-TDI systems were defined by a totally unimodular matrix, see [56] for examples. When the constraint matrix is not totally unimodular, proving that a given system is box-TDI can be quite a challenge: one has to prove its TDIness, and then to deal with the addition of box-contraints that perturb the combinatorial interpretation of the underlying min-max relation. Ding, Feng, and Zang prove in [24] that it is co-NP-complete to recognize box-TDI systems.

They are the center of a renewed interest since the last decade and many deep results appeared about them. This renewed interest in box-TDI systems might be due to the development of new tools to prove box-TDIness, such as the ESP property of Chen, Chen, and Zang [10], which is a sufficient condition for some systems to be box-TDI. Due to its purely combinatorial nature, the ESP property is successfully used to characterize: box-Mengerian matroid ports in [10], the box-TDIness of the matching polytope in [25], subclasses of box-perfect graphs in [26]. Prior to the development of the ESP property, the main tool to prove box-TDIness was [55, Theorem 22.9] of Cook. Its pratical application turns out to be quite technical as one has to combine polyhedral and combinatorial considerations, such as in [12] where the box-TDIness of a system describing the 2-edge-connected spanning subgraph polytope on series-parallel graphs is proved.

Despite the recent progress, Ding, Tan, and Zang [25] write in 2018 that there still “lacks a proper tool for establishing box-total dual integrality”. The characterizations we explain next might be useful in that regard as they allow to prove the box-TDIness of systems in two disjoint steps: find a TDI system describing the polyhedron on the one hand, and, on the other hand, apply one of the characterizations to prove the box-TDIness of the polyhedron. In particular, when a TDI system that describes the polyhedron is already known, these characterizations allow to pick whichever system — TDI or not — describing the polyhedron, and to use algebraic tools to prove the “box” part.

3.2 Characterizations of Box-TDI Polyhedra

3.2.1 Preliminaries

In this section, we provide the definitions and a few results that will be used in this chapter. This section might feel a bit terse, and some readers could prefer skipping it and coming back to it punctually, when needed. Throughout, all entries will be rational.

Lattices

The *lattice* generated by a set V of vectors of $\mathbb{Q}^n$ is the set of integer combinations of these vectors, and is denoted by $\text{lattice}(V) = \{\sum_{v \in V} \lambda_v v : \lambda_v \in \mathbb{Z} \text{ for all } v \in V\}$. The lattice generated by the column vectors of a matrix A is denoted by $\text{lattice}(A)$. A *basis of a lattice* $\mathcal{L}$ is a set B of independent vectors such that $\mathcal{L} = \text{lattice}(B)$. When a lattice in dimension n is generated by n independent vectors, it is well-known [55, Page 49] that all the bases of this lattice yield matrices with the same determinant in absolute value.

Matrices

An element A of $\mathbb{Q}^{m \times n}$ will be thought of as a matrix with m rows and n columns, and an element b of $\mathbb{Q}^m$ as a column vector. When all their entries belong to $\mathbb{Z}$, we will call them *integer*. The i^{th} row vector of A will be denoted by $a_i^\top$. When $\text{rank}(A) = m$, we say that A has *full row rank*.

A matrix is *totally unimodular* if the determinants of its square submatrices are equal to -1, 0, or 1. A rational $r \times n$ matrix is *equimodular* if it has full row rank and its nonzero $r \times r$ determinants all have the same absolute value. Equimodular matrices are studied under the name of matrices with the Dantzig property in [42] or as unimodular sets of vectors in [41]. In particular, we have the following.

Theorem 3.1 (Heller [41]). *For a full row rank $r \times n$ matrix A, the following statements are equivalent.*

1. *A is equimodular.*
2. *For each nonsingular $r \times r$ submatrix D of A,* $\text{lattice}(D) = \text{lattice}(A)$.
3. *For each nonsingular $r \times r$ submatrix D of A, $D^{-1}A$ is totally unimodular.*
4. *There exists a nonsingular $r \times r$ submatrix D of A such that $D^{-1}A$ is totally unimodular.*

The last two statements say that equimodular matrices are totally unimodular matrices written in another basis (of the columns).

Polyhedra

Polyhedra and faces. Given A in $\mathbb{Q}^{m \times n}$ and b in $\mathbb{Q}^m$, the set $P = \{x \in \mathbb{R}^n : Ax \leq b\} = \{x \in \mathbb{R}^n : a_i^\top x \leq b_i, i = 1, \ldots, m\}$ is a *polyhedron*. We will often simply write $P = \{x : Ax \leq b\}$. The matrix A is the *constraint matrix* of P. The *translation of P by* $w \in \mathbb{R}^n$ is $P + w = \{x + w : x \in P\}$.

A *face* of P is a nonempty[1] set obtained by imposing equality on some inequalities in the description of P, that is, a nonempty set of the form $F = \{x : a_i^\top x = b_i, i \in I\} \cap P$ where $I \subseteq \{1, \dots, m\}$. A row $a_i^\top$ or an inequality $a_i^\top x \leq b_i$ with $F \subseteq \{x : a_i^\top x = b_i\}$ is *tight for* F, and $A_F x \leq b_F$ will denote the inequalities from $Ax \leq b$ that are tight for F. The set of points contained in F and in no face $F' \subset F$ forms the *relative interior of* F. Let $\text{lin}(F) = \{x : A_F x = \mathbb{0}\}$ and $\text{aff}(F) = \{x : A_F x = b_F\}$. The *dimension* $\dim(F)$ of a face F is the dimension of its affine hull $\text{aff}(F)$. A *facet* is a face that is inclusionwise maximal among all faces distinct from P. A face is *minimal* if it contains no other face of P. Minimal faces are affine spaces. A face of dimension 0 is called a *vertex*.

A polyhedron is *integer* if each of its minimal faces contains an integer point. An *integer box* is a polyhedron of the type $\{x : \ell \leq x \leq u\}$, with ℓ and u in $\mathbb{Z}^n$. A polyhedron is *box-integer* if its intersection with any integer box $\{\ell \leq x \leq u\}$ is integer[2].

Cones. A *polyhedral cone* is a polyhedron of the form $C = \{x : Ax \leq \mathbb{0}\}$. Since all the cones involved here are polyhedral, we simply write *cone*. A cone C can also be described as the set of nonnegative combinations of a finite set of vectors $R \subseteq \mathbb{R}^n$, and we say that $C = \text{cone}(R)$ is *generated by* R. By Minkowski-Weyl's theorem [18, Theorem 3.13], P is a polyhedron if and only if there exists a finite set of points V and of vectors R such that $P = \text{conv}(V) + \text{cone}(R)$. The set $\text{cone}(R)$ is called the *recession cone* of P, and is denoted by $\text{rec}(P)$. When $P = \{x : Ax \leq b\}$, its recession cone is described by $\text{rec}(P) = \{x : Ax \leq \mathbb{0}\}$. A *conic polyhedron* is a rational translation of a cone, that is, a set of the form $t + \{x : Ax \leq \mathbb{0}\}$ for some $t \in \mathbb{Q}^n$.

The *polar cone* of a cone $C = \{x : Ax \leq \mathbb{0}\}$ is the cone $C^* = \{x : z^\top x \leq 0 \text{ for all } z \in C\}$. Equivalently, C^* is the cone generated by the columns of $A^\top$. Note that $C^{**} = C$.

Given a face F of a polyhedron $P = \{x : Ax \leq b\}$, the *tangent cone associated to* F is the conic polyhedron $C_F = \{x : A_F x \leq b_F\}$. When F is a minimal face of P, its associated tangent cone is a *minimal tangent cone of* P. Informally speaking, the tangent cone associated with a face F is what you see of the polyhedron when you sit at a point which is in the relative interior of F.

Face-defining matrices. Let $P = \{x : Ax \leq b\}$ be a polyhedron of $\mathbb{R}^n$ and F be a face of P. A full row rank matrix M such that $\text{aff}(F)$ can be written $\{x : Mx = d\}$ for some d is *face-defining for* F. Such matrices are called *face-defining matrices of* P[3]. Note that face-defining matrices need not correspond to valid inequalities for the polyhedron. A face-defining matrix for a facet of P is called *facet-defining*.

In practice, we shall use the following observation to prove that a matrix is face-defining. It says that for a matrix M to be face-defining for a given face, sufficiently many affinely independent points of the face have to belong to an affine space defined by M.

Observation 3.2. *A full row rank matrix $M \in \mathbb{Q}^{k \times n}$ is face-defining for a face F of a polyhedron $P \subseteq \mathbb{R}^n$ if and only if there exist a vector $d \in \mathbb{Q}^k$ and a family $\mathcal{H} \subseteq F \cap \{x : Mx = d\}$ of* $\dim(F) + 1$ *affinely independent points such that* $|\mathcal{H}| + k = n + 1$.

[1] In the standard definition, the emptyset is a face. It is not the case here in order to lighten some statements.

[2] Allowing infinite values for ℓ and u yields an equivalent definition.

[3] When we write that a face F has a face-defining matrix M, we mean that M is face-defining for the face F of P, which is more restrictive than being a face-defining matrix of the polyhedron F.

Linear systems

TDI systems. A linear system $Ax \leq b$ is *totally dual integral* (*TDI*) if the minimum in the linear programming duality equation $\max\{w^\top x : Ax \leq b\} = \min\{b^\top y : A^\top y = w,\, y \geq \mathbb{0}\}$ has an integer optimal solution for all integer vectors w for which the optimum is finite. Every polyhedron can be described by a TDI system [55, Theorem 22.6]. Moreover, the right hand side of such a TDI system can be chosen integer if and only if the polyhedron is integer [28], as stated in the following theorem.

Theorem 3.3 ([55, Theorem 22.6(i)])**.** *For each polyhedron P there exists a TDI-system $Ax \leq b$ with A integer and $P = \{x : Ax \leq b\}$. Here b can be chosen to be integer if and only if P is integer.*

We will also use that dilation preserves TDIness [55, Page 312], that is, if $Ax \leq b$ is TDI, then so is $Ax \leq \alpha b$ for all α in $\mathbb{Q}_+$.

Box-TDI systems. A linear system $Ax \leq b$ is a *box-totally dual integral* if $Ax \leq b$, $\ell \leq x \leq u$ is TDI for each pair of rational vectors ℓ and u. In other words, $Ax \leq b$ is box-TDI if

$$\min\{b^\top y + u^\top r - \ell^\top s : A^\top y + r - s = w,\, y \geq \mathbb{0},\, r, s \geq \mathbb{0}\} \tag{3.2}$$

has an integer solution for all integer vectors w and all rational vectors ℓ, u[1] for which the optimum is finite.

Since dilation preserves TDIness and since box-TDIness involves all rational bounds, dilation also preserves box-TDIness: if $Ax \leq b$ is box-TDI, then so is $Ax \leq \alpha b$ for all α in $\mathbb{Q}_+$. It is well-known that box-TDI systems are TDI [55, Theorem 22.7]. General properties of such systems can be found in [19], [56, Chap. 5.20] and [55, Chap. 22.4]. Though not every polyhedron can be described by a box-TDI system, the result of Cook [19] below proves that being box-TDI is a property of the polyhedron.

Theorem 3.4 (Cook [19, Corollary 2.5])**.** *If a system is box-TDI, then any TDI system describing the same polyhedron is also box-TDI.*

This theorem justifies the following definition due to Cook [19]: a polyhedron that can be described by a box-TDI system is called a *box-totally dual integral* polyhedron. Note that dilation also preserves box-TDIness of a polyhedron.

3.2.2 Box-Integer Cones

The results presented in this section and the next are taken from [14]. The exposition proposed in these sections is quite different from that of the paper as we chose a more geometric approach.

By definition of box-TDI systems and by Theorem 3.3, given a box-TDI cone $C = \{x : Ax \leq \mathbb{0}\}$ of $\mathbb{R}^n$ with $Ax \leq \mathbb{0}$ box-TDI, the polyhedron $\{x : Ax \leq \mathbb{0}, \ell \leq x \leq u\}$ is integer for all ℓ and u in $\mathbb{Z}^n$. In other words, C is box-integer. It turns out that the converse holds, that is, box-integer cones are precisely box-TDI cones. The proof of the following can be found in [14].

[1] Allowing infinite values for ℓ and u yields an equivalent definition.

Lemma 3.5 ([14]). *A cone is box-TDI if and only if it is box-integer.*

In this section, we will focus on box-integer cones because box-integrality is easier to handle than box-TDIness. In particular, we will characterize box-integer cones with a matricial point of view. First, let us investigate the two dimensional case: what does destroy the box-integrality of a two-dimensional cone?

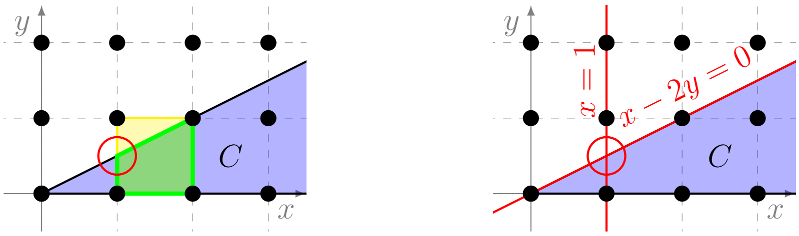


Figure 3.1: A two-dimensional cone that is not box-integer.

In the figure above, the cone C is not box-integer. Indeed, as we can see in the left figure, when intersected with the integer box $\{(x, y) : 1 \leq x \leq 2, 0 \leq y \leq 1\}$, a noninteger vertex appears, $(1, \frac{1}{2})$ to be precise. The figure on the right suggests another way of looking at this phenomenon. First, this vertex lies on the facet of the cone defined by $x - 2y = 0$. Since the coefficient of x and that of y differ in absolute value, the point obtained from this facet by imposing $x = 1$ is the noninteger point $(1, \frac{1}{2})$. Note that $\{(x, y) : x = 1\}$ is an integer box.

This argument holds in general: if a two-dimensional cone has a facet with different coefficients in absolute value, then one can build a noninteger vertex as we just did. Moreover, all vertices of a two-dimensional cone intersected with an integer box are obtained in such a manner. Therefore, a two-dimensional cone is box-integer if and only if all its facet-defining inequalities have their nonzero coefficients equal in absolute value.

Let us confront this first intuition in a higher dimensional case: let us try dimension three. Clearly, the condition we saw in dimension two is still necessary: if a facet has two variables which have different nonzero coefficients in absolute value, then one can build a noninteger vertex as above. This does not happen in the example below, and yet the cone is not box-integer.

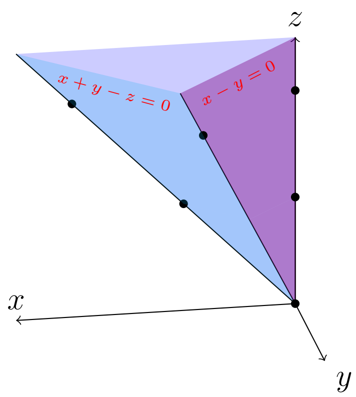
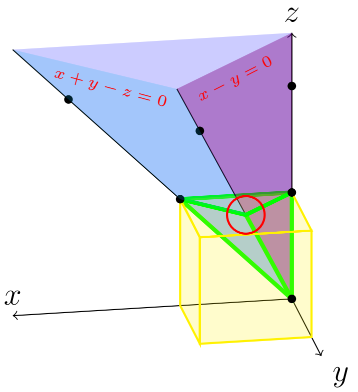


Figure 3.2: The cone C.

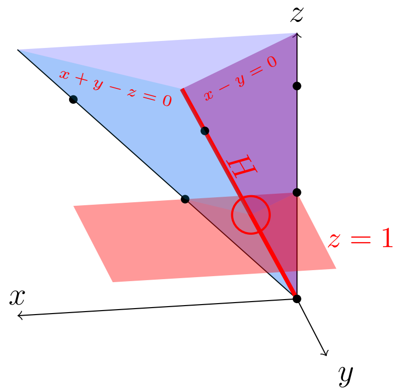
The cone C in Figure 3.2 is described by

$$\begin{aligned} x + y - z &\leq 0, \\ x - y &\leq 0, \\ y &\geq 0. \end{aligned}$$

When intersected with the unit cube $\{x : \mathbb{0} \leq x \leq \mathbb{1}\}$, we get a noninteger vertex $(\frac{1}{2}, \frac{1}{2}, 1)$, as highlighted on the left figure below.



The cone C intersected with the $0/1$ box.



The face H intersected with $\{z = 1\}$.

Let us interpret that in the light of the two-dimensional case. This noninteger vertex lies on the face H of the cone which is the intersection of the facets defined by $x + y - z = 0$ and $x - y = 0$. Written in a matricial form, the face H is the set of solution of the system below that satisfy $y \geq 0$.

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

In the matrix above, the two first column have determinant -2, whereas the two last columns have determinant -1. This means that these sets of columns do not generate the same lattice, and for instance here the third column cannot be written as an integer combinations of the two first columns. Thus, imposing $z = 1$ on this face yields a noninteger point, which is precisely the vertex $(\frac{1}{2}, \frac{1}{2}, 1)$ seen above.

Again, this observation holds in general. When some face of the cone has its affine hull described by $Mx = d$ and M has two nonzero determinants of different absolute values, then the cone is not box-integer. Indeed, it means that the corresponding sets of columns do not generate the same lattice. Thus, one of these columns is a (unique) noninteger combination of a set B of the other columns. Imposing value 1 on the variable associated to this column and 0 to the variables not in B yields a noninteger point, intersection of the cone with an integer box. Actually, there is a little technicality that has to be taken care of: it might be that the resulting point lies outside the cone. But if that happens, then we can always find a suitable integer translation bringing the point back in the cone. This preserves the point's nonintegrality and ensures that the cone is not box-integer.

We recall two definitions introduced in Section 3.2.1: a full row rank matrix is face-defining if it describes the affine hull of some face, and equimodular when all its nonzero maximal determinants have the same absolute value. The above necessary condition for a cone to be box-integer can then be phrased as follows: each of its face-defining matrices has to be equimodular. It turns out that the converse holds, and we refer to [14] for the complete proof of the following result.

Corollary 3.6 ([14]). *A cone is box-integer if and only if all its face-defining matrices are equimodular.*

Since rational translations preserve both box-TDIness and the equimodularity of the face-defining matrices, this result combined with Lemma 3.5 has the following consequence.

Corollary 3.7 ([14]). *A conic polyhedron is box-TDI if and only if all its face-defining matrices are equimodular.*

We mention that Lemma 3.5 has a polar counterpart. Indeed, its proof involves the polar of the cone and the following is implied therein, see [14]: a cone is box-integer if and only if its polar is. Given this specific behaviour, the recognition of box-integer cones might have a different complexity status than the following related problems, which are all co-NP-complete: deciding whether a given polytope is integer [50], deciding whether a given system is TDI or box-TDI [24], deciding whether a given conic system is TDI [49].

Open problem 3.8. *Can box-integer cones be recognized in polynomial time?*

We observe that the general case is hard.

Observation 3.9. *Deciding whether a polyhedron is box-integer is co-NP-complete.*

Indeed, for a polytope contained in the $0/1$ box, box-integrality is equivalent to integrality. The above observation follows because deciding whether a polytope contained in the $0/1$ box is integer is co-NP-complete [24].

3.2.3 Characterizations of Box-TDI Polyhedra

In this section, we shall see how the previous characterizations of box-TDI cones and conic polyhedra yield characterizations of box-TDI polyhedra.

A first connection between the box-TDIness of a polyhedron and that of a specific set of conic polyhedra is given in the following result, which can be proved straightforwardly using only the definition of box-TDIness, see [14].

Lemma 3.10 ([14]). *A polyhedron is box-TDI if and only if all its minimal tangent cones are.*

This lemma will allow us to transfer the previous characterizations of box-TDI cones to box-TDI polyhedra: there will be matricial and geometric counterparts.

Matricial Characterizations

Since minimal tangent cones are conic polyhedra, their box-TDIness is characterized in Corollary 3.7, namely, box-TDI conic polyhedra are the ones for which every face-defining matrix is equimodular. As in Figure 3.4, note that each face of P is contained in a face of some minimal tangent cone of P having the same affine hull. Conversely, each face of a minimal tangent cone of P contains some face of P having the same affine hull. Therefore, Corollary 3.7 and Lemma 3.10 give the following matricial characterization of box-TDI polyhedra.

Theorem 3.11 ([14])**.** *A polyhedron is box-TDI if and only if all its face-defining matrices are equimodular.*

This characterization gives an easy way to disprove box-TDIness: it is enough to exhibit a face-defining matrix having two maximal nonzero determinants of different absolute values. In particular, this provides a simple co-NP certificate for the box-TDIness of a polyhedron.

We mentioned that, if a face has an equimodular face-defining matrix, then all the matrices defining this face are equimodular. Indeed, the other face-defining matrices for this face are obtained by a basis change of the columns, and such a basis change preserves equimodularity as it multiplies all the maximal determinants by the same value. Among all these possible changes of basis, as Statement 3 of Theorem 3.1 says, if we chose a basis within the given matrix, then we obtain a totally unimodular matrix. Therefore, Theorem 3.11 can be reformulated using totally unimodular matrices.

Theorem 3.12 ([14])**.** *A polyhedron is box-TDI if and only if each of its faces has a totally unimodular face-defining matrix.*

We conclude the matricial characterizations with a remark about polarity. The *polar of a polyhedron* P is the set $P^\circ = \{x : x^\top y \leq 1 \text{ for all } y \in P\}$. The polarity result we mentioned for cones in the previous section does not extend to polyhedra, that is, the polar of a box-TDI polyhedron need not be box-TDI. For instance, the polyhedron conv $((2,-1),(-2,-1),(0,1))$ is integer and box-TDI, and its polar conv $((1,1),(-1,1),(0,-1))$ is integer but not box-integer.

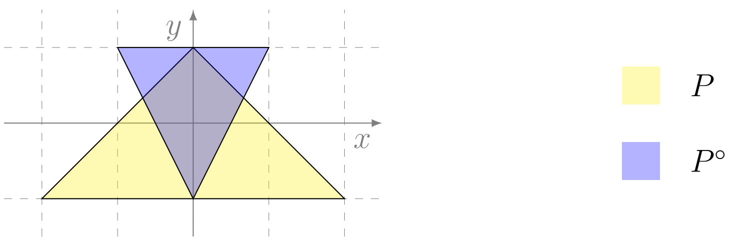


Figure 3.3: An integer box-TDI polyhedron P with a polar P° that is not box-TDI.

Nevertheless, since the affine hull of each face of P is the translation of a cone, Theorem 3.12 and polarity applied to these cones imply the following characterization.

Theorem 3.13 ([14])**.** *A polyhedron P is box-TDI if and only if for every face F of P, there exists a basis of* $\text{lin}(F)$ *which forms a totally unimodular matrix.*

A Geometric Characterization

If one combines Lemma 3.10 with Lemma 3.5, that is, with the fact that the box-TDI cones are precisely the box-integer cones, one gets another characterization of box-TDI polyhedra, essentially more geometric. To see how, let us first show that:

$$\text{An integer polyhedron } P \text{ is box-TDI if and only if } kP \text{ is box-integer for all } k \in \mathbb{Z}_{>0}. \quad (3.3)$$

Let P be an integer polyhedron. Suppose that P is box-TDI and described by the box-TDI system $Ax \leq b$ with A and b integer, which exists by Theorem 3.3. Recall that dilation preserves box-TDIness: for $k \in \mathbb{Z}_{>0}$, the polyhedron kP is integer, box-TDI, and described by the box-TDI system $Ax \leq kb$. In particular, by definition of box-TDI systems and by Theorem 3.3, the polyhedron $\{x : Ax \leq kb, \ell \leq x \leq u\}$ is integer for all ℓ and u in $\mathbb{Z}^n$. In other words, kP is box-integer for all $k \in \mathbb{Z}_{>0}$.

Let us mention a more geometric way of proving this direction, without using Theorem 3.3. The following picture shows that every polyhedron is the intersection of its minimal tangent cones.

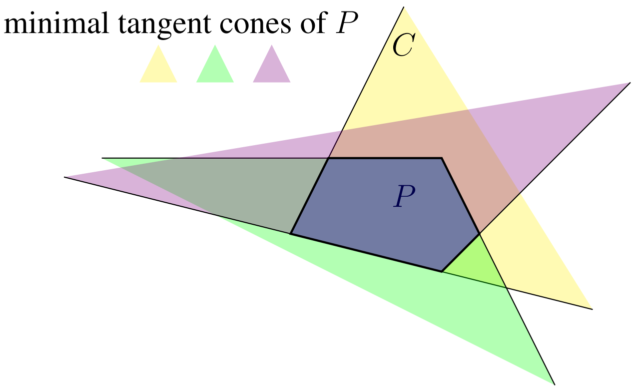


Figure 3.4: Every polyhedron is the intersection of its minimal tangent cones.

Actually, as Figure 3.5 suggests, this somewhat also holds for kP with $k \in \mathbb{Z}_{>0}$.

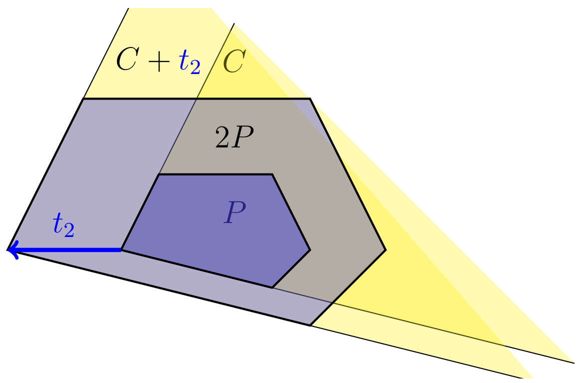


Figure 3.5: kP is the intersection of integer translations of the minimal tangent cones of P.

More precisely, since P was assumed to be integer, kP is the intersection of integer translations of the minimal tangent cones of P. Now, suppose that kP is not box-integer for some $k \in \mathbb{Z}_{>0}$, that is, $kP \cap \{x : \ell \leq x \leq u\}$ has a noninteger vertex z, for some $\ell, u \in \mathbb{Z}^n$. Let C_z be a minimal tangent cone of kP having z on its boundary. Clearly, $C_z \cap \{x : \ell \leq x \leq u\}$ has z as a noninteger vertex, thus C_z is not box-integer. But C_z is the integer translation of some minimal tangent cone of P, therefore some minimal tangent cone of P is not box-integer. Since integer translations preserve box-integrality, this minimal tangent cone is not box-TDI by Lemma 3.5, and hence P is not box-TDI by Lemma 3.10.

For the other direction, suppose that P is not box-TDI. Since P is integer, and since integer translations preserve box-integrality, Lemmas 3.5 and 3.10 implies that some minimal cone C

of P is not box-integer. As the following picture shows, the cone C is the union of integer translations of the dilations of P.

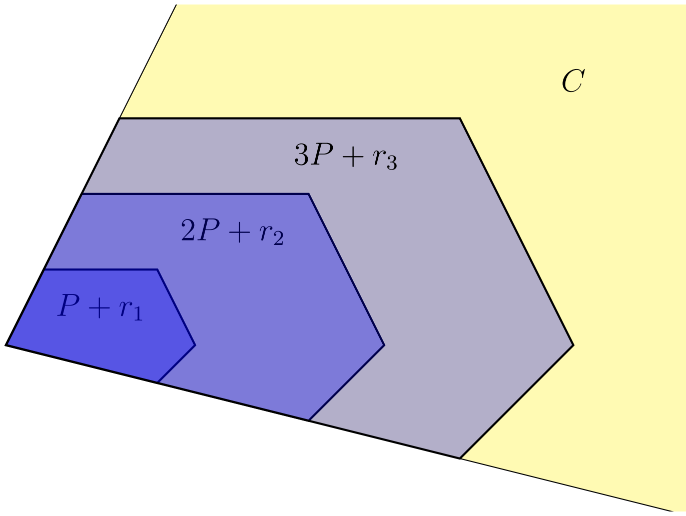


Figure 3.6: C is the union of integer translations of kP over $k \in \mathbb{Z}_{>0}$.

Therefore, since $C \cap \{x : \ell \leq x \leq u\}$ has a noninteger vertex z, for some $\ell, u \in \mathbb{Z}^n$, there exists a integer translation $kP + r_k$ of a dilation kP of P such that $kP + r_k \cap \{x : \ell \leq x \leq u\}$ has a noninteger vertex z. That is, $kP+r_k$ is not box-integer. Since integer translations preserve box-integrality, kP is not box-integer.

When the polyhedron P is not integer, by the rationality of the entries, there exists a smallest $q \in \mathbb{Z}_{>0}$ such that qP is an integer polyhedron. Note that P is box-TDI if and only if qP is box-TDI. Moreover, the minimality of q implies that the set of $k \in \mathbb{Z}_{>0}$ such that kP is integer is $q\mathbb{Z}_{>0}$ (this is easy to see using Bézout's Lemma, see [14, Proposition 1]). Thus, applying (3.3) to the integer polyhedron qP, we have the following characterization of box-TDI polyhedra.

Theorem 3.14 ([14])**.** *A polyhedron P is box-TDI if and only if the polyhedron kP is box-integer for each k such that kP is integer.*

3.2.4 Consequences

Here, we review several known results about box-TDI polyhedra that can be derived from the results presented in Sections 3.2.2 and 3.2.3. We provide the proofs because almost each of them is immediate. The *dominant* of a polyhedron P of $\mathbb{R}^n$ is $\text{dom}(P) = P + \mathbb{R}^n_+$.

Consequence 3.15 ([19, Theorem 3.6] or [55, Theorem 22.11])**.** *The dominant of a box-TDI polyhedron is box-TDI.*

Proof. A face of $\text{dom}(P)$ is the sum of a face of P and a cone generated by unit vectors. By Theorem 3.13, and since adding unit vectors preserves total unimodularity, the dominant of a box-TDI polyhedron is box-TDI. □

Consequence 3.16 ([55, Remark 2.21])**.** *If P is a box-TDI polyhedron, then* $\text{aff}(P) = \{x : Cx = d\}$ *for some totally unimodular matrix C.*

Proof. If P is a box-TDI polyhedron, then by Theorem 3.12, since P is a face of P, its affine hull can be described using a totally unimodular matrix. $\square$

Consequence 3.17 ([55, Remark 2.22]). *Each edge and each extremal ray of a pointed box-TDI polyhedron is in the direction of a* $\{0, \pm 1\}$*-vector.*

Proof. This is Theorem 3.13 applied to the faces of dimension one of the polyhedron. $\square$

Using Theorem 3.12 instead of Theorem 3.13 in the above proof shows that every full-dimensional box-TDI polyhedron can be described using a $\{0, \pm 1\}$ matrix. Edmonds and Giles prove in [28] that it is still true without the full-dimensional hypothesis.

Consequence 3.18 ([28, Theorem 2.16]). *If* P *is a box-TDI polyhedron, then* $P = \{x : Ax \leq b\}$ *for some* $\{0, \pm 1\}$ *matrix* A *and some vector* b.

Proof. Let P be a box-TDI polyhedron. By Consequence 3.16, we have aff$(P) = \{x : Cx = d\}$ for some full row rank totally unimodular matrix C. By Theorem 3.12, for each facet F of P, there exists a totally unimodular matrix D_F such that aff$(F) = \{x : D_F x = d_F\}$. Then, one of the rows $a_F x = b_F$ of $D_F x = d_F$ does not contain aff(P). Possibly multiplying by -1, we may assume that $a_F x \leq b_F$ is valid for P because F is a facet of P. Then, the matrix A whose rows are those of C and every a_F yields a description of P as desired. $\square$

The projection of a box-integer polyhedron along coordinate axes is box-integer. Indeed, if $P = \{x : \text{ there exists } y \text{ such that } (x, y) \in Q\}$ with Q box-integer, then P intersected with an integer box is integer because it is the projection onto the x variables of the intersection of Q with the same integer box. Let us skip a few details: this, together with the geometric characterization of box-TDI polyhedra, yields the following well-known result [55, Page 323].

Consequence 3.19 ([55, Page 323]). *The projection of a box-TDI polyhedron along coordinate axes is box-TDI.*

The following is new and came up thanks to conversations with András Frank. Since it is unpublished, a proof is provided.

Consequence 3.20. *The recession cone of a box-TDI polyhedron is box-TDI.*

Proof. Let $P = Q + \text{rec}(P)$ be a box-TDI polyhedron of $\mathbb{R}^n$, where Q is a polytope. The result comes from the connection between faces of rec(P) and faces of P explicited in Claims 3.21 and 3.22.

Claim 3.21 ([36]). *Each face of* rec(P) *is the recession cone of some face of* P.

Proof. Let F be a face of rec(P). There exists a in $\mathbb{R}^n$ such that F is the face of rec(P) maximizing $a^\top x$ over rec(P), that is, F is the set of points of rec(P) achieving $\max\{a^\top x : x \in \text{rec}(P)\}$.

Let H be the face of P maximizing $a^\top x$ over P. Then, $H = R + F$, where R is the face of Q maximizing $a^\top x$ over Q. Indeed, let h in H. Since h is in P, $h = q + c$ for some q in Q and c in rec(P). Note that q maximizes $a^\top x$ over Q, as if some q' in Q had $aq' > aq$, then $h' = q' + c$ would be in P and would satisfy $a^\top x' > a^\top h$, contradicting the fact that h is in H. That is, q is in R, and similarly rec(P) is in F. Since $R + F \subseteq H$ is immediate, we have $H = R + F$, that is, F is the recession cone of H. $\square$

Claim 3.22. *For each face of* rec(P)*, there is a face of* P *with the same affine full, up to translation.*

Proof. Let F be a face of rec(P). By Claim 3.21, there exists a face H of P such that F is the recession cone of H, that is, $H = R + F$ where R is a polytope. Take such an H of minimum dimension, and let us prove that then aff(H) = aff(F), up to translation. First, since $H = R + F$, one can replace P by some translation of P in order to assume aff$(F) \subseteq$ aff(H).

We proceed by contradiction and assume that equality does not hold. Then, there exists u in aff$(H) \setminus$ aff(F), and we can take such a u to be orthogonal to aff(F). Let H_u be the face of H maximizing $u^\top x$ over H. Then, H_u is a proper face of H. Indeed, H_u is nonempty because u is orthogonal to aff(F) and F is the recession cone of H; and $H_u \neq H$ because u is in aff(H). In particular, $\dim(H_u) < \dim(H)$.

Beside, F is the recession cone of H_u. Indeed, since H_u is a face of H, we have rec$(H_u) \subseteq$ rec$(H) = F$. Moreover, for y in H_u and z in F, the choice of u implies $u^\top(y + z) = u^\top y$. Since F is the recession cone of H, we have $y + z$ in H, and since y maximizes $u^\top x$ over H, we have $y + z$ in H_u. Therefore, rec$(H_u) \supseteq$ rec(H), and we get the equality.

Since H_u is a face of H, and the latter is a face of P, we have that H_u is a face P. Then, $\dim(H_u) < \dim(H)$ contradicts the minimality assumption on the dimension of H. □

We now apply twice Theorem 3.11. Since P is box-TDI, each face of P has an equimodular face-defining matrix. This is a property of the affine hull of the face, and it is preserved under translation. By Claim 3.22, each face of rec(P) inherits this property. Therefore, rec(P) is box-TDI. □

3.3 Totally Equimodular Matrices

Here, we introduce a generalization of totally unimodular matrices for which the associated polyhedra still have nice properties: they are all box-TDI.

Hoffman and Kruskal [43] proved that an integer matrix A is totally unimodular if and only if the polyhedron $\{x : Ax \leq b\}$ is box-integer for all b in $\mathbb{Z}^m$. As noticed by Schrijver [55, Page 318], their result implies the following characterization.

Theorem 3.23 ([55, Page 318])**.** *A matrix A of $\mathbb{Z}^{m\times n}$ is totally unimodular if and only if the system $Ax \leq b$ is box-TDI for all $b \in \mathbb{Z}^m$.*

An equivalent definition of total unimodularity is to ask for every set of linearly independent rows to be unimodular. In this light, it is natural to define *totally equimodular matrices* as those for which all sets of linearly independent rows form an equimodular matrix.

Let $A \in \mathbb{Q}^{m\times n}$ be such a matrix. For all $b \in \mathbb{Q}^n$, each face of the polyhedron $\{x : Ax \leq b\}$ has a face-defining matrix which is formed by a subset of rows of A. By definition of total equimodularity, this matrix is equimodular, hence each face of this polyhedron has an equimodular face-defining matrices. This implies that the polyhedron is box-TDI by Theorem 3.11 and the remark thereafter. In fact, this characterizes totally equimodular matrices. Indeed, suppose that $\{x : Ax \leq b\}$ is box-TDI for all $b \in \mathbb{Q}^m$. Let B be a collection of rows of A of full row rank. By chosing b to be 0 on the coordinates associated to B and a sufficiently large number elsewhere, we ensure that B is face-defining for $\{x : Ax \leq b\}$. The latter being box-TDI, B is equimodular by Theorem 3.11.

Thus, we have the following, which says that relaxing total unimodularity to total equimodularity might not preserve the box-TDIness of the associated linear systems but maintains the box-TDIness of the associated polyhedra.

Theorem 3.24 ([14])**.** *A matrix A of $\mathbb{Q}^{m\times n}$ is totally equimodular if and only if the polyhedron $\{x : Ax \leq b\}$ is box-TDI for all $b \in \mathbb{Q}^m$.*

Since deciding whether a given matrix is totally unimodular can be done in polynomial time, see *e.g.* [55, Chapter 20], Statement 3 of Theorem 3.1 implies that deciding whether a given matrix is equimodular can be done in polynomial time. However, for totally equimodular matrices, the recognition problem remains open.

Open problem 3.25. *Can totally equimodular matrices be recognized in polynomial time?*

Interestingly, it is enough to study totally equimodular matrices with 0, ± 1 coefficients. Indeed, in a totally equimodular matrix, the nonzero coefficients of a given row all have the same absolute value. Thus, such a matrix can be scaled row by row into a 0, ± 1 matrix. This scaling preserves total equimodularity and does not change the family of associated polyhedra.

For totally unimodular matrices, the positive answer to their recognition problem comes from Seymour's decomposition theorem [55, Theorem 19.6]. The starting point of this decomposition theorem is the study of operations preserving total unimodularity. With P. Chervet, M. Lacroix, F. Pisanu, L.-H. Robert, and R. Wolfler Calvo, we started the study of which operations preserve total equimodularity, with the following question in mind.

Open problem 3.26. *Is there a decomposition theorem for totally equimodular matrices?*

We conclude this section by a exhibiting a class of totally equimodular matrices. We believe that this class makes the two open problems above substantial.

The *edge-vertex incidence* matrix of a graph $G = (V, E)$ is the matrix M of $\{0, 1\}^{E \times V}$ such that $M_{ev} = 1$ if and only if the edge e is incident to the node v. It is well-known that the edge-vertex incidence matrix of a graph is totally unimodular if and only if the graph is bipartite. Hence, in this case, it is totally equimodular. We show below that the edge-vertex incidence matrix of a graph is totally equimodular even when the graph is not bipartite. This result is unpublished and is a part of the ongoing work mentioned above, which is why a proof is provided.

Theorem 3.27 ([13]). *The edge-vertex incidence matrix of a graph is totally equimodular.*

Proof. Let $G = (V, E)$ be a graph and let M be a full row rank matrix formed by a subset of k rows of the edge-vertex incidence matrix of G. Let us prove that M is equimodular by induction on its number of rows: the base case is when M has one row, and then M is equimodular since a row has only values in $\{0, 1\}$. The matrix M encodes a subgraph $H = (V, F)$ of G with $k = |F|$ edges. Let W be the vertices of H incident to an edge of F.

We have $|W| \geq |F|$, as otherwise M would have too many columns of zeros to have full row rank. If $|W| = |F|$, then M has exactly one nonsingular $k \times k$ submatrix, hence M is equimodular. If $|W| > |F|$, then H has a vertex u of degree one. Indeed, if every vertex of W had degree at least two we would have $2|F| = \sum\limits_{w \in W} d(w) \geq 2|W|$, a contradiction.

The column of u in M contains a single one, in uv's row, where v is the neighbor of u in H. Let M' be the matrix obtained from M by removing uv's row. A nonsingular $k \times k$ submatrix N of M has to contain at least one of u and v, as otherwise it has only zeros in uv's row. When N contains exactly one of them, then develop with respect to uv's row. When N contains both of them, then develop with respect to u's column. In both cases, this yields a $(k-1) \times (k-1)$ nonzero determinant of N, up to the sign. By the induction hypothesis, N is equimodular, hence all these determinants are equal in absolute value. Therefore, so are the nonzero $k \times k$ determinants of M, and M is equimodular. $\square$

Let A_G be the edge-vertex incidence matrix of a graph $G = (V, E)$. Since multiplying a row by -1 preserves total equimodularity, Theorem 3.24 implies that every polyhedron of the form $\{x \in \mathbb{R}^E : A_G x \lesseqqgtr b\}$ with b rational is box-TDI, where "$\lesseqqgtr$" means that each inequality can be of type "$\leq$", "$\geq$", or "$=$".

In particular, $\{x \in \mathbb{R}^E_+ : A_G x \leq \mathbb{1}\}$ is box-TDI. This polyhedron is called the *edge relaxation* of the stable set polytope of G, because its integer points are precisely the stable sets of G. Since finding a maximal stable set in a given graph is NP-complete [46], Theorem 3.27 implies that integer programming over a box-TDI polyhedron is NP-complete.

Corollary 3.28 ([13]). *Given a box-TDI polyhedron P and a cost vector c, finding an integer point x maximing $c^\top x$ over P is NP-complete.*

Chapter 4

Examples of Box-TDI Polyhedra

This chapter is devoted to examples of box-TDI systems and polyhedra. The results presented here heavily rely on the characterizations of box-TDI polyhedra of Chapter 3.

First, we disprove a conjecture of Ding, Zang, and Zhao [26] about box-perfect graphs. Then, we discuss possible connections between box-total dual integrality and the integer decomposition property. We also shed lights on the equivalence between two results concerning Mengerian clutters. Finally, we provide several box-TDI systems in series-parallel graphs.

4.1 Box-Perfect Graphs

Recall that the stable set polytope of a graph is the convex hull of the incidence vectors of its stable sets, and a clique is a set of pairwise adjacent nodes. *Perfect graphs* [47] are known to be those whose stable set polytope is described by the system composed of the clique inequalities and the nonnegativity constraints:

$$\begin{cases} x(C) \leq 1 & \text{for all cliques } C, \\ x \geq \mathbb{0}. \end{cases}$$

A *box-perfect graph* is a graph for which this system is box-TDI. Since this system is known to be TDI if and only if the graph is perfect [47], a graph is box-perfect if and only if it is perfect and its stable set polytope is box-TDI. The characterization of box-perfect graphs is a long standing open question raised by Cameron and Edmonds [9] in 1982. Recent progress has been made on this topic by Ding, Zang, and Zhao [26]. They exhibit several new subclasses of box-perfect graphs, and in particular prove the conjecture of Cameron and Edmonds [9] that parity graphs are box-perfect. They also propose a conjecture for the characterization of box-perfect graphs.

Before stating their conjecture, we provide the necessary definitions. Given an undirected graph $G = (V, E)$ and a node v in V, we denote by $G \setminus \{v\}$ the graph obtained from G by removing v. For a node subset $W \subseteq V$, the graph $G[W]$ is the *graph induced by* W, that is, the graph obtained by removing all nodes not in W. The graph G is *bipartite* when V can be partitionned into two nonempty sets U and W such that all its edges lie between U and W. In this case, we denote G by (U, W, E). The *biadjacency matrix* of such a graph is the matrix M of $\{0,1\}^{U \times W}$ such that $M_{uw} = 1$ if and only if uw is in E.

The conjecture of Ding, Zang, and Zhao [26] involves the class of graphs $\mathcal{R}$ built as follows. Let $G = (U, W, E)$ be a bipartite graph whose biadjacency matrix M is not totally unimodular but all submatrices of M are. Add a set of edges F between nodes of W such that the neighborhood $N_{G'}(u)$ of u in $G' = (U \cup W, E \cup F)$ is a clique for all $u \in U$. If there exists $u \in U$ such that $N_{G'}(u) = W$, then the graph $G' \setminus \{u\}$ is in $\mathcal{R}$, otherwise G' is in $\mathcal{R}$.

Conjecture 4.1 (Ding, Zang, and Zhao [26])**.** *A perfect graph is box-perfect if and only if it contains no graph from* $\mathcal{R}$ *as an induced subgraph.*

In [14], we provide a construction which preserves non box-perfection, and use it to build the graph below, which is not box-perfect and has no graph from $\mathcal{R}$ as an induced subgraph. It is a counter-example to Conjecture 4.1.

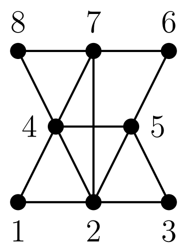


Figure 4.1: A counter-example to Conjecture 4.1.

It can be checked that this graph is perfect. Let us exhibit a nonequimodular face-defining matrix for the stable set polytope of this graph, which will imply that it is not box-perfect.

$$M = \begin{array}{c} \\ \scriptstyle\{1,2,4\} \\ \scriptstyle\{2,3,5\} \\ \scriptstyle\{5,6\} \\ \scriptstyle\{6,7\} \\ \scriptstyle\{4,7,8\} \end{array} \begin{array}{c} \begin{array}{cccccccc} \scriptstyle 1 & \scriptstyle 2 & \scriptstyle 3 & \scriptstyle 4 & \scriptstyle 5 & \scriptstyle 6 & \scriptstyle 7 & \scriptstyle 8 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

The matrix M is not equimodular because the determinant formed by the columns 2,5,6,7,4 is that of a 5-cycle, thus equals 2 in absolute value, whereas the determinant of columns 1,2,5,6,7 equals 1, as the corresponding matrix is triangular with ones on the diagonal.

By Observation 3.2, to prove that M is face-defining, it is enough to exhibit $8 - 5 + 1 = 4$ affinely independent stable sets intersecting every clique defining the matrix, namely the cliques $\{1, 2, 4\}, \{2, 3, 5\}, \{5, 6\}, \{6, 7\}, \{4, 7, 8\}$. Here are 4 stables sets intersecting each of these cliques: $\{1, 3, 6, 8\}, \{1, 5, 7\}, \{2, 6, 8\}, \{3, 4, 6\}$. Note that each of them contains a node that is not in the other ones, hence the are independent. Thus, M is face-defining for the stable set polytope of the above graph. Since M is not equimodular, this graph is not box-perfect by Theorem 3.11.

We refer to [14] for the proof that this graph has no graph from $\mathcal{R}$ as an induced subgraph, and hence is indeed a counter-example to Conjecture 4.1.

We conclude with the following problem. To the best of my knowledge, there are no nice conjectures about which graphs should be forbidden.

Open problem 4.2. *Characterize box-perfect graphs by a list of forbidden induced subgraphs.*

4.2 Integer Decomposition Property

In this section, we discuss possible connections between box-total dual integrality and the integer decomposition property. This property arises in various fields such as integer programming, algebraic geometry, combinatorial commutative algebra. Several classes of polyhedra are known to have the integer decomposition property, as for instance: projections of polyhedra defined by totally unimodular matrices [57], polyhedra defined by nearly totally unimodular matrices [34], certain polyhedra defined by k-balanced matrices [64], the stable set polytope of claw-free t-perfect graphs and h-perfect line-graphs [7].

A polyhedron P has the *integer decomposition property* if for any natural number k and any integer vector x in kP, there exist k integer vectors $x_1, \dots, x_k$ in P such that $x_1 + \cdots + x_k = x$. A stronger property is when the polyhedron P has the *Integer Carathéodory Property*, that is, if for every positive integer k and every integer vector x in kP, there exist $n_1, \dots, n_t$ in $\mathbb{Z}_{\geq 0}$ and affinely independent integer points $x_1, \dots, x_t$ in P such that $n_1 + \cdots + n_t = k$ and $x = \sum_i n_i x_i$.

In [35], Gijswijt and Regts introduce a class $\mathcal{P}$ of polyhedra and show that they have the Integer Carathéodory Property. They define $\mathcal{P}$ to be the set of polyhedra P such that for any k in $\mathbb{Z}_{\geq 0}$, r in $\{0, \dots, k\}$, and w in $\mathbb{Z}^n$ the intersection $rP \cap (w - (k - r)P)$ is box-integer. They also show [35, Proposition 4] that every P in $\mathcal{P}$ is box-integer. Given the definition of $\mathcal{P}$, note that if a polyhedron is in $\mathcal{P}$, then so are all its integer dilations. Therefore, by Theorem 3.14, this has the following consequence.

Corollary 4.3. *Every P in $\mathcal{P}$ is box-TDI.*

Box-TDI polyhedra do not inherit the Integer Carathéodory Property. Actually, they do not even inherit the integer decomposition property, as the classical example of polytope without the integer decomposition property $P = \text{conv}\,((0,0,0),(1,1,0),(1,0,1),(0,1,1))$ is box-TDI. To see that P is box-TDI, note that in the minimal linear description of P given below, the matrix of constraints is totally equimodular, and apply Theorem 3.24. However, the point $(1,1,1)$ is in $2P$ and cannot be writen as an integer combination of the integer points of P, hence P does not have the integer decomposition property.

$$P = \left\{ x \in \mathbb{R}^3 : \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix} x \leq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix} \right\}$$

Nevertheless, given the strong integrality properties of box-TDI polyhedra, it might be that many of them have the integer decomposition property. In this area, a long standing open question is known as Oda's question [48]: is it true that every smooth polytope has the integer decomposition property? A full-dimensional integer polytope of $\mathbb{R}^n$ is *smooth* if for every vertex v, there are exactly n generators of the associated minimal tangent cone, and these generators form a basis of the lattice $\mathbb{Z}^n$.

The polyhedron of the example above is not smooth, and the following special case of Oda's question is a reasonable first step to determine which box-totally dual integral polyhedra have the integer decomposition property.

Open problem 4.4. *Do smooth box-totally dual integral polyhedra have the integer decomposition property?*

4.3 Box-Mengerian Clutters

Here, we explain that the equivalence between the main result of Gerards and Laurent [33] and that of Chen, Ding, and Zang [11] about binary clutters is actually a special case of Theorem 3.14.

We briefly introduce the definitions we need about clutters. A collection $\mathcal{C}$ of subsets of a set E is a *clutter* if none of its sets contains another one. We denote by $A_\mathcal{C}$ the $\mathcal{C} \times E$ incidence matrix of $\mathcal{C}$ and by $P_\mathcal{C} = \{x \in \mathbb{R}^E : A_\mathcal{C} x \geq \mathbb{1}, x \geq \mathbb{0}\}$ the associated covering polyhedron. A clutter $\mathcal{C}$ is *binary* if the symetric difference of any three elements of $\mathcal{C}$ contains an element of $\mathcal{C}$. A clutter $\mathcal{C}$ is *box-$\frac{1}{k}$-integral* if for all $\ell, u \in \frac{1}{k}\mathbb{Z}^E$, each vertex of $P_\mathcal{C} \cap \{\ell \leq x \leq u\}$ belongs to $\frac{1}{k}\mathbb{Z}^E$. Equivalently, the polyhedron $kP_\mathcal{C}$ is box-integer. A matrix $A \in \{0,1\}^{m \times n}$ is called *(box-)Mengerian* if the system $Ax \geq \mathbb{1}, x \geq \mathbb{0}$ is (box-)TDI. A clutter $\mathcal{C}$ is *(box-)Mengerian* if $A_\mathcal{C}$ is (box-)Mengerian. Note that a clutter $\mathcal{C}$ is box-Mengerian if and only if it is Mengerian and $P_\mathcal{C}$ is box-TDI. *Deleting* an element $e \in E$ means replacing $\mathcal{C}$ by $\mathcal{C} \setminus e = \{X \in \mathcal{C} : e \notin X\}$ and *contracting* an element $e \in E$ means replacing $\mathcal{C}$ by $\mathcal{C}/e$ which is composed of the inclusionwise minimal members of $\{X \setminus \{e\} : X \in \mathcal{C}\}$. The *minors* of a clutter are the clutters obtained by repeatedly deleting and contracting elements of E. The clutter Q_6 is defined on the set E_4 of the edges of the complete graph K_4, and its elements are the triangles of K_4, see Figure 4.2. The clutter Q_7 is defined on $E_4 \cup e$ where $e \notin E_4$, and its elements are $X \cup \{e\}$ for each triangle or perfect matching X of K_4.

In 1995, Gerards and Laurent [33] characterized the binary clutters that are box-$\frac{1}{k}$-integral for all $k \in \mathbb{Z}_{>0}$ by forbidding minors.

Theorem 4.5 ([33, Theorem 1.2])**.** *A binary clutter is box-$\frac{1}{k}$-integral for all $k \in \mathbb{Z}_{>0}$ if and only if neither Q_6 nor Q_7 is its minor.*

In 2008, Chen, Ding, and Zang [11] characterized box-Mengerian binary clutters by forbidding minors. In [10], Chen, Chen, and Zang provide a simpler proof of this characterization, based on the so called ESP property. We mention that none of the proofs of Theorem 4.6 rely on Theorem 4.5.

Theorem 4.6 ([11, Corollary 1.2])**.** *A binary clutter is box-Mengerian if and only if neither Q_6 nor Q_7 is its minor.*

The combination of Theorems 4.5 and 4.6 implies that a binary clutter is box-Mengerian if and only if it is box-$\frac{1}{k}$-integral for all $d \in \mathbb{Z}_{>0}$. We show in the following how this equivalence is a special case of Theorem 3.14.

By Theorem 3.3 and (3.3) in Section 3.2.3, if a clutter $\mathcal{C}$ is box-Mengerian, then $kP_\mathcal{C}$ is box-integer for all $k \in \mathbb{Z}_{>0}$. Since, by definition, for $kP_\mathcal{C}$ to be box-integer is exactly the same as $\mathcal{C}$ being box-$\frac{1}{k}$-integral, this gives the first direction of the equivalence.

Conversely, by definition and by Theorem 3.14, if a clutter $\mathcal{C}$ is box-$\frac{1}{k}$-integral for all $k \in \mathbb{Z}_{>0}$, then $P_\mathcal{C}$ is integer and box-TDI. To conclude, it is enough to prove that if $P_\mathcal{C}$ is box-TDI, then $\mathcal{C}$ is Mengerian. We apply Seymour's characterization [58]: a binary clutter is Mengerian if and only if it has no Q_6 minor. The property of $P_\mathcal{C}$ being box-TDI is closed under taking minors since $P_{\mathcal{C}/e}$ and $P_{\mathcal{C}\setminus e}$ are respectively obtained from $P_\mathcal{C} \cap \{x_e = 0\}$ and $P_\mathcal{C} \cap \{x_e = 1\}$ by deleting e's coordinate. Furthermore, P_{Q_6} is not box-TDI. Indeed, the first three rows of the matrix A_{Q_6} of Figure 4.2 form a nonequimodular matrix M, as the determinant of the three first columns equals 2 and that of the three last columns equals 1. Moreover, M is face-defining for P_{Q_6}, by Observation 3.2 and because, if χ^i denotes the i^{th} unit vector, $\chi^1 + \chi^6$, $\chi^2 + \chi^5$, $\chi^3 + \chi^4$, and $\chi^4 + \chi^5 + \chi^6$ are affinely independent, belong to P_{Q_6}, and satisfy $Mx = \mathbb{1}$.

$$A_{Q_6} = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Figure 4.2: The matrix representation of the clutter Q_6.

Therefore, if $P_{\mathcal{C}}$ is box-TDI, then it has no minor Q_6, hence $\mathcal{C}$ is Mengerian. That is, $\mathcal{C}$ is box-Mengerian.

4.4 Box-TDI Systems in Series-Parallel Graphs

As we shall see, the cuts defined by the nodes of the complete graph on four vertices K_4 form a nonequimodular matrix. Thus, having the graph K_4 as a minor will have a good chance to forbid the box-TDIness of a polyhedron in which the cuts define either inequalities or form a basis of some face. We shall see several examples where that happens. For each these examples, it is then natural to wonder whether one gets box-TDIness when there is no minor K_4. It will be the case for each one, and we will provide several box-TDI polyhedra in the class of graphs having no K_4 minors: series-parallel graphs. We will also provide a box-TDI system with integer coefficients describing each of these polyhedra.

In Section 4.4.1, we provide a box-TDI system describing the cut cone of a series-parallel graph, joint work with D. Cornaz and M. Lacroix [21].

In Section 4.4.2, we provide the Schrijver system of the flow cone of a series-parallel graph, the flow cone being the polar of the cut cone. The Schrijver system is the (unique) TDI system having a minimum number of inequalities describing a given full-dimensional polyhedron. This is joint work with M. Barbato, M. Lacroix, E. Lancini, and R. Wolfler Calvo [4].

In Section 4.4.3, we study the polyhedron related to k-edge-connected spanning subgraphs of a given graph: they are the subgraphs that remain connected after the removal of any $k-1$ edges. We first prove that this polyhedron is box-TDI if and only if the graph is series-parallel, and then provide a integer box-TDI system for it. This is joint work with M. Barbato, M. Lacroix, and E. Lancini [3] that has been accepted for publication recently.

We provide a few definitions for the subsequent sections. Let $G = (V, E)$ be an undirected graph. Two edges of G are *parallel* if they share the same endpoints, and G is *simple* if it has neither parallel edges nor loops. A graph is 2-*connected* if it remains connected after the removal of any node. A 2-*connected component of* G is a maximal 2-connected subgraph of G. A 2-connected graph is *trivial* if it is composed of a single edge.

A subset of edges of G is called a *circuit* if, together with the nodes it covers, it forms a connected graph in which these nodes all have degree two. Given a subset U of V, the *cut* $\delta(U)$ is the set of edges having exactly one endpoint in U. A *bond* is a minimal nonempty cut. Given a partition $\{V_1, \dots, V_n\}$ of V, the set of edges having endpoints in two distinct V_i's is called a *multicut* and is denoted by $\delta(V_1, \dots, V_n)$. For every multicut M, there exists a unique partition $\{V_1, \dots, V_{d_M}\}$ of the nodes of V such that $M = \delta(V_1, \dots, V_{d_M})$, and $G[V_i]$ is connected for all $i = 1, \dots, d_M$. We say that d_M is the *order* of M and $V_1, \dots, V_{d_M}$ are the *classes* of M. From now on, we will assume that multicuts are described in this way. Note that $\delta(V_1, \dots, V_{d_M}) = \delta(V_1) \cup \cdots \cup \delta(V_{d_M})$. Moreover, the incidence vector of a multicut $\delta(V_1, \dots, V_{d_M})$ is the half sum of the incidence vectors of the cuts $\delta(V_1), \dots, \delta(V_{d_M})$. Multicuts are characterized in terms of circuits as follows: a set of edges M is a multicut if and only if $|M \cap C| \neq 1$ for all circuits C of G.

A graph is *series-parallel* if its nontrivial 2-connected components can be constructed from a circuit of length 2 by repeatedly adding edges parallel to an existing one, and subdividing edges, that is, replacing an edge by a path of length two. Equivalently, series-parallel graphs are those having no K_4-minor [27].

4.4.1 The Cut Cone

The *cut cone* of an undirected graph $G = (V, E)$ is the set of nonnegative combinations of incidence vectors of cuts of G. Equivalently, it is the set of nonnegative combinations of incidence vectors of multicuts of G, since cuts are multicuts and since the incidence vectors of multicuts are the half sum of incidence vectors of some cuts. This cone satisfies the inequalities

$$\begin{cases} x(C \setminus \{e\}) - x_e \geq 0 & \text{for each circuit } C \text{ and each edge } e \in C, \\ x \geq \mathbb{0}. \end{cases} \tag{4.1}$$

Indeed, multicuts are the sets of edges that intersect no circuit exactly once, and the constraints above imply that, if a $0/1$ vector contains an edge e of a circuit C, then it contains at least another edge of C. Actually, the set of x in $\mathbb{R}^E$ satisfying (4.1) is exactly the cut cone of G when G has no K_5-minor [60]. Schrijver [56, Corollary 29.9c] showed that System (4.1) is TDI if and only if the graph is series-parallel.

In [21], with D. Cornaz and M. Lacroix[1], we strenghten this TDIness result as follows.

Theorem 4.7 ([21])**.** *A graph G is series-parallel if and only if System* (4.1) *is box-TDI.*

We mention that this also strenghtens Corollary 4.1 of [20], which proves that adding $x \leq \mathbb{1}$ to (4.1) preserves TDIness if and only if the graph is series-parallel. Previously, Chopra [15] showed that adding $x \leq \mathbb{1}$ to (4.1) preserves integrality if and only if the graphs is series-parallel. Schrijver [56, Corollary 29.9c] also proved that a graph is series-parallel if and only if the standard systems describing the cycle cone, the T-join polytope, the cut polytope, the multicut polytope, and the T-join dominant are TDI. We also strenghten this in [21] by proving that one can replace TDI by box-TDI for each of the systems in [56, Corollary 29.9c].

Therein, we also explicit a min-max relation implied by Theorem 4.7, and we refer the interested reader to [21, Corollary 6].

4.4.2 The Flow Cone

The flow cone of a graph $G = (V, E)$ is the polar of the cut cone. When G has no K_5-minor [59], it is described by $x(C) \geq 0$, for all cuts C of G. As mentioned in Section 3.2.2, polarity preserves box-TDIness for cones. Thus, the flow cone is box-TDI when the graph is series-parallel. In fact, this is an equivalence.

Theorem 4.8 ([14])**.** *The flow cone of a graph is box-TDI if and only if the graph is series-parallel.*

Indeed, when the graph is not series-parallel, it has K_4 as a minor. By Theorem 3.11, the flow cone of K_4 is not box-TDI: it is easy to check that the following nonequimodular matrix is face-defining for its flow cone. Here, $1, 2, 3, 4$ are the nodes of K_4 and e_{ij} denotes the edge ij.

$$M = \begin{array}{c} \\ \scriptstyle\delta(1) \\ \scriptstyle\delta(2) \\ \scriptstyle\delta(3) \end{array} \begin{array}{c} \begin{matrix} e_{12} & e_{13} & e_{23} & e_{14} & e_{24} & e_{34} \end{matrix} \\ \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

[1] Little tear of emotion: this was my first box-TDIness result!

This transfers straightforwardly to the case when the graph has K_4 as a minor.

To the best of my knowledge, this is the first proof of the box-TDIness of some polyhedron that does not rely on a box-TDI system describing the polyhedron.

Moreover, when the graph G is series-parallel, we provided with P. Chervet and L.-H. Robert [14] the following box-TDI system describing its flow cone:

$$\frac{1}{2}x(B) \geq 0 \quad \text{for all bonds } B \text{ of } G, \tag{4.2}$$

and asked for a box-TDI system with integer coefficients, which exists by [55, Theorem 22.6(i)] and [19, Corollary 2.5].

We answer this question in [4] with M. Barbato, M. Lacroix, E. Lancini, and R. Wolfler Calvo by exhibiting the *Schrijver system* [54] describing the flow cone of a series-parallel graph. The Schrijver system of a full-dimensional polyhedron P is the unique minimal TDI system $Ax \leq b$ describing P with A integer, minimal meaning that it has a minimum number of inequalities.

A few definitions are needed to state the result: the *reduced graph* of a multicut $\delta(V_1, \dots, V_d)$ is the graph obtained by contracting each V_i into a single node; a graph is *chordal* when each cycle has a chord; a multicut is *chordal* if every circuit of length four or more has a chord. Note that bonds are chordal multicuts.

Theorem 4.9 ([4])**.** *The Schrijver system for the flow cone of a series-parallel graph G is the following:*

$$x(M) \geq 0 \quad \text{for all chordal multicuts } M \text{ of } G. \tag{4.3}$$

Moreover, this system is box-TDI.

Moreover, it is proved in [4] that the system $x(M) \geq 0$ for all multicuts M of G is TDI if and only if G is series-parallel.

This last result and Theorem 4.9 can be rephrased using Hilbert basis. A set of vectors $\{v^1, \dots, v^k\}$ is a *Hilbert basis* if each integer vector that is a nonnegative combination of $v^1, \dots, v^k$ can be expressed as a nonnegative integer combination of them. The link between Hilbert basis and TDIness is stated in the following theorem.

Theorem 4.10 (Theorem 22.5 of [55])**.** *A system $Ax \geq b$ is TDI if and only if for every face F of $P = \{x : Ax \geq b\}$, the rows of A associated with tight constraints for F form a Hilbert basis.*

Then, in terms of Hilbert bases, the above results answer to the following questions:

- When do multicuts form a Hilbert basis? When the graph is series-parallel.
- Which multicuts form a Hilbert basis? When they contain all chordal multicuts (provided that the graph is series-parallel).

4.4.3 The k-Edge-Connected Spanning Subgraph Polyhedron

In this section, we are interested in integrality properties of systems related to k-edge-connected spanning subgraphs.

A *k-edge-connected spanning subgraph* of a graph $G = (V, E)$ is a graph $H = (V, F)$, with F being a collection of elements of E where each element can appear several times, that remains connected after the removal of any $k-1$ edges. The incidence vector of a family F of E is the vector χ^F of $\mathbb{Z}_+^E$ such that e's coordinate is the multiplicity of e in F for all e in E. The convex hull of incidence vectors of all the families of E inducing a k-edge-connected spanning subgraph of G forms a polyhedron, hereafter called the *k-edge-connected spanning subgraph polyhedron* of G and denoted by $P_k(G)$. Note that $P_k(G)$ is the dominant of the convex hull of incidence vectors of all the families of E containing at most k copies of each edge and inducing a k-edge-connected spanning subgraph of G. Since the dominant of a polyhedron is a polyhedron, $P_k(G)$ is a full-dimensional polyhedron even though it is the convex hull of an infinite number of points.

For $k = 1$, this polyhedron is the dominant of the spanning tree polytope, and is box-TDI for all graphs. From now on, we assume that $k \geq 2$. When G is series-parallel, Cornuejols, Fonlupt, and Naddef [22] provided a description of $P_2(G)$, that was later generalized to $P_{2h}(G)$ by Didi Biha and Mahjoub [23]. In fact, Didi Biha and Mahjoub [23] gave a complete description of $P_k(G)$ for all $k \geq 2$.

Theorem 4.11 ([23])**.** *Let G be a series-parallel graph and h be a positive integer. Then, $P_{2h}(G)$ is described by:*

$$\begin{cases} \frac{1}{2}x(D) \geq h & \text{for all cuts } D \text{ of } G, \\ x \geq \mathbb{0}, \end{cases} \tag{4.4}$$

and $P_{2h+1}(G)$ is described by:

$$\begin{cases} x(M) \geq (h+1)d_M - 1 & \text{for all multicuts } M \text{ of } G, \\ x \geq \mathbb{0}. \end{cases} \tag{4.5}$$

Chen, Ding, and Zang [12] proved that, for $h = 1$, System (4.4) is box-TDI if and only if G is series-parallel. Their result was our starting point on this topic, with two questions in mind, in the light of Section 4.4.2:

- When the graph is series-parallel, is the polyhedron $P_k(G)$ box-TDI for all $k \geq 2$?
- Can we provide a box-TDI system describing $P_k(G)$ with only integer coefficients, starting with $P_2(G)$?

We answer the first question by the affirmative.

Theorem 4.12 ([3])**.** *For $k \geq 2$, $P_k(G)$ is a box-TDI polyhedron if and only if G is series-parallel.*

The case k even is obtained using the box-TDIness for $k = 2$ and the fact that dilations maintain box-TDIness. For the case k odd, the proof relies entirely on Theorem 3.11. On the contrary to what is generally done, the proof does not exhibit a box-TDI system describing $P_k(G)$. For this case, the proof is by induction on the number of edges of G. We prove that series-parallel operations preserve the box-TDIness of the polyhedron. We first prove that adding a parallel edge maintains equimodularity of the face-defining matrices. The most technical part of the proof is the subdivision of an edge uw into two edges uv and vw. We proceed by contradiction: by Theorem 3.11, we suppose that there exists a face F of $P_k(G)$ defined by a nonequimodular matrix. We study the structure of the inequalities corresponding to this matrix. In particular, we show that they are all associated with multicuts, and that these multicuts contain either both uv and vw, or none of them. These last results allow us to build a nonequimodular face-defining matrix for the smaller graph, which contradicts the induction hypothesis. The detailed proof can be found in [3].

To the best of my knowledge, this is the second proof of the box-TDIness of a polyhedron that does not exhibit a box-TDI system describing it. The first such proof is the polar one mentioned in Section 4.4.2.

Concerning the second question, our idea was to start with $P_2(k)$ and to mimic the approach we had for the flow cone: replace halfs of cuts by multicuts. Indeed, the description of $P_{2h}(G)$ given in Theorem 4.11 and the decomposition of multicuts into cuts yields the following.

Observation 4.13. *Let G be a series-parallel graph. The polyhedron $P_2(G)$ is described by:*

$$\begin{cases} x(M) \geq d_M & \textit{for all multicuts } M \textit{ of } G, \\ x \geq \mathbb{0}. \end{cases} \tag{4.6}$$

We could prove that System (4.6) is TDI, and it turns out that it implies the same for all even $k \geq 2$. That is, we proved the following, where System (4.7) describes $P_{2h}(G)$ for G series-parallel graph and h positive.

Theorem 4.14 ([3]). *For a series-parallel graph G and a positive integer h, the system*

$$\begin{cases} x(M) \geq hd_M & \textit{for all multicuts } M \textit{ of } G, \\ x \geq \mathbb{0} \end{cases} \tag{4.7}$$

is TDI.

Combining Theorem 4.12 and Theorem 4.14, we have that, for h positive, System (4.7) is box-TDI if and only if the graph is series-parallel.

Then, the proof of Theorem 4.14 uses Hilbert bases. Indeed, it is based on the TDIness of System (4.4) when $h = 1$ and the structure of inequalities of System (4.6). Their right-hand sides are proportional to k, hence it is enough to prove the case $k = 2$. This allows us to use the box-TDIness result of Chen, Ding, and Zang [12] to obtain a TDI system for $P_2(G)$, namely System (4.4) with $h = 1$. In terms of Hilbert bases, the TDIness of this system implies that, given a face F of $P_2(G)$, the integer points of the associated cone are the half sum of the cuts tight for F. The technical part of the proof is to show that each integer point of this cone is also the sum of incidence vectors of the multicuts tight for F. We refer the interested reader to [3] for the details.

For the case when $k \geq 2$ is odd, we prove that System (4.5) is TDI if and only if G is a series-parallel graph.

Theorem 4.15 ([3]). *For h positive and integer, System* (4.5) *is TDI if and only if G is series-parallel.*

Proving the TDIness for k odd is considerably more involved than for k even. The first difference with the even case is the lack of a known TDI system describing $P_k(G)$ when k is odd, even a noninteger one. Thus, no property of the Hilbert bases associated with $P_k(G)$ is known, and the approach used to prove Theorem 4.14 cannot be applied. Instead, following the definition of TDIness, we prove the existence of an integer optimal solution to each feasible dual problem.

Another difference with the case when k is even follows from the structure of the inequalities in System (4.5). In particular, the presence of the constant "-1" in the right-hand sides perturbs the structure of tight multicuts. Indeed, when k is odd, the tightness of $\delta(V_1, \dots, V_n)$ does not imply that of $\delta(V_1), \dots, \delta(V_n)$. Consequently, it is not clear how the contraction of an edge impacts the tightness of a multicut $\delta(V_1, \dots, V_n)$: merging adjacent V_i's is not sufficient to obtain new tight multicuts. Due to the link between tight multicuts and positive dual variables, the structure of the optimal solutions to the dual problem is completely modified when subdividing an edge. Proving directly that subdivision preserves TDIness turned out to be challenging, and we overcome this difficulty by deriving new properties of series-parallel graphs. More precisely, we prove that, when G is a nontrivial simple 2-connected series-parallel graph, at least one of the following holds:

- two nodes of degree two are adjacent,
- a node of degree two belongs to a circuit of length three,
- two nodes of degree two belong to the same circuit of length four.

Then, the proof of Theorem 4.15 focuses on properties of nodes of degree two in a minimal counterexample to the TDIness of System (4.5). In particular, we prove that no two nodes of degree two are adjacent, or in the same circuit of length four. Moreover, no triangle contains nodes of degree two. By the properties of series-parallel graphs mentioned above, this implies that the graph is not series-parallel, the contradiction we desired. To derive these properties, we study the interplay between cuts associated with nodes of degree two and dual optimal solutions. Again, we refer to [3] for the complete proof.

We conclude this section with a question that Mourad Baïou asked me some time ago: since many systems and polyhedra seem to be box-TDI for series-parallel graphs, what about the Steiner 2-edge-connected subgraph polyhedron?

Given a graph $G = (V, E)$ and a subset of nodes $S \subseteq V$, the *Steiner 2-edge survivable network problem* is the problem of finding a minimum cost subgraph of G spanning S such that between every two nodes of S there are at least two edge-disjoint paths. The *Steiner 2-edge-connected subgraph polyhedron* is the convex hull of the incidence vectors of such subgraphs. In [2], it is proved that for a series-parallel graph $G = (V, E)$ and $S \subseteq V$, this polyhedron is described by:

$$\begin{cases} x(\delta(W)) \geq 2 & \text{for all } W \subseteq V \text{ such that } W \neq S \text{ and } W \cap S \neq \emptyset, \\ x \leq \mathbb{1}, \\ x \geq \mathbb{0}. \end{cases} \tag{4.8}$$

Open problem 4.16. *Is the Steiner 2-edge-connected subgraph polyhedron box-TDI when the graph is series-parallel?*

Conclusion

Let me summarize the notable things, in my opinion, that we came accross throughout this document. First, we discussed the complexity of polyhedra with the angle of how many inequalities are required to describe them. This made slack matrices appear, and we saw that their nonnegative rank could be handled using randomized communication protocols. These protocols are now a standard tool to derive the existence of some extended formulation. Recently for instance, upper bounds for the extension complexity of matroid polytopes were given by Aprile [1] using randomized communication protocols.

In a second part, we thoroughly studied box-TDI polyhedra. We saw that despite their rather involved definition, we could characterize them in ways that are helpful in practice. Nevertheless, these characterizations do not yield insights towards their recognition problem[1]. Moreover, as we saw with several intriguing open problems, the topic is well alive and many things remain to be understood. Among these problems, one of my favorites these days concerns the matrices I called totally equimodular, and I currently advise a PhD student on the subject.

Incidentally, beside these two topics, several other subjects winked at me these past years. The one I got into the most concerns volumes of polyhedra. With T. Milcent, we are currently revisiting the Moment-Of-Fluid method, which simulates interactions between fluids submitted to a velocity field. This is a completely different area, which involves partial differential equation and numerical simulations, yet polyhedra come into play because the domain in which the fluids interact is discretized into small polyhedral cells.

To conclude this document, in which we discussed two seemingly unrelated notions about polyhedra, let us try to question how they intersect.

As mentioned in Section 3.2.4, projections along coordinate axes preserve box-total dual integrality. Therefore, when a polyhedron has a box-TDI extended formulation and the projection is along coordinate axes, then the polyhedron is box-TDI. This might fail for other directions of projection. For instance, we saw in Section 2.1 the lexicographical polytope as the projection of a flow polyhedron. The latter is box-TDI as we mentioned in Section 3.1. Yet, because of the direction of the projection (2.1), lexicographical polytopes do not inherit this box-TDIness: the example in Figure 2.1 is not box-integer, hence not box-TDI. Thus, having a box-TDI extended formulation does not guarantee being box-TDI.

On the other hand, one could wonder whether for a polyhedron to be box-TDI impacts its extension complexity. Unfortunately, Rothvoß proves in [52] that there exist matroid polytopes having an extension complexity exponential in their dimension. Since matroid polytopes are

[1]Spoiler alert: recently, with P. Chervet, M. Lacroix, F. Pisanu, L.-H. Robert, and R. Wolfler Calvo, we proved that deciding whether a given polyhedron is box-TDI is co-NP-complete. Yet, Open Problem 3.8, the recognition problem of box-TDI cones, remains open.

box-TDI, there exists box-TDI polyhedra with exponential extension complexity. The proof of [52] uses a counting argument, thus is purely existential: it does not provide an explicit family of matroid polytopes having exponentially high extension complexity. Producing explicitly such a family would answer an almost thirty year old open question in communication complexity [40, Page 174].

Maybe the question is easier in the more general class of box-TDI polyhedra, so let me conclude with an open problem involving the two main notions of the document.

Open problem 4.17. *Is there an explicit family of box-totally dual integral polyhedra having exponential extension complexity?*

List of Open Problems

Open problem 3.8 (Page 38)**.** *Can box-integer cones be recognized in polynomial time?*

Open problem 3.25 (Page 44)**.** *Can totally equimodular matrices be recognized in polynomial time?*

Open problem 3.26 (Page 44)**.** *Is there a decomposition theorem for totally equimodular matrices?*

Open problem 4.2 (Page 49)**.** *Characterize box-perfect graphs by a list of forbidden induced subgraphs.*

Open problem 4.4 (Page 50)**.** *Do smooth box-totally dual integral polyhedra have the integer decomposition property?*

Open problem 4.16 (Page 58)**.** *Is the Steiner 2-edge-connected subgraphs polyhedron box-TDI when the graph is series-parallel?*

Open problem 4.17 (Page 60)**.** *Is there an explicit family of box-totally dual integral polyhedra having exponential extension complexity?*

Definitions Index

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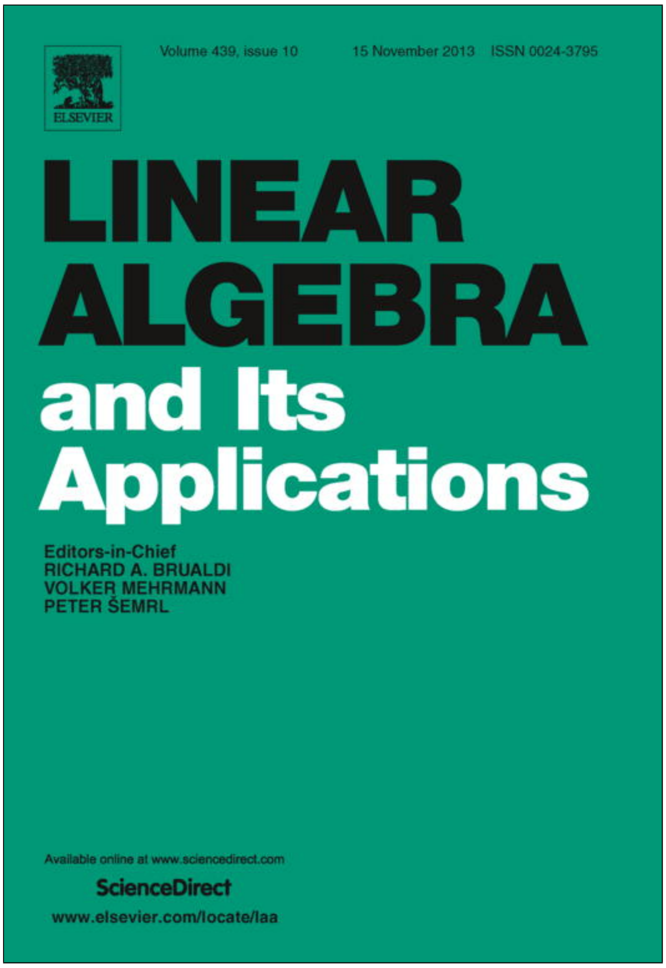
Appendix: Full Length Papers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Which nonnegative matrices are slack matrices? ☆

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ABSTRACT

In this paper we characterize the slack matrices of cones and polytopes among all nonnegative matrices. This leads to an algorithm for deciding whether a given matrix is a slack matrix. The underlying decision problem is equivalent to the *polyhedral verification problem* whose complexity is unknown.

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1. Introduction

This paper concerns a class of nonnegative matrices with real entries, called *slack matrices*, that arise naturally from polyhedral cones and polytopes. Given a polytope $P \subset \mathbb{R}^n$ with vertices $v_1, \dots, v_p$ and facet inequalities $a_j^T x \leqslant \beta_j$ for $j = 1, \dots, q$, a *slack matrix* of P is the $p \times q$ nonnegative matrix whose (i, j)-entry is $\beta_j - a_j^T v_i$, the *slack* (distance from equality), of the ith vertex v_i in the jth facet inequality $a_j^T x \leqslant \beta_j$ of P. A similar definition holds for polyhedral cones.

Slack matrices form an interesting class of nonnegative matrices with many special properties. Most obviously, if M is a slack matrix of a polytope P, then the zeros in M record the face lattice of P and hence the combinatorial structure of P. In its entirety, M specifies an embedding of P up to affine transformation. However, slack matrices carry much more (and surprising) information about P. In [14], Yannakakis proved that the *nonnegative rank* of a slack matrix of P is the minimum k such that P is the linear image of an affine slice of the positive orthant $\mathbb{R}^k_+$. We use $\mathbb{R}_+$ to denote the set of nonnegative real numbers. The nonnegative rank of a matrix $M \in \mathbb{R}^{p\times q}_+$ is the smallest k such that there exist vectors $a_1, \dots, a_p \in \mathbb{R}^k_+$ and $b_1, \dots, b_q \in \mathbb{R}^k_+$ such that $M_{ij} = a_i^T b_j$. Affine slices of positive orthants that project onto P are called *polyhedral lifts* or *polyhedral extended formulations* of P and the smallest k such that $\mathbb{R}^k_+$ admits a lift of P is called the (*polyhedral*) *extension complexity* or nonnegative rank of P. If the extension complexity of P is small (polynomial in the dimension of P), then usually it is possible to optimize a linear function over P in polynomial time by optimizing an appropriate function on the lift. This is a powerful technique in optimization that yields polynomial time algorithms for linear optimization over complicated polytopes. There are many instances of n-dimensional polytopes with exponentially many (in n) facets that allow small polyhedral lifts.

Yannakakis' result was generalized in [5] to lifts of convex sets by affine slices of convex cones via *cone factorizations* of *slack operators*. Even in the larger context of cone lifts of convex sets, the case of polytopes is the simplest and the theory relies on slack matrices of polytopes and their factorizations through cones. Thus, understanding the structure of these matrices is fundamental for this theory. There are several phenomena that occur in the class of nonnegative matrices that have not yet been observed for slack matrices. For instance, an important open question is whether there exists a family of slack matrices of polytopes that exhibit an exponential gap between nonnegative rank and *positive semidefinite rank*. (If $\mathcal{S}^k_+$ denotes the cone of $k \times k$ real symmetric positive semidefinite matrices, then the positive semidefinite rank of a matrix $M \in \mathbb{R}^{p\times q}_+$ is the smallest k such that there exist matrices $A_i \in \mathcal{S}^k_+$, $i = 1, \dots, p$, and $B_j \in \mathcal{S}^k_+$, $j = 1, \dots, q$, such that $M_{ij} = \langle A_i, B_j\rangle$.) While there are simple families of matrices that exhibit even arbitrarily large gaps between nonnegative and positive semidefinite ranks [5, Example 5], no family of slack matrices with this property is known. Such a family would be a clear witness for the power of *semidefinite programming* over linear programming in lifts of polytopes.

This paper was motivated by the many open questions about slack matrices which rely on understanding the structure of these matrices. We establish two main characterizations of slack matrices of polyhedral cones and polytopes. In Section 2 we establish linear algebraic characterizations: Theorem 1 for cones and Theorem 6 for polytopes. In Section 4 we give combinatorial characterizations: Theorem 22 for polytopes and Theorem 24 for polyhedral cones. In Section 3 we use our characterization from Section 2 to give an algorithm for recognizing slack matrices. The computational complexity of this problem is unknown and is equivalent to the *polyhedral verification problem*. There are several further geometric and complexity results about slack matrices throughout the paper.

Notation: For a set of vectors $\mathcal{A} = \{a_1, \dots, a_p\}$, $\mathrm{cone}(\mathcal{A}) := \{\sum \lambda_i a_i\colon\ \lambda_i \geqslant 0\}$ is the cone spanned by $\mathcal{A}$; $\mathrm{conv}(\mathcal{A}) := \{\sum \lambda_i a_i\colon\ \lambda_i \geqslant 0,\ \sum \lambda_i = 1\}$ is the convex hull of $\mathcal{A}$; $\mathrm{lin}(\mathcal{A}) := \{\sum \lambda_i a_i\colon\ \lambda_i \in \mathbb{R}\}$ is the linear span of $\mathcal{A}$, and $\mathrm{aff}(\mathcal{A}) := \{\sum \lambda_i a_i\colon\ \sum \lambda_i = 1\}$ is the affine span of $\mathcal{A}$. The above sets can also be defined for an infinite subset $\mathcal{A} \subset \mathbb{R}^n$ by taking unions over all finite subsets of $\mathcal{A}$. For an $n \times q$ matrix M, we let $\mathrm{rows}(M)$ and $\mathrm{cols}(M)$ denote the sets of all rows and columns, respectively, of M. We let $\mathcal{A} \cdot M$ be the set of vectors $\{x^T M\colon\ x \in \mathcal{A}\}$. For a set $K \subset \mathbb{R}^n$, $\mathrm{lineal}(K)$ is the largest subspace contained in K, known as the *lineality space* of K. The dimension of a polytope P, $\dim(P)$ is the dimension of $\mathrm{aff}(P)$, the affine hull of P, and the dimension of a cone K is the dimension of

$\mathrm{lin}(K)$. For any matrix $M \in \mathbb{R}^{p\times q}$ of rank k, we will call a factorization of the form $M = AB$ with $A \in \mathbb{R}^{p\times k}$, $\mathrm{rank}(A) = k$ and $B \in \mathbb{R}^{k\times q}$, $\mathrm{rank}(B) = k$ a *rank factorization* of M.

2. Geometric characterizations of slack matrices

2.1. Slack matrices of polyhedral cones

Consider the polyhedral cone

$$K = \{x \in \mathbb{R}^n:\ x^T B \geqslant \mathbb{O}\} = \mathbb{R}^p_+ \cdot A$$

in $\mathbb{R}^n$ constrained by the columns of the matrix $B \in \mathbb{R}^{n\times q}$ and generated by the rows of the matrix $A \in \mathbb{R}^{p\times n}$. We call (the set of rows of) A a $\mathcal{V}$*-representation* and (the set of columns of) B an $\mathcal{H}$*-representation* of K. The *slack matrix* of K with respect to the representation (A, B) is $S = AB \in \mathbb{R}^{p\times q}_+$. Its (i, j)-entry records the "slack" of the ith generator of K with respect to the jth inequality of K in the given description of K.

Let $\mathcal{S}_K$ denote the set of all slack matrices of K. For $S \in \mathcal{S}_K$, any matrix obtained by scaling the rows and columns of S by positive reals is again in $\mathcal{S}_K$ since scaling the vectors in a $\mathcal{V}$ and/or $\mathcal{H}$-representation of K does not change K. Also, $\mathcal{S}_K$ can have matrices of different sizes as adding redundant inequalities and/or generators to the representations of K does not change K. From

$$\begin{aligned}(\mathbb{R}^n \cdot B) \cap \mathbb{R}^q_+ = K \cdot B = (\mathbb{R}^p_+ \cdot A) \cdot B = \mathbb{R}^p_+ \cdot S \\ \subseteq (\mathbb{R}^p \cdot S) \cap \mathbb{R}^q_+ = (\mathbb{R}^p \cdot AB) \cap \mathbb{R}^q_+ \subseteq (\mathbb{R}^n \cdot B) \cap \mathbb{R}^q_+\end{aligned}$$

we find that $\mathbb{R}^p_+ \cdot S = \mathbb{R}^p \cdot S \cap \mathbb{R}^q_+$ which says that the *cone generated by the rows of* S coincides with the *nonnegative part of the row span of* S. In fact, this relation characterizes slack matrices of cones as we now show.

Theorem 1. *A nonnegative matrix* $M \in \mathbb{R}^{p\times q}_+$ *is a slack matrix of a polyhedral cone if and only if*

$$\mathbb{R}^p_+ \cdot M = \mathbb{R}^p \cdot M \cap \mathbb{R}^q_+, \tag{1}$$

or in other words, the cone spanned by the rows of M *coincides with the nonnegative part of the row span of* M.

Proof. It remains to show that every matrix $M \in \mathbb{R}^{p\times q}_+$ with $\mathbb{R}^p_+ \cdot M = \mathbb{R}^p \cdot M \cap \mathbb{R}^q_+$ is a slack matrix of some cone. Let $k = \mathrm{rank}(M)$ and consider a rank factorization $M = AB$ with $A \in \mathbb{R}^{p\times k}$ and $B \in \mathbb{R}^{k\times q}$. Let $K = \mathrm{cone}(\{a_1, \ldots, a_p\})$ and $\widetilde{K} = \{x \in \mathbb{R}^k:\ x^T b^j \geqslant 0,\ j = 1, \ldots, q\}$ where a_i is the ith row of A and b^j is the jth column of B. We need to show that $K = \widetilde{K}$.

Since M is nonnegative, we get that $K \subseteq \widetilde{K}$. In order to show the inclusion $\widetilde{K} \subseteq K$, consider a vector x from $\widetilde{K}$. Since the matrix $A \in \mathbb{R}^{p\times k}$ has full column rank, the vector x^T lies in $\mathbb{R}^p \cdot A$, and thus $x^T B$ lies in $\mathbb{R}^p \cdot M \cap \mathbb{R}^q_+$. Thus, due to Eq. (1) the vector $x^T B$ lies in $\mathbb{R}^p_+ \cdot M$, i.e. $x^T B$ lies in $\mathbb{R}^p_+ \cdot AB$. Since the matrix B has full row rank the vector x^T hence lies in $\mathbb{R}^p_+ \cdot A$, i.e. x lies in K. □

Recall that the dual cone of K is the cone

$$K^\star = \{y \in \mathbb{R}^n:\ x^T y \geqslant 0 \text{ for all } x \in K\} = \{y \in \mathbb{R}^n:\ Ay \geqslant 0\} = B \cdot \mathbb{R}^q_+.$$

Hence, S^T is a slack matrix of $K^\star$ and we get the following result.

Proposition 2. *A nonnegative real matrix is a slack matrix of a polyhedral cone if and only if its transpose is also the slack matrix of a polyhedral cone.*

In particular, we obtain the following consequence of Theorem 1.

Corollary 3. *A nonnegative matrix $M \in \mathbb{R}_+^{p\times q}$ is a slack matrix of a polyhedral cone if and only if*

$$M \cdot \mathbb{R}_+^q = M \cdot \mathbb{R}^q \cap \mathbb{R}_+^p, \tag{2}$$

or in other words, the cone spanned by the columns of M coincides with the nonnegative part of the column span of M.

We say that a matrix M satisfies the *row cone generating condition* (*RCGC*) if (1) holds and the *column cone generating condition* (*CCGC*) if (2) holds.

Corollary 4. *For a nonnegative matrix $M \in \mathbb{R}_+^{p\times q}$ the following statements are pairwise equivalent*:

- *M is a slack matrix of a polyhedral cone.*
- *M satisfies the RCGC.*
- *M satisfies the CCGC.*

The equivalence of RCGC and CCGC for a general nonnegative matrix is not obvious. However, its proof becomes transparent via the theory of slack matrices of polyhedral cones and cone duality.

For a nonnegative M with RCGC/CCGC, the proof of Theorem 1 showed how to produce a cone K such that $M \in \mathcal{S}_K$, which is captured by the next lemma.

Lemma 5. *Let $M \in \mathbb{R}_+^{p\times q}$ be the slack matrix of a polyhedral cone and let $M = AB$ be a rank factorization of M. Then if K is the cone generated by the rows of A, the columns of B form an $\mathcal{H}$-representation of K. In particular, $M \in \mathcal{S}_K$.*

Proof. For the slack matrix M Eq. (1) is valid, and thus the statement follows from the proof of Theorem 1. $\square$

2.2. Slack matrices of polytopes

We now investigate the slack matrices of polytopes. Let $V \in \mathbb{R}^{p\times n}$ and $P = \operatorname{conv}(\operatorname{rows}(V))$ be the polytope in $\mathbb{R}^n$ that is the convex hull of the rows of V. Suppose also that $P = \{x \in \mathbb{R}^n\colon Wx \leqslant w\}$ with $W \in \mathbb{R}^{q\times n}$ and $w \in \mathbb{R}^q$. To avoid unnecessary inconveniences, we assume that $\dim(P) \geqslant 1$. We call (the set of rows of) V a $\mathcal{V}$-representation and (the set of columns of) $[w, -W]^T$ an $\mathcal{H}$-representation of P. The *slack matrix* of P with respect to the representation (V, W, w) is then

$$S = [\mathbb{1}, V] \cdot [w, -W]^T \in \mathbb{R}_+^{p\times q}. \tag{3}$$

We denote the set of all slack matrices of P by $\mathcal{S}_P$. Clearly, scaling the columns of a slack matrix of P by positive scalars yields another slack matrix of P, because scaling the vectors in an $\mathcal{H}$-representation of P yields another $\mathcal{H}$-representation of P. However, we cannot scale the rows of a matrix $S \in \mathcal{S}_P$ and still stay in $\mathcal{S}_P$.

The matrix S is also the slack matrix of the *homogenization cone* of P:

$$P^h = \mathbb{R}_+^p \cdot [\mathbb{1}, V] = \left\{(x_0, x) \in \mathbb{R} \times \mathbb{R}^n\colon Wx \leqslant x_0 w\right\} \tag{4}$$

with respect to the representation $\left([\mathbb{1}, V], \left[\begin{smallmatrix} w^T \\ -W^T \end{smallmatrix}\right]\right)$. Since $\dim(P) \geqslant 1$, there is some $c \in \mathbb{R}^n$ with

$$\max\left\{c^T x\colon x \in P\right\} - \min\left\{c^T x\colon x \in P\right\} = 1,$$

and hence, due to LP-duality, we get

$$\left(1, \mathbb{O}^T\right) \in \mathbb{R}^q \cdot (w, W) \quad \text{and so also,} \quad \left(1, \mathbb{O}^T\right) \in \mathbb{R}^q \cdot (w, -W). \tag{5}$$

From (3) and (5) we get that $\mathbb{1} \in S \cdot \mathbb{R}^q$, the column span of S. These properties characterize the slack matrices of polytopes of dimension at least one:

Theorem 6. *A matrix $M \in \mathbb{R}_+^{p\times q}$ with $\operatorname{rank}(M) \geqslant 2$ is a slack matrix of a polytope if and only if M is a slack matrix of a polyhedral cone and $\mathbb{1} \in M \cdot \mathbb{R}^q$.*

Proof. It suffices to show that a matrix $M \in \mathbb{R}_+^{p\times q}$ with $\mathbb{1} \in M \cdot \mathbb{R}^q$ that is the slack matrix of some cone $K \subseteq \mathbb{R}^n$ with respect to a representation (A, B) is also the slack matrix of some polytope. To construct such a polytope, choose any $\mu \in \mathbb{R}^q$ such that $\mathbb{1} = M\mu$ and define $c = B\mu$. Then $Ac = \mathbb{1}$ since $M = AB$. Define $P = \operatorname{conv}(\operatorname{rows}(A))$. Then we have:

$$\begin{aligned} P &= \{y^T A\colon\ y^T \mathbb{1} = 1,\ y \in \mathbb{R}_+^p\} = \{y^T A\colon\ y^T Ac = 1,\ y \in \mathbb{R}_+^p\} \\ &= \{x \in K\colon\ x^T c = 1\} = \{x \in \mathbb{R}^n\colon\ x^T B \geqslant \mathbb{0},\ x^T c = 1\}. \end{aligned}$$

Mapping the hyperplane in $\mathbb{R}^n$ defined by $x^T c = 1$ isometrically to $\mathbb{R}^{n-1}$ (as in the proof of Theorem 1), we find that M is a slack matrix of the resulting image of P. $\square$

Corollary 7. *A matrix $M \in \mathbb{R}_+^{p\times q}$ with $\operatorname{rank}(M) \geqslant 2$ is a slack matrix of some polytope if and only if it satisfies the RCGC (or, equivalently, the CCGC) and $\mathbb{1} \in M \cdot \mathbb{R}^q$ holds.*

Theorem 1 geometrically characterizes the slack matrices of cones as those matrices $M \in \mathbb{R}_+^{p\times q}$ that satisfy

$$\operatorname{cone}(\operatorname{rows}(M)) = \operatorname{lin}(\operatorname{rows}(M)) \cap \mathbb{R}_+^q. \tag{6}$$

There is an analogous geometric characterization of slack matrices of polytopes.

Corollary 8. *A matrix $M \in \mathbb{R}_+^{p\times q}$ with $\operatorname{rank}(M) \geqslant 2$ is a slack matrix of some polytope if and only if*

$$\operatorname{conv}(\operatorname{rows}(M)) = \operatorname{aff}(\operatorname{rows}(M)) \cap \mathbb{R}_+^q. \tag{7}$$

Proof. First, suppose that M is a slack matrix of some polytope. Then by Corollary 7, we have that M satisfies (6) and $\mathbb{1} \in M \cdot \mathbb{R}^q$. Hence, there exists some $c \in \mathbb{R}^q$ such that $Mc = \mathbb{1}$ and the affine hyperplane $L = \{x \in \mathbb{R}^q\colon\ x^T c = 1\}$ contains the rows of M. Intersecting L with both sides of (6), we obtain (7).

For the reverse implication, let $M \in \mathbb{R}_+^{p\times q}$ be a nonnegative matrix satisfying (7). Using any isometry φ between the d-dimensional affine subspace $\operatorname{aff}(\operatorname{rows}(M))$ and $\mathbb{R}^d$, we find that M is a slack matrix of the φ-image of the polytope defined in (7). $\square$

We have seen above that every slack matrix of a polytope P has the all-ones vector in its column span and is also a slack matrix of the homogenization cone P^h of P. The next example shows that not all slack matrices of P^h are slack matrices of P, in fact, this does not even hold for the slack matrices of P^h that have the all-ones vector in their column span.

Example 9. Let P be the square $[-1, 1]^2$. The matrix

$$M = \begin{pmatrix} \frac{4}{3} & 0 & \frac{4}{3} & 0 \\ 2 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 \\ 0 & 4 & 0 & 4 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & \frac{2}{3} & \frac{2}{3} \\ 1 & 1 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & -2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

is in $\mathcal{S}_{P^h}$ and $\mathbb{1}$ is in the column span of M. It is clear, however, that M is not in $\mathcal{S}_P$ since each facet of $[-1, 1]^2$ is equidistant from the two vertices not on the facet. On the other hand, since M has the RCGC/CCGC and $\mathbb{1}$ is in its column span, it is the slack matrix of some other polytope Q. To obtain it, write a new rank factorization of M (note that $\operatorname{rank}(M) = 3$) so that the first factor contains the all-ones vector as its first column as follows:

$$M = \begin{pmatrix} \frac{2}{3} & \frac{2}{3} & \frac{2}{3} \\ 1 & 1 & -1 \\ 1 & -1 & 1 \\ 2 & -2 & -2 \end{pmatrix} UU^{-1} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 0 & 0 \\ 1/4 & 1 & 0 \\ 1/4 & 0 & 1 \end{pmatrix}$$

to get

$$M = \begin{pmatrix} \frac{4}{3} & 0 & \frac{4}{3} & 0 \\ 2 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 \\ 0 & 4 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 1 & \frac{2}{3} & \frac{2}{3} \\ 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & -2 & -2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 3/4 & -5/4 & -1/4 & -1/4 \\ -1/4 & -1/4 & 3/4 & -5/4 \end{pmatrix}.$$

By Lemma 5, M is the slack matrix of the cone with $\mathcal{V}$-representation the rows of the first factor and $\mathcal{H}$-representation the columns of the second factor. Assuming the coordinates of this three-dimensional cone are x_0, x_1, x_2, and slicing the cone with the hyperplane $\{(x_0, x_1, x_2)\colon\ x_0 = 1\}$ gives a polytope Q with vertices $(2/3, 2/3), (1, -1), (-1, 1), (-2, -2)$ and $\mathcal{H}$-representation given by the columns of the second factor. Then $M \in \mathcal{S}_Q$.

2.3. *Further results on slack matrices of cones and polytopes*

In this section we derive some more insight into the geometric relations between cones, polytopes, and their slack matrices that will be useful in later parts of the paper. We return to the setup used earlier: K is assumed to be a cone and S the slack matrix of K with respect to its representation (A, B) where $A \in \mathbb{R}^{p\times n}$ and $B \in \mathbb{R}^{n\times q}$.

First, we will show that every slack matrix of a cone is the slack matrix of some pointed cone. Recall that we use $\operatorname{lin}(K)$ to denote the linear hull of K and $\operatorname{lineal}(K)$ to denote the lineality space of K. Then we have $\operatorname{lin}(K) = \mathbb{R}^p \cdot A$ and $\operatorname{lineal}(K) = \operatorname{leftkernel}(B)$. A cone K is *pointed* if $\operatorname{lineal}(K) = \{\mathbb{O}\}$. Define

$$L := \operatorname{lin}(K) \cap \operatorname{lineal}(K)^{\perp} = \bigl(\mathbb{R}^p \cdot A\bigr) \cap \bigl(B \cdot \mathbb{R}^q\bigr).$$

Then we have

$$\operatorname{lin}(K) = L + \operatorname{lineal}(K)$$

(where the summands are orthogonal to each other) and

$$K = (K \cap L) + \operatorname{lineal}(K),$$

where $K \cap L \subseteq L$ is a pointed (i.e., having trivial lineality space) cone with $\dim(K \cap L) = \dim(L)$. Denoting by $A' \in \mathbb{R}^{p\times n}$ the matrix obtained from A by orthogonal projections of all rows to L, we have

$$K \cap L = \mathbb{R}^p_+ \cdot A' \quad \text{and} \quad S = A'B.$$

By mapping L isometrically to $\mathbb{R}^{\dim(L)}$, we thus find that S is a slack matrix of the pointed cone that is the image of $K \cap L$ under that map and we get the following:

Lemma 10. *A matrix is a slack matrix of a polyhedral cone if and only if it is a slack matrix of some pointed polyhedral cone.*

If the cone K is pointed, then for every zero-row of $S = AB$ the corresponding row of A is a zero-row as well. Hence, removing any zero-row from S results in another slack matrix of K. A similar statement clearly holds for adding zero-rows.

Lemma 11. *If a matrix S is a slack matrix of a pointed polyhedral cone K then every matrix obtained from S by adding or removing zero-rows is a slack matrix of K as well.*

Lemmas 10 and 11 together also imply this statement:

Lemma 12. *If a matrix is a slack matrix of some polyhedral cone then every matrix obtained from it by adding or removing zero-rows is a slack matrix of some polyhedral cone as well.*

Let us further investigate the linear map $x \mapsto x^T B$. It induces the isomorphism

$$L \xrightarrow[\text{isomorphism}]{\cdot B} \mathbb{R}^p \cdot S \tag{8}$$

between the linear space L and the row span of S because of the relations:

$$L \subseteq \operatorname{lineal}(K)^\perp = \operatorname{leftkernel}(B)^\perp$$

and

$$L \cdot B = \big(L + \operatorname{lineal}(K)\big) \cdot B = \operatorname{lin}(K) \cdot B = \big(\mathbb{R}^p \cdot A\big) \cdot B = \mathbb{R}^p \cdot S.$$

It also induces the isomorphism

$$K \cap L \xrightarrow[\text{isomorphism}]{\cdot B} \mathbb{R}^p_+ \cdot S$$

between the cone $K \cap L$ and the cone spanned by the rows of S since $(K \cap L) \cdot B = ((K \cap L) + \operatorname{lineal}(K)) \cdot B = K \cdot B = (\mathbb{R}^p_+ \cdot A) \cdot B = \mathbb{R}^p_+ \cdot S$. In particular, we have shown the following result:

Lemma 13. *A polyhedral cone K is pointed if and only if* $\dim(K) = \operatorname{rank}(S)$ *for any slack matrix S of K.*

Recall that if P is a polytope with representation (V, W, w) and slack matrix $S = [\mathbb{1}, V] \cdot B$ where

$$B = \begin{bmatrix} w^T \\ -W^T \end{bmatrix},$$

then the homogenization P^h of P is a pointed cone that also has S as a slack matrix. Since P^h is pointed, L contains the entire cone and we can restrict the isomorphism in (8) to the set $\{1\} \times P = \operatorname{conv}(\operatorname{rows}([\mathbb{1}, V]))$. Thus we have that $\{1\} \times P$ is isomorphic to $\operatorname{conv}(\operatorname{rows}([\mathbb{1}, V])) \cdot B = \operatorname{conv}(\operatorname{rows}(S))$. This establishes the first part of the following:

Theorem 14. *If S is a slack matrix of the polytope P, then P is isomorphic to* $\operatorname{conv}(\operatorname{rows}(S))$. *In addition, we have* $\dim(P) = \operatorname{rank}(S) - 1$.

Proof. To prove the second statement, note that $\dim(P^h) = \dim(P) + 1$. By Lemma 13, we have that $\dim(P^h) = \operatorname{rank}(S)$. $\square$

In the conic case, we had that $M \in \mathcal{S}_K$ if and only if $M^T \in \mathcal{S}_{K^*}$. This correspondence breaks down for polytopes as we see in the example below. The reason behind this is that we cannot scale $\mathcal{V}$-representations of polytopes by positive scalars.

Example 15. The matrix

$$M = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

is a slack matrix for the triangular prism in $\mathbb{R}^3$. Thus, by Corollary 7, M satisfies both the RCGC and the CCGC, and the all-ones vector is in the column span of M. However, the all-ones vector is not in the row span of M, so M^T is not the slack matrix of any polytope.

Despite this complication, we can still derive some results for transposes of slack matrices of polytopes. Recall that the *polar* of a polytope $P \subset \mathbb{R}^n$ is

$$P^\circ = \left\{y \in \mathbb{R}^n \colon x^T y \leqslant 1 \text{ for all } x \in P\right\}.$$

Then P° is a polytope whenever $0 \in \operatorname{int}(P)$, the interior of P. Since translating P does not change its slack matrices, we may assume that $0 \in \operatorname{int}(P)$. Therefore, P has an $\mathcal{H}$-representation of the form $P = \{x \in \mathbb{R}^n \colon\ Wx \leqslant \mathbb{1}\}$ and $P^\circ = \operatorname{conv}(\operatorname{rows}(W))$. Similarly, if $P = \operatorname{conv}(\operatorname{rows}(V))$, then $P^\circ = \{x \in \mathbb{R}^n \colon\ Vx \leqslant \mathbb{1}\}$. This implies that the slack matrix of P with respect to the representation $(V, W, \mathbb{1})$ is the transpose of the slack matrix of P° with respect to the representation $(W, V, \mathbb{1})$ and we get the following result that is analogous to Proposition 2 for cones.

Proposition 16. *For any polytope P, there exists a slack matrix $M \in \mathcal{S}_P$ such that M^T is also a slack matrix of a polytope.*

In the light of Theorem 6, this says that slack matrices of polytopes (which already have $\mathbb{1}$ in their column span) allow positive scalings of their columns that puts $\mathbb{1}$ into their row span as well. This is false for general nonnegative matrices.

Example 17. Continuing Example 15, we see that the following matrix M' obtained by scaling the columns of M is also a slack matrix of the same prism and does have $\mathbb{1}$ in its row span:

$$M' = \begin{pmatrix} 2 & 2 & 0 & 0 & 0 \\ 2 & 0 & 4 & 0 & 0 \\ 2 & 0 & 0 & 4 & 0 \\ 0 & 2 & 0 & 0 & 2 \\ 0 & 0 & 4 & 0 & 2 \\ 0 & 0 & 0 & 4 & 2 \end{pmatrix}.$$

The prism has vertices:

$$(0, 1, -1), (2, -1, -1), (-2, -1, -1), (0, 1, 1), (2, -1, 1), (-2, -1, 1)$$

and M' comes from the facet description:

$$z \leqslant 1, -y \leqslant 1, -x + y \leqslant 1, x + y \leqslant 1, -z \leqslant 1.$$

Therefore, P° has vertices $(0, 0, 1), (0, -1, 0), (-1, 1, 0), (1, 1, 0), (0, 0, -1)$ and is a bisimplex with slack matrix M'^T.

We can also show a converse to Proposition 16.

Proposition 18. *Suppose $M \in \mathbb{R}_+^{p\times q}$ such that M and M^T are both slack matrices of polytopes. Then there exists a polytope P, with $0 \in \operatorname{int}(P)$, such that $M \in \mathcal{S}_P$ and $M^T \in \mathcal{S}_{P^\circ}$.*

Proof. Since M^T is a slack matrix of a polytope, we have that $\mathbb{1} \in \mathbb{R}_+^p \cdot M$. Without loss of generality, we can scale M by a positive scalar so that $\mathbb{1} \in \operatorname{conv}(\operatorname{rows}(M))$.

Let M be a slack matrix of a polytope R with $\dim(R) = d$. By Theorem 14, $\operatorname{rank}(M) = d + 1$. Since the convex hull of the rows of M is isomorphic to R, we have that the affine hull of the rows of M has dimension d. Let J denote the all-ones matrix of dimension $p \times q$. Since $\mathbb{1}$ is contained in the affine hull of the rows of M, we have that the affine hull of the rows of $M - J$ passes through the origin and has dimension d. Hence, $\operatorname{rank}(M - J) = d$. This implies that we can write $M - J = AB$ with $A \in \mathbb{R}^{p\times d}$ and $B \in \mathbb{R}^{d\times q}$.

Let $A' = (\mathbb{1}, A)$ and let $B' = (\mathbb{1}, B^T)^T$. Then $M = A'B'$ is a rank factorization of M. Let $P := \operatorname{conv}(\operatorname{rows}(A))$ and $Q := \{x \in \mathbb{R}^d \colon\ \mathbb{1} + x^T B \geqslant \mathbb{O}\}$. Then the rows of A' form a $\mathcal{V}$-representation of P^h and the columns of B' form an $\mathcal{H}$-representation for $Q^h = \{(x_0, x) \in \mathbb{R}^{d+1} \colon\ \mathbb{1}x_0 + x^T B \geqslant \mathbb{O}\}$. By Lemma 5, $P^h = Q^h$ which implies that $P = Q$. Therefore, M is a slack matrix of P and M^T is a slack matrix of P°. $\square$

3. An algorithm to recognize slack matrices

In this section, we discuss the algorithmic problem of deciding whether a given nonnegative matrix has the RCGC (or, equivalently, the CCGC). According to Corollaries 4 and 7 this is the crucial step to be performed in order to decide whether a given matrix is a slack matrix of a cone or a polytope.

We start with a promising result:

Theorem 19. *The problem to decide whether a nonnegative matrix satisfies the RCGC* (*or the CCGC*) *is in coNP. In particular, the same holds for checking the property of being a slack matrix* (*of a cone or of a polytope*).

Proof. If the given matrix $M \in \mathbb{R}_+^{p \times q}$ does not satisfy the RCGC, then there is some point $x \in \mathbb{R}^p \cdot M \cap \mathbb{R}_+^q \setminus \mathbb{R}_+^p \cdot M$ (which can be chosen to have coordinates whose encoding lengths are bounded polynomially in the encoding length of M). The fact that $x \notin \mathbb{R}_+^p \cdot M$ can be certified by the help of some separating hyperplane whose normal vector can be chosen to have coordinates with encoding length bounded polynomially in the encoding length of M as well. $\square$

Next, we are going to describe an algorithm to check the CCGC (equivalently, the RCGC) for a nonnegative matrix. By Corollary 4, this algorithm will then provide a method to check if a given nonnegative matrix is a slack matrix of a cone. To check if the matrix is the slack matrix of a polytope (see Corollary 7), we can add the additional step of checking if the all-ones vector is in the column span of the matrix which is doable in polynomial time. A SAGE worksheet implementing this code can be found at `http://www.math.washington.edu/~rzr`.

Algorithm to check if a nonnegative matrix has the CCGC
Input: A matrix $M \in \mathbb{R}_+^{p \times q}$.
Output: `True` if M has the CCGC and `False` otherwise.

(1) Compute a basis L for the left kernel of M. For each vector ℓ in L, generate the equation $\ell^T x = 0$.
(2) Generate an $\mathcal{H}$-representation of the cone K with the equations from the previous step and the inequalities $x_1 \geqslant 0, \ldots, x_p \geqslant 0$.
(3) Compute a minimal $\mathcal{V}$-representation of K.
(4) Normalize the vectors in the $\mathcal{V}$-representation and the columns of M.
(5) Check that each normalized vector in the $\mathcal{V}$-representation is a normalized column of M. If so, return `True`. If not, return `False`.

Proof. We have $K = M \cdot \mathbb{R}^q \cap \mathbb{R}_+^p$ and $M \cdot \mathbb{R}_+^q \subseteq K$ due to the nonnegativity of M. Thus, M satisfies the CCGC if and only if $K \subseteq M \cdot \mathbb{R}_+^q$ holds, which is what the algorithm checks in the last three steps (note that all cones involved are pointed because they are contained in $\mathbb{R}_+^p$). $\square$

The only computationally challenging part of the algorithm is converting from the $\mathcal{H}$-representation of K to a $\mathcal{V}$-representation. There are several algorithms to do this, and we refer to [6,9,11] for information on the different techniques. No polynomial time algorithm for this conversion exists, since the $\mathcal{V}$-representation may have size exponential in that of the $\mathcal{H}$-representation. If the dimension of the cone is fixed, however, then there do exist polynomial time algorithms for the conversion [3]. Thus, we obtain the following complexity results.

Theorem 20. *For fixed r, checking whether a rank r matrix satisfies the RCGC* (*CCGC*) *can be done in polynomial time. In particular, checking whether matrices of fixed rank are slack matrices of cones or polytopes can be done in polynomial time.*

Given an $\mathcal{H}$-polyhedron P and a $\mathcal{V}$-polytope Q contained in P, the problem of deciding whether $P = Q$ is known as the *polyhedral verification problem*. The complexity of this problem is unknown [8]. However, a polynomial time algorithm for the polyhedral verification problem would yield an *output* *sensitive* algorithm for the problem of computing the facets of a polytope given in $\mathcal{V}$-representation, and thus solve a decades old open problem in computational geometry (see [7]).

Clearly, given a $\mathcal{V}$-polytope it is easy to check whether it is contained in an $\mathcal{H}$-polyhedron. The reverse problem of checking whether an $\mathcal{H}$-polyhedron is contained in a $\mathcal{V}$-polytope is known to be coNP-complete [4]. Note that the polyhedral verification problem is the restriction of the latter problem to those instances in which the $\mathcal{V}$-polytope is contained in the $\mathcal{H}$-polyhedron (see also http://www.inf.ethz.ch/personal/fukudak/polyfaq/node21.html, [8] and [12]).

Theorem 21. *The following problems can be reduced in polynomial time to each other*:

(1) *The polyhedral verification problem.*
(2) *Is a given matrix a slack matrix of a polytope?*
(3) *Is a given matrix a slack matrix of a cone?*
(4) *Does a given matrix satisfy the RCGC/CCGC?*

Proof. Corollary 7 shows that (2) can be reduced (in polynomial time) to (4) (since checking whether $\mathbb{1}$ is contained in the column space can be done in polynomial time) and Corollary 4 shows that (4) can be reduced to (3).

We can also reduce (3) to (2): Suppose we need to check whether a given matrix M is a slack matrix of a cone. By Lemma 11, we can assume that M has no zero rows. We can also scale the rows of M by positive scalars without effect on M being a slack matrix of a cone. Using these two facts, we can assume that $\mathbb{1}$ is in the column span of M. Then, being a slack matrix of a cone is equivalent to being a slack matrix of a polytope due to Theorem 6.

Since Corollary 8 shows how to reduce (2) to (1), it thus remains to establish a reduction of (1) to (2). Let $Q = \operatorname{conv}(\operatorname{rows}(V))$ with $V \in \mathbb{R}^{p\times n}$ and $P = \{x \in \mathbb{R}^n\colon\ Wx \leqslant w\}$ with $W \in \mathbb{R}^{q\times n}$ and $w \in \mathbb{R}^q$ with $Q \subseteq P$. Suppose we need to decide whether $P = Q$. First, we check whether P is pointed (i.e., W has a trivial right kernel) and $\dim(P) = \dim(Q)$ (both checks can be done in polynomial time, the second one using linear programming). If either check fails, then $P \neq Q$.

So let us assume $\dim(P) = \dim(Q)$ and that P is pointed. The latter fact implies that the affine map $\varphi : \mathbb{R}^n \to \mathbb{R}^q$ defined via $\varphi(x) = w - Wx$ is injective. Let M be the matrix arising from V by applying φ to each row. Then, due to $Q \subseteq P$, we have that M is nonnegative. According to Corollary 8, the matrix M is a slack matrix of a polytope if and only if

$$\operatorname{conv}\bigl(\operatorname{rows}(M)\bigr) = \operatorname{aff}\bigl(\operatorname{rows}(M)\bigr) \cap \mathbb{R}^q_+. \tag{9}$$

Since we have

$$\operatorname{conv}\bigl(\operatorname{rows}(M)\bigr) = \varphi\bigl(\operatorname{conv}\bigl(\operatorname{rows}(V)\bigr)\bigr) = \varphi(Q)$$

and

$$\begin{aligned}\operatorname{aff}\bigl(\operatorname{rows}(M)\bigr) \cap \mathbb{R}^q_+ &= \varphi\bigl(\operatorname{aff}\bigl(\operatorname{rows}(V)\bigr)\bigr) \cap \mathbb{R}^q_+ = \varphi\bigl(\operatorname{aff}(Q)\bigr) \cap \mathbb{R}^q_+ \\ &= \varphi\bigl(\bigl\{x \in \operatorname{aff}(Q)\colon\ \varphi(x) \geqslant \mathbb{O}\bigr\}\bigr) = \varphi\bigl(P \cap \operatorname{aff}(Q)\bigr) = \varphi(P)\end{aligned}$$

(here we used that $\dim(P) = \dim(Q)$), condition (9) is equivalent to $\varphi(P) = \varphi(Q)$. In turn, this is equivalent to $P = Q$ since φ is injective. Thus, $P = Q$ is equivalent to M being the slack matrix of a polytope. $\square$

4. A combinatorial characterization of slack matrices

Our second characterization of slack matrices of cones and polytopes relies on incidence structures. For a (nonnegative) matrix M, we denote by M_{inc} the 0/1-matrix with $(M_{\text{inc}})_{ij} = 1$ if and only if $M_{ij} = 0$. The matrices M_{inc} arising from slack matrices M of a polyhedral cone K or of a polytope P are called the *incidence matrices* of K or P, respectively.

We start by characterizing the slack matrices of polytopes, since the corresponding statement for cones can easily be deduced from the one for polytopes. The characterization is restricted to nonnegative matrices of rank at least two. It is easy to see that no matrix of rank one is a slack matrix of a non-trivial polytope. One may (or may not) want to consider a rank-zero matrix as a slack matrix of the polytope consisting of the zero-vector in $\mathbb{R}^0$.

Theorem 22. *A nonnegative matrix M with $\operatorname{rank}(M) \geqslant 2$ is a slack matrix of some polytope if and only if M_{inc} is an incidence matrix of some $(\operatorname{rank}(M) - 1)$-dimensional polytope and $\mathbb{1}$ is contained in the column span of M.*

Proof. If M is a slack matrix of a polytope P, then $\mathbb{1}$ is contained in the column span of M (Theorem 6), and by Theorem 14, $\dim(P) = \operatorname{rank}(M) - 1$.

In order to establish the non-trivial implication of the claim, let $M \in \mathbb{R}_+^{p\times q}$ be a nonnegative matrix with $\operatorname{rank}(M) = d+1 \geqslant 2$, $\mathbb{1} \in M \cdot \mathbb{R}^q$ and M_{inc} an incidence matrix of some d-dimensional polytope R. Denote by $V \subseteq \mathbb{R}_+^q$ the set of rows of M and define the polytope $P := \operatorname{conv}(V)$ and the polyhedron $Q := \operatorname{aff}(V) \cap \mathbb{R}_+^q$. Clearly, $P \subseteq Q$, and since $\mathbb{1} \in M \cdot \mathbb{R}^q$, $\dim(Q) = \dim(P) = d$. By Corollary 8, in order to show that M is a slack matrix of a polytope, it suffices to prove $P = Q$.

In order to establish $Q \subseteq P$, let us define

$$V_i = \{v \in V\colon\ v_i = 0\} \quad \text{and} \quad F_i = \operatorname{conv}(V_i) \quad \text{for } 1 \leqslant i \leqslant q.$$

The set

$$F = \bigcup_{i=1}^{q} F_i$$

is contained in the relative boundary ∂Q of Q. Note that as an incidence matrix of some polytope of dimension at least one, M_{inc} does not have an all-ones column. Since $Q = \operatorname{conv}(\partial Q)$ (note that Q is a *pointed* polyhedron of dimension $d \geqslant 2$, which is important here in case of Q being unbounded), if we show that $F = \partial Q$, then we will have that $Q = \operatorname{conv}(F) \subseteq P$.

Thus, our goal is to establish $F = \partial Q$. As mentioned above, we have $F \subseteq \partial Q$. It suffices to show that F is homotopy-equivalent to a $(d-1)$-dimensional sphere,[1] because then F cannot be *properly* contained in the $(d-1)$-dimensional connected (recall $\dim(Q) \geqslant 2$) manifold ∂Q. This follows, e.g., from [2, Cor. 8.5] together with the fact that the $(d-1)$-st cohomology group of a $(d-1)$-dimensional sphere is non-trivial.

To show that F is homotopy-equivalent to a $(d-1)$-dimensional sphere, observe that for every subset $I \subseteq \{1,\dots,q\}$, we have $\bigcap_{i\in I} F_i \neq \emptyset$ if and only if the submatrix of M_{inc} formed by the columns indexed by I has an all-ones row. Now let R be a polytope of which M_{inc} is an incidence matrix. Let $G_1,\dots,G_q$ be the faces of R that correspond to the columns of M_{inc}. Then $\bigcap_{i\in I} G_i \neq \emptyset$ holds if and only if the submatrix of M_{inc} formed by the columns indexed by I has an all-ones row.

Therefore, the abstract simplicial complexes

$$\left\{I \subseteq \{1,\dots,q\}\colon \bigcap_{i\in I} F_i \neq \emptyset\right\}, \quad \text{and} \quad \left\{I \subseteq \{1,\dots,q\}\colon \bigcap_{i\in I} G_i \neq \emptyset\right\}$$

(known as the *nerves* of the polyhedral complexes induced by $F_1,\dots,F_q$ and by $G_1,\dots,G_q$, respectively) are identical. Since all intersections $\bigcap_{i\in I} F_i$ and $\bigcap_{i\in I} G_i$ are contractible (in fact, they are even convex), this simplicial complex is homotopy equivalent to both F and to the $(d-1)$-dimensional (polyhedral) sphere ∂R (see, e.g., [1, Thm. 10.6]). $\square$

Since polygons have a very simple combinatorial structure, Theorem 22 readily yields a simple characterization of their slack-matrices. Here, a *vertex-facet slack matrix* of a polytope P is a slack

[1] Our proof of this is inspired by [7].

matrix of P whose rows and columns are in one-to-one correspondence with the vertices and facets of P, respectively.

Corollary 23. *A matrix $M \in \mathbb{R}_+^{n\times n}$ ($n \geqslant 3$) is a vertex-facet slack matrix of an n-gon if and only if its rows span an affine space of dimension exactly two and its rows and columns can be permuted such that the non-zero entries appear exactly at the positions (i,i) (for $1 \leqslant i \leqslant n$), and $(i,i-1)$ (for $2 \leqslant i \leqslant n$), and $(1,n)$.*

Steinitz's theorem [13] says that a graph G is the 1-skeleton of a three-dimensional polytope if and only if G is planar and three-connected. Using this, one can check in polynomial time whether a given 0/1-matrix is an incidence matrix of a three-dimensional polytope. For every fixed $d \geqslant 4$, however, it is NP-hard to decide whether a given 0/1-matrix is an incidence matrix of a d-dimensional polytope [10].

In the following combinatorial characterization of slack matrices of cones we restrict our attention to matrices of rank at least two as for polytopes. Clearly, every nonnegative matrix of rank one is a slack matrix of the ray $\mathbb{R}^1_+$, and, we may consider a matrix of rank zero as a slack matrix of the trivial cone $\{0\}$ in $\mathbb{R}^0$.

Theorem 24. *A nonnegative matrix M with $\mathrm{rank}(M) \geqslant 2$ is a slack matrix of a polyhedral cone if and only if M_{inc} is an incidence matrix of some $\mathrm{rank}(M)$-dimensional pointed polyhedral cone.*

Proof. If M is a slack matrix of some polyhedral cone then, by Lemma 10, M is a slack matrix (and hence M_{inc} is an incidence matrix) of a pointed polyhedral cone K. By Lemma 13 this cone has dimension $\mathrm{rank}(M)$.

In order to prove the reverse implication, we can assume by the results in Section 2.3 that M does not have any zero-row. Since M is also nonnegative, there exists a positive diagonal matrix D such that DM contains $\mathbb{1}$ in its column span.

Given a pointed cone K, we can slice K by an affine hyperplane L such that the slice is a polytope of dimension $\dim(K)-1$ and the incidence structures of K and $K \cap L$ are identical. Thus, $(DM)_{\mathrm{inc}}$ is an incidence matrix of some $(\mathrm{rank}(M)-1)$-dimensional polytope. By Theorem 22, we have that DM is a slack matrix of a polytope. Hence, M is a slack matrix of the homogenization cone of this polytope. $\square$

Note that dropping *pointed* from the formulation of Theorem 24 makes the statement false. Indeed,

$$M = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{with } M_{\mathrm{inc}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$$

and $\mathrm{rank}(M) = 2$ is not a slack matrix (since M does not satisfy the RCGC), but M_{inc} is the incidence matrix of the non-pointed cone $\{(x_1, x_2)\colon\ x_2 \geqslant 0\}$ with $\mathcal{V}$-representation $(0,1)$, $(0,1)$, $(1,0)$, $(-1,0)$ and $\mathcal{H}$-representation $(0,1)$, $(0,1)$.

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FULL LENGTH PAPER

Extended formulations, nonnegative factorizations, and randomized communication protocols

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Abstract An extended formulation of a polyhedron P is a linear description of a polyhedron Q together with a linear map π such that $\pi(Q) = P$. These objects are of fundamental importance in polyhedral combinatorics and optimization theory, and the subject of a number of studies. Yannakakis' factorization theorem (Yannakakis in J Comput Syst Sci 43(3):441–466, 1991) provides a surprising connection between extended formulations and communication complexity, showing that the smallest size of an extended formulation of P equals the nonnegative rank of its slack matrix S. Moreover, Yannakakis also shows that the nonnegative rank of S is at most 2^c, where c is the complexity of any *deterministic* protocol computing S. In this paper,

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we show that the latter result can be strengthened when we allow protocols to be *randomized*. In particular, we prove that the base-2 logarithm of the nonnegative rank of any nonnegative matrix equals the minimum complexity of a randomized communication protocol computing the matrix in expectation. Using Yannakakis' factorization theorem, this implies that the base-2 logarithm of the smallest size of an extended formulation of a polytope P equals the minimum complexity of a randomized communication protocol computing the slack matrix of P in expectation. We show that allowing randomization in the protocol can be crucial for obtaining small extended formulations. Specifically, we prove that for the spanning tree and perfect matching polytopes, small variance in the protocol forces large size in the extended formulation.

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1 Introduction

Extended formulations are a powerful tool for minimizing linear or, more generally, convex functions over polyhedra (see, e.g., Ziegler [28] for background on polyhedra and polytopes). Consider a polyhedron P in $\mathbb{R}^d$ and a convex function $f : \mathbb{R}^d \to \mathbb{R}$, that has to be minimized over P. If a small size linear description of P is known, then minimizing f over P can be done efficiently using an interior point algorithm, or the simplex algorithm if f is linear and theoretical efficiency is not required.

However, P can potentially have many facets. Or worse: it can be that no explicit complete linear description of P is known. This does not necessarily make the given optimization problem difficult. A fundamental result of Grötschel, Lovász and Schrijver [11] states that if there exists an efficient algorithm solving the separation problem for P, then optimizing over P can be done efficiently. However, this result uses the ellipsoid algorithm, which is not very efficient in practice. Thus it is desirable to avoid using the ellipsoid algorithm.

Now suppose that there exists a polyhedron Q in a higher dimensional space $\mathbb{R}^e$ such that P is the image of Q under a linear projection $\pi : \mathbb{R}^e \to \mathbb{R}^d$. The polyhedron Q together with the projection π defines an *extension* of P, while we call *extended formulation* of P any description of Q by means of linear inequalities and equations, together with the map π. Minimizing f over P amounts to minimizing $f \circ \pi$ over Q. If Q has few facets, then we can resort to an interior point algorithm or the simplex algorithm to solve the optimization problem. Of course, one should also take into account the size of the coefficients in the linear description of Q and in the matrix of π. But this can essentially be ignored for 0/1-polytopes P [21].

The success of extended formulations is due to the fact that a moderate increase in dimension can result in a dramatic decrease in the number of facets. For instance, P can have exponentially many facets, while Q has only polynomially many. We will see examples of this phenomenon later in this paper. For more examples, and background, see the recent surveys by Conforti, Cornuéjols and Zambelli [4], Kaibel [13] and Wolsey [26].

Extensions provide an interesting measure of how "complex" a polyhedron is: define the *size* of an extension Q of P as the number of facets of Q and the *extension* *complexity* of a polyhedron P as the minimum size of any extension of P. Following [9], we denote this number by $\mathrm{xc}(P)$. The *size* of an extended formulation of P is the number of inequalities of the linear system (hence, neither equations nor variables are taken into account). Note that the size of an extended formulation is at least the size of the associated extension, and any extension Q has an extended formulation describing Q with the same size.

This paper builds on Yannakakis' seminal paper [24]. We briefly review his contribution, postponing formal definitions to Sect. 2. Because we mainly consider polytopes, we assume from now on that P is bounded, that is, P is a polytope. (This is not a major restriction.) Yannakakis' *factorization theorem* (Theorem 1) states that to each size-r extension of a polytope P corresponds a *rank-r nonnegative factorization* of some matrix $S(P)$ associated to P, called the *slack matrix*, and conversely to each rank-r nonnegative factorization of $S(P)$ corresponds a size-r extension of P. In particular, the extension complexity $\mathrm{xc}(P)$ equals the smallest rank of a nonnegative factorization of $S(P)$, that is, the *nonnegative rank* of $S(P)$.

In [24], Yannakakis also shows that every $\lg r$-complexity *deterministic* protocol computing a nonnegative matrix M determines a rank-r nonnegative factorization of M.[1] By the aforementioned factorization theorem, this implies that one can produce extended formulations (and hence upper bounds to the extension complexity) via deterministic communication protocols. Yannakakis used this to obtain a quasi-polynomial $n^{O(\log n)}$-size extension for the stable set polytope of a n-vertex perfect graph.

Our contribution The main goal of this paper is to strengthen the connection between nonnegative rank of matrices (and hence, extension complexity of polytopes) and communication protocols. First we give a brief overview of our results and then provide more details along with an outline of the paper. Our contribution is threefold:

- We pinpoint the "right" model of communication protocol, that exactly corresponds to nonnegative factorizations. We remark that this was done independently by Zhang [27]. Proving such a correspondence is an important conceptual step since it gives a third equivalent way to think about extensions of polytopes, besides projections of polytopes and nonnegative factorizations. Communication protocols are very versatile and we hope that this paper will convince discrete optimizers to add this tool to their arsenal.
- We provide examples of already known extensions, seen as communication protocols, and also of new extensions obtained from communication protocols.
- We prove that the randomization allowed in our protocols is sometimes necessary for obtaining small size extensions. We give a general condition under which small variance in the protocol implies that the size of the corresponding extension is large, which in particular applies to the perfect matching polytope and spanning tree polytope. This indicates that Yannakakis' approach for the stable set polytope of a perfect graph *cannot work* for the perfect matching polytope or spanning tree polytope, since his protocol is deterministic and hence the corresponding variance zero.

[1] Throughout this paper, we use lg for binary logarithm.

More specifically, we define a new model of *randomized communication protocols computing the matrix in expectation*. This generalizes the one used by Yannakakis in [24] (see Sect. 3; our definition differs substantially from the usual notion of of random protocol computing a matrix *with high probability*, which can be found e.g. in [16]). Our protocols perfectly model the relation between the nonnegative factorization of a matrix and communication complexity: in fact, we show that the base-2 logarithm of the nonnegative rank of any nonnegative matrix (rounded up to the next integer) *equals* the minimum complexity of a randomized communication protocol computing the matrix in expectation (Theorem 2). By Yannakakis' factorization theorem, this implies a new characterization of the extension complexity of polytopes (Corollary 3).

We then provide evidence that these protocols are substantially more powerful than the deterministic ones used, e.g., by Yannakakis. In fact, one can associate to each protocol a *variance* (see Sect. 3.3) which, roughly speaking, indicates the "amount of randomness" of the protocol: protocols with variance zero are deterministic protocols. We show that no compact formulation for the spanning tree polytope arises from protocols with small variance (see Sect. 6.3), while we provide a randomized protocol that produces the $O(n^3)$ formulation for the spanning tree polytope of K^n due to Martin [19] (see Sect. 5.2).

We also investigate the existence of compact extended formulation for the matching polytope—a fundamental open problem in polyhedral combinatorics. Yannakakis [24] (see also [14]) proved that every *symmetric* extension of the perfect matching polytope of the complete graph K^n has exponential size (we do not formally define *symmetric* here, since we shall not need it; the interested reader may refer to [24]). We show that a negative result similar to the one of the spanning tree polytope holds true for matchings: no compact formulation for the matching polytope arises from protocols with small variance (see Sect. 6). Thus, in particular, deterministic protocols cannot be used to provide compact extended formulations for the perfect matching polytope. We also provide a randomized protocol that produces a $O(1.42^n)$ formulation for the matching polytope implicit in Kaibel, Pashkovich and Theis [14] (see Sect. 5.3). The negative results on both the spanning tree and the matching polytopes are obtained via a general technique that exploits known negative results on the communication complexity of the set disjointness problem.

We would like to remark that the results contained in this paper were, at a conceptual level, an important stepping stone for the strong lower bounds on the extension complexities of the cut, stable set and TSP polytopes of Fiorini, Massar, Pokutta, Tiwary and de Wolf [8].

2 Preliminary definitions and results

2.1 The factorization theorem and related concepts

Consider a polytope P in $\mathbb{R}^d$ with m facets and n vertices. Let $h_1, \ldots, h_m$ be m affine functions on $\mathbb{R}^d$ such that $h_1(x) \geqslant 0, \ldots, h_m(x) \geqslant 0$ are all the facet-defining inequalities of P. Let also $v_1, \ldots, v_n$ denote the vertices of P. The *slack matrix* of P is the nonnegative $m \times n$ matrix $S = S(P) = (s_{ij})$ with $s_{ij} = h_i(v_j)$. Also note that the facet-defining inequalities can be defined up to any positive scaling factor. It should be clear that such a scaling does not alter the non-negative rank of a matrix. To see this let $S = AB$ and let S' be a matrix obtained by multiplying the i-th row of S by $\lambda > 0$. Then, $S' = A'B$ where A' is obtained by multiplying the i-th row of A by λ.

A *rank-r nonnegative factorization* of a nonnegative matrix S is an expression of S as a product $S = AB$ where A and B are nonnegative matrices with r columns and r rows, respectively. The *nonnegative rank* of S, denoted by $\text{rank}_+(S)$, is the minimum nonnegative integer r such that S admits a rank-r nonnegative factorization [3]. Observe that the nonnegative rank of S can also be defined as the minimum nonnegative integer r such that S is the sum of r nonnegative rank-1 matrices.

In a seminal paper, Yannakakis [24] proved, among other things, that the extension complexity of a polytope is precisely the nonnegative rank of its slack matrix (see also [9]).

Theorem 1 (Yannakakis' factorization theorem) *For all polytopes P that are neither empty or a point,*

$$\text{xc}(P) = \text{rank}_+(S(P)).$$

Before going on, we sketch the proof of half of the theorem. Assuming $P = \{x \in \mathbb{R}^d : Ex \leqslant g\}$, consider a rank-$r$ nonnegative factorization $S(P) = FV$ of the slack matrix of P. Then it can be shown that $Q := \{(x, y) \in \mathbb{R}^{d+r} : Ex + Fy = g,\ y \geqslant \mathbf{0}\}$ is an extension of P. Notice that Q has at most r facets, and r extra variables.[2] Taking $r = \text{rank}_+(S(P))$ implies $\text{xc}(P) \leqslant \text{rank}_+(S(P))$. Moreover, since P is a polytope, one can also assume that Q is bounded, as shown by the following lemma.

Lemma 1 *Let $P = \{x \in \mathbb{R}^d : Ex \leqslant g\}$ be a polytope, let $S(P) = FV$ be a rank-r nonnegative factorization of the slack matrix of P with $r := \text{rank}_+(S(P))$, and let $Q := \{(x, y) \in \mathbb{R}^{d+r} : Ex + Fy = g,\ y \geqslant \mathbf{0}\}$. Then Q is bounded.*

Proof The polyhedron Q is unbounded if and only if its recession cone $\text{rec}(Q) = \{(x, y) \in \mathbb{R}^{d+r} : Ex + Fy = \mathbf{0}, y \geqslant \mathbf{0}\}$ contains some nonzero vector. Since P is bounded and the image of Q under the projection $(x, y) \mapsto x$ is P, we have $x = \mathbf{0}$ for every point $(x, y) \in \text{rec}(Q)$. Therefore, Q is unbounded if and only if the system $Fy = \mathbf{0}$, $y \geqslant \mathbf{0}$ has a solution $y \neq \mathbf{0}$. But any such y represents $\mathbf{0}$ as a non-trivial conical combination of the column vectors of F. Since F is nonnegative, this is only possible if one of the columns of F is identically zero, which would contradict the minimality of r. □

2.2 Polytopes relevant to this work

Now we describe briefly various families of polytopes relevant to this paper. For a more detailed description of these polytopes, we refer the reader to Schrijver [22].

[2] The extended formulation for Q given above potentially has a large number of equalities, but recall we only consider the number of inequalities in the size of the extended formulation. The reasons for this are twofold: first, one can ignore most of the equalities after picking a small number of linearly independent equalities; and second, our concern in this paper is mainly the *existence* of certain extensions.

Let I be a finite ground set. The *characteristic vector* of a subset $J \subseteq I$ is the vector $\chi^J \in \mathbb{R}^I$ defined as

$$\chi_i^J = \begin{cases} 1 & \text{if } i \in J \\ 0 & \text{if } i \notin J \end{cases}$$

for $i \in I$. For $x \in \mathbb{R}^I$, we let $x(J) := \sum_{i \in J} x_i$.

Throughout this section, $G = (V, E)$ denotes a (finite, simple, undirected) graph. For a subset of vertices $U \subseteq V$, we denote the edges of the subgraph induced by U as $E[U]$. The *cut* defined by U, denoted as $\delta(U)$, is the set of edges of G exactly one of whose endpoints is in U. That is,

$$\begin{aligned} E[U] &= \{uv \in E : u \in U, v \in U\}, \text{ and} \\ \delta(U) &= \{uv \in E : u \in U, v \notin U\}. \end{aligned}$$

Later in this paper, we will often take G to be the *complete graph* K^n with vertex set $V(K^n) = [n] := \{1, \ldots, n\}$ and edge set $E(K^n) = \{ij : i, j \in [n], i \neq j\}$.

2.2.1 Spanning tree polytope

A *spanning tree* of G is a tree $T = (V(T), E(T))$ (i.e., a connected graph without cycles) whose set of vertices and edges respectively satisfy $V(T) = V$ and $E(T) \subseteq E$. The *spanning tree polytope* of G is the convex hull of the characteristic vectors of the spanning trees of G, i.e.,

$$\mathrm{P}_{\text{spanning tree}}(G) = \operatorname{conv}\{\chi^{E(T)} \in \mathbb{R}^E : T \text{ spanning tree of } G\}.$$

Edmonds [6] showed that this polytope admits the following linear description (see also [22, page 861]):

$$\begin{aligned} x(E[U]) &\leqslant |U| - 1 && \text{for nonempty } U \subsetneq V, \\ x(E) &= |V| - 1, && \\ x_e &\geqslant 0 && \text{for } e \in E. \end{aligned}$$

This follows, e.g., from the fact that the spanning tree polytope of G is the base polytope of the graphic matroid of G.

2.2.2 Perfect matching polytope

A *perfect matching* of G is set of edges $M \subseteq E$ such that every vertex of G is incident to exactly one edge in M. The *perfect matching polytope* of the graph G is the convex hull of the characteristic vectors of the perfect matchings of G, i.e.,

$$\mathrm{P}_{\text{perfect matching}}(G) = \operatorname{conv}\{\chi^M \in \mathbb{R}^E : M \text{ perfect matching of } G\}.$$

Edmonds [5] showed that the perfect matching polytope of G is described by the following linear constraints (see also [22, page 438]):

$$\begin{aligned} x(\delta(U)) &\geqslant 1 \quad \text{for } U \subseteq V \text{ with } |U| \text{ odd, } |U| \geqslant 3 \\ x(\delta(\{v\})) &= 1 \quad \text{for } v \in V, \\ x_e &\geqslant 0 \quad \text{for } e \in E. \end{aligned}$$

2.2.3 Stable set polytope

A *stable set* S (often also called an *independent set*) of G is a subset of the vertices such that no two of them are adjacent. A *clique* K of G is a subset of the vertices such that every two of them are adjacent. The *stable set polytope* $\mathrm{STAB}(G)$ of a graph $G(V, E)$ is the convex hull of the characteristic vectors of the stable sets in G, i.e.,

$$\mathrm{STAB}(G) = \mathrm{conv}\{\chi^S \in \mathbb{R}^V : S \text{ stable set of } G\}.$$

No complete linear description of the stable set polytope for arbitrary graphs is known. It is, however, known that the following inequalities are valid for $\mathrm{STAB}(G)$ for any graph G:

$$x(K) \leqslant 1 \quad \text{for cliques } K \text{ of } G, \tag{1}$$
$$x_v \geqslant 0 \quad \text{for } v \in V. \tag{2}$$

Inequalities (1) are called the *clique inequalities*. See Schrijver [22] for details.

A graph G is called *perfect* if the chromatic number of every induced subgraph equals the size of the largest clique of that subgraph. It is known that G is perfect if and only if inequalities (1) and (2) completely describe $\mathrm{STAB}(G)$ [2].

3 Communication complexity

We start by an overview of the standard model of deterministic communication protocols, as described in detail in the book by Kushilevitz and Nisan [16]. We follow this with a detailed description of our notion of a randomized protocol (with private random bits and nonnegative outputs) computing a function *in expectation*. This differs significantly from the standard definition in the literature where randomized protocols usually compute a function exactly *with high probability*.

3.1 Deterministic protocols

Let X, Y, and Z be arbitrary finite sets with $Z \subseteq \mathbb{R}_+$, and let $f : X \times Y \to Z$ be a function. Suppose that there are two players Alice and Bob who wish to compute $f(x, y)$ for some inputs $x \in X$ and $y \in Y$. Alice knows only x and Bob knows only y. They must therefore exchange information to be able to compute $f(x, y)$. (We assume that each player possesses unlimited computational power.)

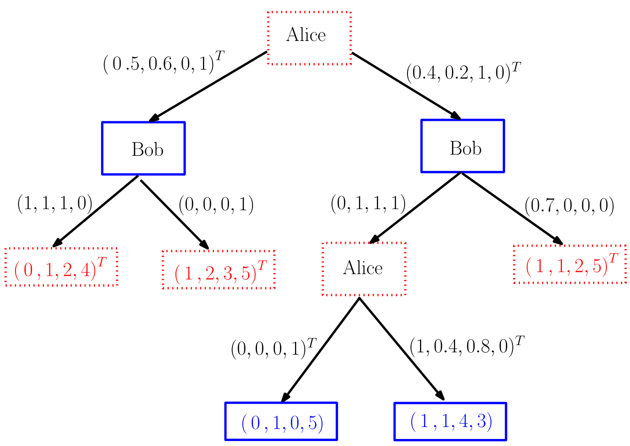
The communication is carried out as a protocol that is agreed upon beforehand by Alice and Bob, on the sole basis of the function f. At each step of the protocol, one of the players has the token. Whoever has the token sends a bit to the other player, that depends only on their input and on previously exchanged bits. This is repeated until the value of f on (x, y) is known to both players. The minimum number of bits exchanged between the players in the worst case to be able to evaluate f by any protocol is called the *communication complexity* of f.

3.2 Randomized protocols and computation in expectation

A protocol can be viewed as a rooted binary tree where each node is marked either Alice or Bob. The leaves have vectors associated with them. An execution of the protocol on a particular input is a path in the tree starting at the root. At a node owned by Alice, following the path to the left subtree corresponds to Alice sending a zero to Bob and taking the right subtree corresponds to Alice sending a one to Bob; and similarly for nodes owned by Bob.

More formally, we define a *randomized protocol (with private random bits and* nonnegative *outputs)* as a rooted binary tree with some extra information attached to its nodes. Let X and Y be finite sets, as above. Each node of the tree has a *type*, which is either X or Y. To each node v of type X are attached two function $p_{0,v}, p_{1,v} : X \to [0, 1]$; to each node v of type Y are attached two functions $q_{0,v}, q_{1,v} : Y \to [0, 1]$; and to each leaf v is attached a nonnegative vector Λ_v that is a column vector of size $|X|$ for leaves of type X and a row vector of size $|Y|$ for leaves of type Y. The functions $p_{i,v}$ and $q_{j,v}$ define *transition probabilities*, and we assume that $p_{0,v}(x)+p_{1,v}(x) \leqslant 1$ and $q_{0,v}(y)+q_{1,v}(y) \leqslant 1$. Figure 1 shows an example of a protocol.

An *execution* of the protocol on input $(x, y) \in X \times Y$ is a random path that starts at the root and descends to the left child of an internal node v with probability



(a)

$X \backslash Y$	y_1	y_2	y_3	y_4
x_1	0.28	0.40	1.60	1.70
x_2	0.74	0.68	0.92	1.44
x_3	1.40	0.80	3.20	2.40
x_4	4.00	4.00	4.00	5.00

(b)

Fig. 1 Illustration of a (non-optimal) randomized protocol computing a matrix in expectation, **a** protocol as a tree, **b** the associated communication matrix

$p_{0,v}(x)$ if v is of type X and $q_{0,v}(y)$ if v is of type Y, and to the right child of v with probability $p_{1,v}(x)$ if v is of type X and $q_{1,v}(y)$ if v is of type Y. With probability $1 - p_{0,v}(x) - p_{1,v}(x)$ and $1 - q_{0,v}(y) - q_{1,v}(y)$ respectively, the execution stops at v. For an execution stopping at leaf v with vector Λ_v, the *value* of the execution is defined as the entry of Λ_v that corresponds to input $x \in X$ if v is of type X, and $y \in Y$ if v is of type Y. For an execution stopping at an internal node, the *value* is defined to be 0.

For each fixed input $(x, y) \in X \times Y$, the value of an execution on input (x, y) is a random variable. If we let $Z \subseteq \mathbb{R}_+$ as before, we say that the protocol *computes* a function $f : X \times Y \to Z$ *in expectation* if the expectation of this random variable on each $(x, y) \in X \times Y$ is precisely $f(x, y)$.

The *complexity* of a protocol is the height of the corresponding tree.

Given an ordering $x_1, \ldots, x_m$ of the elements of X, and $y_1, \ldots, y_n$ of the elements of Y, we can visualize the function $f : X \times Y \to Z$ as a $m \times n$ nonnegative matrix $S = S(f)$ such that $S_{i,j} = f(x_i, y_j)$ for all $(i, j) \in [m] \times [n]$. The matrix S is called the *communication matrix* of f. Below, as is natural, we will not always make a distinction between a function and its communication matrix.

These formal definitions capture the informal ones given above. Observe that the nodes of type X are assigned to Alice, and those of type Y to Bob. Observe also that Alice and Bob have unlimited resources for performing their part of the computation. It is only the communication between the two players that is accounted for. When presenting a protocol, we shall often say that one of the two players sends an integer k rather than a binary value. This should be interpreted as the player sending the binary encoding of k or, as a (sub)tree of height $\lceil \lg k \rceil$. Finally, our definitions are such that the complexity of a protocol equals the number of bits exchanged by Alice and Bob.

3.3 Normalized variance

Since the output of a randomized protocol—as defined above—is a random variable, one can define its variance. However, we would like to refine the notion of variance so that protocols computing different scalings of the same matrix have the same variance. This is essential since the nonnegative rank of a matrix is an invariant under scaling and, as we will see in the next section, there is an equivalence between the nonnegative rank of a matrix S and the smallest complexity protocol computing S in expectation.

Let S be a nonnegative matrix and suppose there exists a protocol of complexity c computing S in expectation. Let $\xi_{i,j}$ denote the random variable corresponding to the output of the protocol on input $(x_i, y_j) \in X \times Y$. That is $\mathbb{E}[\xi_{i,j}] = S_{i,j}$. The *normalized variance* σ^2 of the protocol is defined as the maximum variance of the random variables $\xi'_{i,j} = \frac{\xi_{i,j}}{S_{i,j}}$ for the nonzero entries of S. That is

$$\sigma^2 = \max_{(i,j) \mid S_{i,j} \neq 0} \operatorname{Var}(\xi_{i,j}/S_{i,j})$$

4 Factorizations versus protocols

Theorem 2 *If there exists a randomized protocol of complexity c computing a matrix $S \in \mathbb{R}_+^{X \times Y}$ in expectation, then* $\lg \operatorname{rank}_+(S) \leqslant c$. *Conversely, if the nonnegative rank of matrix $S \in \mathbb{R}_+^{m \times n}$ is r, then there exists a randomized protocol computing S in expectation, whose complexity is at most $\lceil \lg r \rceil$. In other words, if $c_{\min}(S)$ denotes the minimum complexity of a randomized protocol computing S in expectation, we have*

$$c_{\min}(S) = \lceil \lg \operatorname{rank}_+(S) \rceil.$$

Proof Suppose there exists a protocol of complexity c computing S in expectation. Each node v of the protocol has a corresponding *traversal probability matrix* $P_v \in \mathbb{R}_+^{X \times Y}$ such that, for all inputs $(x, y) \in X \times Y$, the entry $P_v(x, y)$ is the probability that an execution on input (x, y) goes through node v.

Let $v_1, \ldots, v_k$ denote the nodes of type X on the unique path from the root to the parent of v, and let $w_1, \ldots, w_\ell$ denote the nodes of type Y on this path. Then we have

$$P_v(x, y) = \prod_{i=1}^{k} p_{\alpha_i, v_i}(x) \cdot \prod_{j=1}^{\ell} q_{\beta_j, w_j}(y),$$

where α_i is either 0 or 1 depending on if the path goes the left or right subtree at v_i, and similarly for β_j. We immediately see that P_v is a rank one matrix of the form $a_v b_v$ where a_v is a column vector of size $|X|$ and b_v is a row vector of size $|Y|$.

Finally, let L_X and L_Y be the set of all leaves of the protocol that are of type X and Y respectively and let Λ_v denote the (column or row) vector of values at a leaf $v \in L_X \cup L_Y$. Because the protocol computes S in expectation, for all inputs $(x, y) \in X \times Y$ we have $S(x, y) = \sum_{v \in L_X} \Lambda_v(x) P_v(x, y) + \sum_{w \in L_Y} P_w(x, y) \Lambda_w(y)$. Thus, $S = \sum_{v \in L_X} (\Lambda_v \circ a_v) b_v + \sum_{v \in L_Y} a_w (b_w \circ \Lambda_w)$, where $\circ$ denotes the Hadamard product. Therefore, it is possible to express S as a sum of at most $|L_X \cup L_Y| \leqslant 2^c$ nonnegative rank one matrices. Hence, $\operatorname{rank}_+(S) \leqslant 2^c$, that is, $\lg \operatorname{rank}_+(S) \leqslant c$.

To prove the other part of the theorem, let $A \in \mathbb{R}_+^{m \times r}$ and $B \in \mathbb{R}_+^{r \times n}$ be nonnegative matrices such that $S = AB$. By scaling, we can assume that the maximum row sum of A is 1. Otherwise, we replace A and B by $\Delta^{-1} A$ and ΔB respectively, where Δ denotes the maximum row sum of A.

The protocol is as follows: Alice knows a row index i, and Bob knows a column index j. Together they want to compute $S_{i,j}$ in expectation, by exchanging as few bits as possible. They proceed as follows. Let $\delta_i := \sum_k A_{i,k} \leqslant 1$. Alice selects a column index $k \in [r]$ according to the probabilities found in row i of matrix A, sends this index to Bob, and Bob outputs the entry of B in row k and column j. With probability $1 - \delta_i$ Alice does not send any index to Bob and the computation stops with implicit output zero (see Sect. 3.2).

This randomized protocol computes the matrix S in expectation. Indeed, the expected value on input (i, j) is $\sum_{k=1}^{r} A_{i,k} B_{k,j} = S_{i,j}$. Moreover, the complexity of the protocol is precisely $\lceil \lg(r) \rceil$. □

We would like to remark that our contruction of a factorization from a protocol is similar to the one used by Krause [17] to construct an approximate factorization from a protocol. However his discussion was limited to traditional definitions of a randomized protocol and hence could not produce exact factorizations.

The above theorem together with Theorem 1 gives us the following corollary:

Corollary 3 *Let P be a polytope with associated slack matrix $S = S(P)$, such that P is neither empty or a point. If there exists a randomized protocol of complexity c computing S in expectation, then $\mathrm{xc}(P) \leqslant 2^c$. Conversely, if $\mathrm{xc}(P) = r$, then there exists a randomized protocol computing S in expectation, whose complexity is at most $\lceil \lg r \rceil$. In other words, if $c_{\min}(S)$ denotes the minimum complexity of a randomized protocol computing S in expectation, we have*

$$c_{\min}(S(P)) = \lceil \lg \mathrm{xc}(P) \rceil.$$

The concrete polytopes considered in this paper have some facet-defining inequalities enforcing nonnegativity of the variables along with other facet-defining inequalities. The next lemma and its corollary will allow us to ignore the rows corresponding to nonnegativity inequalities, and focus on the non-trivial parts of the slack matrices.

Lemma 2 *Let S be a nonnegative matrix. Let R_1, R_2 be a partition of the rows of S defining partition of S into S_1 and S_2. If there exist randomized protocols computing S_1 and S_2 in expectation with complexity c_1 and c_2 respectively, then there exists a randomized protocol complexity computing S with complexity $1 + \max\{c_1, c_2\}$.*

Proof When Alice gets a row index of S she sends a bit to Bob to indicate whether the corresponding row lies in R_1 or R_2. Now that both Alice and Bob know whether they want to compute an entry in S_1 or S_2, they use the protocol for that particular submatrix. □

Corollary 4 *Let $P \subseteq \mathbb{R}^d_+$ be a polytope and let $S'(P)$ denote the submatrix of $S(P)$ obtained by deleting the rows corresponding to nonnegativity inequalities. If there is a complexity c randomized protocol for computing $S'(P)$ in expectation, then there is a complexity $1+\max\{c, \lceil \lg d \rceil\}$ randomized protocol for computing $S(P)$ in expectation.*

Proof For computing the part of $S(P)$ that is deleted in $S'(P)$, which corresponds to nonnegativity inequalities, we use the obvious protocol where Alice sends her row number to Bob and Bob computes the slack. Since at most d facets of P are defined by nonnegativity inequalities, this protocol has complexity $\lceil \lg d \rceil$. The corollary thus follows from Lemma 2. □

For the protocols constructed here, we will always have $c \geqslant \lceil \lg d \rceil$. Because of Corollary 4, we can thus ignore the nonnegativity inequalities without blowing up the size of any extension by more than a factor of 2. Moreover, in terms of lower bounds, it is always safe to ignore inequalities because the nonnegative rank of a matrix cannot increase when rows are deleted.

5 Examples

In this section, we give three illustrative examples of protocols defining nonnegative factorizations of various slack matrices, and thus (via Corollary 3) extensions of the corresponding polytopes. The first one gives a $O(n^3)$-size extension of the stable set polytope of a claw-free perfect graph. The second one is a reinterpretation of a well-known $O(n^3)$-size extended formulation for the spanning tree polytopes due to Martin [19]. Our interpretation allows for a more general result. In particular we prove new upper bounds for the spanning tree polytopes for minor-free graphs. The third one concerns the perfect matching polytopes and is implicit in Kaibel, Pashkovich and Theis [14].

5.1 The stable set polytope of a claw-free perfect graph

A graph G is called *claw-free* if no vertex has three pairwise non-adjacent neighbors. Even though the separation problem for $\mathrm{STAB}(G)$ for claw-free graphs is polynomial-time solvable, no explicit description of all its facets is known (see, e.g., [22, page 1216]). Recently Faenza, Oriolo, and Stauffer [7] provided (non-compact) extended formulations for this polytope, while Galluccio et al. [10] gave a complete description of the facets for claw-free graphs with at least one stable set of size greater than or equal to four, and no clique-cutsets. Also, recall that for a perfect graph G the facets of $\mathrm{STAB}(G)$ are defined by inequalities (1) and (2) (see Sect. 2.2.3).

Let G be a claw-free, perfect graph with n vertices. We give a deterministic protocol that computes the slack matrix of the stable set polytope $\mathrm{STAB}(G)$ of G. Because G is perfect, the (non-trivial part of the) slack matrix of $\mathrm{STAB}(G)$ has the following structure: it has one column per stable set S in G, and each one of its rows corresponds to a clique K in G. The entry for a pair (K, S) equals 0 if K and S intersect (in which case they intersect in exactly one vertex) and 1 if K and S are disjoint (note that we are ignoring the $|V|$ rows that correspond to nonnegativity inequalities (2). This can be done safely, see Corollary 4).

Consider the communication problem in which Alice is given a clique K of G, Bob is given a stable set S of G, and Alice and Bob together want to compute $1 - |K \cap S|$. Alice starts and sends the name of any vertex u of her clique K to Bob. Then Bob sends the names of all the vertices of his stable set S that are in $N(u) \cup \{u\}$ to Alice, where $N(u)$ denotes the neighborhood of u in G. Finally, Alice can compute $K \cap S$ because this intersection is contained in $N(u) \cup \{u\}$ and Alice knows all vertices of $S \cap (N(u) \cup \{u\})$. She outputs $1 - |K \cap S|$. Because G is claw-free, there are at most two vertices in $S \cap (N(u) \cup \{u\})$, thus at most $3 \lg n + O(1)$ bits are exchanged by Alice and Bob. It follows that there exists an extension (and hence, an extended formulation) of $\mathrm{STAB}(G)$ of size $O(n^3)$. Notice that the normalized variance of our protocol is zero, because it is deterministic.

We obtain the following result.

Proposition 1 *For every perfect, claw-free graph G with n vertices,* $\mathrm{STAB}(G)$ *has an extended formulation of size* $O(n^3)$.

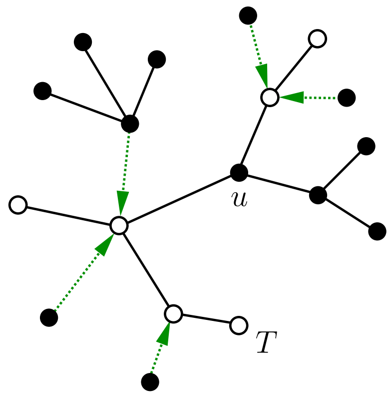


Fig. 2 Illustration of the protocol for the slack of MST polytope. The *black vertices* are those in U. The *green* directed edges are those for which Alice outputs a non-zero value. The number of such edges is the number of connected components of $T[U]$ minus one (color figure online)

5.2 The spanning tree polytope

Let $\mathrm{P}_{\text{spanning tree}}(G)$ denote the spanning tree polytope of a graph $G = (V, E)$ (see Sect. 2.2.1). The (non-trivial part of the) slack matrix of P has one column per spanning tree T and one row per proper nonempty subset U of vertices. The slack of T with respect to the inequality that corresponds to U is the number of connected components of the subgraph of T induced by U (denoted by $T[U]$ below) minus one.

In terms of the corresponding communication problem, Alice has a proper nonempty set U and Bob a spanning tree T. Together, they wish to compute the slack of the pair (U, T). Alice sends the name of some (arbitrarily chosen) vertex u in U. Then Bob picks an edge e of T uniformly at random and sends to Alice the endpoints v and w of e as an ordered pair of vertices (v, w), where the order is chosen in such a way that w is on the unique path from v to u in the tree. That is, he makes sure that the directed edge (v, w) "points" towards the root u. Then Alice checks that $v \in U$ and $w \notin U$, in which case she outputs $n - 1$; otherwise she outputs 0.

The resulting randomized protocol is clearly of complexity $\lg |V| + \lg |E| + O(1)$. Moreover, it computes the slack matrix in expectation because for each connected component of $T[U]$ distinct from that which contains u, there is exactly one directed edge (v, w) that will lead Alice to output a non-zero value, see Fig. 2 for an illustration. Since she outputs $(n-1)$ in this case, the expected value of the protocol on pair (U, T) is $(n - 1) \cdot (k - 1)/(n - 1) = k - 1$, where k is the number of connected components of $T[U]$. Therefore, we obtain the following result.

Proposition 2 *For every graph G with n vertices and m edges,* $\mathrm{P}_{\text{spanning tree}}(G)$ *has an extended formulation of size $O(mn)$.*

The above result is implicit in Martin [19], although the paper only states the following corollary. More specifically, variables $z_{i,j,k}$ such that ij is not an edge of G can be deleted from his $O(n^3)$-size extended formulation, so that the resulting formulation has size $O(mn)$.

Corollary 5 $\mathrm{P}_{\text{spanning tree}}(K^n)$ *has extended formulation of size $O(n^3)$, where K^n is the complete graph on n vertices.*

Corollary 6 *Let G be an H minor-free graph, where H is a graph with h vertices, then* $\mathrm{P}_{\text{spanning tree}}(G)$ *has extended formulation of size $O(n^2 h\sqrt{\lg h})$.*

Proof It is known that any H minor-free graph G with n vertices has at most $O(nh\sqrt{\lg h})$ edges, where h is the number of vertices of H [23]. The result follows. □

We remark that when G is planar, $\mathrm{P}_{\text{spanning tree}}(G)$ has an extended formulation of size $O(n)$ [25]. It is natural to ask whether a linear size extended formulation also exists for general H minor-free graphs. So far, the best that seems to be known is the upper bound in Corollary 6.

Finally, it can be easily verified that the normalized variance of the protocol given above is $\sigma^2 = n - 2$, which is large compared to the previous protocol.

5.3 Perfect matching polytope

For the next example, we will need the fact that one can cover K^n with $k = O(2^{n/2}\mathrm{poly}(n))$ balanced complete bipartite graphs $G_1,\ldots, G_k$ in such a way that every perfect matching of K^n is a perfect matching of at least one of the G_i's. We say that $X \subseteq [n]$ is an $(n/2)$-*subset* of $[n]$ if $|X| = n/2$. Given a matching M of K^n and a $(n/2)$-subset X of $[n]$, we say that X is *compatible* with M if all the edges of M have exactly one end in X.

Lemma 3 *Let n be an even positive integer. Then, there exists a collection of $k = O(2^{n/2}\sqrt{n}\ln n)$ $(n/2)$-subsets $X_1,\ldots, X_k$ of $[n]$ such that for every perfect matching M of K^n at least one of the subsets X_i is compatible with M.*

Proof Finding a minimum size such collection $X_1, \ldots, X_k$ amounts to solving a set covering instance that we formulate by an integer linear program. For each $(n/2)$-subset X, we define a variable binary variable $\lambda(X)$. For each perfect matching M, these variables have to satisfy the constraint $\sum\{\lambda(X) : X \text{ is compatible with } M\} \geqslant 1$. The goal is to minimize $\sum \lambda(X)$, the sum of all variables $\lambda(X)$.

A feasible fractional solution to this linear program is to let $\lambda^*(X) = 1/2^{n/2}$. This gives a feasible fractional solution because each perfect matching M is compatible with exactly $2^{n/2}$ $(n/2)$-subsets X, so $\sum\{\lambda^*(X) : X \text{ is compatible with } M\} = 2^{n/2}(1/2^{n/2}) = 1$. (By symmetry considerations, it is in fact possible to argue that this solution is actually optimal.) The cost of this fractional solution λ^* is

$$\sum \lambda^*(X) = \frac{1}{2^{n/2}}\binom{n}{n/2} \leqslant \frac{2^{n/2}}{\sqrt{n}},$$

for n sufficiently large. By Lovász's analysis of the greedy algorithm for the set covering problem [18], there exists a feasible integer solution λ of cost at most $(1 + \ln u)$ times the fractional optimum, where u is the number of elements to cover. By what precedes, this is at most

$$\left(1 + \ln \frac{n!}{2^{n/2}(n/2)!}\right)\frac{2^{n/2}}{\sqrt{n}} = O(2^{n/2}\sqrt{n}\lg n),$$

from which the result follows directly. □

Assume that n is even and let P denote the perfect matching polytope of the complete graph K^n with vertex set $[n]$, see Sect. 2.2.2. The (non-trivial part of the) slack matrix of P has one column per perfect matching M, and its rows correspond to odd sets $U \subseteq [n]$. The entry for a pair (U, M) is $|\delta(U) \cap M| - 1$ (recall that $\delta(U)$ denotes the set of edges that have one endpoint in U and the other endpoint in $\overline{U}$, the complement of U).

We describe a randomized protocol for computing the slack matrix in expectation, of complexity at most $(1/2 + \varepsilon)n$, where $\varepsilon > 0$ can be made as small as desired by taking n large. First, Bob finds an $(n/2)$-subset $X \subseteq [n]$ that is compatible with his matching M, and tells the name of this subset to Alice, see Lemma 3. Then Alice checks which of X and $\overline{X}$ contains the least number of vertices of her odd set U. Without loss of generality, assume it is X. If $U \cap X = \varnothing$ then, because $U \subseteq \bar{X}$ and X is compatible with M, Alice can correctly infer that the slack is $|U| - 1$, and outputs this number. Otherwise, she picks a vertex u of $U \cap X$ uniformly at random and send its name to Bob. He replies by sending the name of u', the mate of u in the matching M. Alice then checks whether u' is in U or not. If u' is not in U, then she outputs $|U| - 1$. Otherwise u' is in U, and she outputs $|U| - 1 - 2|U \cap X|$. Telling the name of X can be done in at most $n/2 + \lg \sqrt{n} + \lg \lg n + O(1)$ bits, see Lemma 3. The extra amount of communication is $2 \lg n + O(1)$ bits. In total, at most $(1/2 + \varepsilon)n$ bits are exchanged, for n sufficiently large ($\varepsilon > 0$ can be chosen arbitrarily).

Now, we check that the protocol correctly computes the slack matrix of the perfect matching polytope. Letting $E[U]$ denote the edges of the complete graph with both endpoints in U, the expected value output by Alice (in the case $U \cap X \neq \varnothing$) is

$$
\begin{aligned}
&(|U| - 1)\frac{|U \cap X| - |E[U] \cap M|}{|U \cap X|} + (|U| - 1 - 2|U \cap X|)\frac{|E[U] \cap M|}{|U \cap X|} \\
&= |U| - 1 - 2|U \cap X|\frac{|E[U] \cap M|}{|U \cap X|} \\
&= |U| - 2|E[U] \cap M| - 1 \\
&= |\delta(U) \cap M| - 1.
\end{aligned}
$$

We obtain the following result.

Proposition 3 *Let $\varepsilon > 0$. For every large enough even nonnegative integer n, the polytope $\mathrm{P}_{\text{perfect matching}}(K^n)$ has an extended formulation of size at most $2^{(1/2+\varepsilon)n}$.*

We remark that our extension has size at most $2^{(1/2+\varepsilon)n} \leqslant (1.42)^n$, whereas the main result of Yannakakis [24] gives a lower bound of $\binom{n}{n/4} \geqslant (1.74)^n$ for the size of any *symmetric* extension.

6 When low variance forces large size

We have seen that every extension of a polytope P corresponds to a randomized protocol computing its slack matrix $S = S(P)$ in expectation and vice-versa. Now we show that if the set disjointness matrix can be embedded in a certain way in a matrix S (see below for definitions), then efficient protocols computing S in expectation necessarily have large variance. We prove that such an embedding can be found for the slack matrices of the perfect matching polytope and also, surprisingly, of the spanning tree polytope.

6.1 Embedding the set disjointness matrix

The *set disjointness problem* is the following communication problem: Alice and Bob each are given a subset of $[n]$. They wish to determine whether the two subsets intersect or not. In other words, Alice and Bob have to compute the *set disjointness matrix* DISJ defined by $\mathrm{DISJ}(A, B) = 1$ if A and B are disjoint subsets of $[n]$, and $\mathrm{DISJ}(A, B) = 0$ if A and B are non-disjoint subsets of $[n]$. The set disjointness problem plays a central role in communication complexity, comparable to the role played by the satisfiability problem in NP-completeness theory [1].

It is known that any randomized protocol that computes the disjointness function *with high probability* (that is, the probability that the value output by the protocol is correct is, for each input, bounded from below by a constant strictly greater than $1/2$) has $\Omega(n)$ complexity [15,20].

Consider a matrix $S \in \mathbb{R}_+^{X\times Y}$. An *embedding* of the set disjointness matrix on $[n]$ in S is defined by two maps $\alpha : 2^{[n]} \to X$ and $\beta : 2^{[n]} \to Y$ such that

$$\forall A, B \subseteq [n] : \mathrm{DISJ}(A, B) = 1 \iff S(\alpha(A), \beta(B)) = 0. \tag{3}$$

Notice that this kind of embedding could be called "negative" because zeros in the set disjointness matrix correspond to non-zeros in S.

We remark that "positive" embeddings of the set disjointness matrix force up the rank of S, because the rank of any matrix with the same support as the set disjointness matrix on $[n]$ is at least 2^n [12]. This is not desirable because the nonnegative rank of S is always at least its rank. Thus the lower bound on the nonnegative rank of S obtained from such a "positive" embedding would be useless in our context (the rank of the slack matrix $S(P)$ of polytope P equals $\dim(P) + 1$).

However, "positive" embeddings the *unique set disjointness matrix*, that is the restriction of the set disjointness matrix to pairs (A, B) such that $|A \cap B| \leqslant 1$, do not have this problem of forcing up the rank. Actually, "positive" embeddings of the unique set disjointness matrix led to the main result of Fiorini et al. [8].

Theorem 7 *Let $S \in \mathbb{R}_+^{X\times Y}$ be a matrix in which the set disjointness matrix on $[n]$ can be embedded. Consider a randomized protocol computing S in expectation. If the probability that the protocol outputs a non-zero value, given an input (x, y) with $S(x, y) > 0$, is at least $p = p(n)$, then the protocol has complexity $\Omega(np)$. In particular, by Chebyshev's inequality, the complexity is $\Omega(n(1-\sigma^2))$, where σ^2 denotes the normalized variance of the protocol.*

Proof Let c be the complexity of the protocol computing S in expectation. From this protocol, we obtain a new protocol, this time for the set disjointness problem, by mapping each input pair $(A, B) \in 2^{[n]} \times 2^{[n]}$ to the corresponding input pair $(\alpha(A), \beta(B)) \in X \times Y$ (Alice and Bob can do this independently of each other), running the original protocol $\lceil 1/p \rceil$ times, and outputting 0 if at least one of the executions led to a non-zero value or 1 otherwise.

The new protocol always outputs 1 for every disjoint pair (A, B) because of (3) (remember that our protocols have nonnegative outputs), and outputs 0 most of the times for non-disjoint pairs (A, B). More precisely, the probability of outputting 0 in case (A, B) is non-disjoint is at least $1 - (1 - p)^{\frac{1}{p}} \geqslant 1 - \mathrm{e}^{-1} > 1/2$, where e is Euler's number. The theorem follows then directly from the fact that the new protocol has complexity $O(c/p)$ and from the fact that the set disjointness problem has randomized communication complexity $\Omega(n)$. □

6.2 The perfect matching polytope

First, we construct an embedding of the set disjointness matrix in the slack matrix of the perfect matching polytope. Then, we discuss implications for extensions of the perfect matching polytope.

Lemma 4 *There exists an embedding of the set disjointness matrix on* $[n]$ *in the slack matrix of the perfect matching polytope for perfect matchings of* K^ℓ*, where* $\ell \leqslant 3n+14$.

Proof Let $k \leqslant n + 4$ denote the first multiple of 4 that is strictly greater than n, and let $\ell := 3k + 2 \leqslant 3n + 14$.

For two subsets A and B of $[n]$, we define an odd set $U := \alpha(A)$ and a perfect matching $M := \beta(B)$ as follows.

First, we add the dummy element $n + 1$ to B in case $|B|$ is odd, so that both B and $[k] - B$ contain an even number of elements. Note that this does not affect the intersection of A and B because A is contained in $[n]$. Then, we let $U := \{i : i \in A\} \cup \{i + k : i \in A\} \cup \{3k + 1\}$.

Second, we define M by adding matching edges to the partial matching $\{\{i, i+k\} : i \in [k] - B\} \cup \{\{i + k, i + 2k\} : i \in B\} \cup \{\{3k + 1, 3k + 2\}\}$ in such a way that each of the extra edges matches two consecutive unmatched vertices both in $\{i : i \in [k]\}$ or both in $\{i + 2k : i \in [k]\}$. See Fig. 3 for an example.

It can be easily verified that A and B are disjoint if and only if the slack for (U, M) is zero. Hence, the maps $\alpha : A \mapsto U$ and $\beta : B \mapsto M$ define the desired embedding of the set disjointness matrix. □

Let P denote the perfect matching polytope of K^n. Consider a size-r extension of P and a corresponding complexity-$\lceil \lg r \rceil$ protocol computing $S(P)$ in expectation (the existence of such a protocol is guaranteed by Theorems 1 and 2). Lemma 4 and Theorem 7 together imply that $r = 2^{\Omega(n(1-\sigma^2))}$, where σ^2 is the normalized variance of the protocol. For instance, deterministic protocols for computing the slack matrix of the perfect matching polytope give rise to exponential size extensions ($\sigma^2 = 0$ in this case). The same holds if σ^2 is a constant with $0 < \sigma^2 < 1$. When σ^2 is about $(n - 1)/n$ or more, the bound given by Theorem 7 becomes trivial.

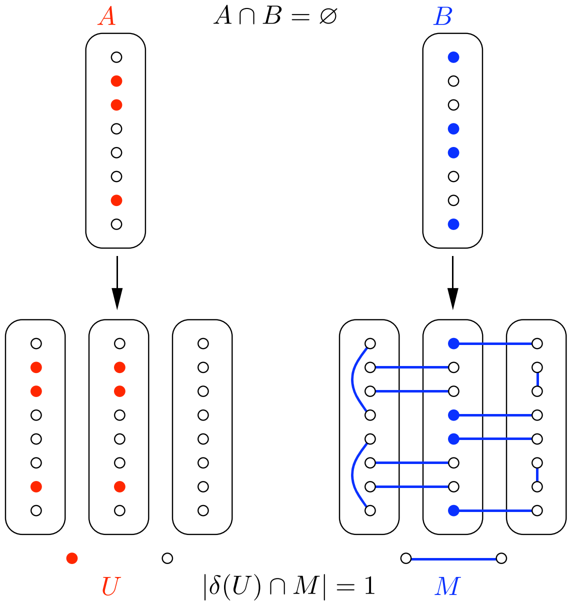


Fig. 3 Constructing an odd set and a perfect matching from a set disjointness instance

6.3 Spanning tree polytopes

We prove that similar results hold for the spanning tree polytope of K^n as well. This is surprising, because for this polytope an extension of size $O(n^3)$ exists.

Lemma 5 *There exists an embedding of the set disjointness matrix on* $[n]$ *in the slack matrix of the spanning tree polytope of* K^{2n+1}.

Proof Let $\ell := 2n+1$. Recall that the rows and columns of (the non-trivial part of) the slack matrix of the spanning tree polytope of K^ℓ respectively correspond to subsets U and spanning trees T. The entry for a pair (U, T) is zero iff the subgraph of T induced by U is connected.

Given an instance of the set disjointness problem with sets $A, B \subseteq [n]$, we define $U := \alpha(A)$ and $T := \beta(B)$ as follows. For every $i \in [n]$ add the edge $\{i, 2n+1\}$ to T. For every $i \in B$ add the edge $\{n+i, i\}$ to T and for every $i \in [n] - B$ add the edge $\{n+i, 2n+1\}$ to T. See Fig. 4 for an example.

Finally, we let $U := \{n+i : i \in A\} \cup \{2n+1\}$. As is easily seen, $T[U]$ is connected iff $A \cap B = \varnothing$. Indeed, if $i \in A \cap B$ then $n+i$ and $2n+1$ are in different connected components of $T[U]$. Moreover, if $A \cap B = \varnothing$ then $T[U]$ is a star with $2n+1$ as center. $\square$

Therefore, the "low variance forces large size" phenomenon we exhibited for the perfect matching polytope also holds for the spanning tree polytope. Incidentally, the $O(n^3)$-size extension for the spanning tree polytope of K^n can be obtained via randomized protocols, but not via deterministic ones. This is because Lemma 5 and Theorem 7 implies that any extension for the spanning tree polytope that corresponds to a deterministic protocol must have exponential size. (Notice that the value of $p = p(n)$ for the protocol given in Sect. 5.2 is roughly $1/n$.)

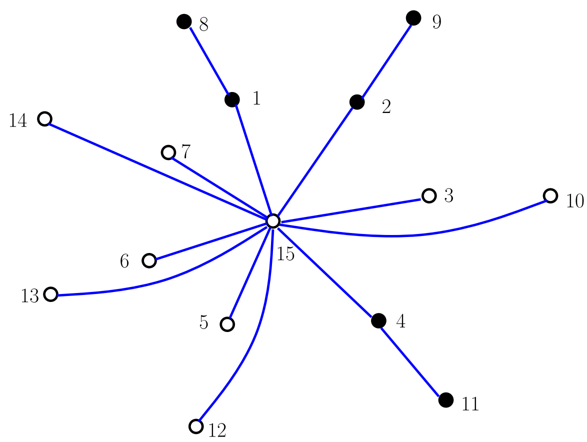


Fig. 4 The spanning tree T for $B = \{1, 2, 4\}$ and $n = 7$. *Black vertices* are those of the form i or $n + i$ where $i \in B$

7 Concluding remarks

Given a perfect matching M and an odd set U as above there is always an edge in $\delta(U) \cap M$. But it is not clear if such an edge can be found using a protocol with sublinear communication. Now we show that if such an edge can be found using few bits then the perfect matching polytope has an extension of small size. As one of the referees pointed out, this fact can be considered as folklore.

Theorem 8 *Suppose Alice is given an odd set* $U \subseteq [n]$ *and Bob is given a perfect matching* M *of* K^n. *Furthermore, suppose that Bob knows an edge* $e \in \delta(U) \cap M$. *Then, there exists a randomized protocol of complexity* $2 \lg n + O(1)$ *that computes the slack for the pair* (U, M) *in expectation.*

Proof The protocol works as follows. Bob picks an edge e' from $M \setminus \{e\}$ uniformly at random and sends it to Alice. She outputs $|M| - 1 = n/2 - 1$ if $e' \in \delta(U)$ and 0 otherwise. The expected value of the protocol is $(|M|-1)\cdot(|\delta(U)\cap M|-1)/(|M|-1) = |\delta(U) \cap M| - 1$, as required. Bob needs to send the endpoints of the edge e' to Alice and this requires $2 \lg n + O(1)$ bits. □

The theorem above implies that if an edge in $\delta(U) \cap M$ can be computed using a protocol requiring $o(n)$ bits, then there exists an extension for the perfect matching polytope of subexponential size. We leave it as an open question to settle the existence of such a protocol.

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Lexicographical polytopes

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Within a fixed integer box of $\mathbb{R}^n$, lexicographical polytopes are the convex hulls of the integer points that are lexicographically between two given integer points. We provide their descriptions by means of linear inequalities.

Throughout, ℓ, u, r, s will denote integer points satisfying $\ell \leq r \leq u$ and $\ell \leq s \leq u$, that is r and s are within $[\ell, u]$. A point $x \in \mathbb{Z}^n$ is *lexicographically smaller than* $y \in \mathbb{Z}^n$, denoted by $x \preccurlyeq y$, if $x = y$ or the first nonzero coordinate of $y - x$ is positive. We write $x \prec y$ if $x \preccurlyeq y$ and $x \neq y$. The *lexicographical polytope* $P^{r \preccurlyeq s}_{\ell,u}$ is the convex hull of the integer points within $[\ell, u]$ that are lexicographically between r and s:

$$P^{r \preccurlyeq s}_{\ell,u} = conv\{x \in \mathbb{Z}^n : \ell \leq x \leq u, r \preccurlyeq x \preccurlyeq s\}.$$

The *top-lexicographical polytope* $P^{\preccurlyeq s}_{\ell,u} = conv\{x \in \mathbb{Z}^n : \ell \leq x \leq u, x \preccurlyeq s\}$ is the special case when $r = \ell$. Similarly, the *bottom-lexicographical polytope* is $P^{r \preccurlyeq}_{\ell,u} = conv\{x \in \mathbb{Z}^n : \ell \leq x \leq u, r \preccurlyeq x\}$.

Given $a, u \in \mathbb{R}^n_+$ and $b \in \mathbb{R}_+$, the *knapsack polytope* defined by $K^{a,b}_u = conv\{x \in \mathbb{Z}^n : \mathbf{0} \leq x \leq u, ax \leq b\}$ is *superdecreasing* if:

$$\sum_{i>k} a_i u_i \leq a_k \quad \text{for } k = 1, \ldots, n. \tag{1}$$

Close relations between top-lexicographical and superdecreasing knapsack polytopes appear in the literature. For the 0/1 case, that is when $\ell = \mathbf{0}$ and $u = \mathbf{1}$, Gillmann and Kaibel [2] first noticed that top-lexicographical polytopes are special cases of superdecreasing knapsack ones, and the converse has been later established by Muldoon et al. [5]. Recently, Gupte [3] generalized the latter result by showing that all superdecreasing knapsacks are top-lexicographical polytopes.

To prove this last statement, Gupte [3] observes that a superdecreasing knapsack $K^{a,b}_u$ is the top-lexicographical polytope $P^{\preccurlyeq s}_{\mathbf{0},u}$, where s the lexicographically greatest integer point of $K^{a,b}_u$. The non trivial inclusion actually holds because every integer point x of $P^{\preccurlyeq s}_{\mathbf{0},u}$ satisfies $ax \leq as$. Indeed, by definition, if $x \prec s$, there exists $k \in \{1, \ldots, n\}$ such that $x_k + 1 \leq s_k$ and $x_i = s_i$

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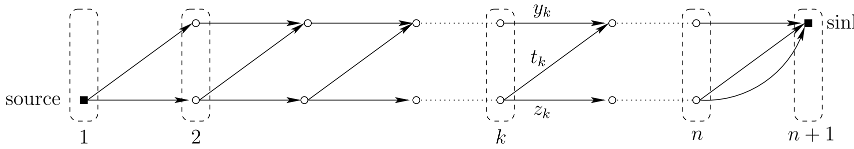

Fig. 1. Path representation of the points of $X^{\preccurlyeq s}_{\ell,u}$.

for $i < k$. Hence, we have $b - ax \geq as - ax \geq \sum_{i>k} a_i(s_i - x_i) + a_k \geq \sum_{i>k} a_i(s_i - x_i + u_i) \geq 0$, because of (1), $s_i \geq 0$ and $u_i \geq x_i$.

It turns out that top-lexicographical polytopes are superdecreasing knapsack polytopes. Indeed, let $P^{\preccurlyeq s}_{\ell,u}$ be a top-lexicographical polytope for some s within $[\ell, u]$. Possibly after translating, we may assume $\ell = \mathbf{0}$. Define a by $a_k = \sum_{i>k} a_i u_i + 1$, for $k = 1, \ldots, n$, and let $b = as$. Since the associated knapsack polytope $K^{a,b}_u$ is superdecreasing, if $x \preccurlyeq s$ then $ax \leq as = b$, for all x within $[\mathbf{0}, u]$. Moreover, the converse holds because, inequalities (1) being all strict, $s \prec x$ implies $b = as < ax$. Therefore, $P^{\preccurlyeq s}_{\mathbf{0},u} = K^{a,b}_u$. These observations are summarized in the following.

Observation 1. *Superdecreasing knapsacks are top-lexicographical polytopes, and conversely (up to translations).*

Motivated by a wide range of applications, such as knapsack cryptosystems [6] or binary expansion of bounded integer variables (*e.g.*, [8, p. 477]), several papers are devoted to the polyhedral description of these families of polytopes. For the 0/1 case, the description appeared in [4] from the knapsack point of view. It was later rediscovered from the lexicographical point of view in [2,5]. Moreover, Muldoon et al. [5] and Angulo et al. [1] independently showed that intersecting a 0/1 top-with a 0/1 bottom-lexicographical polytope yields the description of the corresponding lexicographical polytope. Recently, these results were generalized for the bounded case by Gupte [3].

In this paper, we provide the description of the lexicographical polytopes using extended formulations. Our approach provides alternative proofs of the aforementioned results of Gupte [3].

The outline of the paper is as follows. In Section 1, we provide a flow based extended formulation of the convex hull of the componentwise maximal points of a top-lexicographical polytope. Projecting this formulation is surprisingly straightforward, and thus we get the description in the original space. In Section 2, using the fact that a top-lexicographical polytope is, up to translation, the submissive of the above convex hull, we derive the description of top-lexicographical polytopes. We then show that a lexicographical polytope is the intersection of its top- and bottom-lexicographical polytopes.

1. Convex hull of componentwise maximal points

From now on, $X^{\preccurlyeq s}_{\ell,u}$ will denote the set of the points $p^i = (s_1, \ldots, s_{i-1}, s_i - 1, u_{i+1}, \ldots, u_n)$, for $i = 1, \ldots, n+1$ such that $s_i > \ell_i$, where $p^{n+1} = s$ by definition. Note that $X^{\preccurlyeq s}_{\ell,u}$ consists of the componentwise maximal integer points of $P^{\preccurlyeq s}_{\ell,u}$, to which we added, for later convenience, the point $p^n = (s_1, \ldots, s_{n-1}, s_n - 1)$ if $s_n > \ell_n$.

1.1. A flow model for $X^{\preccurlyeq s}_{\ell,u}$

We first model the points of $X^{\preccurlyeq s}_{\ell,u}$ as paths from 1 to $n+1$ in the digraph given in Fig. 1.

Our digraph is composed of $n+1$ layers, each containing two nodes except the first and the last ones. There are three arcs connecting the layer k to the layer $k+1$, an upper arc y_k, a diagonal arc t_k and a lower arc z_k. The only exception concerns the first level, which does not have the upper arc.

The arcs connecting two successive layers correspond to a coordinate of $x \in X^{\preccurlyeq s}_{\ell,u}$. More precisely, given a directed path P from 1 to $n+1$, we define the point x by setting, for $k = 1, \ldots, n$,

$$x_k = \begin{cases} u_k & \text{if } y_k \in P, \\ s_k - 1 & \text{if } t_k \in P, \\ s_k & \text{if } z_k \in P. \end{cases}$$

As shown in Observation 2, the set of (x, y, z, t) satisfying the following set of inequalities is an extended formulation of $conv(X^{\preccurlyeq s}_{\ell,u})$:

$$x_i = u_i y_i + (s_i - 1)t_i + s_i z_i \qquad \text{for } i = 1, \ldots, n, \tag{2}$$
$$y_1 = 0 \tag{3}$$
$$y_i = y_{i-1} + t_{i-1} \qquad \text{for } i = 2, \ldots, n, \tag{4}$$
$$z_i = z_{i+1} + t_{i+1} \qquad \text{for } i = 1, \ldots, n-1, \tag{5}$$
$$t_i = 0 \qquad \text{whenever } s_i = \ell_i, \tag{6}$$
$$y_n + t_n + z_n = 1 \tag{7}$$
$$y_i, t_i, z_i \geq 0 \qquad \text{for } i = 1, \ldots, n. \tag{8}$$

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Observation 2. *$conv(X_{\ell,u}^{\preccurlyeq s}) = proj_x\{(x, y, z, t)$ satisfying (2)–(8)$\}$.*

Proof. First, note that there is a one-to-one correspondence between the points of $X_{\ell,u}^{\preccurlyeq s}$ and the paths from layer 1 to layer $n + 1$ of the digraph. This implies that $X_{\ell,u}^{\preccurlyeq s}$ is the projection onto the x variables of the integer points of $Q = \{(x, y, z, t)$ satisfying (2)–(8)$\}$. The digraph being acyclic, the set of (y, z, t) satisfying (3)–(8) is a path polytope and thus is an integral polytope [7, Theorem 13.10]. The integrality of u and s implies that Q is integer, hence so is its projection onto the x variables, which concludes the proof. □

1.2. Description of $conv(X_{\ell,u}^{\preccurlyeq s})$

In the following result, we use Observation 2 to provide a linear description of $conv(X_{\ell,u}^{\preccurlyeq s})$.

Lemma 3. *$conv(X_{\ell,u}^{\preccurlyeq s})$ is described by the inequalities:*

$$\sum_{i=1, s_i>\ell_i}^{n} A_i(x) \geq -1 \tag{9}$$

$$A_k(x) \leq 0 \qquad \text{for } k = 1, \ldots, n, \tag{10}$$

$$A_k(x) \geq 0 \qquad \text{when } s_k = \ell_k, \tag{11}$$

where, for $k = 1, \ldots, n$,

$$A_k(x) := (x_k - s_k) + (u_k - s_k) \sum_{i=1, s_i>\ell_i}^{k-1} \left(\prod_{j=i+1, s_j>\ell_j}^{k-1} (u_j - s_j + 1) \right) (x_i - s_i).$$

Proof. By Observation 2, it suffices to project onto the x variables of the set of x, y, t, z satisfying (2)–(8).

For $k = 1, \ldots, n$, we get $y_k = \sum_{i=1}^{k-1} t_i$ by (3) and (4). This, combined with (5) and (7), yields $z_k = 1 - \sum_{i=1}^{k} t_i$. Using those two equations in (2), and $t_k = 0$ whenever $s_k = \ell_k$, we obtain

$$t_k = s_k - x_k + (u_k - s_k) \sum_{i=1, s_i>\ell_i}^{k-1} t_i, \quad \text{for } k = 1, \ldots, n. \tag{12}$$

We now show by induction on k that, for all $k = 1, \ldots, n$,

$$\sum_{i=1, s_i>\ell_i}^{k} t_i = \sum_{i=1, s_i>\ell_i}^{k} (s_i - x_i) \prod_{j=i+1, s_j>\ell_j}^{k} (u_j - s_j + 1). \tag{13}$$

By definition of t_k, (13) holds for $k = 1$. Let us suppose that (13) holds for $k < n$ and show that it holds for $k + 1$. The result is immediate if $s_{k+1} = \ell_{k+1}$, hence assume that $s_{k+1} > \ell_{k+1}$. We have

$$\sum_{i=1, s_i>\ell_i}^{k+1} t_i = (s_{k+1} - x_{k+1}) + (u_{k+1} - s_{k+1}) \sum_{i=1, s_i>\ell_i}^{k} t_i + \sum_{i=1, s_i>\ell_i}^{k} t_i \tag{14}$$

$$= (s_{k+1} - x_{k+1}) + (u_{k+1} - s_{k+1} + 1) \sum_{i=1, s_i>\ell_i}^{k} (s_i - x_i) \prod_{j=i+1, s_j>\ell_j}^{k} (u_j - s_j + 1) \tag{15}$$

$$= \sum_{i=1, s_i>\ell_i}^{k+1} (s_i - x_i) \prod_{j=i+1, s_j>\ell_j}^{k+1} (u_j - s_j + 1).$$

Above, equality (14) follows from (12) applied to t_{k+1} and equality (15) follows using (13).

Injecting (13) in (12) yields

$$t_k = s_k - x_k + (u_k - s_k) \sum_{i=1, s_i>\ell_i}^{k-1} (s_i - x_i) \prod_{j=i+1, s_j>\ell_j}^{k-1} (u_j - s_j + 1) \quad \text{for } k = 1, \ldots, n. \tag{16}$$

Up to now, we only used linear transformations, thus projecting out the variables y, z gives us (16), $\sum_{i=1, s_i>\ell_i}^{n} t_i \leq 1$, $t_k = 0$ whenever $s_k = \ell_k$ and $t_k \geq 0$ otherwise. Then, projecting onto the x variable gives the desired result. □

Note that the following derives from the above proof by combining (12) and the fact that, by (16), we have $t_k = -A_k$:

$$A_k(x) = (x_k - s_k) + (u_k - s_k) \sum_{i=1, s_i > \ell_i}^{k-1} A_i(x), \quad \text{for } k = 1, \ldots, n. \tag{17}$$

2. Lexicographical polytopes

In this section, we first provide the description of top-lexicographical polytopes. We then show that a lexicographical polytope is the intersection of its top- and bottom-lexicographical polytopes.

2.1. Description of top-lexicographical polytopes

The following observation unveils the polyhedral relation between a top-lexicographical polytope and the convex hull of its componentwise maximal points.

Observation 4. $P_{\ell,u}^{\preccurlyeq s} = (conv(X_{\ell,u}^{\preccurlyeq s}) + \mathbb{R}_-^n) \cap \{x \geq \ell\}$.

Proof. Since $conv(X_{\ell,u}^{\preccurlyeq s})$ is integer and contained in $\{x \geq \ell\}$, the polyhedron on the right is integer. Seen the definitions, the observation follows. □

Remark that, when $\ell = \mathbf{0}$, $P_{\ell,u}^{\preccurlyeq s}$ is precisely the submissive of $conv(X_{\ell,u}^{\preccurlyeq s})$. Now, we derive from Lemma 3 and Observation 4 the linear description of top-lexicographical polytopes.

Theorem 5. $P_{\ell,u}^{\preccurlyeq s} = \{x \in \mathbb{R}^n : \ell \leq x \leq u, A_k(x) \leq 0,$ *for* $k = 1, \ldots, n\}$.

Proof. Theorem 5 immediately follows from Observation 4 and the following description of $conv(X_{\ell,u}^{\preccurlyeq s}) + \mathbb{R}_-^n$,

$$conv(X_{\ell,u}^{\preccurlyeq s}) + \mathbb{R}_-^n = \{x \in \mathbb{R}^n : x \leq u \text{ and } A_k(x) \leq 0, \text{ for } k = 1, \ldots, n\}. \tag{18}$$

To prove (18), denote by Q its right hand side. By Lemma 3, the above inequalities are valid for $conv(X_{\ell,u}^{\preccurlyeq s})$. Since their coefficients for x are nonnegative, they also hold for $conv(X_{\ell,u}^{\preccurlyeq s}) + \mathbb{R}_-^n$. Note that the latter and Q have the same recession cone, thus it remains to show that the vertices of Q are vertices of $conv(X_{\ell,u}^{\preccurlyeq s})$. Let us prove it by induction on the dimension, the base case being immediate. We may assume that $u_n > s_n$, as otherwise $A_n(x) = x_n - s_n$ and the induction concludes. Let $\bar{x}$ be a vertex of Q.

Claim 6. $\sum_{i=1, s_i > \ell_i}^{n} A_i(\bar{x}) \geq -1$.

Proof. The indices i of $A_i(x)$ involved in sums throughout this proof satisfy $s_i > \ell_i$, yet to ease the reading, we will omit the subscripts "$s_i > \ell_i$". By contradiction, assume that $\sum_{i=1}^{n} A_i(\bar{x}) < -1$. Since $\bar{x}$ is a vertex, and x_n appears only in $x_n \leq u_n$ and $A_n(x) \leq 0$, at least one of them holds with equality. If the latter does, then by (17) and $u_n > s_n$, we get the contradiction $0 = A_n(\bar{x}) \leq (u_n - s_n)(1 + A_1(\bar{x}) + \cdots A_{n-1}(\bar{x})) < (u_n - s_n)(1 - 1) = 0$. Therefore $A_n(\bar{x}) < 0$ and $\bar{x}_n = u_n$. For $x \in \mathbb{R}^n$, we denote $x' := (x_1, \ldots, x_{n-1})$. Necessarily, $\bar{x}'$ satisfies to equality $n-1$ linearly independent of the remaining inequalities, and hence $\bar{x}'$ is a vertex of $\{x \in \mathbb{R}^{n-1} : x_k \leq u_k, A_k(x) \leq 0, \text{ for } k = 1, \ldots, n-1\}$. By the induction hypothesis, $\bar{x}'$ is a vertex of $conv(X_{\ell',u'}^{\preccurlyeq s'}) + \mathbb{R}_-^{n-1}$, hence $\sum_{i=1}^{n-1} A_i(\bar{x}') \geq -1$. But now $A_n(\bar{x}) < 0$, $\bar{x}_n = u_n$ and (17) imply $A_1(\bar{x}') + \cdots + A_{n-1}(\bar{x}') < -1$, a contradiction. ■

Let us show that $A_k(\bar{x}) = 0$ whenever $s_k = \ell_k$. Indeed, in this case, $\bar{x}_k$ only appears in $A_k(\bar{x}) \leq 0$ and $\bar{x}_k \leq u_k$, and one is satisfied with equality since $\bar{x}$ is a vertex. If $\bar{x}_k = u_k$, then by (17), Claim 6 and $A_i(\bar{x}) \leq 0$, for $i = 1 \ldots, n$, we get $0 \geq A_k(\bar{x}) = (u_k - s_k)(1 + \sum_{i=1, s_i > \ell_i}^{k-1} A_i(\bar{x})) \geq 0$. Consequently, $\bar{x}$ belongs to $conv(X_{\ell,u}^{\preccurlyeq s})$ and this proves (18). □

Symmetrically, bottom-lexicographical polytopes are described as follows.

Corollary 7. $P_{\ell,u}^{r \preccurlyeq} = \{x \in \mathbb{R}^n : \ell \leq x \leq u, B_k(x) \leq 0,$ *for* $k = 1, \ldots, n\}$, *where, for* $k = 1, \ldots, n$,

$$B_k(x) = (r_k - x_k) + (r_k - \ell_k) \sum_{i=1, r_i < u_i}^{k-1} \left(\prod_{j=i+1, r_j < u_j}^{k-1} (r_j - \ell_j + 1) \right) (r_i - x_i).$$

2.2. Lexicographical polytopes

By definition, we have $P^{r\preccurlyeq s}_{\ell,u} \subseteq P^{r\preccurlyeq}_{\ell,u} \cap P^{\preccurlyeq s}_{\ell,u}$. It turns out that the converse holds, see Theorem 8. In particular, $P^{r\preccurlyeq}_{\ell,u} \cap P^{\preccurlyeq s}_{\ell,u}$ is an integer polytope.

Theorem 8. *A lexicographical polytope is the intersection of its top- and bottom-lexicographical polytopes.*

Proof. It remains to prove that $P^{r\preccurlyeq s}_{\ell,u} \supseteq Q$, where $Q = P^{r\preccurlyeq}_{\ell,u} \cap P^{\preccurlyeq s}_{\ell,u}$. Let us prove it by induction on the dimension, the one-dimensional case being immediate.

If $r_1 = s_1$, then the problem reduces to the $(n-1)$-dimensional case, and using induction concludes.

If $r_1 + 1 \leq \pi \leq s_1 - 1$ for some integer π, then let ℓ' be obtained from ℓ by replacing ℓ_1 by π. By $s_1 > \ell'_1$ and the definition of $A_k(x)$, applying Theorem 5 gives $P^{\preccurlyeq s}_{\ell,u} \cap \{x_1 \geq \pi\} = P^{\preccurlyeq s}_{\ell',u}$. Moreover, since $\pi > r_1$, the latter is contained in $P^{r\preccurlyeq}_{\ell,u}$. Therefore $Q \cap \{x_1 \geq \pi\} = P^{\preccurlyeq s}_{\ell',u}$ is integer. Similarly, $Q \cap \{x_1 \leq \pi\}$ is integer, hence so is Q, and we are done.

The remaining case is when $r_1 = s_1 - 1$. Let $\bar{x} \in P^{r\preccurlyeq}_{\ell,u} \cap P^{\preccurlyeq s}_{\ell,u}$. If $\bar{x}_1 = s_1$, when $\bar{x}$ is written as a convex combination of integer points of $P^{\preccurlyeq s}_{\ell,u}$, all of them have their first coordinate equal to s_1, and hence belong to $P^{r\preccurlyeq s}_{\ell,u}$. By convexity, so does $\bar{x}$ and we are done. A similar argument may be applied if $\bar{x}_1 = r_1$. Therefore, we may assume that $r_1 < \bar{x}_1 < s_1$.

Let $\lambda = \bar{x}_1 - r_1$, and define y by $y_1 = s_1$ and $y_k = u_k + \frac{\bar{x}_k - u_k}{\lambda}$ for $k = 2, \dots, n$. Similarly, define z by $z_1 = r_1$ and $z_i = \ell_i + \frac{\bar{x}_i - \ell_i}{1-\lambda}$, for $i = 2, \dots, n$. The following claim finishes the proof, where, given two points v and w of $\mathbb{R}^n$, $\max(v, w)$ (resp. $\min(v, w)$) will denote the point of $\mathbb{R}^n$ whose ith coordinate is $\max\{v_i, w_i\}$ (resp. $\min\{v_i, w_i\}$) for $i = 1, \dots, n$.

Claim 9. *$\bar{x}$ is a convex combination of $\bar{y} = \max(y, \ell)$ and $\bar{z} = \min(z, u)$ which both belong to $P^{r\preccurlyeq s}_{\ell,u}$.*

Proof. First, let us show that $y \in conv(X^{\preccurlyeq s}_{\ell,u}) + \mathbb{R}^n_-$. As $\bar{x} \leq u$, we have $y \leq u$. Moreover, $A_1(y) = y_1 - s_1 = 0$. Now, we prove by induction that $A_k(y) = \frac{1}{\lambda} A_k(\bar{x})$ for $k = 2, \dots, n$. Using (17), $A_1(y) = 0$, the definition of y_k, and the induction hypothesis, we have $A_k(y) = \frac{1}{\lambda}[\bar{x}_k - s_k + (\lambda - 1)(u_k - s_k) + (u_k - s_k) \sum_{i=2, s_i > \ell_i}^{k-1} A_i(\bar{x})]$. Since $\lambda - 1 = \bar{x}_1 - s_1 = A_1(\bar{x})$ and $s_1 = r_1 + 1 > \ell_1$, we get by (17) that $A_k(y) = \frac{1}{\lambda} A_k(\bar{x})$, for $k = 2, \dots, n$. Since $A_k(\bar{x}) \leq 0$, we have $A_k(y) \leq 0$. Hence, $y \in conv(X^{\preccurlyeq s}_{\ell,u}) + \mathbb{R}^n_-$. Therefore, there exists y^+ of $conv(X^{\preccurlyeq s}_{\ell,u})$ with $y^+ \geq y$. Clearly, $y^+ \geq \ell$ hence $y^+ \geq \max(y, \ell)$. Thus, $\max(y, \ell)$ belongs to $conv(X^{\preccurlyeq s}_{\ell,u}) + \mathbb{R}^n_-$ and, by Observation 4, to $P^{\preccurlyeq s}_{\ell,u}$. Moreover, as its first coordinate equals s_1, $\max(y, \ell)$ belongs to $P^{r\preccurlyeq s}_{\ell,u}$. Similarly, $\min(z, u)$ also belongs to $P^{r\preccurlyeq s}_{\ell,u}$.

Finally, we have $(1-\lambda)\bar{z}_1 + \lambda \bar{y}_1 = (1-\lambda)(s_1 - 1) + \lambda s_1 = s_1 - 1 + \lambda = \bar{x}_1$. For $i \in \{2, \dots, n\}$, we have $(1-\lambda)\bar{z}_i + \lambda \bar{y}_i = \min(\bar{x}_i - \lambda \ell_i, (1-\lambda)u_i) + \max((\lambda-1)u_i + \bar{x}_i, \lambda \ell_i) = \bar{x}_i - \max(\lambda \ell_i, (\lambda-1)u_i + \bar{x}_i) + \max((\lambda-1)u_i + \bar{x}_i, \lambda \ell_i) = \bar{x}_i$. Therefore, $\bar{x} = (1-\lambda)\bar{z} + \lambda \bar{y}$ and we are done. ■ □

Note that the above result implies that the family of lexicographical polytopes defined on a fixed box $[\ell, u]$ is closed by intersection. Beside, combined with Theorem 5 and Corollary 7, it provides the description of lexicographical polytopes.

Corollary 10. *The lexicographical polytope $P^{r\preccurlyeq s}_{\ell,u}$ is described as follows:*

$$P^{r\preccurlyeq s}_{\ell,u} = \left\{ \begin{array}{lll} x \in \mathbb{R}^n : & A_k(x) \leq 0 & \text{for } k = 1, \dots, n \\ & B_k(x) \leq 0 & \text{for } k = 1, \dots, n \\ & \ell \leq x \leq u & \end{array} \right\}.$$

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Circuit and bond polytopes on series–parallel graphs[☆]

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ABSTRACT

In this paper, we describe the circuit polytope on series–parallel graphs. We first show the existence of a compact extended formulation. Though not being explicit, its construction process helps us to inductively provide the description in the original space. As a consequence, using the link between bonds and circuits in planar graphs, we also describe the bond polytope on series–parallel graphs.

In an undirected graph, a *circuit* is a subset of edges inducing a connected subgraph in which every vertex has degree two. In the literature, a circuit is sometimes called *simple cycle*. Given a graph and costs on its edges, the *circuit problem* consists in finding a circuit of maximum cost. This problem is already NP-hard in planar graphs [1], yet some polynomial cases are known, for instance when the costs are non-positive.

Although characterizing a polytope corresponding to an NP-hard problem is unlikely, a partial description may be sufficient to develop an efficient polyhedral approach. Concerning the *circuit polytope*, which is the convex hull of the (edge-)incidence vectors of the circuits of the graph, facets have been exhibited by Bauer [2] and Coullard and Pulleyblank [3], and the cone has been characterized by Seymour [4]. Several variants of cardinality constrained versions have been studied, such as [5–8].

For a better understanding of the circuit polytope on planar graphs, a natural first step is to study it in smaller classes of graphs. For instance, in [3], the authors provide a complete description in Halin graphs.

Another interesting subclass of planar graphs are the series–parallel graphs. Due to their nice decomposition properties, many problems NP-hard in general are polynomial for these graphs, in which case it is quite

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standard to (try to) characterize the corresponding polytopes. Results of this flavor were obtained for various combinatorial optimization problems, such as the stable set problem [9], graph partitioning problem [10], 2-connected and 2-edge-connected subgraph problems [11,12], k-edge-connected problems [13], Steiner-TSP problem [14].

Since a linear time combinatorial algorithm solves the circuit problem in series–parallel graphs, an obvious question arising is the description of the corresponding polytope. Surprisingly, it does not appear in the literature, and we fill in this gap with Theorem 11.

The main ingredient for the proof of our main theorem is the existence of a compact extended formulation for the circuit polytope on series–parallel graphs. An *extended formulation* of a given polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ is a polyhedron $Q = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m : Bx + Cy \leq d\}$ whose projection onto the x variables $proj_x(Q) = \{x \in \mathbb{R}^n : \text{ there exists } y \in \mathbb{R}^m \text{ such that } (x, y) \in Q\}$ is P. The *size* of a polyhedron is the number of inequalities needed to describe it. An extended formulation is called *compact* when its size is polynomial. We refer to [15] for further insights on this topic.

The past few years, extended formulations proved to be a powerful tool for polyhedral optimization, and thus received a growing interest in the community. Indeed, describing a polytope directly in its original space is often pretty challenging, and by looking for an extended formulation one has more tools at disposal. As an example, for most combinatorial optimization polytopes in series–parallel graphs, Martin et al. [16] proposed a general technique to derive extended formulations from dynamic programming algorithms, but the corresponding descriptions in the original space remain unknown.

Recently, it has been shown that the perfect matching polytope admits no compact extended formulation [17]. It means, even if an optimization problem is polynomial, there may not exist such a formulation. Here, though we are not able to explicitly construct a compact extended formulation for the circuit polytope on series–parallel graphs, we show that there exists one, see Section 2.1.1. The construction process of this extended formulation relies on a straightforward inductive description of the circuits of series–parallel graphs, combined with a theorem of Balas [18,19]. It allows us to prove by induction that the circuit polytope on series–parallel graphs is completely described by three families of inequalities. We provide examples where exponentially many of these inequalities define facets, see Corollary 19. Thus, the circuit polytope on series–parallel graphs is another example of polytope having exponentially many facet-defining inequalities that admits a compact extended formulation.

A graph is series–parallel if and only if, given any planar drawing of the graph, its dual is series–parallel. The dual of a circuit is a *bond*, that is a cut containing no other nonempty cut. These bonds play an important role e.g. in multiflow problems [20]. By planar duality and the description of the circuit polytope on series–parallel graphs, we get the description of the bond polytope on series–parallel graphs, see Theorem 13.

The paper is organized as follows. In Section 1, we fix graph related notation and definitions, and review some known and new auxiliary results about circuits in series–parallel graphs. Section 2 deals with the circuit polytope on series–parallel graphs. First, we get a polyhedral description of the latter for non trivial 2-connected series–parallel graphs, by providing the existence of a compact extended formulation, and then inductively projecting it. By applying standard techniques, the polyhedral description for general series–parallel graphs follows, which has exponential size in general. In Section 3, using the planar duality, we describe the bond polytope on series–parallel graphs, and then we study facet-defining inequalities, which have counterparts for the circuit polytope as well.

1. Circuits in series–parallel graphs

Throughout, $G = (V, E)$ will denote a connected undirected graph with $n = |V|$ vertices and $m = |E|$ edges. The graph *induced* by a subset W of V is the graph $G[W]$ obtained by removing the vertices of $V \setminus W$, and $\delta_G(W)$ is the set of edges having exactly one extremity in W. Given disjoint $U, W \subset V$, $\delta_G(U, W)$ is the set of edges having one extremity in each of U and W. When it is clear from the context, we will omit the subscript G. Given a set of edges $F \subseteq E$, $V(F)$ denotes the set of vertices incident to any edge of F. We denote by $A\Delta B = (A \cup B) \setminus (A \cap B)$ the symmetric difference of A and B.

A subset F of E is called a *cut* if $F = \delta_G(W)$ for some $W \subseteq V$. If $u \in W$ and $v \in V \setminus W$, the cut *separates* u and v. A cut defined by a singleton is a *star*. A *bond* is a cut containing no other nonempty cut. One can check that a nonempty cut $\delta_G(W)$ is a bond if and only if both $G[W]$ and $G[V \setminus W]$ are connected. In the literature, a bond is sometimes called a *central cut*. A *bridge* is an edge whose removal disconnects the graph, that is a bond of size one. Note that the symmetric difference of bonds is a cut.

A subset of edges is called a *cycle* if it induces a subgraph where every vertex has even degree. A connected cycle with every vertex of degree two is a *circuit*. If e is a circuit, it is called a *loop*. Let $\mathcal{C}(G)$ denote the set of circuits of G. Note that the symmetric difference of circuits is a cycle.

By definition, the emptyset is both a bond and a circuit.

When no removal of a single vertex disconnects a graph, the latter is said *2-connected*. Loops and bridges are called *trivial 2-connected* graphs. The *non trivial 2-connected components* of a graph are the maximal 2-connected subgraphs of the graph, *i.e.*, the components obtained after removing the loops and bridges.

A graph is *series–parallel* if all its non trivial 2-connected components can be built, starting from the circuit of length two C_2, by repeatedly applying the following operations: add a parallel edge to an existing edge; or subdivide an existing edge, that is replace the edge by a path of length two. This construction gives an inductive description of the circuits of such graphs.

Observation 1. *Let $G = (V, E)$ be a non trivial 2-connected series–parallel graph.*

(i) *If G is obtained from a graph H by subdividing an edge $e \in E(H)$ into e, f, then the circuits of G are obtained from those of H as follows:*
- *C, for $C \in \mathcal{C}(H)$ not containing e,*
- *$C \cup f$, for $C \in \mathcal{C}(H)$ containing e.*

(ii) *If G is obtained from a graph H by adding a parallel edge f to an edge $e \in E(H)$, then the circuits of G are obtained from those of H as follows:*
- *C, for $C \in \mathcal{C}(H)$ not containing e,*
- *C and $C \setminus e \cup f$, for $C \in \mathcal{C}(H)$ containing e,*
- *$\{e, f\}$.*

A well-known characterization of cuts is that they are the sets of edges intersecting every circuit an even number of times. In series–parallel graphs, we have the following property [20].

Observation 2 ([20]). *In a series–parallel graph, a bond and a circuit intersect in zero or two edges.*

If the graph is also 2-connected, then this property becomes a characterization of circuits, see below. Note that the following does not hold if the series–parallel graph is not 2-connected.

Lemma 3. *In a non trivial 2-connected series–parallel graph, a set of edges is a circuit if and only if it intersects every bond in zero or two edges.*

Proof. We prove the non trivial direction. By contradiction, let G be a minimal counter-example, and let F be a set of edges intersecting every bond in zero or two edges that is not a circuit. First, suppose that G is build from H by adding a parallel edge f to an edge $e \in E(H)$. Necessarily, we have $f \in F$ as otherwise H would be a smaller counter-example. Similarly, $e \in F$. Suppose there exists $g \in F \setminus \{e, f\}$. Since G is planar and 2-connected, so is its dual. Any pair of edges in a 2-connected graph being contained in a circuit, the planar duality between circuits and bonds implies that there exists a bond B of G containing both g and e. Hence, B also contains f, which provides the contradiction $|F \cap B| \geq 3$. Now, assume that G is build from H by subdividing $e \in E(H)$ into $\{f, g\}$. Since G is 2-connected, $\{f, g\}$ is a bond, hence F contains either both f and g or none of them. In both cases, H is clearly a smaller counter-example, a contradiction. □

For an ordering $v_1, \ldots, v_n$ of V such that $\delta(\{v_1, \ldots, v_i\})$ is a bond for all $i = 1, \ldots, n-1$, the partition $\mathcal{S} = \{S_1, \ldots, S_{n-1}\}$ of E defined by $S_\ell = \delta(v_\ell, \{v_{\ell+1}, \ldots, v_n\})$, for $\ell = 1, \ldots, n-1$, is a *star decomposition*. We will denote the initial star $\delta(v_1, \{v_2, \ldots, v_n\})$ by $I_{\mathcal{S}}$. Equivalently, a star decomposition is obtained by partitioning the edgeset by iteratively removing stars of the graph such that, at each step, the vertex to be removed is adjacent to some removed vertex, and the set of remaining vertices induces a connected graph. $I_{\mathcal{S}}$ is the unique element of the star decomposition which is a star of the original graph.

Using induction and the construction of non trivial 2-connected series–parallel graphs, one can see that in these graphs any vertex is the initial vertex of some star decomposition. In particular, such decompositions exist.

Lemma 4. *Given a star decomposition $\mathcal{S}$ of a series–parallel graph G, the following holds:*

(a) *a circuit intersects each member of $\mathcal{S}$ at most twice,*
(b) *a circuit does not intersect two members of $\mathcal{S}$ twice.*

Proof. Let C be a circuit of G and $v_1, \ldots, v_n$ an ordering of V such that $\mathcal{S} = \{S_1, \ldots, S_{n-1}\}$ with $S_\ell = \delta(v_\ell, \{v_{\ell+1}, \ldots, v_n\})$, for $\ell = 1, \ldots, n-1$.

Since every member of $\mathcal{S}$ is contained in a star of G and a circuit goes through each vertex at most once, Lemma 4(a) holds. Let us show Lemma 4(b) by contradiction, and let $i < j$ be such that $|S_i \cap C| = |S_j \cap C| = 2$ and $|S_k \cap C| \leq 1$ for all $k < j$, $k \neq i$. By construction of star decompositions, we have $C \cap S_\ell = \emptyset$, for all $\ell < i$, and $C \setminus (\bigcup_{\ell=1}^{j-1} S_\ell)$ is a path of which S_j contains two edges, hence $|\delta(\{v_1, \ldots, v_j\}) \cap C| = 4$. Since $\delta(\{v_1, \ldots, v_j\})$ is a bond, this contradicts Observation 2. □

Two sequences of edge subsets $\mathcal{M} = (M_0, \ldots, M_k)$ and $\mathcal{N} = (N_1, \ldots, N_k)$ form a *star-cut collection* if $\{M_0, \ldots, M_k\} \subseteq \mathcal{S}$ and $M_0 = I_{\mathcal{S}}$, for some star decomposition $\mathcal{S}$ of G, and $M_i \Delta N_i$ is a cut of G, for $i = 1, \ldots, k$. Note that the elements of $\mathcal{N}$ are not required to be disjoint.

2. Circuit polytope on series–parallel graphs

Given a graph $G = (V, E)$ and $F \subseteq E$, $\chi^F \in \mathbb{R}^E$ denotes the incidence vector of F, that is χ^F_e equals 1 if $e \in F$ and 0 otherwise. Since there is a bijection between edge sets and their incidence vectors, we will often use the same terminology for both. Let $C(G)$ be the convex hull of the incidence vectors of the circuits of G, that is $C(G) = \text{conv}\{\chi^C : C \in \mathcal{C}(G)\}$. In this section, we give an external description of the circuit polytope on series–parallel graphs.

Note that the circuit polytope of the graph is the union of the circuit polytopes of its loops, bridges, and non trivial 2-connected components. Therefore, we start by studying the circuit polytope for this latter case, and then derive the description for general series–parallel graphs.

Throughout, we will use the following theorem of Balas [18,19]. His result holds for any finite union of polyhedra, yet we only state what we need in this paper, the union of two polytopes.

Theorem 5 (*Balas [18,19]*). *Given two polytopes $P_1 = \{x \in \mathbb{R}^n : A^1 x \leq b^1\}$ and $P_2 = \{x \in \mathbb{R}^n : A^2 x \leq b^2\}$, we have $conv\{P_1 \cup P_2\} = proj_x(Q)$, where $Q = \{x = x^1 + x^2, A^1 x^1 \leq (1-\lambda) b^1, A^2 x^2 \leq \lambda b^2, 0 \leq \lambda \leq 1\}$.*

Note that Theorem 5 applied to integral polytopes yields an extended formulation which is also integral. Furthermore, it also implies the following.

Corollary 6. *Given two polytopes P_1 and P_2, there exists an extended formulation of conv$\{P_1 \cup P_2\}$ whose size is two plus the sizes of P_1 and P_2.*

Later on, we shall use this corollary when P_2 is a vertex, in which case we get an extended formulation of conv$\{P_1 \cup P_2\}$ with two more inequalities than the one of P_1.

2.1. 2-connected series–parallel graphs

In this section, we describe the circuit polytope for non trivial 2-connected series–parallel graphs. The main ingredient of our proof is the existence of a compact extended formulation for this polytope, based on Observation 1. Though this extended formulation is not explicit, we use its construction process to prove inductively that the circuit polytope is described by the inequalities given in Theorem 10. Let us mention that there are examples where exponentially many of these inequalities are facet-defining, see Corollary 19.

In this section, $G = (V, E)$ is a non trivial 2-connected series–parallel graph.

2.1.1. Existence of a compact extended formulation

We show the existence of a compact extended formulation by induction on the construction of G. First, note that $C(C_2) = \text{conv}\{(0,0),(1,1)\} = \{x \in \mathbb{R}^2_+ : x_e = x_f, x_e + x_f \le 2\}$, where e and f denote the edges of C_2. Next, let us describe how to get an extended formulation for $C(G)$ when G is obtained from a 2-connected series–parallel graph H by either subdividing an edge or adding a parallel edge.

When G is obtained from H by subdividing an edge $e \in E(H)$ into e, f, the following immediately derives from Observation 1(i).

Observation 7. *Suppose G is obtained from H by subdividing an edge $e \in E(H)$ into e, f. Then, adding a variable x_f to any extended formulation of $C(H)$ and imposing $x_e = x_f$ provides an extended formulation for $C(G)$.*

When G is obtained from H by adding a parallel edge f to $e \in E(H)$, an extended formulation for $C(G)$ can be obtained as follows.

Lemma 8. *Suppose G is obtained from H by adding a parallel edge f to an edge $e \in E(H)$ and let $Q(H)$ be an integral polyhedron which is an extended formulation of $C(H)$. Then,*

(a) *The polytope $S(G)$ obtained by replacing x_e by $x_e + x_f$ in $Q(H)$ and setting $0 \le x_e$ and $0 \le x_f$ is an extended formulation of the convex hull of the incidence vectors of all the circuits of G different from $\chi^{e,f}$.*
(b) *The convex hull of $S(G)$ union $\chi^{e,f}$ is an extended formulation of $C(G)$.*

Proof. (a) Let $R(G)$ denote the convex hull of incidence vectors of all the circuits of G except $\{e, f\}$. By Observation 1(ii), since $proj_x Q(H) = C(H)$, we have $proj_x S(G) \cap \mathbb{Z}^m = R(G) \cap \mathbb{Z}^m$. Since $Q(H)$ is integral, so is $S(G)$, which implies the integrality of $proj_x S(G)$.

(b) By (a), $proj_x S(G)$ is integral, hence so is $\text{conv}\big\{proj_x S(G) \cup \chi^{e,f}\big\}$. Since the projection of the convex hull of a set of points is the convex hull of its projected points, $proj_x\big(\text{conv}\{S(G) \cup \chi^{e,f}\}\big)$ is integral, and we are done. □

Note that the operations involved in Observation 7 and Lemma 8 preserve integrality. By construction of non trivial 2-connected series–parallel graphs, and since $C(C_2)$ is integral, we get an extended formulation for $C(G)$ by repeatedly applying Observation 7 and Lemma 8. Moreover, the extended formulation given by Lemma 8(a) yields two new inequalities, and that applying Corollary 6 in Lemma 8(b) provides an extended formulation with two more inequalities. Thus, if G is obtained from H by adding a parallel edge, then an extended formulation for $C(G)$ has 4 more inequalities than an extended formulation for $C(H)$. Furthermore, if G is obtained from H by subdividing an edge, then an extended formulation for $C(G)$ has the size of an extended formulation for $C(H)$. The following corollary stems from these observations and the fact that $C(C_2)$ is described by 3 inequalities.

Corollary 9. *There exists an extended formulation for $C(G)$ of size $O(|E(G)|)$.*

We mention here that a polytope closely related to the circuit polytope is, given a vertex r, the *r-circuit polytope*, that is the convex hull of the circuits containing r. Indeed, the circuit polytope of a graph can be seen as the union of all its r-circuit polytopes. In series–parallel graphs, the latter have been thoroughly studied by Baïou and Mahjoub in [14] who provide, in particular, their description into the original space. Therefore, an explicit extended formulation for the circuit polytope on series–parallel graphs can be obtained by applying Balas' Theorem [18,19] for the union of polyhedra together with their description. However, since the description of the r-circuit polytope has exponentially many inequalities, this approach yields an exponential-size extended formulation. Moreover, projecting such a formulation to get a description into the original space usually requires tremendous efforts. In contrast, our approach allows to project step by step, which is done in the next section.

2.1.2. Description in the original space

In this section, we show that the inequalities (1)–(3) given below describe the circuit polytope on non trivial 2-connected series–parallel graphs, see Theorem 10. Throughout, for a sequence $\mathcal{M} = (M_0, \dots, M_k)$ of edge sets, $x(\mathcal{M})$ will stand for $\sum_{i=1}^{k} x(M_i)$.

$$x_e \geq 0 \qquad \text{for all } e \in E. \tag{1}$$

$$x_e \leq x(B \setminus e) \qquad \text{for all bonds } B \text{ of } G, \text{ for all } e \in B, \tag{2}$$

$$x(\mathcal{M}) - x(\mathcal{N}) \leq 2 \qquad \text{for all } \mathcal{M}, \mathcal{N} \text{ star-cut collections of } G, \tag{3}$$

Inequalities (1) are called *non-negativity constraints*, (2) are *bond constraints*, and (3) are *star-cut constraints*.

Theorem 10. $C(G) = \{x \in \mathbb{R}^m_+$ satisfying (2) and (3)$\}$.

Proof. Let us first show that (1)–(3) are valid for $C(G)$. Clearly, every incidence vector of a circuit satisfies the non-negativity constraints (1). The validity of bond constraints (2) comes from Observation 2. To show the validity of star-cut constraints (3), let $\mathcal{M}, \mathcal{N}$ be a star-cut collection and C a circuit of G. Since M_0 and $M_i \Delta N_i$ are cuts for $i \in \{1, \dots, k\}$, each of them intersects C an even number of times. Therefore, if C intersects $M_i \in \mathcal{M}$ at most once, then $\chi^C(M_i) - \chi^C(N_i) \leq 0$ if $i \geq 1$ and $\chi^C(M_i) = 0$ if $i = 0$. The validity of $x(\mathcal{M}) - x(\mathcal{N}) \leq 2$ follows since, by Lemma 4, at most one member of $\mathcal{M}$ intersects C twice, the other ones intersecting C at most once.

Let us prove the theorem by induction. The first step of the induction comes from $C(C_2) = \{x \in \mathbb{R}^2_+ :$ satisfying (2) and $x_e + x_f \leq 2\}$ and the fact that $\{\{e, f\}\}, \emptyset$ forms a star-cut collection, where $C_2 = \{e, f\}$. Suppose now that $C(H)$ is given by inequalities (1)–(3) for a non trivial 2-connected series–parallel graph H, and let us show that $C(G)$ is also described by (1)–(3) when G is obtained from H by subdividing an edge or by adding a parallel edge in H.

First, remark that if G is obtained from H by subdividing e into e, f, then $C(G)$ is given by the inequalities of $C(H)$ and $x_e = x_f$. The inequalities of $C(H)$ of type (1)–(3) remain of the same type in G, and $x_e = x_f$ is implied by the two inequalities of type (2) associated with the bond $\{e, f\}$.

Now, let G be obtained from H by adding a parallel edge f to $e \in E(H)$. By the induction hypothesis, we have $C(H) = \{x^H \in \mathbb{R}^{m-1} : A^H x^H \leq b^H\}$ where A^H is given by the non-negativity (1), bond (2), and star-cut (3) constraints for H. Denote by $\bar{A}^H$ and $\bar{x}^H$ the matrix and vector obtained from A^H and x^H by, respectively, removing the column A_e^H corresponding to e and the component x_e^H. The application of Lemma 8(a) introduces a new variable y and provides the following description of $S(G)$:

$$\{(x^H, y) \in \mathbb{R}^{m-1} \times \mathbb{R} : \bar{A}^H \bar{x}^H + A_e^H (x_e^H + y) \leq b^H, 0 \leq x_e^H, 0 \leq y\}.$$

Lemma 8(b) implies that $C(G)$ is the convex hull of the union of $S(G)$ and $\chi^{\{e,f\}}$. Let us apply Theorem 5 to $P_1 = S(G)$ and $P_2 = \{\chi^{\{e,f\}}\}$. The latter being a vertex, we can get rid of x^1 and x^2 to get the following extended formulation of $C(G)$, where $\bar{x}$ denotes the vector x after the removal of x_e and x_f.

$$\{(\bar{x}, x_e, x_f, \lambda) \in \mathbb{R}^{m-2} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} : \bar{A}^H \bar{x} + A_e^H (x_e + x_f - 2\lambda) \leq (1-\lambda) b^H, \lambda \leq x_e, \lambda \leq x_f, 0 \leq \lambda \leq 1\}.$$

To project it by Fourier–Motzkin's method [21], we only need to consider the inequalities where λ appears, and since A^H is given by (1)–(3) for H, we may write them down explicitly, implicitly using the fact that if e belongs to a cut of G, then so does f, and conversely:

$$0 \leq \lambda \tag{4}$$

$$-1 \leq -\lambda \tag{5}$$

$$-x_h \leq -\lambda \quad \text{for } h = e, f \tag{6}$$

$$x_\ell - x(B \setminus \ell) \leq -2\lambda \quad \text{for all bonds } B \text{ of } G \text{ containing } e, f \text{ and } \ell \in B \setminus \{e, f\} \tag{7}$$

$$x_e + x_f - x(D \setminus \{e, f\}) \leq 2\lambda \quad \text{for all bonds } D \text{ of } G \text{ containing } e, f \tag{8}$$

$$x(\mathcal{M}) - x(\mathcal{N}) \leq 2 - 2(\alpha_e(\mathcal{M}, \mathcal{N}) + 1)\lambda \quad \text{for all star-cut collections } \mathcal{M}, \mathcal{N} \text{ of } G, \text{ with } \alpha_e(\mathcal{M}, \mathcal{N}) \geq 0, \tag{9}$$

where $\alpha_e(\mathcal{M}, \mathcal{N}) = |\{N \in \mathcal{N} : e \in N\}| - |\{M \in \mathcal{M} : e \in M\}|$.

We now prove that the inequalities obtained by projecting out λ are either contained or implied by the non-negativity constraints (1) and bond constraints (2) and star-cut constraints (3) for G, which implies our theorem. Recall that, to get rid of λ, one has to combine every inequality where λ's coefficient is negative with every inequality where it is positive [21]. Combinations with $0 \leq \lambda$ immediately give rise to inequalities of type (1), (2) or (3) for G. Thus, it remains to combine (8) with every other inequality.

First, remark that adding twice inequality (5) to any inequality (8) leads to an inequality obtained by adding non-negativity constraints and the star-cut constraint $x(M_0) \leq 2$ where M_0 is a star of G containing e, f. Moreover, adding (8) to twice (6) gives $x_h - x(D \setminus h) \leq 0$ for all bonds D containing e, f, and $h \in \{e, f\}$, which are inequalities of type (2).

Adding (8) to (7) gives $x_\ell \leq x(B \setminus \{e, f, \ell\}) + x(D \setminus \{e, f\})$. If D contains ℓ, the latter is a sum of non-negativity constraints (1). Otherwise, $B \Delta D$ is a cut contained in $B \cup D \setminus \{e, f\}$ and thus contains a bond J containing ℓ but not e, f, since a cut is a disjoint union of bonds. Hence, the inequality is the sum of $x_\ell \leq x(J \setminus \ell)$ and non-negativity constraints (1).

For a bond D containing e, f and a star-cut collection $\mathcal{M} = (M_0, \dots, M_k)$, $\mathcal{N} = (N_1, \dots, N_k)$ with $\alpha_e(\mathcal{M}, \mathcal{N}) \geq 0$, combining (8) and (9) gives $x(\mathcal{M}) - x(\mathcal{N}) + (\alpha_e(\mathcal{M}, \mathcal{N}) + 1)(x_e + x_f - x(D \setminus \{e, f\})) \leq 2$. If e and f belong to a member of $\mathcal{M}$, then $\alpha_e(\mathcal{M}, \mathcal{N}) + 1 = |\{N \in \mathcal{N} : e \in N\}|$. Moreover, considering separately the elements of $\mathcal{N}$ containing e and f from the other ones, the inequality can be rewritten as:

$$x(\mathcal{M}) - \sum_{N \in \mathcal{N} : e, f \in N} \Big(x(N \setminus \{e, f\}) + x(D \setminus \{e, f\}) \Big) - \sum_{N \in \mathcal{N} : e, f \notin N} x(N) \leq 2.$$

Since $x(N\Delta D) \leq x(N \setminus \{e,f\}) + x(D \setminus \{e,f\})$ for all $N \in \mathcal{N}$ containing e and f, the above inequality is implied by $x(\mathcal{M}) - x(\mathcal{N}') \leq 2$, where $\mathcal{N}' = (N'_1, \dots, N'_k)$ with N'_i equals $N_i \Delta D$ if $e \in N_i$ and N_i otherwise, for $i = 1, \dots, k$. Moreover, since D and $M_i \Delta N_i$ are cuts, so is $M_i \Delta N'_i$, for $i = 1, \dots, k$, as the symmetric difference of two cuts is a cut. Therefore, $\mathcal{M}, \mathcal{N}'$ is a star-cut collection.

Suppose now that no member of $\mathcal{M}$ contains e and f. Applying the previous argument leads to the inequality $x(\mathcal{M}) - x(\mathcal{N}') + x_e + x_f - x(D \setminus \{e,f\}) \leq 2$, where $\mathcal{M}, \mathcal{N}'$ is the star-cut collection defined above. Moreover, there exists M_{k+1} containing e, f such that $\{M_0, \dots, M_k, M_{k+1}\}$ is contained in a star decomposition. Let $\tilde{\mathcal{M}} = (M_0, \dots, M_{k+1})$ and $\tilde{\mathcal{N}} = (N'_1, \dots, N'_{k+1})$, where $N'_{k+1} = D \Delta M_{k+1}$. The symmetric difference being associative, $M_{k+1} \Delta N_{k+1}$ equals D, and hence $\tilde{\mathcal{M}}, \tilde{\mathcal{N}}$ is a star-cut collection of G. Moreover, the associated star-cut constraint (3) implies the inequality obtained by combination of (8) and (9). □

Let us mention a few simple constraints implied by the ones of Theorem 10. First, whenever $\{k, \ell\}$ is a bond, we have $x_k = x_\ell$, which is implied by the inequalities (2) for $\{k, \ell\}$. These will turn out to be the only hyperplanes containing $C(G)$. We postpone the proof of this fact to Section 3.3, see Corollary 18 . For every edge $uv \in E$, $\delta_G(u)$ is a bond since G is 2-connected. Then, we obtain the inequality $x_{uv} \leq 1$ by adding x_{uv} to each side of $x_{uv} \leq x(\delta_G(u) \setminus uv)$ and by applying $x(\delta_G(u)) \leq 2$, which is a special case of (3). We also mention that, given a bond B, the inequality $x(B) \leq 2$ is implied by a suitable star-cut constraint.

We will see at the end of Section 3 a family of examples where exponentially many of the inequalities of Theorem 10 define facets.

2.2. General series–parallel graphs

In this section, we provide a polyhedral description of the circuit polytope on general series–parallel graphs, see Theorem 11. The result is obtained by applying a standard union technique and the fact that the circuit polytope of a graph is the convex hull of the union of the circuit polytopes of its 2-connected components.

Theorem 11. *Let G be a series–parallel graph, $G_1, \dots, G_k$ its non trivial 2-connected components, $\mathfrak{L}$ its set of loops, and $\mathfrak{B}$ its set of bridges. Then*

$$C(G) = \left\{ \begin{array}{ll} x \in \mathbb{R}^m_+ \text{ satisfying (2)}, \ x(\mathfrak{B}) = 0 \text{ and} & \\ \displaystyle\sum_{i=1}^{k} (x(\mathcal{M}_i) - x(\mathcal{N}_i)) + 2x(\mathfrak{L}) \leq 2, & \begin{array}{l}\text{for all } i = 1, \dots, k, \\ \text{for all star-cut collections } \mathcal{M}_i, \mathcal{N}_i \text{ of } G_i\end{array} \end{array} \right\}.$$

Proof. We prove the result by induction on the number of 2-connected components.

Let us see the first step. Since no bridge b belongs to a circuit, its circuit polytope is described by $\{x_b = 0\}$. Moreover, the circuit polytope of a loop ℓ is described by $\{0 \leq 2x_\ell \leq 2\}$. Finally, the circuit polytope of a non trivial 2-connected series–parallel component is given by Theorem 10.

Suppose that the result holds for two series–parallel graphs I and $H = \bigcup_{i=1}^{k-1} H_i$, where I is 2-connected and $H_i, i = 1, \dots, k-1$ are the 2-connected components of H, and let $G = I \bigcup \ (\cup_{i=1}^{k-1} H_i)$ be the graph obtained by identifying a vertex of I and a vertex of H. Then, $C(G) = \text{conv}\{C(H) \cup C(I)\}$. Remark that $C(H) = \{x \in \mathbb{R}^{E(H)}_+ : A_H x_H \leq b_H\}$ and $C(I) = \{x \in \mathbb{R}^{E(I)}_+ : A_I x_I \leq b_I\}$ live in different spaces. Extend them to polytopes of $\mathbb{R}^{E(H)} \times \mathbb{R}^{E(I)}$ by setting the new coordinates to zero, and apply Theorem 5 to get $C(G) = proj_x\{x = (x_H, x_I), A_H x_H \leq \lambda b_H, A_I x_I \leq (1-\lambda) b_I, 0 \leq \lambda \leq 1\}$.

Let us get rid of λ in the above extended formulation. Combinations with $0 \leq \lambda$ or $\lambda \leq 1$ immediately give desired inequalities. It remains to combine $a_H x_H \leq \lambda b_H$ and $a_I x_I \leq (1-\lambda) b_I$ when $b_H \neq 0$ and $b_I \neq 0$.

Since in this case the induction hypothesis says that both inequality are of the new type, we get $b_H = b_I = 2$, thus the resulting inequality is $a_H x_H + a_I x_I \leq b_H$, and the theorem follows. □

Since every 2-connected component of a series–parallel graph has a compact extended formulation, using the Theorem of Balas [18,19] for the union of several polytopes, one can extend Corollary 9 as follows.

Corollary 12. *If G is series–parallel, there exists a compact extended formulation of $C(G)$ in size $O(|E|)$.*

3. The bond polytope on series–parallel graphs

In this section, as a consequence of Theorem 11 and the planar duality, we describe the bond polytope on series–parallel graphs. We also provide examples where the latter contains exponentially many facets. Before stating these results, we introduce a few definitions.

3.1. Definitions

Given a series–parallel graph G, we denote its set of bonds by $\mathcal{B}(G)$, and the convex hull of their incidence vectors by $B(G)$.

If G is a non trivial 2-connected series–parallel graph, an *open nested ear decomposition* [22] $\mathcal{E}$ of G is a partition of $E(G)$ into a sequence $E_0, \ldots, E_k$ such that E_0 is a circuit of G and the *ears* E_i, $i \in \{1, \ldots, k\}$, are paths with the following properties:

- the two endpoints of each ear are distinct and appear in an E_j with $j < i$,
- no interior point of an ear E_i belongs to E_j for all $j < i$,
- if two ears E_i and $E_{i'}$ have both their endpoints in the same E_j, then any two paths contained in E_j, one between the endpoints of E_i and the other between the endpoints of $E_{i'}$, are either disjoint or contained one in another.

We will denote by $C_{\mathcal{E}}$ the unique circuit of an open nested ear decomposition $\mathcal{E}$. Two sequences of edge subsets $\mathcal{F} = (F_0, F_1, \ldots, F_k)$ and $\mathcal{P} = (P_1, \ldots, P_k)$ form an *ear-cycle collection* if $\{F_0, F_1, \ldots, F_k\}$ is contained in an open nested ear decomposition $\mathcal{E}$ of G, $F_0 = C_{\mathcal{E}}$, and $F_i \Delta P_i$ is a cycle for $i = 1, \ldots, k$. Note that the elements of $\mathcal{P}$ are not required to be disjoint.

A graph H is a *minor* of G if H arises from G by contractions and deletions of edges and deletions of vertices, where *contracting* an edge uv of E corresponds to deleting e and identifying u and v. A graph is series–parallel if and only if it does not contain a K_4-minor [23], where K_4 denotes the complete graph on four vertices.

3.2. The bond polytope on series–parallel graphs

K_4 being its own dual, a graph is series–parallel if and only if, given any planar drawing of the graph, its dual is series–parallel. It is immediate that the circuits of such a graph are precisely the bonds of its dual, thus the bond polytope of the graph is the circuit polytope of its dual. Then, applying Theorem 11 provides a description of the bond polytope on series–parallel graphs.

Given a circuit C and $e \in C$, $x_e \leq x(C \setminus e)$ is a *circuit constraint*, and given an ear-cycle collection, $x(\mathcal{F}) - x(\mathcal{P}) \leq 2$ is an *ear-cycle constraint*.

Theorem 13. *Let G be a series–parallel graph, $G_1, \ldots, G_k$ its non trivial 2-connected components, $\mathfrak{L}$ its set of loops and $\mathfrak{B}$ its set of bridges.*

$$B(G) = \left\{ \begin{array}{ll} \multicolumn{2}{l}{x \in \mathbb{R}_+^m \text{ satisfying } x_e \le x(C \setminus e) \text{ for all circuits } C \text{ and } e \in C,\ x(\mathfrak{L}) = 0 \text{ and}} \\ \displaystyle\sum_{i=1}^{k} (x(\mathcal{F}_i) - x(\mathcal{P}_i)) + 2x(\mathfrak{B}) \le 2, & \begin{array}{l} \text{for all } i = 1, \ldots, k, \\ \text{for all ear-cycle collections } \mathcal{F}_i, \mathcal{P}_i \text{ of } G_i \end{array} \end{array} \right\}.$$

Proof. Fix a planar drawing of G, and let $\tilde{G}$ be the dual graph of G. The edgesets of G and $\tilde{G}$ are in bijection, and $\tilde{e}$ will denote the edge of $E(\tilde{G})$ corresponding to $e \in E(G)$. As noted above, the bond polytope of G is precisely the circuit polytope of $\tilde{G}$. First, recall that bridges of G are in bijection with loops of $\tilde{G}$, and conversely. Then, by Theorem 11, to get the desired result, we just need to show that the bond polytope on non trivial 2-connected series–parallel graphs is given by non-negativity, circuit and ear-cycle constraints.

Let G be a non trivial 2-connected series–parallel graph. Then, $\tilde{G}$ is also a non trivial 2-connected series–parallel graph. Since, by Theorem 10, $C(\tilde{G})$ is described by (1)–(3), and by the bijection between circuits in G and bonds in $\tilde{G}$, we only have to show that the ear-cycle constraints are valid for $B(G)$ and that a star-cut collection of $\tilde{G}$ is an ear-cycle collection of G.

To see the validity of the constraints, let us show that, given an open nested ear decomposition $\mathcal{E} = \{E_0, \ldots, E_k\}$ and a bond B of G,

$$(*) \text{ if } |B \cap E| = 2 \text{ for some } E \in \mathcal{E}, \text{ then } |B \cap F| \le 1 \text{ for all } F \in \mathcal{E} \setminus E.$$

First, note that, by Observation 2 and the fact that an ear is always contained in a circuit, we have $|B \cap E| \le 2$, for all $E \in \mathcal{E}$. Now, suppose that $E_i, E_j \in \mathcal{E}$ both intersect B twice, with $i < j$. Denote by u and v the extremities of E_j and let e be an edge of $E_i \cap B$. The graph induced by the edges of $E_0 \cup \ldots \cup E_i \cup \{uv\}$ is 2-connected so it contains a circuit containing e and uv. Replacing uv by the ear E_j, we get that G contains a circuit C containing e and E_j. Therefore, $|C \cap B| \ge 3$, yet B is a bond, a contradiction to Observation 2. Therefore $(*)$ holds.

Then, with arguments similar to those proving the validity of star-cut constraints for the circuit polytope (see Theorem 10), we get the validity of the ear-cycle constraints by $(*)$ and the fact that a circuit and a cut intersect each other an even number of times.

We now prove by induction on the number of edges of G that a star decomposition of $\tilde{G}$ corresponds to an open nested ear decomposition of G. We will use edge subdivision and parallel addition operations, thus note that these two operations are dual one of each other. As the dual of C_2 is C_2, one can easily check that a star decomposition of C_2 corresponds to an open nested ear decomposition in its dual.

If G is obtained from H by subdividing an edge $e \in E(H)$ into e, f, then $\tilde{G}$ is obtained from $\tilde{H}$ by adding a parallel edge $\tilde{f}$ to $\tilde{e}$. By induction, any star decomposition $\mathcal{S}$ of $\tilde{H}$ corresponds to an ear decomposition $\mathcal{E}_{\mathcal{S}}$ of H. Adding e to the suitable set of $\mathcal{S}$ (which is the first extremity of e appearing in the star decomposition) gives a star decomposition of $\tilde{G}$, which straightforwardly corresponds to the ear decomposition of G obtained from $\mathcal{E}_{\mathcal{S}}$ by replacing e by $\{e, f\}$ in the member of $\mathcal{E}_{\mathcal{S}}$ containing e.

If G is obtained from H by adding a parallel edge f to $e \in E(H)$, then $\tilde{G}$ is obtained from $\tilde{H}$ by subdividing $\tilde{e}$ into $\tilde{e}, \tilde{f}$. Let u be the vertex that is common to $\tilde{e}$ and $\tilde{f}$, and v, w the other ends of $\tilde{e}$ and $\tilde{f}$. Let $\mathcal{S} = \{\delta_{\tilde{G}_1^+}(v_1), \ldots, \delta_{\tilde{G}_{n-1}^+}(v_{n-1})\}$ be a star decomposition of $\tilde{G}$. We may suppose, without loss of generality, that u and v or u and w are consecutive in the star decomposition. Indeed, otherwise, $u = v_i$ for some $i \in \{2, \ldots, n-1\}$, and since G_i^+ and G_i^- are connected, exactly one of v, w is in G_i^- that is, equals some v_j for $j < i$. In this case, the star decomposition $\mathcal{S}'$ obtained from $\mathcal{S}$ by removing u, and then inserting

u right after v_j, without changing the rest, gives the same partition of E as $\mathcal{S}$. Thus $\mathcal{S}$ and $\mathcal{S}'$ are in bijection with the same partition of the edgesets of G.

Since u and one of v, w appear consecutively in the star decomposition, contracting them gives a star decomposition of $\tilde{H}$. By induction, the latter corresponds to an ear decomposition $\mathcal{E}$ of H. Now, possibly having exchanged the role of e and f because of the contraction, $\mathcal{E} \cup \{f\}$ is an ear decomposition of G, and we are done.

To finish the proof, if suffices to apply Theorem 11 and the fact that the dual of a cut is a cycle. □

It turns out that there are more ear-cycle collections of G than star-cut collections of $\tilde{G}$, and it is unclear which restrictions are to be made in order to get a bijection. As a consequence, if $B(G)$ can be deduced from $C(\tilde{G})$ in a rather simple manner, the converse seems more challenging.

3.3. Facet-defining inequalities

Determining directly which inequalities are facet-defining for the circuit polytope is not that easy. Surprisingly, the bond polytope is much simpler to study polyhedrally. The main reason is that we can safely remove parallel edges, see Observation 14. Thus Theorem 13 is not only a standard planar duality result, but also a tool to prove polyhedral results for the original polytope.

First, we provide the dimension of the bond polytope, see Lemma 15. Then, we characterize which of the non-negativity and circuit constraints are facet-defining, see Lemma 16. Unfortunately, it seems challenging to exhibit the structures for which ear-cycle inequalities define facets. We provide an example where exponentially many of them are facet-defining, see Claim 17.

Seen the structure of the inequalities given by Theorem 13, it is enough to study the facet-defining ones for non trivial 2-connected series–parallel graphs. In this section, let $G = (V, E)$ be such graph.

Observation 14. *The set of bonds of G is unchanged if we remove parallel edges.*

Proof. Whenever two edges of the graph are parallel, every bond contains either both or none. □

By the above observation and the construction of non trivial 2-connected series–parallel graphs, we may assume there are no parallel edges. This is emphasized by the following lemma.

Lemma 15. *The dimension of $B(G)$ is the number of edges of the graph obtained from G by removing every parallel edge.*

Proof. By Observation 14, we may assume that G has no parallel edges. Then, since the emptyset is a bond, that is $0 \in B(G)$, the result is equivalent to the existence of $|E(G)|$ linearly independent bonds of G. Clearly, $\dim B(G) \leq |E(G)|$. We prove the result by induction on $|E(G)|$, noting that $\dim B(C_3) = 3$.

Since G has no parallel edges, G is obtained from a non trivial 2-connected series–parallel graph H by subdividing an edge e into f, g.

If H contains no parallel edges, then the induction hypothesis gives a family of $\dim B(H) = |E(H)| = |E(G)| - 1$ linearly independent bonds of H. Replacing e by f, for each member of $\mathcal{F}$ containing e, and then adding $\{f, g\}$, gives a linearly independent family of $|E(G)|$ bonds of G, and we are done.

Since G did not contain parallel edges, if H does, then these parallel edges are $\{e, h\}$ for some $h \in E(G) \setminus \{f, g\}$. In this case, we have $B(H) \subseteq \{x_e = x_h\}$. By the induction hypothesis, there exists a family $\mathcal{L}$ of $\dim B(H) = |E(H)| - 1 = |E(G)| - 2$ linearly independent bonds of H. We may assume that $e \in B$ for some $B \in \mathcal{L}$. Define $D = B \setminus e \cup f$, we get the family $\mathcal{L} \cup D \cup \{f, g\}$ of bonds of G. Let us prove that they are linearly independent, by showing that the corresponding matrix A has full column rank. Since $D = B \setminus e \cup f$, by basic column operations we get that A has the same rank as the matrix whose columns are composed of χ^e, χ^f and the elements of $\mathcal{L}$. Thus A has full column rank if and only if the matrix obtained from $\mathcal{L}$ by deleting the coordinate corresponding to e has. It is indeed the case because $\mathcal{L}$ is a family of linearly independent circuits of H, and they all satisfy $x_e = x_h$. $\square$

The following lemma characterizes which of the non-negativity and circuit inequalities are facet-defining.

Lemma 16. *The inequality*

1. $x_\ell \geq 0$ *defines a facet of* $B(G)$ *if and only if* ℓ *is not contained in a triangle.*
2. $x_\ell \leq x(C \setminus \ell)$ *defines a facet of* $B(G)$ *if and only if* C *has no chord and* $|C| \geq 3$.

Proof. By Observation 14, we may assume that G has no parallel edges. We prove both results by a maximality argument.

(1) First, suppose that ℓ is contained in a triangle, say $\{\ell, e, f\}$. The two circuit inequalities $x_e - x_f - x_\ell \leq 0$ and $-x_e + x_f - x_\ell \leq 0$ give $x_\ell \geq 0$ so the latter is not facet-defining.

Suppose now that ℓ is not contained in a triangle. Consider the face F defined by $x_\ell \geq 0$ and suppose that it is not a facet, that is, there exists a face F' defined by an inequality $ax \leq b$ of $B(G)$ such that $F \subsetneq F'$. Since $\emptyset \in F$, we have $b = 0$. For every edge uv non incident to ℓ, the bonds $\delta(u)$, $\delta(v)$ and $\delta(\{u, v\})$ belong to F, implying that $a(\delta(u)) = a(\delta(v)) = a(\delta(\{u, v\})) = 0$, leading to $a_{uv} = 0$. Finally, for any edge $f = uv$ incident to ℓ at node u, $\delta(v) \in F$. By hypothesis, f is the only edge of $\delta(v)$ incident to ℓ, implying that $a_f = 0$. Thus, $a = \rho\chi_\ell$, for some $\rho < 0$ and F defines a facet.

(2) If $|C| = 2$, then C is two parallel edges e and f, and $B(G) \subseteq \{x_e = x_f\}$. If C has a chord c, let C' and C'' be the two circuits defined by $C' \cup C'' \setminus c = C$, and assume $c \in C'$. Then, the inequality $x_\ell \leq x(C \setminus \ell)$ is obtained applying the circuit inequalities for ℓ and C' and then for c and C'', $x_\ell \leq x(C' \setminus \ell) = x(C' \setminus \{\ell, c\}) + x_c \leq x(C' \setminus \{\ell, c\}) + x(C'' \setminus c) = x(C \setminus \ell)$.

Suppose that C has no chord, $|C| \geq 3$, and $F' = \{x_\ell \leq x(C \setminus \ell)\} \subseteq \{ax \leq b\} = F$, where F is facet-defining. Since $0 \in F'$, we have $b = 0$. Let $uv \in E$ with $u, v \notin V(C)$. Since $\{uv\} = (\delta_G(u) \cup \delta_G(v)) \setminus \delta_G(\{u, v\})$, and $\delta_G(u), \delta_G(v), \delta_G(\{u, v\})$ are bonds, and are contained in F', we have

$$(\#)\ a_{uv} = 0, \text{ for all } uv \in E \text{ such that } u, v \notin V(C).$$

Denote the vertices of C by $\{v_1, \ldots, v_k\}$ where $\ell = v_k v_1$ and $v_i v_{i+1} \in C$ for $i = 1, \ldots, k-1$. Let $u \in V \setminus V(C)$. Note that there are at most two edges between u and $\{v_1, \ldots, v_k\}$. If there is exactly one, say uv_i, then, by (#) and $\delta_G(u) \in F'$, we have $a_{uv_i} = 0$. If there are two, say uv_i and uv_j, since G is series–parallel, every uv_i-path not containing v_j does not intersect $V(C)$. Therefore, since G is 2-connected, there exists a bond $B = \delta_G(W)$ containing uv_i and ℓ such that $B' = \delta_G(W \cup \{u\})$ is also a bond. Since the edges of $B \Delta B'$ are uv_i, uv_j, and edges not in $\delta_G(C)$, and then by $B, B' \in F'$ and (#), we get $a_{uv_i} = a_{uv_j}$. By $\delta_G(u) \in F'$, we have $a_{uv_i} + a_{uv_j} = 0$. Therefore, $a_{uv_i} = a_{uv_j} = 0$. Since C had no chord, we proved $a_{uv} = 0$ whenever $uv \notin C$.

To finish the proof, since G is 2-connected, there exists a bond B_i containing ℓ and $v_i v_{i+1}$ for all $i = 1, \ldots, k-1$. By Observation 2, $B_i \cap C = \{\ell, v_i v_{i+1}\}$, thus $B_i \in F'$. Therefore, we have $a_\ell = -a_f$ for all $f \in C \setminus \ell$. Since $ax \leq 0$ is valid for the bond $\delta_G(v_2)$, we have $0 \geq a_{v_1 v_2} + a_{v_2 v_3} = 2a_{v_1 v_2}$, thus we may assume $a_\ell = 1$, and then we get $F' = F$. $\square$

We now provide a family of series–parallel graphs where an exponential number of ear-cycle constraints are facet-defining.

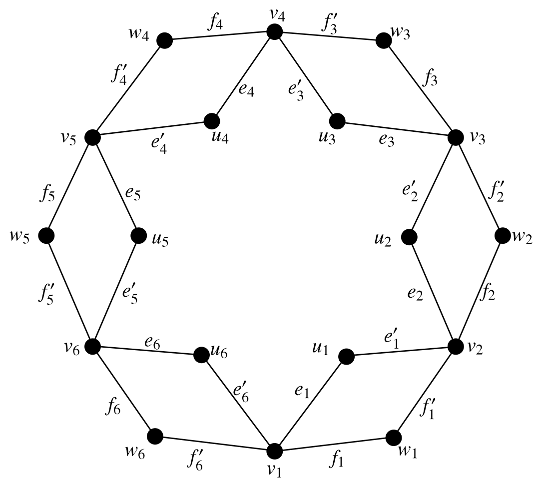

Fig. 1. An example of graph obtained from C_6 by parallel addition and subdivision of all the edges.

Example

The graph G_k we consider is built from the circuit on k edges C_k where a parallel edges is added to every edge and then all edges are subdivided. Fig. 1 shows the construction of such a graph from C_6. Denote by e_i and e_i' (resp. f_i and f_i') the edges obtained by subdividing one parallel edge (resp. the other one). Let u_i (resp. w_i) be the vertex incident to e_i and e_i' (resp. f_i and f_i') for all $i = 1, \dots, k$.

Claim 17. *$x(C) \le 2$ is facet-defining for $B(G_k)$ if C is a circuit of $2k$ edges.*

Proof. Without loss of generality, suppose that $C = \{e_i, e_i' : i = 1, \dots, k\}$. Let F' be the face induced by the inequality and suppose that $F' \subseteq F$ where F is a facet induced by the constraint $ax \le b$. Since $\{e_i, e_i'\} \in F'$, we have $a_{e_i} + a_{e_i'} = b$ for $i = 1, \dots, n$. Moreover, $\{e_i, f_i, f_j, e_j\}$ and $\{e_i', f_i, f_j, e_j\}$ belong to F', for all $j \ne i$, from which we get $a_{e_i} = a_{e_i'}$. Combining these two remarks give $a_{e_i} = a_{e_i'} = b/2$, for $i = 1, \dots, n$. Now, since both $\{e_i, f_i, f_j, e_j\}$ and $\{e_i, f_i', f_j, e_j\}$ belong to F', we have $a_{f_i} = a_{f_i'} = 0$, for $i = 1, \dots, n$. The emptyset being a bond, we have $b \ge 0$. In fact, $b > 0$ because otherwise $(a, b) = 0$. Therefore, without loss of generality, we may assume that $b = 2$. Then, we get $F = F'$, and we are done. $\square$

If we set $E_i = \{e_i, e_i'\}$ and $F_i = \{f_i, f_i'\}$ for all $i = 1, \dots, k$, one can also prove that the inequalities $x(E_j \cup F_j) + \sum_{i \ne j}(x(M_i) - x(E_i \cup F_i \setminus M_i)) \le 2$ for all $j \in \{1, \dots, k\}$, where $M_i \in \{E_i, F_i\}$ for all $i = 1, \dots, k$ are facet-defining. In fact, together with the inequalities of Claim 17 and Lemma 16, this gives all the facet-defining inequalities for the example. However, other examples show that ear-cycle constraints are not always that nicely structured.

Let us mention some dual consequences of the results of Section 3.3 for the circuit polytope on series–parallel graphs.

First, we get its dimension by planar duality and Lemma 15.

Corollary 18. *Let F be a minimal set of edges intersecting every size two bond of $G = (V, E)$. Then, the dimension of $C(G)$ is $|E \setminus F|$.*

Moreover, Claim 17 implies the following.

Corollary 19. *There are examples for which exponentially many of the star-cut constraints (3) define facets of the circuit polytope on series–parallel graphs.*

Thus, the circuit polytope on series–parallel graphs is another example of polytope having exponentially many facet-defining inequalities that admits a compact extended formulation.

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FULL LENGTH PAPER

Series A

Box-total dual integrality, box-integrality, and equimodular matrices

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Abstract

Box-totally dual integral (box-TDI) polyhedra are polyhedra described by systems which yield strong min-max relations. We characterize them in several ways, involving the notions of principal box-integer polyhedra and equimodular matrices. A polyhedron is *box-integer* if its intersection with any integer box $\{\ell \leq x \leq u\}$ is integer. We define *principally box-integer* polyhedra to be the polyhedra P such that kP is box-integer whenever kP is integer. A rational $r \times n$ matrix is *equimodular* if it has full row rank and its nonzero $r \times r$ determinants all have the same absolute value. A *face-defining* matrix is a full row rank matrix describing the affine hull of a face of the polyhedron. Our main result is that the following statements are equivalent.

- The polyhedron P is box-TDI.
- The polyhedron P is principally box-integer.
- Every face-defining matrix of P is equimodular.
- Every face of P has an equimodular face-defining matrix.
- Every face of P has a totally unimodular face-defining matrix.
- For every face F of P, $\text{lin}(F)$ has a totally unimodular basis.

Along our proof, we show that a polyhedral cone is box-TDI if and only if it is box-integer, and that these properties are carried over to its polar. We illustrate these charaterizations by reviewing well known results about box-TDI polyhedra. We also provide several applications. The first one is a new perspective on the equivalence between two results about binary clutters. Secondly, we refute a conjecture of Ding, Zang, and Zhao about box-perfect graphs. Thirdly, we discuss connections with an abstract class of polyhedra having the Integer Carathéodory Property. Finally, we characterize the box-TDIness of the cone of conservative functions of a graph and provide a corresponding box-TDI system.

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1 Introduction

Box-totally dual integral systems are systems which yield strong min-max relations. These systems are useful to prove strong min-max combinatorial theorems and are known to be difficult to handle. A polyhedron that can be described by a box-totally dual integral system is called a *box-totally dual integral polyhedron* [14]. In this paper, we characterize box-totally dual integral polyhedra in several new ways. The key idea is to introduce and study the abstract class of principally box-integer polyhedra—see Definition 1 below. Indeed, the completely geometric nature of principally box-integer polyhedra makes them easier to be studied, and it turns out that this class coincides with that of box-totally dual integral polyhedra.

We characterize principally box-integer polyhedra in several ways. In this regard, some matrices play an important role. They generalize unimodular matrices and we call them equimodular matrices—see Definition 2 below. We show that the notion of principal box-integrality is strongly intertwined with that of equimodularity: equimodular matrices are characterized using principal box-integrality and, in turn, principally box-integer polyhedra are characterized by the equimodularity of a family of matrices. This sheds new lights on fundamental results in combinatorial optimization and integer programming. For instance, the classical characterization of unimodular matrices by Veinott and Dantzig [45] and that of totally unimodular matrices due to Hoffman and Kruskal [31] can be reformulated and extended using these notions.

More importantly, these notions bring a geometric and matricial perspective about box-totally dual integral polyhedra. Since the class of principally box-integer polyhedra coincides with that of box-totally dual integral polyhedra our results provide several new characterizations of the latter. We believe that these characterizations fill in “the lack of a proper tool for establishing box-total dual integrality”—to quote Ding et al. [17]—and we illustrate their use.

Main definitions Before going deeper into the details of our contributions, let us give the main definitions relevant to this paper.

A polyhedron $P = \{x \colon Ax \leq b\}$ of $\mathbb{R}^n$ is *integer* if each of its faces contains an integer point and *box-integer* if $P \cap \{\ell \leq x \leq u\}$ is integer for all $\ell, u \in \mathbb{Z}^n$. For $k \in \mathbb{Z}_{>0}$, the kth *dilation* of P is $kP = \{kx \colon x \in P\} = \{x \colon Ax \leq kb\}$.

Definition 1 A polyhedron P is *principally box-integer* if kP is box-integer for all $k \in \mathbb{Z}_{>0}$ such that kP is integer.

A full row rank $r \times n$ matrix is *unimodular* if it is integer and its nonzero $r \times r$ determinants have value 1 or -1 [38, Page 267]. There is a strong connection between principally box-integer polyhedra and the following generalization of unimodular matrices. Note that equimodular matrices are studied under the name of matrices with the Dantzig property in [29] or as unimodular sets of vectors in [28].

Definition 2 A rational $r \times n$ matrix is *equimodular* if it has full row rank and its nonzero $r \times r$ determinants all have the same absolute value.

A linear system $Ax \leq b$ is *totally dual integral* (*TDI*) if the minimum in the linear programming duality equation $\max\{w^\top x \colon Ax \leq b\} = \min\{b^\top y \colon A^\top y = w,\ y \geq \mathbf{0}\}$ has an integer optimal solution for all integer vectors w for which the optimum is finite. Every polyhedron can be described by a TDI system [38, Theorem 22.6]. Moreover, the right-hand side of such a TDI system can be chosen integer if and only if the polyhedron is integer [22]. A linear system $Ax \leq b$ is a *box-TDI system* if $Ax \leq b$, $\ell \leq x \leq u$ is TDI for each pair of rational vectors ℓ and u. In other words, $Ax \leq b$ is box-TDI if

$$\min\{b^\top y + u^\top r - \ell^\top s \colon A^\top y + r - s = w,\ y \geq \mathbf{0},\ r, s \geq \mathbf{0}\} \tag{1}$$

has an integer solution for all integer vectors w and all rational vectors ℓ, u for which the optimum is finite. It is well-known that box-TDI systems are TDI [38, Theorem 22.7]. General properties of such systems can be found in [14], [39, Chap. 5.20] and [38, Chap. 22.4]. Though not every polyhedron can be described by a box-TDI system, the result of Cook [14] below proves that being box-TDI is a property of the polyhedron.

Theorem 1 (Cook [14, Corollary 2.5]) *If a system is box-TDI, then any TDI system describing the same polyhedron is also box-TDI.*

This theorem justifies the following definition [14].

Definition 3 A polyhedron that can be described by a box-TDI system is called a *box-TDI polyhedron*.

Let us now review results from the literature related to these notions.

Unimodular matrices The notion of unimodularity dates back to Smith [43] and ensures that a linear system has an integral solution for each integer right-hand side. Hoffman and Kruskal [31] proved that integral solutions still exist under the weaker condition that (*) the gcd of the $r \times r$ determinants equals 1. Condition (*) and equimodularity are complementary generalizations of unimodularity, in the sense that if an integer matrix is equimodular and satisfies (*), then it is unimodular. Hoffman and Oppenheim [30] introduced variants of unimodularity, which were afterward studied by Truemper [44]. In [7,28], it is proved that equimodular matrices ensure that all basic solutions are integer, provided that one of them is—see also Barnett [5, Chap. 7].

The stronger notion of total unimodularity plays a central role in combinatorial optimization. A matrix is *totally unimodular* when all its subdeterminants have value in $\{0, \pm 1\}$. Examples of such matrices are network matrices and incidence matrices of bipartite graphs. Hoffman and Kruskal [31] characterized totally unimodular matrices to be the matrices for which the associated polyhedra are all box-integer. Several other characterizations were obtained since then—see e.g. [10,25]. Totally unimodular matrices are now well understood due to the decomposition theorem of Seymour [40]. For a survey of related results, we refer to [38, Chap. 4 and 19]. More recently, Appa [2] and Appa and Kotnyek [3] generalized total unimodularity to rational matrices, their goal being to ensure the integrality of the associated family of polyhedra for a specified set of right-hand sides, such as those with only even coordinates. In another direction, Lee [33] generalized totally unimodular matrices by considering the associated linear spaces. The connections between his results and the previous ones are discussed in Kotnyek's thesis [32, Chap. 11].

We will see how principal box-integrality fits within the characterization of unimodular matrices by Veinott and Dantzig [45] and that of totally unimodular matrices due to Hoffman and Kruskal [31]. Then, these results are naturally extended to characterize equimodular matrices. Also, a new generalization of totally unimodular matrices appears in Sect. 4.1, the notion of totally equimodular matrices, which still have nice polyhedral properties.

Box-integrality In combinatorial optimization and integer programming, a desirable property for polyhedra is to be integer, as then the vertices can be seen as combinatorial objects. Henceforth, many results in those fields are devoted to the study of properties and descriptions of integer polyhedra. The stronger property of being box-integer is far less studied. Nevertheless, some important classes of polyhedra are known to be box-integer, such as polymatroids [21], and more generally box-totally dual integer polyhedra [38]. Box-integrality plays some role for polyhedra to have the Integer Carathéodory Property in [27]. Binary clutters being $\frac{1}{k}$-box-integer for all $k \in \mathbb{Z}_{>0}$ are characterized in [24].

Actually, all these examples of box-integer polyhedra are principally box-integer. Our characterizations then yield new insights towards their properties.

Box-total dual integrality Box-TDI systems and polyhedra received a lot of attention from the combinatorial optimization community around the 80s. These systems yield strong combinatorial min-max relations with a geometric interpretation. A renewed interest appeared in the last decade and since then many deep results appeared involving such systems. The famous MaxFlow-MinCut theorem of Ford and Fulkerson [23] is a typical example of min-max relation implied by the box-TDIness of a system. Other examples of fundamental box-TDI systems appear for polymatroids and for systems with a totally unimodular matrix of constraints.

Originally, box-TDI systems were closely related to totally unimodular matrices. Indeed, any system with a totally unimodular matrix of constraint is box-TDI. Actually, until recently, the vast majority of known box-TDI systems were defined by a totally unimodular matrix, see [39] for examples. When the constraint matrix is not totally unimodular, proving that a given system is box-TDI can be quite a challenge: one has to prove its TDIness, and then to deal with the addition of box-contraints that perturb the combinatorial interpretation of the underlying min-max relation. Ding, Feng, and Zang prove in [16] that it is *NP*-hard to recognize box-TDI systems.

Based on an idea of Ding and Zang [18], Chen, Chen, and Zang provide in [11] a sufficient condition for some systems to be box-TDI, namely the ESP property. Due to its purely combinatorial nature, the ESP property is successfully used to characterize:

box-Mengerian matroid ports in [11], the box-TDIness of the matching polytope in [17], subclasses of box-perfect graphs in [19]. Prior to the development of the ESP property, the main tool to prove box-TDIness was [38, Theorem 22.9] of Cook. Its practical application turns out to be quite technical as one has to combine polyhedral and combinatorial considerations, such as in [13] where the box-TDIness of a system describing the 2-edge-connected spanning subgraph polytope on series-parallel graphs is proved. In [15], Cornaz, Grappe, and Lacroix prove that a number of standard systems are box-TDI if and only if the graph is series-parallel.

Contributions Our results provide a framework within which the notions of equimodularity, principal box-integrality, and box-TDIness are all connected. The point of view obtained from principally box-integer polyhedra unveils new properties and simplifies the approach.

We now state our main result. A *face-defining* matrix for a polyhedron is a full row rank matrix describing the affine hull of a face of the polyhedron—see Sect. 4.2 for more details.

Theorem 2 *For a polyhedron P, the following statements are equivalent.*

1. *The polyhedron P is box-TDI.*
2. *The polyhedron P is principally box-integer.*
3. *Every face-defining matrix of P is equimodular.*
4. *Every face of P has an equimodular face-defining matrix.*
5. *Every face of P has a totally unimodular face-defining matrix.*

Along our proof, we show that a polyhedral cone is box-TDI if and only if it is box-integer, and that these properties are carried over to its polar. We use this to derive a polar version of Theorem 2—see Corollary 6.

These new results allow us to prove the box-TDIness of systems by making full use of Theorem 1: find a TDI system describing the polyhedron on the one hand, and, on the other hand, apply one of the characterizations of principally box-integer polyhedra to prove the box-TDIness of the polyhedron. In particular, when a TDI system that describes the polyhedron is already known, our characterizations allow us to pick whichever system—TDI or not—describing the polyhedron, and to use algebraic tools to prove the "box" part. The drawback of our characterization is that it does not provide a box-TDI system describing the polyhedron. Nevertheless, one of our characterizations gives an easy way to disprove box-TDIness: it is enough to exhibit a face-defining matrix having two maximal nonzero determinants of different absolute values. In particular, this provides a simple co-NP certificate for the box-TDIness of a polyhedron.

We show how known results on box-TDI polyhedra are simple consequences of our characterizations—see Sect. 5.2. We also explain how our results are connected with Schrijver's sufficient condition [39, Theorem 5.35] and Cook's characterization [14], [38, Theorem 22.9].

We illustrate the use of our characterizations on several examples—see Sect. 6. First, we explain the equivalence between the main result of Gerards and Laurent [24] and that of Chen et al. [12] about binary clutters. As a second application, we disprove a conjecture of Ding et al. [19] about box-perfect graphs. Then, we discuss Gijswijt and Regts [27]'s abstract class of polyhedra having the Integer Carathéodory Property and possible connections between principal box-integrality and the integer decomposition property. Finally, we prove that the cone of conservative functions of a graph is box-TDI if and only if the graph is series-parallel and we provide a box-TDI system describing it.

Outline Section 2 contains standard definitions. In Sect. 3, we study general properties of principally box-integer polyhedra. Section 4 shows how equimodularity and principal box-integrality are intertwined: each notion is characterized using the other one. In Sect. 5, we first prove that a polyhedron is box-TDI if and only if it is principally box-integer, and then discuss the connections between our characterizations and existing results about box-TDI polyhedra. In Sect. 6, we illustrate the use of our characterizations on several examples.

2 Definitions

Matrices Throughout the paper, all entries will be rational. The ith unit vector of $\mathbb{R}^n$ will be denoted by χ^i. For $I \subseteq \{1, \dots, n\}$, let $\chi^I = \sum_{i \in I} \chi^i$. An element A of $\mathbb{R}^{m\times n}$ will be thought of as a matrix with m rows and n columns, and an element b of $\mathbb{R}^m$ as a column vector. When all their entries belong to $\mathbb{Z}$, we will call them *integer*. The row vectors of A will be denoted by $a_i^\top$, the column vectors of A by A^i. When $\operatorname{rank}(A) = m$, we say that A has *full row rank*. A matrix is *totally unimodular*, or *TU*, if the determinants of its square submatrices are equal to -1, 0 or 1.

Lattices The *lattice* generated by a set V of vectors of $\mathbb{Q}^n$ is the set of integer combinations of these vectors, and is denoted by $\operatorname{lattice}(V) = \{\sum_{v\in V} \lambda_v v \colon \lambda_v \in \mathbb{Z} \text{ for all } v \in V\}$. The lattice generated by the column vectors of a matrix A is denoted by $\operatorname{lattice}(A)$.

Polyhedra Given $A \in \mathbb{Q}^{m\times n}$ and $b \in \mathbb{Q}^m$, the set $P = \{x \in \mathbb{R}^n \colon Ax \le b\} = \{x \in \mathbb{R}^n \colon a_i^\top x \le b_i, i = 1, \dots, m\}$ is a *polyhedron*. We will often simply write $P = \{x \colon Ax \le b\}$. The matrix A is the *constraint matrix* of P. The *translation of P by $w \in \mathbb{R}^n$* is $P + w = \{x + w \colon x \in P\}$.

A *face* of P is a nonempty[1] set obtained by imposing equality on some inequalities in the description of P, that is, a nonempty set of the form $F = \{x \colon a_i^\top x = b_i, i \in I\} \cap P$ where $I \subseteq \{1, \dots, m\}$. A row $a_i^\top$ or an inequality $a_i^\top x \le b_i$ with $F \subseteq \{x \colon a_i^\top x = b_i\}$ is *tight for* F, and $A_F x \le b_F$ will denote the inequalities from $Ax \le b$ that are tight for F. The set of points contained in F and in no face $F' \subset F$ forms the *relative interior of* F. Let $\operatorname{lin}(F) = \{x \colon A_F x = \mathbf{0}\}$ and $\operatorname{aff}(F) = \{x \colon A_F x = b_F\}$. The dimension $\dim(F)$ of a face F is the dimension of its affine hull $\operatorname{aff}(F)$. A *facet* is a face that is inclusionwise maximal among all faces distinct from P. A face is *minimal* if it contains

[1] In the standard definition, the emptyset is a face. It is not the case in this paper in order to lighten the statements.

no other face of P. Minimal faces are affine spaces. A minimal face of dimension 0 is called a *vertex*. Note that a polyhedron is integer if and only if each of its minimal faces contains an integer point.

Cones A *polyhedral cone* is a polyhedron of the form $C = \{x \colon Ax \leq \mathbf{0}\}$. Since all the cones involved in this paper are polyhedral, we simply write *cone*. A cone C can also be described as the set of nonnegative combinations of a finite set of vectors $R \subseteq \mathbb{R}^n$, and we say that $C = \operatorname{cone}(R)$ is *generated by* R. A *conic polyhedron* is a rational translation of a cone, that is, a set of the form $t + \{x \colon Ax \leq \mathbf{0}\}$ for some $t \in \mathbb{Q}^n$.

The *polar cone* of a cone $C = \{x \colon Ax \leq \mathbf{0}\}$ is the cone $C^* = \{x \colon z^\top x \leq 0 \text{ for all } z \in C\}$. Equivalently, C^* is the cone generated by the columns of $A^\top$. Note that $C^{**} = C$.

Given a face F of a polyhedron $P = \{x \colon Ax \leq b\}$, the *tangent cone associated to* F is the conic polyhedron $C_F = \{x \colon A_F x \leq b_F\}$. When F is a minimal face of P, its associated tangent cone is a *minimal tangent cone of* P. The cone of $\mathbb{R}^n$ generated by the columns of $A_F^\top$ is the *normal cone associated to* F. Note that the normal cone associated to F is the polar of $\{x \colon A_F x \leq \mathbf{0}\}$.

For more details, we refer the reader to Schrijver's book [38].

3 Generalities on principally box-integer polyhedra

This section is devoted to the basic properties of box-integer and principally box-integer polyhedra. In particular, we study the behavior of these notions with respect to dilation and translation.

3.1 Box-integer polyhedra

Recall that a polyhedron P is box-integer if $P \cap \{\ell \leq x \leq u\}$ is integer for all $\ell, u \in \mathbb{Z}^n$. Frequently, the following characterization will be more convenient to use than the definition.

Lemma 1 *A polyhedron P is box-integer if and only if for each face F of P, $I \subseteq \{1, \ldots, n\}$, and $p \in \mathbb{Z}^I$ such that $\operatorname{aff}(F) \cap \{x_i = p_i, i \in I\}$ is a singleton v, if v belongs to F then v is integer.*

Proof Let $P = \{x \in \mathbb{R}^n \colon Ax \leq b\}$. Suppose that P is not box-integer. Then, $P \cap \{\ell \leq x \leq u\}$ has a noninteger vertex v for some $\ell, u \in \mathbb{Z}^n$. In particular, v belongs to $P \cap \{\ell \leq x \leq u\}$ and is the unique solution of a nonsingular system $a_j x = b_j$, $j \in J$, $x_i = p_i$, $i \in I$ where $p_i \in \{\ell_i, u_i\}$. Now, $F = \{x \colon a_j x = b_j, j \in J\} \cap P$ is a face of P, and $\{v\} = \operatorname{aff}(F) \cap \{x_i = p_i, i \in I\}$ is not integer.

Conversely, suppose that $\{v\} = \operatorname{aff}(F) \cap \{x_i = p_i, i \in I\}$ belongs to F and is not integer, for some $p \in \mathbb{Z}^I$. Define ℓ and u as follows: $\ell_i = u_i = p_i$ for $i \in I$, and $\ell_i = \lfloor v_i \rfloor$ and $u_i = \lceil v_i \rceil$ otherwise. Then, v is a noninteger vertex of $P \cap \{\ell \leq x \leq u\}$ and P is not box-integer. $\square$

Note that, if I is such that the set $\operatorname{aff}(F) \cap \{x_i = p_i, i \in I\}$ is a singleton for some $p \in \mathbb{R}^I$, then this set is either empty or a singleton for all $p \in \mathbb{R}^I$. If I is moreover assumed inclusionwise minimal, then $\mathrm{aff}(F) \cap \{x_i = p_i, i \in I\}$ is a singleton for all $p \in \mathbb{R}^I$.

The following two results seem to be known in the literature, we provide a proof for the sake of completeness.

Corollary 1 *If a polyhedron P is box-integer, then P is integer.*

Proof Let F be a minimal face of P. There exists an inclusionwise minimal set I as above, hence setting $\{x_i = p_i, i \in I\}$ for some $p \in \mathbb{Z}^I$ yields a singleton in $\mathrm{aff}(F)$. Since $\mathrm{aff}(F) = F$, this singleton is integer by Lemma 1, and thus F contains an integer point. $\square$

Corollary 2 *Let P be a polyhedron of $\mathbb{R}^n$. The following statements are equivalent.*

1. *P is box-integer.*
2. *$P \cap \{x \geq \ell\}$ is integer for all $\ell \in \mathbb{Z}^n$.*
3. *$P \cap \{\ell \leq x \leq u\}$ is integer for all $\ell, u \in \mathbb{Z} \cup \{-\infty, +\infty\}^n$.*

Proof Statement 3 immediately implies statement 2. Statement 2 implies statement 1 by Lemma 1, as if $\mathrm{aff}(F) \cap \{x_i = p_i, i \in I\}$ is a singleton $v \in F$, then v is a vertex of $P \cap \{x \geq \lfloor v \rfloor\}$. Statement 1 implies statement 3 because if P is box-integer, then for all $\ell, u \in \mathbb{Z} \cup \{-\infty, +\infty\}^n$, $P \cap \{\ell \leq x \leq u\}$ is box-integer—and hence integer by Corollary 1. $\square$

The following lemma shows two operations which preserve box-integrality. The second one will be used in Sect. 5.

Lemma 2 *Let $P = \{x \in \mathbb{R}^n \colon Ax \leq b\}$ be a polyhedron.*

1. *P is box-integer if and only if $\widetilde{P} = \{(y, z) \in \mathbb{R}^n \times \mathbb{R}^n \colon A(y+z) \leq b\}$ is box-integer.*
2. *P is box-integer if and only if $P_{\pm} = \{(y, z) \in \mathbb{R}^n \times \mathbb{R}^n \colon A(y - z) \leq b,\ y, z \geq \mathbf{0}\}$ is box-integer.*

Proof To establish the "only if" part of statement 1, suppose that $\widetilde{P}$ is box-integer. Then, so is $\widetilde{P} \cap \{z = \mathbf{0}\}$. Since P is obtained from $\widetilde{P} \cap \{z = \mathbf{0}\}$ by deleting z's coordinates, P is box-integer. To establish the "if" part of statement 1, we use Lemma 1. Let F be a face of $\widetilde{P}$, of affine space $\mathrm{aff}(F) = \{(y, z) \in \mathbb{R}^n \times \mathbb{R}^n : a_j(y + z) = b_j, j \in J\}$, and let p and q be integer vectors such that $S = \mathrm{aff}(F) \cap \{y_i = p_i, i \in I_y, z_i = q_i, i \in I_z\}$ is a singleton $(\bar{y}, \bar{z})$ which belongs to F. Let us show that $(\bar{y}, \bar{z})$ is integer. By Lemma 1, this implies that $\widetilde{P}$ is box-integer.

We denote by G the face of P of affine space $\{x \in \mathbb{R}^n : a_j x = b_j, j \in J\}$. Then $\mathrm{aff}(G) \cap \{x_i = p_i + q_i, i \in I_y \cap I_z\}$ is the singleton $\bar{x} = \bar{y} + \bar{z}$. Indeed, if it contained an other point $\bar{x}'$, we could set $\bar{y}'_i = p_i, i \in I_y, \bar{z}'_i = q_i, i \in I_z$ and then build $(\bar{y}', \bar{z}')$ in S such that $\bar{y}' + \bar{z}' = \bar{x}' \neq \bar{y} + \bar{z}$, a contradiction. P is box-integer and $\bar{y} + \bar{z}$ belongs to P, thus $\bar{y} + \bar{z}$ is integer by Lemma 1. Since S is a singleton, no $(\bar{y} + \chi_i, \bar{z} - \chi_i)$ belongs to S, and for all i, we have either $y_i = p_i$ or $z_i = q_i$. Since p, q, and $\bar{y} + \bar{z}$ are integer, $(\bar{y}, \bar{z})$ is integer.

To establish the "only if" part of statement 2, suppose that P is box-integer. Then, so is $P_{\pm}$ by statement 1 and because $P_{\pm}$ is obtained from $\widetilde{P} \cap \{y \geq \mathbf{0}, z \leq \mathbf{0}\}$ by replacing z by $-z$. To establish the “if” part of statement 2, suppose now that $P_\pm$ is box-integer. For $t \in \mathbb{R}^n$, define $t_+ = \max\{\mathbf{0}, t\}$ and $t_- = \max\{\mathbf{0}, -t\}$. For $\ell, u \in \mathbb{Z}^n$, we have $u = u_+ - u_-$, $\ell = \ell_+ - \ell_-$, and $u_+, u_-, \ell_+, \ell_- \geq \mathbf{0}$, hence $P \cap \{\ell \leq x \leq u\}$ is the projection onto $x = y - z$ of $P_\pm \cap \{\ell_+ \leq y \leq u_+, -\ell_- \leq -z \leq -u_-\}$. Since the latter is integer, this implies the integrality of $P \cap \{\ell \leq x \leq u\}$. □

3.2 Dilations of box-integer polyhedra

In this section, we investigate how the box-integrality of a polyhedron behaves with respect to dilation. As a preliminary, the following observation describes the behaviour of integrality with respect to dilation.

Proposition 1 *Let P be a polyhedron. There exists $d \in \mathbb{Z}_{>0}$ such that $\{k \in \mathbb{Z}_{>0}\colon kP$ is integer$\} = d\mathbb{Z}_{>0}$.*

Proof When P has vertices, it is enough to choose d as the smallest positive integer d such dv is integer for every vertex v of P. To treat the general case, we prove that if kP and $k'P$ are integer polyhedra, then $\gcd(k, k')P$ is an integer polyhedron too. Then, the smallest positive integer k such that kP is integer divides all the others, and as any dilation of an integer polyhedron is an integer polyhedron too, this proves the observation.

Let $P = \{x\colon Ax \leq b\}$, $i = \gcd(k, k')$, $\overline{k} = k/i$, $\overline{k'} = k'/i$, and F be a minimal face of iP. Since F is a minimal face, F is the affine space $F = \{x\colon A_F x = ib_F\}$. Note that $\overline{k}F$ and $\overline{k'}F$ are minimal faces, respectively of kP and $k'P$, thus contain an integer point, respectively x_k and $x_{k'}$. By Bézout’s lemma, there exist λ and μ in $\mathbb{Z}$ such that $\lambda k + \mu k' = i$. Then $A_F(\lambda x_k + \mu x_{k'}) = ib_F$, hence F contains an integer point. Therefore, $\gcd(k, k')P$ is an integer polyhedron. □

One of the arguments in the previous proof is the fact that the dilations of an integer polyhedron are also integer polyhedra. This does not hold for box-integrality, intuitively because any $0/1$ polytope is box-integer, though its dilations have no reasons to be. Actually, an example of box-integer polyhedron having non box-integer dilations will be provided at the end of this section. For now we prove the following lemma in order to determine, given a polyhedron P, the structure of the set of positive integers k such that kP is box-integer.

Lemma 3 *Let P be a polyhedron and $k \in \mathbb{Z}_{>0}$ such that kP is integer but not box-integer. Then, no dilation $k'P$ with $k' \geq k$ is box-integer.*

Proof Let $k' \geq k$. Assume $k'P$ integer, as otherwise $k'P$ would not be box-integer. By Lemma 1, there exist a face F of kP and an integer vector p such that $\mathrm{aff}(F) \cap \{x_i = p_i, i \in I\}$ is a noninteger singleton $v \in F$. By Proposition 1, kP and $k'P$ are both dilations of an integer polyhedron dP. In particular, there exists an integer point z in F such that $z' = \frac{k'}{k}z$ is an integer point contained in the face $F' = \frac{k'}{k}F$ of $k'P$. Since $k' \geq k$, we have $F - z \subseteq F' - z'$, thus in particular $v - z$ is in $F' - z'$, which implies that $v' = (z' - z) + v$ is in F'. Moreover, $\mathrm{aff}(F') \cap \{x_i = (z'_i - z_i) + p_i, i \in I\}$ is the singleton v' of F', which is not integer, hence $k'P$ is not box-integer by Lemma 1. □

A polyhedron P is *fully box-integer* if kP is box-integer for all $k \in \mathbb{Z}_{>0}$. In other words, P is fully box-integer if and only if P is principally box-integer and integer.

Proposition 2 *For a polyhedron P, the following statements are equivalent.*

1. *P is principally box-integer.*
2. *There exists $d \in \mathbb{Z}_{>0}$ such that $\{k \in \mathbb{Z}_{>0} : kP \text{ is box-integer}\} = d\mathbb{Z}_{>0}$.*
3. *P has a fully box-integer dilation.*

Proof The definition of principal box-integrality and Proposition 1 give (1)⇒(2). To get (2)⇒(3), just note that dP is a fully box-integer polyhedron. To prove (3)⇒(1), suppose that P is not principally box-integer, that is, there exists a positive integer k such that kP is integer but not box-integer. By Lemma 3, this is not compatible with the existence of a fully box-integer dilation of P. □

We mention that relaxing $k \in \mathbb{Z}_{>0}$ to $k \in \mathbb{Z}$ in Definition 1 yields an equivalent definition. Then, the set arising in statement 2 of Proposition 2 is $d\mathbb{Z}$, which is a principal ideal of $\mathbb{Z}$. This explains why we called these polyhedra principally box-integer. The next proposition shows what can happen when a polyhedron is not principally box-integer.

Proposition 3 *For a polyhedron P, exactly one of the following situations holds.*

1. *P is principally box-integer.*
2. *No dilation of P is a box-integer polyhedron.*
3. *There exist $d, q \in \mathbb{Z}_{>0}$ such that kP is box-integer if and only if $k \in \{d, 2d, \ldots, qd\}$.*

Proof If P has a box-integer dilation but is not principally box-integer, then there is a smallest q in $\mathbb{Z}_{>0}$ such that $(q+1)P$ is a polyhedron which is integer but not box-integer. By Lemma 3, no kP with $k > q$ is box-integer. Now, if d is chosen as in Proposition 1, the minimality of q gives $\{k \in \mathbb{Z}_{>0} : kP \text{ is box-integer}\} = \{d, 2d, \ldots, qd\}$. □

Note that the following property, which holds for integrality, also holds for box-integrality: if kP and $k'P$ are box-integer polyhedra, then so is $gcd(k, k')P$.

Remark 1 Though we only considered dilations with positive integer coefficients, all these results can readily be adapted to dilations with rational coefficients.

We conclude this section with an example of polyhedron whose box-integrality is not preserved by dilation.

As $P = \text{conv}\,(\mathbf{0}, (1, 1, 0, 0, 0), (1, 0, 1, 0, 0), (1, 0, 0, 1, 0), (1, 1, 1, 1, 1))$ is a 0/1 polytope, it is box-integer. However, it can be checked that $(2, 1, 1, 1, 1/2)$ is a fractional vertex of $2P \cap \{x_2 = x_3 = x_4 = 1\}$. In particular, P illustrates statement 3 of Proposition 3.

3.3 Translations of principally box-integer polyhedra

Box-integrality is clearly preserved under integer translation. So are principal and full box-integrality.

Observation 1 *Box-integrality, principal box-integrality and full box-integrality are all preserved by integer translation.*

Proof The translation $Q = t + P$ of a box-integer polyhedron P by t in $\mathbb{Z}^n$ is also box-integer because $Q \cap \{\ell \le x \le u\} = t + (P \cap \{\ell - t \le x \le u - t\})$ for all $\ell, u \in \mathbb{Z}^n$. Moreover, since $kQ = kt + kP$ and $kt \in \mathbb{Z}^n$ for all $k \in \mathbb{Z}_{>0}$, principal box-integrality and full box-integrality are also preserved by integer translation. □

Conic polyhedra play an important role in the next sections. One of the reasons is that, up to translation, every dilation of a conic polyhedron is the conic polyhedron itself. Since box-integrality is preserved by integer translation, this has the following consequences.

Observation 2 *Let $D = t + C$ be a conic polyhedron for some cone C of $\mathbb{R}^n$ and some $t \in \mathbb{Q}^n$.*

1. *For C, the three properties of being box-integer, fully box-integer, or principally box-integer are equivalent.*
2. *D is fully box-integer if and only if it is box-integer.*
3. *D is principally box-integer if and only if C is box-integer.*

Proof The fact that $kC = C$ for all $k \in \mathbb{Z}_{>0}$ proves statement 1. When D is box-integer, its minimal face contains an integer point, hence t can be chosen to be an integer. Since $kD = (k-1)t + D$ for all $k \in \mathbb{Z}_{>0}$, and since integer translation preserves box-integrality, statement 2 follows. When $t \in \mathbb{Q}^n$, take k large enough such that kt is integer. Now, $kD = kt + C$ is a fully box-integer dilation of D if and only if C is box-integer, which proves statement 3. □

4 Principally box-integer polyhedra and equimodular matrices

In this section, we show how equimodularity and principal box-integrality are intertwined. First, we characterize equimodular matrices using principal box-integrality. Then, principally box-integer polyhedra are characterized by the equimodularity of a family of matrices.

4.1 Characterizations of equimodular matrices

In this section, we extend to equimodular matrices two classical results about unimodular matrices. We first state the results of Heller [28] about unimodular sets in terms of equimodular matrices—see also [38, Theorem 19.5].

Theorem 3 (Heller [28]) *For a full row rank $r \times n$ matrix A, the following statements are equivalent.*

1. *A is equimodular.*
2. *For each nonsingular $r \times r$ submatrix D of A,* $\mathrm{lattice}(D) = \mathrm{lattice}(A)$.
3. *For each nonsingular $r \times r$ submatrix D of A, $D^{-1}A$ is integer.*

4. *For each nonsingular $r \times r$ submatrix D of A, $D^{-1}A$ is in $\{0, \pm 1\}^{r\times n}$.*
5. *For each nonsingular $r \times r$ submatrix D of A, $D^{-1}A$ is totally unimodular.*
6. *There exists a nonsingular $r \times r$ submatrix D of A such that $D^{-1}A$ is totally unimodular.*

Veinott and Dantzig [45] proved that an integer $r \times n$ matrix A of full row rank is unimodular if and only if the polyhedron $\{x : Ax = b, x \geq \mathbf{0}\}$ is integer for all $b \in \mathbb{Z}^r$. Observe that statement 2 of Corollary 2 allows us to reformulate their result as follows, since $\{x : Ax = kb\} \cap \{x \geq \ell\} = \ell + \{x : Ax = b', x \geq \mathbf{0}\}$, where $b' = kb + kA\ell \in \mathbb{Z}^r$.

Theorem 4 (Veinott and Dantzig [45]) *Let A be a full row rank matrix of $\mathbb{Z}^{r\times n}$. Then, A is unimodular if and only if $\{x : Ax = b\}$ is fully box-integer for all $b \in \mathbb{Z}^r$.*

It turns out that this result can be extended to characterize equimodular matrices.

Theorem 5 *Let A be a full row rank matrix of $\mathbb{Q}^{r\times n}$. Then, A is equimodular if and only if $\{x : Ax = b\}$ is principally box-integer for all $b \in \mathbb{Q}^r$.*

Proof Suppose that A is equimodular and let $b \in \mathbb{Q}^r$, $k \in \mathbb{Z}_{>0}$ be such that $H = \{x : Ax = kb\}$ is integer. Then $b' = kb$ belongs to lattice(A). Let D be a nonsingular $r \times r$ submatrix D of A. By statement 2 of Theorem 3, we have lattice(D) = lattice(A), hence $D^{-1}b'$ is in $\mathbb{Z}^r$. Since A has full row rank, by statement 5 of Theorem 3, $D^{-1}A$ is unimodular. By Theorem 4, we get that $\{x : D^{-1}Ax = D^{-1}b'\}$ is fully box-integer. In particular, H is box-integer.

Conversely, suppose that A is not equimodular. Then, possibly reordering the columns, we may assume that the first r columns of A are linearly independent, and, by statement 3 of Theorem 3, that the $(r+1)$th column A^{r+1} of A is a noninteger combination of those. Let $H = \{x : Ax = A^{r+1}\}$. Then, $\{x : Ax = A^{r+1}\} \cap \{x_j = 0, j \geq r+1\}$ has no integer solution, hence H is not box-integer. However, H is integer as it contains χ^{r+1} as an integer point. Therefore, H is not principally box-integer. □

Veinott and Dantzig [45] devised Theorem 4 in order to get a simpler proof of a characterization of totally unimodular matrices due to Hoffman and Kruskal [31]. This characterization states that an integer matrix A is totally unimodular if and only if $\{x : Ax \leq b\}$ is box-integer for all $b \in \mathbb{Z}^m$. In our context, this can be reformulated as follows.

Theorem 6 (Hoffman and Kruskal [31]) *A matrix A of $\mathbb{Z}^{m\times n}$ is totally unimodular if and only if $\{x : Ax \leq b\}$ is fully box-integer for all $b \in \mathbb{Z}^m$.*

An equivalent definition of total unimodularity is to ask for every set of linearly independent rows to be unimodular. In this light, it is natural to define *totally equimodular matrices* as those for which all sets of linearly independent rows form an equimodular matrix. Theorem 6 then extends to totally equimodular matrices as follows.

Theorem 7 *A matrix A of $\mathbb{Q}^{m\times n}$ is totally equimodular if and only if $\{x : Ax \leq b\}$ is principally box-integer for all $b \in \mathbb{Q}^m$.*

Proof Suppose A totally equimodular and $b \in \mathbb{Q}^m$, and let us prove that $P = \{x: Ax \leq b\}$ is principally box-integer. Let $k \in \mathbb{Z}_{>0}$ be such that kP is an integer polyhedron, and let us prove that kP is box-integer. Let F be a face of kP and p be an integer vector such that $\text{aff}(F) \cap \{x_i = p_i, i \in I\}$ is a singleton $\bar{x}$ in F. By Lemma 1, it remains to show that $\bar{x}$ is integer. There exists a full row rank subset L of rows of A such that $\text{aff}(F) = \{x: A_L x = kb_L\}$. Since A is totally equimodular, A_L is equimodular. By Theorem 5, $\text{aff}(F)$ is principally box-integer. Now, since kP is integer, so is $\text{aff}(F)$. Hence, $\text{aff}(F)$ is box-integer and $\bar{x}$ is integer.

Suppose now that A is not equimodular, that is, there exists a full row rank submatrix A_L of size $r \times n$ of A which is not equimodular. We may assume that the first r columns of A_L are linearly independent, and that the $(r+1)$th column of A_L is a noninteger combination of those. Let $\bar{x}$ be the unique solution of $A_L x = 0, x_{r+1} = -1, x_j = 0, j > r+1$. Then, $\bar{x} \notin \mathbb{Z}^n$. Define $b_L = \mathbf{0}$ and $b_j = 1$ if $j \notin L$, and let us show that $P = \{x : Ax \leq b\}$ is not principally box-integer. There exists $k \in \mathbb{Z}_{>0}$ large enough such that $\bar{x} \in kP$, and such that kP is integer. Then, $kP \cap \{x_{r+1} = -1, x_j = 0, j > r+1\}$ contains $\bar{x}$ as a vertex because $\bar{x}$ satisfies to equality n linearly independent inequalities. Therefore, kP is not box-integer. ◻

Since deciding whether a given matrix is totally unimodular can be done in polynomial time—see e.g. [38, Chapter 20]—statement 5 of Theorem 3 implies that deciding whether a given matrix is equimodular can be done in polynomial time. However, for totally equimodular matrices, the recognition problem remains open.

Open Problem 1 *Can totally equimodular matrices be recognized in polynomial time?*

As we shall see later, totally equimodular matrices are precisely the matrices whose associated polyhedra are all box-TDI—see Corollary 8. Interestingly, it is enough to study totally equimodular matrices with 0, ± 1 coefficients. Indeed, in a totally equimodular matrix, the nonzero coefficients of a given row all have the same absolute value. Thus, such a matrix can be scaled row by row into a 0, ± 1 matrix. This scaling preserves total equimodularity and does not change the family of associated polyhedra.

Remark 2 The full row rank hypothesis made throughout this section is convenient, but not really necessary, provided the notions of unimodularity and equimodularity are correctly extended. Hoffman and Kruskal [31] extend the notion of unimodularity to not necessarily full row rank matrices, and Theorem 4 still holds for those matrices [38, Page 301]. The correct extension of equimodularity to general matrices is to require, for a matrix A of rank r, that each set of r linearly independent rows of A forms an equimodular matrix. Properties of such matrices are studied in [28]. None of the definitions and results of this paper are affected if these extended definitions are adopted and the full row rank hypothesis removed.

4.2 Affine spaces and face-defining matrices

Affine spaces being special cases of conic polyhedra, by statement 3 Observation 2, $\{x: Ax = b\}$ is principally box-integer for all b if and only if $\{x: Ax = \mathbf{0}\}$ is fully box-integer. In particular, one can drop the quantification over all $b \in \mathbb{Q}^n$ from Theorem 5 as follows.

Corollary 3 *Let A be a full row rank matrix of $\mathbb{Q}^{r\times n}$ and $b \in \mathbb{Q}^n$. Then, A is equimodular if and only if the affine space $\{x\colon Ax = b\}$ is principally box-integer.*

An affine space $\{x\colon Ax = b\}$ being integer if and only if b belongs to lattice(A), the previous result has the following immediate consequence.

Corollary 4 *Let A be a full row rank matrix of $\mathbb{Q}^{r\times n}$ and $b \in \mathbb{Q}^n$. The affine space $\{x\colon Ax = b\}$ is fully box-integer if and only if A is equimodular and $b \in$ lattice(A).*

Corollary 3 yields a correspondence between equimodular matrices and principally box-integer affine spaces. We shall see in the next section that this correspondence, when applied to the faces of a polyhedron, provides a characterization of principally box-integer polyhedra. This motivates the following definition.

Face-defining matrices Let $P = \{x\colon Ax \leq b\}$ be a polyhedron of $\mathbb{R}^n$ and F be a face of P. A full row rank matrix M such that aff(F) can be written $\{x\colon Mx = d\}$ for some d is *face-defining for F*. Such matrices are called *face-defining matrices of P*.[2] Note that face-defining matrices need not correspond to valid inequalities for the polyhedron. A face-defining matrix for a facet of P is called *facet-defining*.

Affine spaces are polyhedra whose only face is themselves. The following observation characterizes their principal box-integrality in terms of face-defining matrices.

Observation 3 *For an affine space H, the following statements are equivalent.*

1. *H is principally box-integer.*
2. *H has an equimodular face-defining matrix.*
3. *Every face-defining matrix of H is equimodular.*
4. *H has a totally unimodular face-defining matrix.*

Proof The equivalence among statements 1, 2, and 3 follows from Corollary 3. The equivalence between statements 2 and 4 follows from statement 5 of Theorem 3, because if $A \in \mathbb{Q}^{r\times n}$ is face-defining for H, then so is $D^{-1}A$ for each nonsingular $r \times r$ submatrix D of A. □

Note that, when P is full-dimensional, facet-defining matrices are composed of a single row and are uniquely determined, up to multiplying by a scalar. In general, the number of rows of a face-defining matrix for a face F is $n - \dim(F)$. More precisely, the following immediate observation characterizes face-defining matrices.

Observation 4 *A full row rank matrix $M \in \mathbb{Q}^{k\times n}$ is face-defining for a face F of a polyhedron $P \subseteq \mathbb{R}^n$ if and only if there exist a vector $d \in \mathbb{Q}^k$ and a family $\mathcal{H} \subseteq F \cap \{x\colon Mx = d\}$ of $\dim(F)+1$ affinely independent points such that $|\mathcal{H}| + k = n+1$.*

4.3 Characterizations of principally box-integer polyhedra

In this section, we provide several characterizations of principally box-integer polyhedra, the starting point being the following lemma.

[2] When we write that a face F has a face-defining matrix M, we mean that M is face-defining for the face F, which is more restrictive than being a face-defining matrix of the polyhedron F.

Lemma 4 *A polyhedron P is principally box-integer if and only if* $\operatorname{aff}(F)$ *is principally box-integer for each face F of P.*

Proof Let P be a polyhedron such that the affine spaces generated by its faces are all principally box-integer. Then, when $k \in \mathbb{Z}_{>0}$ is such that kP is integer, all the affine spaces generated by the faces of kP are box-integer. Therefore, by Lemma 1, if F is a face of such a kP and p is an integer vector such that $\operatorname{aff}(F) \cap \{x_i = p_i, i \in I\}$ is a singleton in F, then this singleton is integer. Then, by the other direction of Lemma 1, kP is box-integer, thus P is principally box-integer.

Conversely, let P be a principally box-integer polyhedron and F be a face of P. If F is a singleton, then $\operatorname{aff}(F) = F$ is a singleton, thus obviously principally box-integer. Otherwise, let t be a rational point in the relative interior of F, let $G = F - t$ and $Q = P - t$. By statement 3 of Observation 2, it suffices to show that $\operatorname{aff}(G)$ is box-integer. Let p be an integer vector such that $\operatorname{aff}(G) \cap \{x_i = p_i, i \in I\}$ is a singleton $\bar{x}$ in $\operatorname{aff}(G)$. Since t was chosen in the relative interior of F, there exists $k \in \mathbb{Z}_{>0}$ such that $\bar{x} \in kQ$. Moreover, such a k can be chosen so that kt is integer and kP is an integer polyhedron. Since P is principally box-integer, kP is box-integer and so is $kQ = kP - kt$ by Observation 1. Applying Lemma 1 to the face kG of kQ yields $\bar{x}$ integer. By applying the other direction of Lemma 1 to the unique face $\operatorname{aff}(G)$ of $\operatorname{aff}(G)$, we obtain that $\operatorname{aff}(G)$ is box-integer. □

Theorem 8 *For a polyhedron P, the following statements are equivalent.*

1. *The polyhedron P is principally box-integer.*
2. *Every minimal tangent cone of P is principally box-integer.*
3. *Every face of P has an equimodular face-defining matrix.*

Proof Each face of P is contained in a face of some minimal tangent cone of P having the same affine hull. Conversely, each face of a minimal tangent cone of P contains some face of P having the same affine hull. Therefore, Lemma 4 gives the equivalence between statement 1 and statement 2. The equivalence between statement 1 and statement 3 is immediate by Corollary 3 and Lemma 4. □

The minimal faces of a polyhedron being affine spaces, Lemma 4 has a fully box-integer counterpart. Moreover, by statement 2 of Observation 2, so does the equivalence between statement 1 and statement 3 of Theorem 8. This gives the following corollary.

Corollary 5 *For a polyhedron P, the following statements are equivalent.*

1. *The polyhedron P is fully box-integer.*
2. *Every minimal tangent cone of P is box-integer.*
3. *For each face F of P,* $\operatorname{aff}(F)$ *is fully box-integer.*

5 Box-totally dual integral polyhedra

5.1 New characterizations of box-TDI polyhedra

The main result of this section is that the notions of principal box-integrality and box-TDIness coincide—see Theorem 9 below. Combined with Theorem 8, this provides several new characterizations of box-TDI polyhedra.

Theorem 9 *A polyhedron is box-TDI if and only if it is principally box-integer.*

Proof The proof relies on Lemmas 5 and 6, which are proven below.

Lemma 5 states that a polyhedron is box-TDI if and only if all its minimal tangent cones ares box-TDI. By Theorem 8, a polyhedron is principally box-integer if and only if all its minimal tangent cones are principally box-integer. Hence it is enough to prove Theorem 9 for conic polyhedra.

Lemma 6 states that a cone is box-TDI if and only if it is box-integer. Then, by statement 3 of Observation 2, and since box-TDIness is preserved under rational translation, a conic polyhedron is box-TDI if and only if it is principally box-integer. □

The following lemma seems somewhat implicitly known in the literature, but is not stated explicitly to the best of our knowledge. For the sake of completeness, we provide a proof which relies only on the definitions. It can also be shown using known characterizations of box-TDI polyhedra, such as the one by Cook [38, Theorem 22.9].

Lemma 5 *A polyhedron is box-TDI if and only if all its minimal tangent cones are.*

Proof Let $P = \{x \colon Ax \leq b\}$ be a polyhedron of $\mathbb{R}^n$ and $w \in \mathbb{Z}^n$. We will denote $(P_{\ell,u}) = \max\{wx \colon Ax \leq b, \ell \leq x \leq u\}$ and $(P^F_{\ell,u}) = \max\{wx \colon A_I x \leq b_I, \ell \leq x \leq u\}$ for a minimal face F of P where I is the index set of the tight rows for F.

To establish the "only if" part of the statement, suppose that the system $Ax \leq b$ is box-TDI. Let F be a minimal face of P, $v \in F$ and let $x^\star$ be an optimal solution of $(P^F_{\ell,u})$. Since $a_i v < b_i$ for all $i \notin I$, there exists $\lambda > 0$ such that $y^\star = v + \lambda(x^\star - v)$ belongs to P and $a_i y^\star < b_i$ for all $i \notin I$. Let $\ell' = v + \lambda(\ell - v)$ and $u' = v + \lambda(u - v)$. Then, $y^\star$ is an optimal solution of $(P_{\ell',u'})$, as otherwise $x^\star$ would not be an optimal solution of $(P^F_{\ell,u})$. Let $(z^\star, r^\star, s^\star)$ be an integer optimal solution of the dual of $(P_{\ell',u'})$. By complementary slackness, denoting by $z^\star_I$ the vector obtained from $z^\star$ by deleting the coordinates not in I, without loss of generality we have $z^\star = (z^\star_I, \mathbf{0})$. Now, since $w^\top y^\star = b^\top z^\star + u'^\top r^\star - \ell'^\top s^\star$, one can check that $w^\top x^\star = b_I^\top z^\star_I + u^\top r^\star - \ell^\top s^\star$, by applying the definition of $y^\star$, u' and ℓ', $b^\top z^\star = b_I^\top z^\star_I$, $w = A^\top z^\star + r^\star - s^\star$, $A^\top z^\star = A_I^\top z^\star_I$, and $A_I v = b_I$. Therefore, $(z^\star_I, r^\star, s^\star)$ is an integer optimal solution of the dual $\min\{b_I^\top z + u^\top r - \ell^\top s \colon A_I^\top z_I + r - s = w,\ z_I, r, s \geq \mathbf{0}\}$ of $(P^F_{\ell,u})$.

To establish the "if" part of the statement, let H be the face of P composed of all the optimal solutions of $(P_{\ell,u}) = \max\{wx \colon Ax \leq b, \ell \leq x \leq u\}$ and let F be a minimal face of P contained in H whose tight rows are indexed by I. Let $(z^\star_I, r^\star, s^\star)$ be an integer optimal solution of the dual of $(P^F_{\ell,u})$. Then, one can check that extending $z^\star_I$ to a vector $z^\star = (z^\star_I, \mathbf{0})$ of $\mathbb{R}^m$ yields an integer optimal solution $(z^\star, r^\star, s^\star)$ of the dual of $(P_{\ell,u})$. □

The following result reveals that cones behave nicely with respect to box-TDIness. It is already known that a box-TDI cone is box-integer [39, Equation (5.82)]. Surprisingly, the converse holds and these properties are carried over to the polar.

Lemma 6 *For a cone C, the following statements are equivalent.*

1. *C is box-TDI,*

2. *C is box-integer,*
3. *C^* is box-TDI,*
4. *C^* is box-integer.*

Proof Let $C = \{x \colon Ax \leq \mathbf{0}\}$ be a cone of $\mathbb{R}^n$. By [38, Theorem 22.6(i)], the system $Ax \leq \mathbf{0}$ can be chosen to be TDI.

Suppose that C is box-TDI. By Theorem 1, the system $Ax \leq \mathbf{0}$ is box-TDI. Hence, for all $\ell, u \in \mathbb{Z}^n$, the system $Ax \leq \mathbf{0}, \ell \leq x \leq u$ is TDI. As ℓ and u are integer, this system defines an integer polyhedron by [38, Corollary 22.1c]. Therefore, C is box-integer, and we get (1)⇒(2). This also gives (3)⇒(4).

All that remains to prove is (4)⇒(1). Indeed, applying this implication to the cone C^* and using that $C^{**} = C$ yields (2)⇒(3).

Suppose that C^* is box-integer and let us prove that the dual (D) of the linear program (P) below has an integer solution for all $w \in \mathbb{Z}^n$ and $\ell, u \in \mathbb{Q}^n$ such that the optimum is finite.

$$
(P)\quad \begin{array}{lrcl} \max & w^\top x & & \\ & Ax & \leq & \mathbf{0} \\ & x & \leq & u \\ & -x & \leq & -\ell \end{array}
\qquad\qquad
(D)\quad \begin{array}{lccccccccc} \min & & & u^\top r & - & \ell^\top s & & \\ & A^\top z & + & r & - & s & = & w \\ & z & , & r & , & s & \geq & \mathbf{0} \end{array}
$$

The projection of the set of points (z, r, s) satisfying the constraints of (D) onto the variables r and s is the polyhedron $Q = \{r, s \geq \mathbf{0} \colon v^\top(s - r + w) \leq 0, \text{ for all } v \in K\}$, where K is the projection cone $K = \{v \in \mathbb{R}^n \colon v^\top A^\top \leq \mathbf{0}\}$. That is $K = C$ and therefore $Q = (C^* - w)_\pm$—see Lemma 2.

Since integer translations of box-integer polyhedra are box-integer, $C^* - w$ is box-integer. Thus, by statement 2 of Lemma 2, Q is box-integer. In particular, Q is integer.

Since the optimum of (D) is finite, so is $\min\{u^\top r - \ell^\top s \colon (r, s) \in Q\}$. Since Q is an integer polyhedron, this minimum is achieved by an integer $(\bar{r}, \bar{s}) \in Q$. Let $\bar{w} = w - \bar{r} + \bar{s}$. As $(\bar{r}, \bar{s})$ belongs to Q, there exists a feasible solution $\bar{z}$ of the dual of $\max\{\bar{w}^\top x \colon Ax \leq \mathbf{0}\}$. Recall that $Ax \leq \mathbf{0}$ has been chosen to be TDI. Hence, since $\bar{w}$ is integer, such a $\bar{z}$ can be chosen to be an integer. Then, $(\bar{z}, \bar{r}, \bar{s})$ is an integer optimal solution of (D). □

We are now ready to prove our main result, Theorem 2.

Proof of Theorem 2 Statements 2 and 1 are equivalent by Theorem 9. Statements 2 and 4 are equivalent by the equivalence between statements 1 and 3 of Theorem 8. Finally, the equivalence among statements 3, 4, and 5 comes from Observation 3. □

We now apply polarity to derive additional characterizations of box-TDI polyhedra.

Corollary 6 *For a polyhedron P, the following statements are equivalent.*

1. *The polyhedron P is box-TDI.*
2. *For every face F of P, every basis of* $\mathrm{lin}(F)$ *is the transpose of an equimodular matrix.*

3. *For every face F of P, some basis of* $\text{lin}(F)$ *is the transpose of an equimodular matrix.*
4. *For every face F of P, some basis of* $\text{lin}(F)$ *is a totally unimodular matrix.*

Proof Let F be a face of P. By Corollary 3, F has an equimodular face-defining matrix if and only if $\text{aff}(F)$ is principally box-integer. Equivalently, by Observation 2, $\text{lin}(F)$ is box-integer. By Lemma 6, $\text{lin}(F)$ is box-integer if and only $\text{lin}(F)^*$ is. By Corollary 3, $\text{lin}(F)^*$ is box-integer if and only if $\text{lin}(F)^*$ has an equimodular face-defining matrix M. Note that the columns of $M^\top$ form a basis of $\text{lin}(F)$, therefore F has an equimodular face-defining matrix if and only if some basis of $\text{lin}(F)$ is the transpose of an equimodular matrix.

Since, by Theorem 2, the polyhedron P is box-TDI if and only if each of its faces F has an equimodular face-defining matrix, this proves the equivalence between statements 1 and 3. The equivalence with the two others statements follows from Observation 3. □

Recall that a cone $C = \{x : Ax \leq \mathbf{0}\}$ can also be defined as $C = \text{cone}(R)$ for some set R of generators. Moreover, by Lemma 6, a cone is box-TDI if and only if it is box-integer. Corollary 6 then allows us to check whether cones are box-integer by looking at their generators.

Corollary 7 *A cone $C =$ $\text{cone}(R)$ is box-integer if and only if $S^\top$ is equimodular for each linearly independent subset S of R generating a face of C.*

Consequently, the recognition of box-integer cones might have a different complexity status than the following related problems, which are all co-NP-complete: deciding whether a given polytope is integer [37], deciding whether a given system is TDI or box-TDI [16], deciding whether a given conic system is TDI [36].

Open Problem 2 *What is the complexity of deciding whether a given cone is box-integer?*

We mention that polarity preserves box-integrality only for cones, and does not extend to polyhedra. For instance, the polyhedron $\text{conv}((2, -1), (-2, -1), (0, 1))$ is fully box-integer, and its polar $\text{conv}((1, 1), (-1, 1), (0, -1))$ is integer but not box-integer.

5.2 Connections with existing results

In this section, we investigate the connections of our results with those from the literature about box-TDI polyhedra. We first derive known results about box-TDI polyhedra from our characterizations. Then, we show how Cook's characterization [38, Theorem 22.9] is connected to ours. Finally, we discuss Schrijver's sufficient condition [39, Theorem 5.35].

5.2.1 Consequences

Here, we review several known results about box-TDI polyhedra which can be derived from our results. The *dominant* of a polyhedron P of $\mathbb{R}^n$ is $\text{dom}(P) = P + \mathbb{R}^n_+$.

Consequence 1 ([14, Theorem 3.6] or [38, Theorem 22.11]) *The dominant of a box-TDI polyhedron is box-TDI.*

Proof A face of $\operatorname{dom}(P)$ is the sum of a face of P and a cone generated by unit vectors. By statement 4 of Corollary 6, and since adding unit vectors preserves total unimodularity, the dominant of a box-TDI polyhedron is box-TDI. □

Consequence 2 ([38, Remark 2.21]) *If P is a box-TDI polyhedron, then* $\operatorname{aff}(P) = \{x : Cx = d\}$ *for some totally unimodular matrix C.*

Proof If P is a box-TDI polyhedron, then by statement 5 of Theorem 2, since P is a face of P, its affine hull can be described using a totally unimodular matrix. □

Consequence 3 ([38, Remark 2.22]) *Each edge and each extremal ray of a pointed box-TDI polyhedron is in the direction of a* $\{0, \pm 1\}$*-vector.*

Proof This is statement 4 of Corollary 6 applied to the faces of dimension one of the polyhedron. □

By polarity, the above proof shows that every full-dimensional box-TDI polyhedron can be described using a $\{0, \pm 1\}$-matrix. Edmonds and Giles prove in [22] that it is still true without the full-dimensional hypothesis.

Consequence 4 ([22, Theorem 2.16]) *If P is a box-TDI polyhedron, then* $P = \{x : Ax \le b\}$ *for some* $\{0, \pm 1\}$*-matrix A and some vector b.*

Proof Let P be a box-TDI polyhedron. By Consequence 2, we have $\operatorname{aff}(P) = \{x : Cx = d\}$ for some full row rank totally unimodular matrix C. By statement 5 of Theorem 2, for each facet F of P, there exists a totally unimodular matrix D_F such that $\operatorname{aff}(F) = \{x : D_F x = d_F\}$. Then, one of the rows $a_F x = b_F$ of $D_F x = d_F$ does not contain $\operatorname{aff}(P)$. Possibly multiplying by -1, we may assume that $a_F x \le b_F$ is valid for P because F is a facet of P. Then, the matrix A whose rows are those of C and every a_F yields a description of P as desired. □

5.2.2 Cook's characterization [14], [38, Theorem 22.9]

In order to get a geometric characterization of box-TDI polyhedra, Cook [14] introduced the so-called box property. Schrijver [38, Theorem 22.9] states Cook's characterization with the following equivalent form of the box property: a cone C of $\mathbb{R}^n$ has the *box property* if for all $c \in C$ there exists $\widetilde{c} \in C \cap \mathbb{Z}^n$ such that $\lfloor c \rfloor \le \widetilde{c} \le \lceil c \rceil$. To highlight the connections with our results, we reformulate Schrijver's version as follows.

- A polyhedron is box-TDI if and only if the normal cones of its faces all have the box property (Cook [38, Theorem 22.9]).

The parallel with our work is clear with the following reformulation of one of our characterizations.

- A polyhedron P is box-TDI if and only if every minimal tangent cone of P is box-integer, up to translation (Observation 2 and Theorems 8 and 9).

The first difference between these two results is that the first one involves the normal cones, whereas the second one involves the tangent cones. Recall that the tangent cones are the polars of the normal cones, up to translation. This polarity connection between the two statements is not surprising in light of the polarity result of Lemma 6. The second difference is that the first result involves the box property, whereas the second involves the notion of box-integrality. It is easy to see that box-integer cones have the box property. The converse does not hold. In fact, the lemma below shows that the box property is a local property when the box-integrality is a global one. The third difference is a consequence of this local/global aspect: the first result involves *all* the normal cones, whereas the second involves only the *minimal* tangent cones.

To sum up, the first result is a polar local characterization of box-TDI polyhedra, and the second is a primal global characterization.

Proposition 4 *A cone is box-integer if and only if all its faces have the box property.*

The following lemma proves the proposition, since a cone C is box-integer if and only if $\operatorname{aff}(F)$ is box-integer for all faces F of C.

Lemma 7 *Let F be a face of a cone C.*

- *If C is box-integer, then F has the box property.*
- *If F has the box property, then* $\operatorname{aff}(F)$ *is box-integer.*

Proof Suppose that C is box-integer and let $c \in F$. Since c belongs to $P = F \cap \{\lfloor c \rfloor \le x \le \lceil c \rceil\}$, the latter is nonempty. Since C is box-integer, so is F, hence P has only integer vertices, and any of them forms a suitable $\widetilde{c}$ which shows that F has the box property.

Suppose now that F has the box property. Let $p \in \mathbb{Z}^I$ be such that $\operatorname{aff}(F) \cap \{x_i = p_i, i \in I\}$ is a singleton c in $\operatorname{aff}(F)$. There exists $t \in \mathbb{Z}^n$ such that $c' = c + t \in F$. By the box property of F, there exists $\widetilde{c} \in F \cap \mathbb{Z}^n$ such that $t + \lfloor c \rfloor = \lfloor c' \rfloor \le \widetilde{c} \le \lceil c' \rceil = \lceil c \rceil + t$. Now, $\widetilde{c} - t$ belongs to $\operatorname{aff}(F) \cap \{x_i = p_i, i \in I\}$, hence $c = \widetilde{c} - t$ is integer. By Lemma 1, $\operatorname{aff}(F)$ is box-integer. □

In a way, the above lemma shows that the box property of a cone is sandwiched between the box-integrality of the cone and that of its underlying affine space—an even more local property. This, up to polarity again, further compares Cook's characterization and ours, as the latter property appears in Lemma 4. Figure 1 illustrates some differences between the three properties.

The notion of box-integrality of cones and affine spaces sheds a better light on box-TDI polyhedra by providing insights of how their local, global, and polar properties are connected. Both are preserved by polarity, the global notion yields a global geometric characterization of box-TDI polyhedra, and the most local one allows us to derive matricial counterparts.

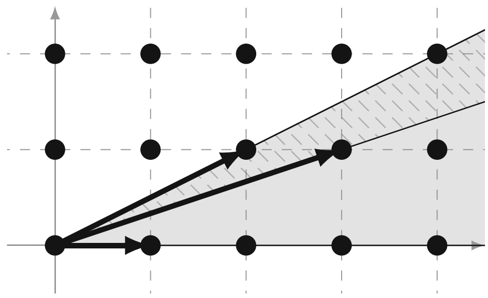


Fig. 1 The cone $C = \text{cone}\{(2,1),(1,0)\}$ has the box property but is not box-integer. The cone $C' = \text{cone}\{(2,1),(3,1)\}$ does not have the box property, yet $\text{aff}(C') = \mathbb{R}^2$ is box-integer. The cone $C'' = \text{cone}\{(2,1)\}$ does not have the box property, yet its polar does

5.2.3 Schrijver's sufficient condition [39, Theorem 5.35]

In this section, we compare our results on box-TDI polyhedra with known results on box-TDI systems. It appears that our results in some sense allow us to split the "box-" from the "-TDI": to prove that a given system is box-TDI, prove that it is TDI on the one hand, and prove that the polyhedron is box-TDI with Theorem 2 on the other hand.

As noticed by Schrijver [38, Page 318], Hoffman and Kruskal's result [31] implies that a matrix A is totally unimodular if and only if the system $Ax \leq b$ is box-TDI for each vector b. Then, by Theorems 7 and 9, the parallel with totally equimodular matrices can be thought of as relaxing the box-TDIness of those systems to that of the associated polyhedra.

Corollary 8 *A matrix A of $\mathbb{Q}^{m\times n}$ is totally equimodular if and only if the polyhedron $\{x: Ax \leq b\}$ is box-TDI for all $b \in \mathbb{Q}^m$.*

Totally unimodular matrices being totally equimodular, the following well-known result is a special case of the above corollary.

Consequence 5 *A polyhedron whose constraint matrix is totally unimodular is box-TDI.*

We mention that there exist box-TDI systems which are not defined by a totally unimodular matrix. By Corollary 8 and Theorem 1, any TDI system defined with a totally equimodular matrix is box-TDI. Therefore, to find a box-TDI system for a polyhedron described by a totally equimodular matrix, there only remains to find a TDI system describing this polyhedron.

Another interesting parallel can be observed with Schrijver's Sufficient Condition. Schrijver proves in [39, Theorem 5.35] that the following weakening of A being totally unimodular already suffices to obtain the box-TDIness of the system $Ax \leq b$.

Theorem 10 ([39, Theorem 5.35]) *Let $Ax \leq b$ be a system of linear inequalities, with A an $m \times n$ matrix. Suppose that ($\star$) for each $c \in \mathbb{R}^n$, $\max\{c^\top x: Ax \leq b\}$ has (if finite) an optimum dual solution $y \in \mathbb{R}^m_+$ such that the rows of A corresponding to positive components of y form a totally unimodular submatrix of A. Then $Ax \leq b$ is box-TDI.*

Note that the property ($\star$) is equivalent to the condition that for every face F of $\{x: Ax \leq b\}$, the system $Ax \leq b$ contains a totally unimodular face-defining matrix for F. Theorem 2 contains a polyhedral version: a polyhedron is box-TDI if and only if each of its faces has a totally unimodular face-defining matrix. This latter condition is weaker than ($\star$), hence does not ensure the box-TDIness of the system. Nevertheless, when satisfied, all that remains to do is to find a TDI system describing the same polyhedron.

In light of our characterizations, one could wonder whether Theorem 10 can be turned into an equivalence, that is: can every box-TDI polyhedron be described by a box-TDI system satisfying ($\star$)? Unfortunately, the answer to this question is negative. Indeed, systems satisfying ($\star$) can be assumed $\{0, \pm 1\}$, and there exist box-TDI polyhedra for which no TDI description is $\{0, \pm 1\}$—see [38, Page 325].

6 Illustrations

In this section, we provide illustrations of our results. The first one is a new perspective on the equivalence between two results about binary clutters. Secondly, we refute a conjecture of Ding et al. [19] about box-perfect graphs. Thirdly, we discuss connections with an abstract class of polyhedra introduced in [27]. Finally, we characterize the box-TDIness of the cone of conservative functions of a graph.

6.1 Box-Mengerian clutters

We briefly introduce the definitions we need about clutters. A collection $\mathcal{C}$ of subsets of a set E is a *clutter* if none of its sets contains another one. We denote by $A_{\mathcal{C}}$ the $\mathcal{C} \times E$ incidence matrix of $\mathcal{C}$ and by $P_{\mathcal{C}} = \{x \in \mathbb{R}^E: A_{\mathcal{C}}x \geq \mathbf{1}, x \geq \mathbf{0}\}$ the associated covering polyhedron. A clutter $\mathcal{C}$ is *binary* if the symmetric difference of any three elements of $\mathcal{C}$ contains an element of $\mathcal{C}$. A clutter $\mathcal{C}$ is *box-$\frac{1}{d}$-integral* if for all $\ell, u \in \frac{1}{d}\mathbb{Z}^E$, each vertex of $P_{\mathcal{C}} \cap \{\ell \leq x \leq u\}$ belongs to $\frac{1}{d}\mathbb{Z}^E$. A matrix $A \in \{0, 1\}^{m \times n}$ is called *(box-)Mengerian* if the system $Ax \geq \mathbf{1}, x \geq \mathbf{0}$ is (box-)TDI. A clutter $\mathcal{C}$ is *(box-)Mengerian* if $A_{\mathcal{C}}$ is (box-)Mengerian. *Deleting* an element $e \in E$ means replacing $\mathcal{C}$ by $\mathcal{C} \backslash e = \{X \in \mathcal{C}: e \notin X\}$ and *contracting* an element $e \in E$ means replacing $\mathcal{C}$ by $\mathcal{C}/e$ which is composed of the inclusionwise minimal members of $\{X \backslash \{e\}: X \in \mathcal{C}\}$. The *minors* of a clutter are the clutters obtained by repeatedly deleting and contracting elements of E. The clutter Q_6 is defined on the set E_4 of the edges of the complete graph K_4, and its elements are the triangles of K_4—see Fig. 2. The clutter Q_7 is defined on $E_4 \cup \{e\}$ where $e \notin E_4$, and its elements are $X \cup \{e\}$ for each triangle or perfect matching X of K_4.

In 1995, Gerards and Laurent [24] characterized the binary clutters that are box-$\frac{1}{d}$-integral for all $d \in \mathbb{Z}_{>0}$ by forbidding minors.

Theorem 11 ([24, Theorem 1.2]) *A binary clutter is box-$\frac{1}{d}$-integral for all $d \in \mathbb{Z}_{>0}$ if and only if neither Q_6 nor Q_7 is its minor.*

In 2008, Chen et al. [12] characterized box-Mengerian binary clutters by forbidding minors. In [11], Chen, Chen, and Zang provide a simpler proof of this characterization,

$$A_{Q_6} = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Fig. 2 The matrix representation of the clutter Q_6

based on the so called ESP property. We mention that none of the proofs of Theorem 12 rely on Theorem 11.

Theorem 12 ([12, Corollary 1.2]) *A binary clutter is box-Mengerian if and only if neither Q_6 nor Q_7 is its minor.*

The combination of Theorems 11 and 12 implies that a binary clutter is box-Mengerian if and only if it is box-$\frac{1}{d}$-integral for all $d \in \mathbb{Z}_{>0}$. We show in the following how this equivalence is actually a special case of Theorem 9.

By definition, a clutter $\mathcal{C}$ is box-$\frac{1}{d}$-integral if and only if $dP_{\mathcal{C}}$ is box-integer, which implies the following reformulation of the class of polyhedra characterized in Theorem 11.

- A clutter $\mathcal{C}$ is box-$\frac{1}{d}$-integral for all $d \in \mathbb{Z}_{>0}$ if and only if $P_{\mathcal{C}}$ is fully box-integer.

Recall that a system is box-TDI if and only if it is TDI and defines a box-TDI polyhedron. Then, by Theorem 9, a clutter is box-Mengerian if and only if it is Mengerian and $P_{\mathcal{C}}$ is principally box-integer. Since $\mathcal{C}$ being Mengerian implies the integrality of $P_{\mathcal{C}}$, we get the following reformulation for the systems involved in Theorem 12.

- A clutter $\mathcal{C}$ is box-Mengerian if and only if it is Mengerian and $P_{\mathcal{C}}$ is fully box-integer.

Therefore, to prove the announced equivalence it is enough to show the following statement.

- If $\mathcal{C}$ is binary and $P_{\mathcal{C}}$ is fully box-integer, then $\mathcal{C}$ is Mengerian.

We apply Seymour's characterization [41]: a binary clutter is Mengerian if and only if it has no Q_6 minor. The property of $P_{\mathcal{C}}$ being fully box-integer is closed under taking minors since $P_{\mathcal{C}/e}$ and $P_{\mathcal{C}\backslash e}$ are respectively obtained from $P_{\mathcal{C}} \cap \{x_e = 0\}$ and $P_{\mathcal{C}} \cap \{x_e = 1\}$ by deleting e's coordinate. Furthermore, P_{Q_6} is not fully box-integer by statement 3 of Theorem 8. Indeed, the first three rows of the matrix A_{Q_6} of Fig. 2 form a nonequimodular matrix M, as the determinant of the three first columns equals 2 and that of the three last columns equals 1. Moreover, M is face-defining for P_{Q_6}, by Observation 4 and because $\chi^1 + \chi^6$, $\chi^2 + \chi^5$, $\chi^3 + \chi^4$, and $\chi^4 + \chi^5 + \chi^6$ are affinely independent, belong to P_{Q_6}, and satisfy $Mx = \mathbf{1}$. Therefore, if $\mathcal{C}$ is binary and $P_{\mathcal{C}}$ is fully box-integer, then $\mathcal{C}$ has no Q_6 minor.

6.2 On box-perfect graphs

In this section, we provide a construction which preserves non box-perfection, and use it to refute a conjecture of Ding et al. [19].

In a graph, a *clique* is a set of pairwise adjacent vertices, and a *stable set* is the complement of a clique. The *stable set polytope* of a graph is the convex hull of the incidence vectors of its stable sets. *Perfect graphs* are known to be those whose stable set polytope is described by the system composed of the clique inequalities and the nonnegativity constraints:

$$\begin{aligned} x(C) &\leq 1 \quad \text{for all cliques } C,\\ x &\geq \mathbf{0}. \end{aligned}$$

A *box-perfect graph* is a graph for which this system is box-TDI. Since this system is known to be TDI if and only if the graph is perfect [34], a graph is box-perfect if and only if it is perfect and its stable set polytope is box-TDI. The characterization of box-perfect graphs is a long standing open question raised by Cameron and Edmonds [8]. Recent progress has been made on this topic by Ding et al. [19]. They exhibit several new subclasses of perfect graphs, and in particular prove the conjecture of Cameron and Edmonds [8] that parity graphs are box-perfect. They also propose a conjecture for the characterization of box-perfect graphs.

To state their conjecture, they introduce the class of graphs $\mathcal{R}$, built as follows. Let $G = (U, V, E)$ be a bipartite graph whose biadjacency matrix is minimally non-TU. Add a set of edges F between vertices of V such that the neighborhood $N_{G'}(u)$ of u in $G' = (U \cup V, E \cup F)$ is a clique for all $u \in U$. If there exists $u \in U$ such that $N_{G'}(u) = V$, then $G' \backslash \{u\}$ is in $\mathcal{R}$, otherwise G' is in $\mathcal{R}$.

Conjecture 1 (Ding et al. [19]) *A perfect graph is box-perfect if and only if it contains no graph from $\mathcal{R}$ as an induced subgraph.*

We introduce the operation of *unfolding* a vertex $v \in V$ in $G = (V, E)$. Take a vertex $v \in V$ and two sets of vertices X and Y such that $X \cup Y = N_G(v)$ and no edge connects $X \backslash Y$ and $Y \backslash X$. Delete v and add two new vertices x and y such that the neighborhoods of x and y are respectively X and Y. Finally, add another vertex z adjacent only to x and y.

We mention that unfolding a vertex might not preserve perfection. Nevertheless, if the starting graph is perfect but not box-perfect, then the graph obtained by unfolding is not box-perfect.

Lemma 8 *Unfolding any vertex in a perfect but not box-perfect graph yields a non box-perfect graph.*

Proof We show that if the stable set polytope of a graph has a nonequimodular face-defining matrix, then so does any graph obtained by unfolding. By Theorem 2, this proves the Lemma.

Let $G = (V, E)$ be a graph which is perfect but not box-perfect, let v be a vertex of G, let H be obtained from G by unfolding v, and x, y, z be the new vertices. Let $n = |V|$. Since G is not box-prefect, its stable set polytope has a nonequimodular face-defining matrix $M \in \mathbb{Q}^{k \times n}$ for a face F. Since G is perfect, we may assume that the rows of M are the incidence vectors of a set $\mathcal{K}$ of cliques of G. Indeed, it can be checked that removing the rows corresponding to nonnegativity constraints yields a

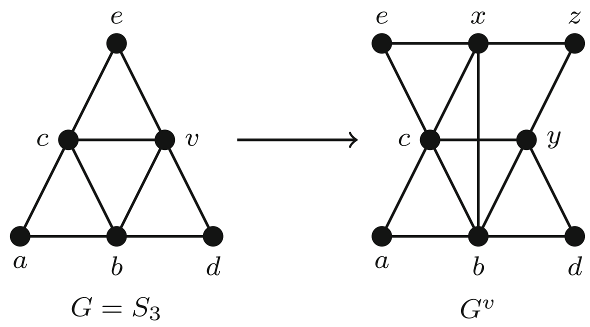


Fig. 3 A non box-perfect graph obtained by unfolding the vertex v in S_3, with $X = \{b, c, e\}$ and $Y = \{b, c, d\}$

smaller nonequimodular face-defining matrix. By Observation 4, there exists a family $\mathcal{S}$ of affinely independent stable sets of F with $|\mathcal{S}| = n - \dim(F) + 1$. Build a family $\mathcal{T}$ of stable sets of H from $\mathcal{S}$ as follows: if $S \in \mathcal{S}$ contains v, then $S \backslash \{v\} \cup \{x, y\} \in \mathcal{T}$, otherwise $S \cup \{z\} \in \mathcal{T}$. All these sets are stable sets and are affinely independent. Build a family $\mathcal{L}$ of $k+2$ cliques of H as follows. For each $K \in \mathcal{K}$,

- If $v \notin K$, then $K \in \mathcal{L}$.
- If $v \in K$, the fact that $X \cup Y = N_G(v)$ and no edge connects $X \backslash Y$ and $Y \backslash X$ ensures that at least one of $K \backslash \{v\} \cup \{x\}$ and $K \backslash \{v\} \cup \{y\}$ is a clique of H. If both are cliques, then add one of them to $\mathcal{L}$, otherwise add the clique.
- Add $\{x, z\}$ and $\{y, z\}$ to $\mathcal{L}$.

Let N denote the $(k+2) \times (n+2)$ matrix whose rows are the incidence vectors of the cliques of $\mathcal{L}$. The matrix N has full row rank and each stable set T of $\mathcal{T}$ satisfies $|T \cap L| = 1$ for all $L \in \mathcal{L}$, hence N is face-defining for the stable set polytope of H by Observation 4. There only remains to show that N is not equimodular. To prove this, we show that each $k \times k$ submatrix of M gives rise to a $(k+2) \times (k+2)$ submatrix of N having the same determinant. Since M is not equimodular, neither is N.

Let A be a $k \times k$ submatrix of M. If A does not contains v's column M^v, then add two rows of zeros and then the two columns N^y and N^z. Note that the determinant has not changed: first develop with respect to $\{x, z\}$'s row, and then with respect to $\{y, z\}$'s row, to obtain the starting matrix. If A contains v's column M^v, then delete it, add two rows of zeros and finally add the three columns N^x, N^y, and N^z. Let A' denote this new matrix. We obtain $\det(A') = \det(A)$ as follows: first replace the column A^x by $A^x + A^y - A^z$, then develop with respect to $\{x, z\}$'s row, and finally with respect to $\{y, z\}$'s row. The resulting matrix is precisely A. $\square$

Unfolding a vertex in S_3 as shown in Fig. 3 yields a graph which is perfect but not box-perfect, and contains no induced subgraphs from $\mathcal{R}$. This disproves Conjecture 1—see Proposition 5.

It is well-known that the graph S_3 in Fig. 3 is not box-perfect [9]. It can also be seen because the nonequimodular matrix M below is face-defining for the stable set polytope of S_3.

$$M = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Indeed, up to reordering the vertices, the rows of M correspond to the three external triangles, and the four affinely independent stable sets $\{a, v\}$, $\{b, e\}$, $\{c, d\}$, $\{a, d, e\}$ belong to the corresponding face. By Observation 4 and Theorem 2, S_3 is not box-perfect.

Proposition 5 *The graph G of Fig. 3 is perfect but not box-perfect and none of its induced subgraphs belongs to $\mathcal{R}$.*

Proof Note that the graphs G and G^v are perfect. By Lemma 8, G^v is not box-perfect. The graph $G^v \backslash \{z\}$ is box-perfect, as one can check that the constraint matrix of its stable set polytope is totally unimodular. Hence, if G^v contains an induced subgraph $H \in \mathcal{R}$, then $z \in V(H)$. As no graph in $\mathcal{R}$ has a vertex of degree one, this contradicts the claim below.

- If $H \in \mathcal{R}$ has a vertex z with only two neighbors x and y, then xy is an edge of H.

Recall that vertices of H are partitioned into sets U and V such that the neighborhood of every vertex of U is a clique of V, and the biadjacency matrix M of the edges between U and V is either minimally non-TU or obtained from such a matrix by removing a row. In particular, every column of M contains at least a one, and every row of M contains at least two ones.

If $z \in U$, then xy is an edge of H. Suppose now $z \in V$. The z-column of M contains a one, so a neighbor of z, say x, belongs to U. The x-row of M contains two ones, so x has an other neighbor in V, which is connected to z. Therefore, this neighbor is y, and xy is an edge of H. □

Note that choosing $X = \{c, e\}$ and $Y = \{b, c, d\}$ when unfolding v in Fig. 3 yields another perfect but not box-perfect graph with no graph from $\mathcal{R}$ as an induced subgraph.

6.3 Integer decomposition property

In this section, we discuss possible connections between full box-integrality and the integer decomposition property. This property arises in various fields such as integer programming, algebraic geometry, combinatorial commutative algebra. Several classes of polyhedra are known to have the integer decomposition property, as for instance: projections of polyhedra defined by totally unimodular matrices [40], polyhedra defined by nearly totally unimodular matrices [26], certain polyhedra defined by k-balanced matrices [46], the stable set polytope of claw-free t-perfect graphs and h-perfect line-graphs [6].

A polyhedron P has the *integer decomposition property*, if for any natural number k and any integer vector $x \in kP$, there exist k integer vectors $x_1, \ldots, x_k \in P$ with $x_1 + \cdots + x_k = x$. A stronger property is when the polyhedron P has the *Integer Carathéodory Property*, that is, if for every positive integer k and every integer vector $x \in kP$, there exist $n_1, \ldots, n_t \in \mathbb{Z}_{\geq 0}$ and affinely independent $x_1, \ldots, x_t \in P \cap \mathbb{Z}^n$ such that $n_1 + \cdots + n_t = k$ and $x = \sum_i n_i x_i$.

In [27], Gijswijt and Regts introduce a class $\mathcal{P}$ of polyhedra and show that they have the Integer Carathéodory Property. They define $\mathcal{P}$ to be the set of polyhedra P such that for any $k \in \mathbb{Z}_{\geq 0}, r \in \{0, \ldots, k\}$, and $w \in \mathbb{Z}^n$ the intersection $rP \cap (w - (k-r)P)$ is box-integer. They also show [27, Proposition 4] that every $P \in \mathcal{P}$ is box-integer. Given the definition of $\mathcal{P}$, note that if a polyhedron is in $\mathcal{P}$, then so are all its dilations. Therefore, every P in $\mathcal{P}$ is fully box-integer. By Theorem 9, this has the following consequence.

Corollary 9 *Every $P \in \mathcal{P}$ is box-TDI.*

The converse of Corollary 9 does not hold. We show below that polyhedra in $\mathcal{P}$ satisfy the stronger property that not only the affine hulls of their faces are principally box-integer, but also the intersection of the affine hulls of any two faces. In terms of matrices, this is phrased as follows.

Proposition 6 *If $P \in \mathcal{P}$, then $\text{aff}(F) \cap \text{aff}(G)$ has an equimodular face-defining matrix for all faces F and G of P.*

Proof Let F and G be faces of P, and let x_F and x_G be rational points in their respective relative interiors. There exists $k \in \mathbb{Z}_{>0}$ such that both kx_F and kx_G are integer. Let $w = k(x_F + x_G)$, and $Q = kP \cap (w - kP) = k(P \cap (x_F + x_G - P))$. Since $P \in \mathcal{P}$, note that rQ is box-integer for all $r \in \mathbb{Z}_{>0}$, that is, Q is fully box-integer. By the choice of x_F and x_G, the minimal face H of Q containing kx_F satisfies $\text{aff}(H) = k\,(\text{aff}(F) \cap -(x_F + x_G + \text{aff}(G)))$. Thus, the latter is a translation of $\text{aff}(F) \cap -\text{aff}(G)$. Since Q is fully box-integer, $\text{aff}(H)$ has an equimodular face-defining matrix by Theorem 8, hence so has $\text{aff}(F) \cap -\text{aff}(G)$ by translation. Since $\text{aff}(F) \cap \text{aff}(G)$ can be described using the matrix of constraints of $\text{aff}(F) \cap \text{aff}(G)$ and multiplying by -1 the right-hand sides corresponding to $\text{aff}(G)$, we get an equimodular face-defining matrix for $\text{aff}(F) \cap \text{aff}(G)$. $\square$

Fully box-integer polyhedra do not inherit the Integer Carathéodory Property. Actually, they do not even inherit the integer decomposition property, as the classical example of polytope without the integer decomposition property $P = \text{conv}\,((0,0,0), (1,1,0), (1,0,1), (0,1,1))$ is fully box-integer. To see that P is fully box-integer, note that in the minimal linear description of P given below, the matrix of constraints is totally equimodular. Since P is also integer, this implies that P is fully box-integer by Theorem 7. The point $(1,1,1)$ is in $2P$ and cannot be written as an integer combination of the integer points of P, hence P does not have the integer decomposition property.

$$P = \left\{ x \in \mathbb{R}^3 \colon \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix} x \leq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix} \right\}$$

Nevertheless, given the strong integrality properties of fully box-integer polyhedra and as the above large subclass $\mathcal{P}$ has the Integer Carathéodory Property, it might be that many of them have the integer decomposition property. In this area, a long standing open question is known as Oda's question [35]: is it true that every smooth polytope has the integer decomposition property? A full-dimensional polytope of $\mathbb{R}^n$ is *simple* if every vertex has n neighbors. A simple integer polytope is *smooth* if for every vertex v the generators of the associated minimal tangent cone form a basis of the lattice $\mathbb{Z}^n$.

The polyhedron of the example above is not smooth, and the following special case of Oda's question is a reasonable first step to determine which fully box-integer polyhedra have the integer decomposition property.

Open Problem 3 *Do smooth fully box-integer polyhedra have the integer decomposition property?*

6.4 Box-TDIness for conservative functions

In [15], the authors prove that the standard system describing the circuit cone is box-TDI if and only if the graph is series-parallel. We illustrate that polarity preserves the box-TDIness of cones by providing a box-TDI system for the cone of conservative function—polar of the circuit cone.

Let $G = (V, E)$ be an undirected graph. The set of edges connecting a given set of vertices and its complement is called a *cut*. A cut containing no other nonempty cut is called a *bond*. A set of edges is called a *circuit* if it induces a connected subgraph where every vertex has degree two. The *minors* of a graph are the graphs obtained by repeatedly contracting edges and deleting edges and isolated vertices. Given $e \in E$, the graphs obtained from G by respectively deleting and contracting e are denoted by $G\backslash e$ and G/e. A graph is *series-parallel* if and only if contains no K_4 minor [20].

The *circuit cone* $C_{\text{circuit}}(G) = \text{cone}\{\chi^C$ for all circuits C of $G\}$ is the cone generated by the incidence vectors of the circuits of G. Seymour [42] proved that $C_{\text{circuit}}(G) = \{x \in \mathbb{R}^E : x \geq \mathbf{0}, x(D\backslash e) \geq x_e$ for all cuts D of G and $e \in D\}$. A function $f : E \to \mathbb{R}$ is *conservative* if $f(C) \geq 0$ for each circuit C of G. These functions form the *cone of conservative functions* $C_{\text{cons}}(G) = \{x \in \mathbb{R}^E : x(C) \geq 0$ for all circuits C of $G\}$. By polarity [39, Corollary 29.2h], we have $C_{\text{cons}}(G) = -C_{\text{circuit}}(G)^\star = \text{cone}\{\chi^e$ for all $e \in E$, $\chi^{D\backslash e} - \chi^e$ for all cuts D of G and $e \in D\}$.

We show that box-TDI systems describing $C_{\text{cons}}(G)$ only exist when G is series-parallel. In this case, we provide such a system in the following proposition.

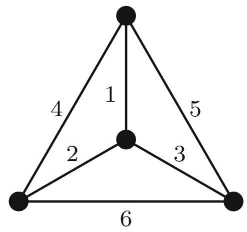
Proposition 7 *For a graph $G = (V, E)$, the following statements are equivalent.*

1. *The graph G is series-parallel.*
2. *The cone of conservative functions of G is box-TDI.*
3. *The system $\frac{1}{2}x(C) \geq 0$ for all circuits C of G is box-TDI.*

Proof Since the cone of conservative functions of G is described by $\frac{1}{2}x(C) \geq 0$ for all circuits C of G, statement 3 implies statement 2.

To prove that statement 1 implies statement 3, suppose that G is series-parallel. Then, [15, Theorem 1] asserts that the system $x \geq \mathbf{0}$, $x(D\backslash e) \geq x(e)$ for all cuts D of G and $e \in C$ is box-TDI. Hence the circuit cone of G is a box-TDI cone. By Lemma 6, $C_{\text{cons}}(G) = -C_{\text{circuit}}(G)^\star$ is box-TDI. By Theorem 1, it remains to show

Fig. 4 The graph K_4 and a face-defining matrix M of $C_{\text{cons}}(K_4)$



$$M = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

that the system $\frac{1}{2}x(C) \geq 0$ for all circuits C of G is TDI. [38, Corollary 22.5a] states that a system $Ax \leq 0$ is TDI if and only if the rows of A form a Hilbert basis. In other words, it remains to show that any integer vector z in the circuit cone of G is a nonnegative integer combination of vectors of $\mathcal{H} = \{\frac{1}{2}\chi^C\colon C \text{ is a circuit of } G\}$. [1, Theorem 1] asserts that, in graphs with no Petersen minors, if p is an integer vector of the circuit cone such that $p(C)$ is even for all cuts C of G, then p is a sum of circuits. Since the Petersen graph contains a K_4 minor, [1, Theorem 1] applies to G. Since $2z$ satisfies the conditions, $2z = \sum_{C\in\mathcal{C}} \chi^C$ for some family $\mathcal{C}$ of circuits of G. Therefore, $z = \sum_{C\in\mathcal{C}} \frac{1}{2}\chi^C$.

To prove that statement 2 implies statement 1, we show that if the graph G is not series-parallel, then its cone of conservative functions is not box-TDI. For $e \in E$, one can see that $C_{\text{cons}}(G\backslash e)$ and $C_{\text{cons}}(G/e)$ are respectively obtained by deleting e's coordinate in $C_{\text{cons}}(G) \cap \{x_e = +\infty\}$ and $C_{\text{cons}}(G) \cap \{x_e = 0\}$. Hence, taking minors preserves the box-TDIness of the cone of conservative functions. It remains to prove that $C_{\text{cons}}(K_4)$ is not box-TDI. Let us apply Theorem 2.

The nonequimodular matrix M of Fig. 4 is the constraint matrix obtained by considering the inequalities associated with the three circuits formed by the three internal triangles of K_4. By Observation 4, M is face-defining for $C_{\text{cons}}(K_4)$ because $\mathbf{0}$ and the three conservative functions $\chi^4+\chi^5-\chi^1$, $\chi^4+\chi^6-\chi^2$ and $\chi^5+\chi^6-\chi^3$ are affinely independent, belong to $C_{\text{cons}}(K_4)$ and satisfy $Mx = \mathbf{0}$. Therefore, by statement 3 of Theorem 2, the cone $C_{\text{cons}}(K_4)$ is not box-TDI. □

Note that the coefficients of the system in Proposition 7 are half-integral. We leave open the question[3] of finding a box-TDI system with integer coefficients, which exists by [38, Theorem 22.6(i)] and Theorem 1.

By planar duality, there is a correspondence between the circuits of a planar graph and the bonds of its planar dual. This is used in [15] to obtain the box-TDIness of the standard system describing the cut cone of a series-parallel graph. Applying planar duality to Proposition 7 provides the following: if the graph is series-parallel, then $\frac{1}{2}x(B) \geq 0$ for all bonds B is a box-TDI system describing the polar of the cut cone. This is in fact an equivalence as one can check that the box-TDIness of the corresponding cone is preserved under taking minors and that the matrix of Fig. 4 is face-defining when $G = K_4$.

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[3] Between the submission and the publication of this paper, the question was answered in [4].

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Trader multiflow and box-TDI systems in series–parallel graphs

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ABSTRACT

Series–parallel graphs are known to be precisely the graphs for which the standard linear systems describing the cut cone, the cycle cone, the T-join polytope, the cut polytope, the multicut polytope and the T-join dominant are TDI. We prove that these systems are actually box-TDI. As a byproduct, our result yields a min–max relation for a new problem: the trader multiflow problem. The latter generalizes both the maximum multiflow and min-cost multiflow problems and we show that it is polynomial-time solvable in series–parallel graphs.

1. Introduction

Throughout the paper, all the entries will be rational. A linear system $Ax \geq b, x \geq 0$ is *totally dual integral* (*TDI* for short) if the maximum in the LP-duality equation

$$\min\{c^\top x : Ax \geq b,\ x \geq \mathbf{0}\} = \max\{b^\top y : A^\top y \leq c,\ y \geq \mathbf{0}\}$$

has an integer optimal solution for all integer vectors c for which the optimum is finite. This property is much sought-after since such systems describe integer polyhedra when b is integer and yield min–max relations [1]. An even stronger property than TDIness is box-TDIness, where a *box-TDI system* is a TDI system $Ax \geq b, x \geq \mathbf{0}$ which remains TDI when adding box-constraints $\ell \leq x \leq u$, for all rational[1] vectors ℓ, u. In other words, it is box-TDI if

$$\max\{b^\top y + \ell^\top z^1 - u^\top z^2 : A^\top y + z^1 - z^2 \leq c,\ y \geq \mathbf{0},\ z^1, z^2 \geq \mathbf{0}\}$$

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[1] Allowed to take infinite values.

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has an integer solution for all integer vectors c and all rational vectors ℓ, u for which the optimum is finite. General properties of such systems can be found in Cook [2] and Chapter 22.4 of Schrijver [3]. Note that, although every rational polyhedron $\{x : Ax \geq b, x \geq \mathbf{0}\}$ is described by a TDI system $\frac{1}{k}Ax \geq \frac{1}{k}b, x \geq \mathbf{0}$, for some integer k, not every polyhedron is described by a box-TDI system.

The book by Schrijver [4] contains numerous min–max relations of combinatorial optimization derived from TDI systems. When such systems are box-TDI, most of the time, the matrix A is totally unimodular. The past few years, this topic has received a renewed interest [5,6], and other box-TDI systems have been studied [7–9], with matrices that are not totally unimodular. A 0–1 matrix A so that the linear system $Ax \geq \mathbf{1}$, $x \geq \mathbf{0}$ is (box-) TDI is called *(box-) Mengerian*. In 1977, Seymour [10] proved that a 0–1 matrix associated with a binary clutter is Mengerian if and only if it does not contain Q_6 as a minor. In 2008, Chen, Ding and Zang [8] proved that such matrices are box-Mengerian if and only if they contain neither Q_6 nor Q_7 as a minor. Recently, Ding, Tan and Zang [11] announced a characterization of the graphs for which a box-TDI system describes the matching polytope.

In 2009, Chen, Ding and Zang [9] proved that a graph is series–parallel if and only if the system $\frac{1}{2}Ax \geq \mathbf{1}$, $x \geq \mathbf{0}$ describing the 2-edge-connected spanning subgraph polytope is box-TDI, where A is the cut-edge incidence matrix of the graph. Another set of characterizations of series–parallel graphs given by Schrijver asserts that they are precisely the graphs for which the standard linear systems describing the cut cone, the cycle cone [12], the cut polytope [13], the T-join polytope [14] and the T-join dominant [15] are TDI — see Corollary 29.9c of [4]. Moreover, it is proved in [16] that a graph is series–parallel if and only if the standard linear system describing its multicut polytope is TDI.

Multiflows are among the most famous NP-hard problems in combinatorial optimization and have been considerably studied, see for instance [4]. We focus on integer multiflows in the present paper. Multiflow problems involve two simple undirected graphs, a *supply graph* $G = (V, E)$ and a *demand graph* $H = (V, R)$, and two vectors, a *capacity vector* $c \in \mathbb{Z}_+^E$ and a *demand vector* $d \in \mathbb{Z}_+^R$. An edge $e \in E$ is a *link* of *capacity* c_e whereas an edge $r \in R$ is a *net* of *demand* d_r. From now on, (G, H, c, d) will refer to such a quadruplet. For a net $r = st$, let $\mathcal{P}(r)$ be the set of all st-paths in G, and let $\mathcal{P}$ be the union of $\mathcal{P}(r)$ for all nets r. A *multiflow of* (G, H, c, d) is an integer vector $y \in \mathbb{Z}^{\mathcal{P}}$ satisfying the following system of linear inequalities:

$$\text{(MFLOW)}\begin{cases} \displaystyle\sum_{P \in \mathcal{P}(r)} y_P \geq d_r & \text{for each net } r \in R, \\ \displaystyle\sum_{P \in \mathcal{P}:\, e \in P} y_P \leq c_e & \text{for each link } e \in E, \\ \qquad\quad y \geq \mathbf{0}. & \end{cases}$$

Two famous NP-hard problems are related to multiflows. Given G, H and c, the *maximum multiflow problem* asks for a demand vector d such that there exists a multiflow for (G, H, c, d) and $\sum_{r \in R} d_r$ is maximum.

Given (G, H, c, d) and some cost vector $w \in \mathbb{Z}_+^E$ on the links, the *min-cost multiflow problem* asks for a multiflow minimizing the sum of $w_e y_e$ over all links $e \in E$, where $y_e := \sum_{P \in \mathcal{P}:\, e \in P} y_P$ is the amount of flow through link e.

A necessary condition for the existence of a multiflow in (G, H, c, d) is the *cut condition* which requires that $d(D \cap R) \leq c(D \cap E)$ for all cuts D of $G + H$, the latter being $G + H = (V, E \cup R)$ where E and R are considered as disjoint, that is, $G + H$ may contain parallel edges. Seymour [14] proved that a graph (V, F) is series–parallel if and only if for all partitions F into E and R, and for all $c \in \mathbb{Z}_+^E$ and $d \in \mathbb{Z}_+^R$, the cut condition implies the existence of a multiflow.

Contribution. In this paper, we investigate some box-TDI systems related to multiflows. Our main result is to strengthen the TDI characterizations of series–parallel graphs mentioned earlier by proving that the standard linear systems describing the cut cone, the cycle cone, the T-join polytope, the cut polytope, the multicut polytope, and the T-join dominant are actually box-TDI systems for series–parallel graphs — see Theorem 1.

From the box-TDIness of the cut cone, we derive a min–max relation for series–parallel graphs that involves a new multiflow problem generalizing both the maximum multiflow and min-cost multiflow problems. Given (G, H, c, d), a *profit* $\ell \in \mathbb{Z}_+^R$ and a *cost* $u \in \mathbb{Z}_+^E$, the *trader multiflow problem* asks to maximize $\ell^\top z^1 - u^\top z^2$ over all $(y, z^1, z^2) \in \mathbb{Z}_+^{\mathcal{P}} \times \mathbb{Z}_+^R \times \mathbb{Z}_+^E$ such that y is a multiflow of $(G, H, \tilde{c}, \tilde{d})$ with $\tilde{c} = c + z^2$ and $\tilde{d} = d + z^1$. Therefore, in this new multiflow problem, we gain ℓ_r for each additional unit of demand on net $r \in R$ that we are able to satisfy, we pay u_e to add a unit of capacity on link $e \in E$, and the goal is to maximize the total benefit. The min–max relation we derive connects the trader multiflow problem and box-multicuts, where box-multicuts are a generalization of multicuts. We also show that the trader multiflow problem is polynomial time solvable in series–parallel graphs.

Outline. In Section 2, we establish our characterization of series–parallel graphs in terms of box-TDI systems. Section 3 is devoted to the trader multiflow problem. We first show how it generalizes both the maximum multiflow and min-cost multiflow problems. Then, we provide our min–max relation for the trader multiflow problem in series–parallel graphs and explain why this problem is polynomial in these graphs. For the sake of clarity, the most technical part of the proof of Theorem 1 is postponed to the Appendix. The rest of this section is devoted to definitions.

Definitions. Throughout, $G = (V, E)$ will denote an undirected graph and $T \subseteq V$ a set of vertices of even cardinality. A graph is *series–parallel* if it is obtained from a forest by repeating the operations of replacing one edge by two edges in parallel, or by two edges in series. Equivalently, these are the graphs without K_4 minor [17]. Then, a series–parallel graph is planar and its planar dual is also series–parallel. Following [4], a *cycle* is a subset $C \subseteq E$ so that every vertex of (V, C) has an even degree. A minimal nonempty cycle is a *circuit*. The *cut* defined by a subset of vertices U, denoted by $\delta(U)$, is the set of edges having one extremity in U and the other one in $V \setminus U$. A minimal nonempty cut is a *bond*. Note that cycles (resp. cuts) are disjoint unions of circuits (resp. bonds). A *multicut* is the set of all the edges between different classes of some partition of the vertex set. A *T-join* is a subset of edges F such that the odd degree vertices of (V, F) are the ones in T. Note that a cycle is an $\emptyset$-join. A *T-cut* is a cut $\delta(U)$ with $|U \cap T|$ odd. For $x \in \mathbb{R}^E$ and $F \subseteq E$, we use the notation $x(F) = \sum_{e \in F} x_e$. We will make no difference between combinatorial objects and their characteristic vectors, that is, for instance, we will speak of nonnegative combinations of cycles instead of nonnegative combinations of characteristic vectors of cycles.

2. Box-TDI systems of series–parallel graphs

In this section, we first provide the systems involved in our main theorem. Then, we state and prove Theorem 1, which establishes that the standard linear systems describing the cut cone, the cycle cone, the T-join polytope, the cut polytope, the multicut polytope and the T-join dominant are box-TDI if and only if the graph is series–parallel. These systems were already known to be TDI [4,16].

2.1. TDI systems of series–parallel graphs...

Let us write now the systems involved in Theorem 1. Let $G = (V, E)$ be an undirected graph and $T \subseteq V$ a set of vertices of even cardinality.

Seymour [12] proved that the *cycle cone* of G, that is, the set of nonnegative combinations of cycles of G, is described by the following set of inequalities.

$$\text{(Cycle cone)} \begin{cases} x(\delta(U) \setminus \{e\}) - x_e \geq 0 & \text{for each } U \subseteq V \text{ and each } e \in \delta(U), \\ x \geq \mathbf{0}. \end{cases}$$

The *T-join polytope* of G is the convex hull of its T-joins. Seymour [14] proved that it is described by the following set of inequalities.

$$(T\text{-join}) \begin{cases} x(F) - x(\delta(U) \setminus F) \leq |F| - 1 & \text{for each } U \subseteq V,\ F \subseteq \delta(U) \\ & \text{with } |U \cap T| + |F| \text{ odd}, \\ \mathbf{0} \leq x \leq \mathbf{1}. \end{cases}$$

The T-join dominant of G is the set of vectors greater than or equal to some T-join of G. This dominant is described by the following set of inequalities, see Corollary 29.2b in [4].

$$(T\text{-join dominant}) \begin{cases} x(C) \geq 1 & \text{for each } T\text{-cut } C, \\ x \geq \mathbf{0}. \end{cases}$$

Sebő [18] provided a minimal TDI system describing the T-join dominant of G.

Let us assume that G is planar and let G^* denote its dual graph. Recall that the cycles of G are the cuts of G^*. Hence,

$$(\text{Cut cone}) \begin{cases} x(C \setminus \{e\}) - x_e \geq 0 & \text{for each circuit } C \text{ and each edge } e \in C, \\ x \geq \mathbf{0}, \end{cases}$$

describes the *cut cone* of G, that is, the set of nonnegative combinations of cuts of G. Moreover, by taking $T = \emptyset$ in system (T-join), and then writing the planar dual, we have the following description of the *cut polytope* of G, that is, the convex hull of its cuts.

$$(\text{Cut}) \begin{cases} x(F) - x(C \setminus F) \leq |F| - 1 & \text{for each circuit } C \text{ and } F \subseteq C \\ & \text{with } |F| \text{ odd}, \\ \mathbf{0} \leq x \leq \mathbf{1}. \end{cases}$$

Actually, the systems (Cut cone) and (Cut) describe the cut cone and the cut polytope for a larger class than planar graphs, namely graphs with no K_5-minor — see [14] and [13], respectively.

Schrijver showed that the systems (Cycle cone), (T-join) and (T-join dominant) are TDI if and only if the graph is series–parallel— see Corollary 29.9c of [4]. A graph is series–parallel if and only if its dual is; this result, combined with the fact that cycles are $\emptyset$-joins, implies that (Cut cone) and (Cut) are TDI if and only if the graph is series–parallel.

Multicuts can be equivalently defined as arbitrary unions of cuts, or as sets of edges $D \subseteq E$ such that $|D \cap C| \neq 1$ for all cycles C. The *multicut polytope* of a graph is the convex hull of its multicuts, and is therefore contained in the polyhedron defined by the inequalities of (Cut cone) and $x \leq \mathbf{1}$. Chopra [19] showed that the following system, called (Multicut), describes the multicut polytope of a graph if and only if the graph is series–parallel.

$$(\text{Multicut}) \begin{cases} x(C \setminus \{e\}) - x_e \geq 0 & \text{for each circuit } C \text{ and each edge } e \in C, \\ \mathbf{0} \leq x \leq \mathbf{1}. \end{cases}$$

Corollary 4.1 of [16] strengthens the result of Chopra [19] by stating that system (Multicut) is TDI if and only if the graph is series–parallel.

2.2. . . . are actually box-TDI

We now strengthen the aforementioned TDIness results. More precisely, we show that each system mentioned in Section 2.1 which is TDI for series–parallel graphs is actually box-TDI for these graphs. Our theorem implies Corollary 4.1 of [16] and Corollary 29.9c of [4].

Theorem 1. *Let $G = (V, E)$ be a graph. The following statements are equivalent.*

(i) G is series–parallel.
(ii) System (Cut cone) is box-TDI.
(iii) System (Cycle cone) is box-TDI.
(iv) System (T-join) is box-TDI, for all $T \subseteq V$ of even cardinality.
(v) System (Cut) is box-TDI.
(vi) System (Multicut) is box-TDI.
(vii) System (T-join dominant) is box-TDI, for all $T \subseteq V$ of even cardinality.

Proof. Proof. Series-parallelness is already necessary for the systems of (ii)–(vii) to be TDI — see [16] for (vi) and Corollary 29.9c of [4] for the others. A box-TDI system being TDI, the necessity of (i) follows. For the other directions, we will show that $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (v)$ and $(ii) \Rightarrow (vi)$ and $(iv) \Rightarrow (vii)$.

$(i) \Rightarrow (ii)$: Let $G = (V, E)$ be series–parallel, $c \in \mathbb{Z}^E$ and $\ell, u \in \mathbb{Q}^E$ with $\ell \leq u$. The primal problem is to optimize over the system (Cut cone) intersected with the box $\{x : \ell \leq x \leq u\}$. Since we have $x \geq \mathbf{0}$, we may suppose that $\ell \geq \mathbf{0}$ and we get:

$$(P) \begin{cases} \min c^\top x \\ x(C \setminus \{e\}) - x_e \geq 0 \qquad \text{for each circuit } C \text{ of } G \text{ and each edge } e \in C, \\ \mathbf{0} \leq \ell \leq x \leq u. \end{cases}$$

To prove box-TDIness, one has to show that if the dual given below has an optimal solution, then it also has an integer one.

$$(D) \begin{cases} \max \ell^\top z^1 - u^\top z^2 \\ \displaystyle\sum_{\text{circuit } C \ni e} \Big(\sum_{f \in C \setminus \{e\}} y_{C,f} \quad - \quad y_{C,e} \Big) \leq c_e - z_e^1 + z_e^2 \qquad \text{for each } e \in E, \\ y \geq \mathbf{0}, \quad z^1, z^2 \geq \mathbf{0}. \end{cases}$$

The feasible set for (D) has the form $Q = \{z^1, z^2 \geq \mathbf{0},\ y \geq \mathbf{0}\colon z^1 - z^2 + Ay \leq c\}$, and its projection onto the space of $z = (z^1, z^2) \in \mathbb{R}^{E \times E}$ is $\mathrm{proj}_z(Q) = \{z^1, z^2 \geq \mathbf{0} : v^\top z^1 - v^\top z^2 \leq v^\top c,$ for each $v \in K\}$ where K is the projection cone $K = \{v \in \mathbb{R}^E : v^\top A \geq \mathbf{0}^\top,\ v \geq \mathbf{0}\}$. Observe that K is the set of $v \in \mathbb{R}^E$ satisfying the inequalities of the system (Cut cone). Since G is series–parallel, K is the cut cone of G [14]. Therefore

$$\mathrm{proj}_z(Q) = \{(z^1, z^2) \in \mathbb{R}_+^{E \times E} : z^1(D) - z^2(D) \leq c(D), \text{ for each cut } D \text{ of } G\}.$$

The following claim states that $\mathrm{proj}_z(Q)$ is an integer polyhedron. It is a direct corollary of a technical result whose statement and proof are postponed to the Appendix.

Claim 2. *$\mathrm{proj}_z(Q)$ is integer whenever c is integer.*

Suppose (D) has an optimal solution. By Claim 2, there exists an integer optimal solution $(\bar{z}^1, \bar{z}^2)$ of $\max \ell^\top z^1 - u^\top z^2$ over $\mathrm{proj}_z(Q)$. We now build an optimal solution $(\bar{y}, \bar{z}^1, \bar{z}^2)$ of (D) as follows.

Let $b := c - \bar{z}^1 + \bar{z}^2$. Then b is integer and satisfies $b(D) \geq 0$ for each cut D of G. Define R as the set of all $e \in E$ with $b_e \leq 0$ and $E' = E \setminus R$. Let $G' = (V, E')$ and $H = (V, R)$. Let $c' \in \mathbb{Z}_+^{E'}$ and $d \in \mathbb{Z}_+^R$ be defined by $c'_e = b_e$ for all $e \in E'$ and $d_r = -b_r$ for all $r \in R$. Then $d(D \cap R) \leq c'(D \cap E')$ for each cut D of $G' + H$. In other words, the cut condition is satisfied in (G', H, c', d). Hence, $G' + H = G$ being series–parallel, Theorem 8.1 of [14] implies that there exists a multiflow $\hat{y}$ of (G', H, c', d). Define $\bar{y}$ as follows:

$$\bar{y}_{C,e} := \begin{cases} \hat{y}_P & \text{if } b_e \leq 0 \text{ and } P = C \setminus \{e\}, \\ 0 & \text{otherwise.} \end{cases}$$

By construction, $(\bar{y}, \bar{z}^1, \bar{z}^2)$ is an integer optimal solution of (D), and we are done.

$(ii) \Rightarrow (iii)$: The system (Cycle cone) of a series–parallel graph is the system (Cut cone) of its planar dual which is also a series–parallel graph. As the latter system is box-TDI precisely for such graphs, we get the desired implication.

$(iii) \Rightarrow (iv)$: In the following, $Ax \leq b$ is a system whose underlying polyhedron $P = \{x : Ax \leq b\}$ is pointed. The *vertex system* associated with a vertex z of $\{x : Ax \leq b\}$ is the system $A_z x \leq b_z$ composed of the inequalities of $Ax \leq b$ satisfied with equality by z.

Claim 3. *The system $Ax \leq b$ is box-TDI if and only if the vertex system associated with each vertex of $P = \{x : Ax \leq b\}$ is box-TDI.*

Proof. Cook proves that a system is box-TDI if and only if, for each face F of the associated polyhedron, the set of active rows for F forms a box Hilbert basis [2, Proposition 2.2].

Suppose that all the vertex systems of P are box-TDI. Let F be a proper face of P and z be a vertex of F. Then, the active rows in $A_z x \leq b_z$ for the minimal face of $\{x : A_z x \leq b_z\}$ containing F are exactly the same as those in $Ax \leq b$ for F. Hence, by [2, Proposition 2.2], the set of active rows for F forms a box Hilbert basis. Since this holds for every face of P, [2, Proposition 2.2] implies that $Ax \leq b$ is box-TDI. The converse can be proved in a similar way. ■

Let $T \subseteq V$. Recall that vertices of the polytope defined by the system (T-join) correspond to T-joins of G, and conversely. Let J be any T-join of G. By Claim 3, it suffices to show that the vertex system of (T-join) associated with vertex J is box-TDI. Let $\phi_J : \mathbb{R}^E \to \mathbb{R}^E$ be defined by

$$[\phi_J(x)]_e := \begin{cases} 1 - x_e & \text{if } e \in J, \\ x_e & \text{if } e \in E \setminus J. \end{cases}$$

The next two claims exhibit properties of ϕ_J.

Claim 4. *The system obtained from (Cycle cone) by replacing x by $\phi_J(x)$ is the vertex system of (T-join) associated with J.*

Proof. Schrijver proves that replacing x by $\phi_J(x)$ in the vertex system of (T-join) associated with J gives the system (Cycle cone) — see (29.61) to (29.63) page 506 in [4] for the details. As $\phi_J(\phi_J(x)) = x$, the assertion follows. ■

Claim 5. *Replacing x by $\phi_J(x)$ preserves box-TDIness.*

Proof. From the definition of box-TDI systems, it follows that replacing some coordinates by their opposite preserves box-TDIness. So does translation, see Theorem 5.34 in [4]. ■

The (Cycle cone) being box-TDI by (iii), Claims 4 and 5 imply the box-TDIness of the vertex system of (T-join) associated with J. Since this holds for any T-join J of G, Claim 3 gives the box-TDIness of (T-join).

$(iv) \Rightarrow (v)$: We have already shown that (T-join) is box-TDI if and only if the graph is series–parallel. Recall that the cuts of a planar graph are the cycles of its planar dual, and that cycles are $\emptyset$-joins. Therefore, (Cut) is nothing but the system ($\emptyset$-join) for the planar dual of the graph, and since planar duality preserves series–parallelness, we get that (iv) implies (v).

$(ii) \Rightarrow (vi)$: This is immediate because (Multicut) is nothing but the box-TDI system (Cut cone) together with the box-constraints $x \leq \mathbf{1}$.

$(iv) \Rightarrow (vii)$: The system describing the T-join polytope being box-TDI, the TDI system (T-join dominant) describing its dominant is also box-TDI — by Theorem 22.11 of [3]. □

Box-TDI systems have the remarkable property that any TDI system describing the same polyhedron is also box-TDI [2]. This gives the following consequence of Theorem 1. The minimal TDI system describing the T-join dominant given by Sebő [18] becomes box-TDI when the graph is series–parallel.

3. Trader multiflow vs box-multicut

In this section, we first explain how the trader multiflow problem generalizes both the min-cost multiflow and maximum multiflow problems. We then provide a min–max relation involving the trader multiflow problem and the so-called box-multicuts. Finally, we briefly explain why the trader multiflow problem is polynomial in series–parallel graphs.

3.1. Related multiflow problems

Recall that an instance (G, H, c, d, ℓ, u) of the trader multiflow problem is composed of two simple undirected graphs $G = (V, E)$ and $H = (V, R)$, a capacity $c \in \mathbb{Z}_+^E$, a demand $d \in \mathbb{Z}_+^R$, a profit $\ell \in \mathbb{Z}_+^R$ and a cost $u \in \mathbb{Z}_+^E$. The trader multiflow problem aims at maximizing $\ell^\top z^1 - u^\top z^2$ over all $(y, z^1, z^2) \in \mathbb{Z}_+^{\mathcal{P}} \times \mathbb{Z}_+^R \times \mathbb{Z}_+^E$ such that y is a multiflow of $(G, H, \tilde{c}, \tilde{d})$ with $\tilde{c} = c + z^2$ and $\tilde{d} = d + z^1$.

This problem contains the maximum multiflow problem as a special case. Let (G, H, c, d, ℓ, u) be an instance of the trader multiflow problem with $d = \mathbf{0}$, $\ell = \mathbf{1}$ and $u = +\infty$. In any optimal solution $(\bar{y}, \bar{z}^1, \bar{z}^2)$, since $u = +\infty$, we have $\bar{z}^2 = \mathbf{0}$, that is, capacities remain unchanged. Since $d = \mathbf{0}$ and $\ell = \mathbf{1}$, the problem reduces to find $\bar{z}^1$ such that $\sum_{r \in R} \bar{z}_r^1$ is maximum and there exists a multiflow in $(G, H, c, \bar{z}^1)$. This is nothing but the maximum multiflow problem associated with (G, H, c).

The trader multiflow problem also contains the min-cost multiflow problem as a special case. Let (G, H, c, d, w) be an instance of the min-cost multiflow problem. It is transformed into an instance $(G', H', c', d', \ell', u')$ of the trader multiflow problem as follows. Let $G' = (V', E')$ be the graph obtained from G by subdividing every link $e \in E$ into two links e_1, e_2 in series. Then, the amount of flow passing by e_1 equals the amount of flow passing by e_2. Let $c'_{e_1} = c_e$ and $u'_{e_1} = +\infty$. The capacity of e_1 is chosen in order to limit the value of the flow passing by e_1, e_2 to c_e. Let $c'_{e_2} = 0$ and $u'_{e_2} = w_e$. The role of e_2 is to charge a fee w_e for each unit of flow passing by e_1, e_2. Let $H' = (V', R)$, $d' = d$ and $\ell' = \mathbf{0}$. In an optimal solution $(\bar{y}, \bar{z}^1, \bar{z}^2)$ of the trader multiflow problem, we may suppose without loss of generality that $\bar{z}^1 = \mathbf{0}$ since $\ell' = \mathbf{0}$. Since $u'_{e_1} = +\infty$, the amount of flow passing by e_1, e_2 is no more than $c'_{e_1} = c_e$. Since $c'_{e_2} = 0$, for each unit of flow passing by e_1, e_2, one has to increase the capacity of e_2 by one at cost $u'_{e_2} = w_e$. Hence, $\bar{y}$ defines a multiflow in (G, H, c, d) minimizing the total cost of the flow.

3.2. Min–max theorem

Given a graph and integer vectors ℓ and u indexed on its edges, the integer vectors x satisfying system (Cut cone) and $\ell \leq x \leq u$ are called *box-multicuts within* $[\ell, u]$. If we are also given a cost vector c defined on the edges, the *minimum box-multicut problem* seeks a box-multicut x within $[\ell, u]$ of minimum cost $c^\top x$.

Box-multicuts are a generalization of multicuts, these latter being box-multicuts within $[\mathbf{0}, \mathbf{1}]$. Box-multicuts also generalize separating multicuts, where, given a supply graph G and a demand graph $H = (V, R)$, a *separating multicut* is a multicut of $G + H$ containing R. Indeed, separating multicuts are box-multicuts of $G + H$ within $[\ell, \mathbf{1}]$ where ℓ equals 1 for every net of R and 0 otherwise.

The min–max relation between the trader multiflow and minimum box-multicut problems given in the following Corollary 6 is a consequence of Theorem 1. Its statement uses the following notation: given a supply graph $G = (V, E)$ and a demand graph $H = (V, R)$ and two vectors $v^1 \in \mathbb{Z}_+^E$ and $v^2 \in \mathbb{Z}_+^R$, the vector associated with the edges of $G + H$ defined by v^1 and v^2 is denoted by (v^1, v^2).

Corollary 6. *The maximum trader multiflow of* (G, H, c, d, ℓ, u) *equals the minimum box-multicut of* $G+H$ *within* $[(\mathbf{0}, \ell), (u, +\infty)]$ *with respect to costs* $(c, -d)$*, if* $G+H$ *is series–parallel.*

Proof. First, set $\hat{c} = (c, -d)$, $\hat{\ell} = (\mathbf{0}, \ell)$ and $\hat{u} = (u, +\infty)$. Consider the linear program (P) of the proof of Theorem 1 where G, c, ℓ and u are replaced by $G+H$, $\hat{c}$, $\hat{\ell}$ and $\hat{u}$, respectively. Since $\hat{\ell}_e = 0$, we may suppose, without loss of generality, that $\bar{z}^1_e = 0$ for all links $e \in E$ in an optimal solution $(\bar{y}, \bar{z}^1, \bar{z}^2)$ of the dual (D). Moreover, as $u_r = +\infty$, $\bar{z}^2_r = 0$ for all nets $r \in R$. The dual can then be written as:

$$(D') \begin{cases} \max \sum_{r \in R} \ell_r z^1_r - \sum_{e \in E} u_e z^2_e \\ \sum_{\text{circuit } C \ni r} \Big(y_{C,r} \quad - \sum_{f \in C \setminus \{r\}} y_{C,f} \Big) \geq d_r + z^1_r & \text{for each } r \in R, \\ \sum_{\text{circuit } C \ni e} \Big(\sum_{f \in C \setminus \{e\}} y_{C,f} \quad - \quad y_{C,e} \Big) \leq c_e + z^2_e & \text{for each } e \in E, \\ y \geq \mathbf{0}, \quad z^1, z^2 \geq \mathbf{0}. \end{cases}$$

By strong duality, the optimal values of (P) and (D') are equal, when finite. In this case, we will show that there exists an integer optimal solution for both problems.

We may suppose that $\bar{y}_{C,f} = 0$ if $f \in E$. Otherwise, one may decrease $\bar{y}_{C,f}$ by some $\epsilon > 0$. If the solution becomes infeasible, then there exist a circuit $C' \ni f$ and link $f' \in C' \setminus \{f\}$ with $\bar{y}_{C',f'} \geq \epsilon$ since $c \geq \mathbf{0}$. Decreasing $\bar{y}_{C',f'}$ by ϵ and increasing $\bar{y}_{C'',f'}$ by ϵ where C'' is the circuit of $C \Delta C'$ containing f' restores its feasibility. Similarly, we may suppose that $\bar{y}_{C,f} = 0$ if $C \setminus f$ intersects R. Thus, for every $\bar{y}_{C,f} > 0$, $f \in R$ and $C \setminus f \in \mathcal{P}(r)$. Since $G+H$ is series–parallel, system (Cut cone) is box-TDI and $(\bar{y}, \bar{z}^1, \bar{z}^2)$ may be assumed integer. The latter then corresponds to an optimal solution to the trader multiflow problem. Finally, since $\hat{\ell}$ and $\hat{u}$ are integer, the box-TDIness of system (Cut cone) implies that the optimal solution of (P) is integer, that is, a box-multicut of $G+H$ within $[\hat{\ell}, \hat{u}]$. $\square$

Min–max relations involving min-cost multiflow and maximum multiflow stem from Corollary 6 since the transformations described in Section 3.1 preserve series-parallelness. In particular, Corollary 6 implies that the two following min–max relations of [16] that hold if $G+H$ is series–parallel:

- the maximum multiflow equals the minimum separating multicut,
- the minimum multiflow loss equals the maximum multicut,

where the *minimum multiflow loss problem* asks to remove a minimum number of demands of H to ensure the existence of a multiflow in $G+H$.

Applying the arguments used in the proof of $(i) \Rightarrow (ii)$ of Theorem 1, it can be shown that optimizing over (D') amounts to optimize over an integer polyhedron similar to $\text{proj}_z(Q)$. For series–parallel graphs, optimizing over such a polyhedron is polynomial-time solvable [20,21]. It yields an increase of capacities and demands which maximizes the objective function and ensures that the cut condition is satisfied. Then, applying Theorem 8.1 of [14] provides an optimal solution to the trader multiflow problem. To sum up, we have the following complexity result.

Corollary 7. *If* $G+H$ *is series–parallel, then the maximum trader multiflow problem on* (G, H, c, d, ℓ, u) *is polynomial-time solvable for all vectors* ℓ *and* u *and for all integer vectors* c *and* d.

As seen in Corollary 7, our approach yields a polynomial algorithm, however it relies on the ellipsoid method. We conclude with the question: is there a combinatorial algorithm that solves the trader multiflow problem in series–parallel graphs?

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Appendix

The proof of Theorem 1 is based on Claim 2 which is a direct consequence of the following result.

Lemma 8. *Let $G = (V, E)$ be a graph. The polyhedron $P(G, c)$ defined by*

$$P(G, c) := \{(x, y) \in \mathbb{R}_+^{E\times E} : x(D) - y(D) \leq c(D), \text{ for each cut } D \text{ of } G\}$$

is integer for all integer weights $c \in \mathbb{Z}^E$ if and only if G is series–parallel.

Proof. *Necessity.* First, note that $P(\hat{G}, \hat{c})$ has a fractional extreme point if $\hat{G}$ is the complete graph K_4 with cost $\hat{c}_e = -1$ on the three edges of a triangle and $\hat{c}_e = +1$ on the remaining star. Indeed, the point $\hat{p} = (\hat{x}, \hat{y})$ defined by $\hat{y}_e = 1/2$ for the edges of the triangle and zero elsewhere is the unique optimal solution of maximizing $\hat{\ell}^\top x - \hat{u}^\top y$ over $P(\hat{G}, \hat{c})$, where $\hat{\ell}$ is zero and $\hat{u}$ is the all-one vector. Now, let $\bar{G}$ be a graph which is not series–parallel, then, by [17], it has a K_4-minor, that is we can remove and contract some edges of $\bar{G}$ to obtain K_4. Let us extend $(\hat{c}, \hat{\ell}, \hat{u})$ to $(\bar{c}, \bar{\ell}, \bar{u})$ by defining $\bar{\ell}_e = -\infty$ and $\bar{u}_e = +\infty$ for the new edges e, with $\bar{c}_e = +\infty$ if e must be contracted, and $\bar{c}_e = 0$ if it must be deleted. Clearly, the point $\bar{p}$ obtained by extending $\hat{p}$ with zero components is the unique optimal solution of maximizing $\bar{\ell}^\top x - \bar{u}^\top y$ over $P(\bar{G}, \bar{c})$.

Sufficiency. By contradiction, let (G, c) be a counter-example with a minimum number of edges. Throughout, $\bar{p} = (\bar{x}, \bar{y})$ will denote some fractional extreme point of $P(G, c)$ and

$$\bar{b} := c - \bar{x} + \bar{y}.$$

Note that $\bar{b}(D) \geq 0$, for each cut D.

First, note that G has no loops or bridges. Indeed, a loop belongs to no cut, and a bridge e appears exactly in three nonredundant constraints, namely $x_e \geq 0$, $y_e \geq 0$ and $y_e - x_e \geq c_e$, two of which are satisfied with equality by any extreme point.

Moreover, $P(G, c)$ is full-dimensional. To see this, observe that the point $p = (x, y) \in \mathbb{R}^{E\times E}$ defined by $x_e = 1$ and $y_e = +\infty$ for all $e \in E$ belongs to $P(G, c)$. Moreover, for each edge $e \in E$, the point p_e^x (resp. p_e^y) obtained from p by resetting x_e to zero (resp. y_e to zero) also belongs to $P(G, c)$ since each cut has size at least two. The $2|E| + 1$ points p, p_e^x, p_e^y, for $e \in E$, are affinely independent, hence the dimension of $P(G, c)$ is $2|E|$.

In consequence, the point $\bar{p}$ is the solution of a system of $2|E|$ equations of the following type, where the left-hand-side forms a full-rank matrix.

$$\bar{x}_e = 0 \quad \text{for some edges } e, \tag{A.1}$$

$$\bar{y}_e = 0 \quad \text{for some edges } e, \tag{A.2}$$

$$\bar{x}(D) - \bar{y}(D) = c(D) \quad \text{for some bonds } D \neq \emptyset. \tag{A.3}$$

Suppose G has two parallel edges $\bar{e}$ and $\bar{f}$. Then, replacing $(\bar{x}_{\bar{e}}, \bar{y}_{\bar{e}})$ by $(\bar{x}_{\bar{e}}, \bar{y}_{\bar{e}}) + (\bar{x}_{\bar{f}}, \bar{y}_{\bar{f}})$ and $(\bar{x}_{\bar{f}}, \bar{y}_{\bar{f}})$ by $(0, 0)$ yields a feasible point $(\tilde{x}, \tilde{y})$ because $\bar{e}$ and $\bar{f}$ belong to the same cuts. This point $(\tilde{x}, \tilde{y})$ satisfies all Eqs. (A.1)–(A.3) except possibly the Eqs. (A.1) and (A.2) associated with $\bar{e}$. But these two equations are not satisfied only if $\bar{x}_{\bar{f}} > 0$ or $\bar{y}_{\bar{f}} > 0$ respectively. This implies that $(\tilde{x}, \tilde{y})$ satisfies $2|E|$ equations among (A.1)–(A.3), $\bar{x}_{\bar{f}} = 0$, and $\bar{y}_{\bar{f}} = 0$. Hence, it is also an extreme point of $P(G, c)$. Therefore resetting $c_{\bar{e}} := c_{\bar{e}} + c_{\bar{f}}$ and removing $\bar{f}$ gives a counter-example with a smaller number of edges, a contradiction. We have just proved the following.

$$G \text{ has no parallel edges.} \tag{A.4}$$

Note that, if both $\bar{x}_e > 0$ and $\bar{y}_e > 0$ for some edge e, then one could reset $\bar{x}_e := \bar{x}_e - \varepsilon$ and $\bar{y}_e := \bar{y}_e - \varepsilon$ (for some $\varepsilon > 0$) and still satisfy (A.1)–(A.3), contradicting the extremality of $\bar{p}$. Thus,

$$\text{for all } e, \text{ either } \bar{x}_e = 0 \text{ or } \bar{y}_e = 0. \tag{A.5}$$

We can choose c so as to minimize the norm of $\bar{p}$ (*e.g.* Euclidean). Consequently, nonzero coordinates of $\bar{p}$ are fractional. Indeed, we have

$$\mathbf{0} \leq \bar{p} < \mathbf{1}, \tag{A.6}$$

as otherwise, if $\bar{x}_e \geq 1$ (resp. $\bar{y}_e \geq 1$) for some edge e, then (A.1)–(A.3) would still be satisfied after resetting $\bar{x}_e := \bar{x}_e - 1$ and $c_e := c_e - 1$ (resp. $\bar{y}_e := \bar{y}_e - 1$ and $c_e := c_e + 1$).

By (A.4) and by construction of series–parallel graphs, there are two edges $\bar{e}$ and $\bar{f}$ in series. We may assume w.l.o.g. that $\bar{b}_{\bar{e}} \leq \bar{b}_{\bar{f}}$. Since $\bar{D} = \{\bar{e}, \bar{f}\}$ is a cut, we have $\bar{b}_{\bar{f}} \geq -\bar{b}_{\bar{e}}$. Denote by $\hat{p} = (\hat{x}, \hat{y}) \in \mathbb{R}^{E\setminus\{\bar{f}\}\times E\setminus\{\bar{f}\}}$ the restriction of $\bar{p}$ to $E\setminus\{\bar{f}\}\times E\setminus\{\bar{f}\}$, and let $\hat{G}$ be the graph obtained from G by contracting $\bar{f}$, and $\hat{c}$ the restriction of c to $E \setminus \{\bar{f}\}$. Clearly, $\hat{p}$ belongs to $P(\hat{G}, \hat{c})$, and the latter is full-dimensional since neither loops nor bridges appeared in $\hat{G}$.

Moreover, since c is integer and $\bar{p}$ fractional, (A.3) and (A.5) imply that at least two edges have a fractional $\bar{x}$ or $\bar{y}$ coordinate. Therefore $\hat{p}$ is fractional, and hence, by minimality of $|E|$, $\hat{p}$ is not an extreme point of $P(\hat{G}, \hat{c})$.

Remark that in fact we have:

$$\bar{b}_{\bar{f}} = |\bar{b}_{\bar{e}}| \tag{A.7}$$

If it is not true, then $\bar{p}$ does not saturate the constraint associated to $\bar{D}$, and moreover $\bar{b}_{\bar{f}} > \bar{b}_{\bar{e}}$. Hence, except maybe for $\bar{x}_{\bar{f}} = 0$ or $\bar{y}_{\bar{f}} = 0$, the edge $\bar{f}$ appears in no equation among (A.1)–(A.3). Then $\hat{p}$ is an extreme point, a contradiction.

By the integrality of c, a direct consequence of (A.5)–(A.7) is that:

$$\text{Exactly one of } \bar{x}_{\bar{e}}, \bar{y}_{\bar{e}} \text{ is fractional} \iff \text{exactly one of } \bar{x}_{\bar{f}}, \bar{y}_{\bar{f}} \text{ is fractional.} \tag{A.8}$$

Since $\hat{p}$ is not extreme, there is a (nonzero) direction $\hat{d} = (\hat{d}^x, \hat{d}^y) \in \mathbb{R}^{E\setminus\{\bar{f}\}\times E\setminus\{\bar{f}\}}$ and an $\varepsilon > 0$ such that

$$\hat{p} = \frac{1}{2}(\hat{p} + \varepsilon \cdot \hat{d}) + \frac{1}{2}(\hat{p} - \varepsilon \cdot \hat{d})$$

where both $\hat{p} + \varepsilon \cdot \hat{d}$ and $\hat{p} - \varepsilon \cdot \hat{d}$ belong to $P(\hat{G}, \hat{c})$. Extend the direction $\hat{d} = (\hat{d}^x, \hat{d}^y) \in \mathbb{R}^{E\setminus\{\bar{f}\}\times E\setminus\{\bar{f}\}}$ to a direction $\bar{d} = (\bar{d}^x, \bar{d}^y) \in \mathbb{R}^{E\times E}$ by arbitrarily defining the two missing components $\bar{d}^x_{\bar{f}}$ and $\bar{d}^y_{\bar{f}}$. So

$$\bar{p} = \frac{1}{2}(\bar{p} + \varepsilon \cdot \bar{d}) + \frac{1}{2}(\bar{p} - \varepsilon \cdot \bar{d}) \qquad \forall \varepsilon > 0$$

where the points $\bar{p}^+ = \bar{p} + \varepsilon \cdot \bar{d}$ and $\bar{p}^- = \bar{p} - \varepsilon \cdot \bar{d}$ are different. Since $\bar{p}$ is extreme, we can assume that $\bar{p}^+ = (\bar{x}^+, \bar{y}^+) \notin P(G, c)$. Clearly, we have

$$\bar{x}_{\bar{e}} = 0 \text{ (resp. } \bar{y}_{\bar{e}} = 0) \text{ implies } \bar{d}^x_{\bar{e}} = 0 \text{ (resp. } \bar{d}^y_{\bar{e}} = 0). \tag{A.9}$$

Define $\bar{b}^+ := c - \bar{x}^+ + \bar{y}^+$. By (A.7), there are two cases.

Case 1: $\bar{b}_{\bar{e}} = \bar{b}_{\bar{f}} \geq 0$.

Define

$$\bar{d}^x_{\bar{f}} = \begin{cases} \bar{d}^x_{\bar{e}} - \bar{d}^y_{\bar{e}} & \text{if } \bar{x}_{\bar{f}} > 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \bar{d}^y_{\bar{f}} = \begin{cases} \bar{d}^y_{\bar{e}} - \bar{d}^x_{\bar{e}} & \text{if } \bar{y}_{\bar{f}} > 0 \\ 0 & \text{otherwise.} \end{cases}$$

By definition of $\bar{d}$, and by (A.8)–(A.9), we have

$$\bar{b}^+_{\bar{e}} - \bar{b}_{\bar{e}} = (\bar{y}^+_{\bar{e}} - \bar{y}_{\bar{e}}) - (\bar{x}^+_{\bar{e}} - \bar{x}_{\bar{e}}) = \varepsilon(\bar{d}^y_{\bar{e}} - \bar{d}^x_{\bar{e}}) = (\bar{y}^+_{\bar{f}} - \bar{y}_{\bar{f}}) - (\bar{x}^+_{\bar{f}} - \bar{x}_{\bar{f}}) = \bar{b}^+_{\bar{f}} - \bar{b}_{\bar{f}}.$$

Therefore, $\bar{b}^+_{\bar{e}} = \bar{b}^+_{\bar{f}}$. By (A.9), choosing a small enough ε ensures the nonnegativity of $\bar{p}^+$. Since $\bar{p}^+$ does not belong to $P(G, c)$, we get that $\bar{p}^+$ violates $x(\bar{D}) - y(\bar{D}) \leq c(\bar{D})$, that is,

$$\bar{b}^+_{\bar{e}} + \bar{b}^+_{\bar{f}} = \bar{b}_{\bar{e}} + \bar{b}_{\bar{f}} + 2\varepsilon(\bar{d}^y_{\bar{e}} - \bar{d}^x_{\bar{e}}) < 0, \qquad \forall \varepsilon > 0 \tag{A.10}$$

Notice that exactly one of $\bar{x}_{\bar{e}}$ and $\bar{y}_{\bar{e}}$ is fractional, as otherwise (A.9) would imply $\bar{d}^x_{\bar{e}} = \bar{d}^y_{\bar{e}} = 0$, and then (A.10) would give the contradiction $\bar{b}(\bar{D}) < 0$. Consequently, we have $\bar{b}_{\bar{e}} + \bar{b}_{\bar{f}} > 0$, a contradiction to the fact that (A.10) holds for all $\epsilon > 0$. This settles Case 1.

Case 2: $\bar{b}_{\bar{e}} = -\bar{b}_{\bar{f}} < 0$.

Define

$$\bar{d}^x_{\bar{f}} = \begin{cases} \bar{d}^y_{\bar{e}} - \bar{d}^x_{\bar{e}} & \text{if } \bar{x}_{\bar{f}} > 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \bar{d}^y_{\bar{f}} = \begin{cases} \bar{d}^x_{\bar{e}} - \bar{d}^y_{\bar{e}} & \text{if } \bar{y}_{\bar{f}} > 0 \\ 0 & \text{otherwise.} \end{cases}$$

By definition of $\bar{d}$, and by (A.8)–(A.9), we have $\bar{b}^+_{\bar{e}} - \bar{b}_{\bar{e}} = \varepsilon(\bar{d}^y_{\bar{e}} - \bar{d}^x_{\bar{e}}) = (\bar{x}^+_{\bar{f}} - \bar{x}_{\bar{f}}) - (\bar{y}^+_{\bar{f}} - \bar{y}_{\bar{f}}) = \bar{b}_{\bar{f}} - \bar{b}^+_{\bar{f}}$. Therefore, $\bar{b}^+_{\bar{f}} = -\bar{b}^+_{\bar{e}}$.

In particular, $\bar{p}^+$ satisfies the constraint of the cut $\bar{D}$, and since nonnegativity is ensured, then $\bar{p}^+$ violates the constraint of a cut D containing $\bar{f}$ but not $\bar{e}$, that is

$$\bar{b}^+(D) = \bar{b}(D) + \varepsilon(\bar{d}^y(D) - \bar{d}^x(D)) < 0 \qquad (\forall \varepsilon > 0) \tag{A.11}$$

Since $D' = D \cup \{\bar{e}\} \setminus \{\bar{f}\}$ is a cut, we have $\bar{b}(D') \geq 0$, thus $\bar{b}(D) = \bar{b}(D') - \bar{b}_{\bar{e}} + \bar{b}_{\bar{f}} > 0$. This contradiction to (A.11) finishes the proof. $\square$

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The Schrijver system of the flow cone in series–parallel graphs

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ABSTRACT

We represent a *flow* of a graph $G = (V, E)$ as a couple (C, e) with C a circuit of G and e an edge of C, and its incidence vector is the $0/\pm 1$ vector $\chi^{C\setminus e} - \chi^e$. The *flow cone* of G is the cone generated by the flows of G and the unit vectors.

When G has no K_5-minor, this cone can be described by the system $x(M) \geq 0$ for all multicuts M of G. We prove that this system is box-totally dual integral if and only if G is series–parallel. Then, we refine this result to provide the Schrijver system describing the flow cone in series–parallel graphs.

This answers a question raised by Chervet et al., (2018).

1. Introduction

Totally dual integral systems were introduced in the late 70s and are strongly connected to min–max relations in combinatorial optimization [16]. A rational system of linear inequalities $Ax \leq b$ is *totally dual integral (TDI)* if the minimization problem in the linear programming duality:

$$\max\{cx\colon Ax \leq b\} = \min\{yb\colon y \geq \mathbf{0}, yA = c\}$$

admits an integer optimal solution for each integer vector c such that the maximum is finite. Such systems describe integer polyhedra when b is integer [13]. Schrijver [15] proved that every full-dimensional polyhedron is described by a unique minimal TDI system $Ax \leq b$ with A integer—its *Schrijver system* [6].

A stronger property is the box-total dual integrality, where a system $Ax \leq b$ is *box-totally dual integral (box-TDI)* if

$$Ax \leq b, \quad \ell \leq x \leq u$$

is TDI for all rational vectors ℓ and u (with possible infinite components). General properties of such systems can be found in Cook [5] and Chapter 22.4 of Schrijver [16]. Note that, although every rational polyhedron $\{x : Ax \leq b\}$ is described by a TDI system $\frac{1}{k}Ax \leq \frac{1}{k}b$, for some integer k, not every polyhedron is described by a box-TDI system. A polyhedron

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described by a box-TDI system is called a *box-TDI polyhedron*. As proved by Cook [5], every TDI system describing such a polyhedron is actually box-TDI.

In the last decade, several new box-TDI systems were exhibited. Chen, Ding, and Zang [1] characterized box-Mengerian matroid ports. In [2], they provided a box-TDI system describing the 2-edge-connected spanning subgraph polyhedron for series–parallel graphs. Ding, Tan, and Zang [10] characterized the graphs for which the TDI system of Cunningam and Marsh [9] describing the matching polytope is actually box-TDI. Ding, Zang, and Zhao [11] introduced new subclasses of box-perfect graphs. Cornaz, Grappe, and Lacroix [8] provided several box-TDI systems in series–parallel graphs. Recently, Chervet, Grappe, and Robert [3] gave new geometric characterizations of box-TDI polyhedra.

As mentioned by Pulleyblank [14], it is not uncommon that the minimal integer system and the Schrijver system of a polyhedron coincide. This is the case of the matching polytope and matroid polyhedra. However, this does not hold in general, as shown by Cook [4] and Pulleyblank [14] for the b-matching polyhedron, and by Sebő [18] for the T-join polyhedron.

In this paper, we are interested in TDI, box-TDI, and Schrijver systems for the flow cone of series–parallel graphs. Given a graph $G = (V, E)$, a *flow* of G is a couple (C, e) with C a circuit of G and e an edge of C. In a flow (C, e), the edge e represents a demand and $C \setminus e$ represents the path satisfying this demand. The *incidence vector* of a flow (C, e) is the $0/\pm1$ vector $\chi^{C\setminus e} - \chi^e$. The *flow cone* of G is the cone generated by the flows of G and the unit vectors χ^e of $\mathbb{R}^E$.

The *cut* $\delta(W)$ is the set of edges having exactly one endpoint in a subset W of V. A *bond* is an inclusionwise minimal nonempty cut. Note that a nonempty cut is the disjoint union of bonds. Given a partition $\{V_1, \dots, V_k\}$ of V, the set of edges having endpoints in two distinct V_i's is called *multicut* and is denoted by $\delta(V_1, \dots, V_k)$. The *cut cone* of G is the cone generated by the incidence vectors of the cuts of G. Equivalently, it is the cone generated by the incidence vectors of the bonds of G, or by those of the multicuts of G.

When G has no K_5-minor, the flow cone of G is the polar of the cut cone and is described by $x(C) \geq 0$, for all cuts C of G [19]. Chervet, Grappe, and Robert [3] proved that the flow cone is a box-TDI polyhedron if and only if the graph is series–parallel. Moreover they provided the following box-TDI system:

$$\frac{1}{2}x(B) \geq 0 \quad \text{for all bonds } B \text{ of } G. \tag{1}$$

Quoting them, they "*leave open the question of finding a box-TDI system with integer coefficients, which exists by [16, Theorem 22.6(i)] and [5, Corollary 2.5]*".

Contribution. The goal of this paper is to answer the question of [3] mentioned above. Throughout, the main concept that we use is that of Hilbert basis, whose definition and connection with TDIness are given at the end of the introduction.

We first prove that

$$x(M) \geq 0 \quad \text{for all multicuts } M \text{ of } G, \tag{2}$$

is a TDI system describing the flow cone if and only if the graph is series–parallel. As the flow cone is a box-TDI polyhedron for such graphs, this implies that System (2) is a box-TDI system if and only if the graph is series–parallel. We then refine this result by providing the corresponding Schrijver system, which is composed of the so-called chordal multicuts—see Corollary 3.4.

This completely answers the question of [3].

Outline. In the next paragraph, we provide definitions and notation. In Section 2, we first characterize the graphs for which multicuts form a Hilbert basis. It follows that System (2) is box-TDI precisely for series–parallel graphs. In Section 3, we provide a minimal integer Hilbert basis for multicuts in series–parallel graphs. This gives the Schrijver system for the flow cone in series–parallel graphs.

Definitions. Given a finite set S and a subset T of S, we denote by $\chi^T \in \{0, 1\}^S$ the incidence vector of T, that is χ^T_s equals 1 if s belongs to T and 0 otherwise, for all $s \in S$. Since there is a bijection between sets and their incidence vectors, we will often use the same terminology for both.

Let $G = (V, E)$ be a loopless undirected graph. Given $U \subseteq V$, the graph $G[U]$ is obtained from G by removing all the vertices not in U. A set of edges M is a multicut if and only if $|M \cap C| \neq 1$ for all circuits C of G—see e.g. [7]. The *reduced graph* of a multicut M is the graph G_M obtained by contracting all the edges of $E \setminus M$. Note that a multicut of G_M is also a multicut of G. We denote respectively by $\mathcal{M}_G$ and $\mathcal{B}_G$ the set of multicuts and the set of bonds of G. A subset of edges of G is called a *circuit* if it induces a connected graph in which every vertex has degree 2. Given a circuit C, an edge of G is a *chord* of C if its endpoints are two nonadjacent vertices of C. A graph is 2-connected if it remains connected whenever a vertex is removed.

A graph is *series–parallel* if its 2-connected components either consist of a single edge or can be constructed from the circuit of length two C_2 by repeatedly adding edges parallel to an existing one, and subdividing edges, that is, replacing an edge by a path of length two. Series–parallel graphs are those having no K_4-minor [12]. A graph is *chordal* if every circuit of length 4 or more has a chord.

The cone $\mathcal{C}$ *generated* by a set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ of $\mathbb{R}^n$ is the set of nonnegative combinations of $\mathbf{v}_1, \dots, \mathbf{v}_k$, that is, $\mathcal{C} = \left\{\sum_{j=1}^k \lambda_j \mathbf{v}_j : \lambda_1, \dots, \lambda_k \geq 0\right\}$. A set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is a *Hilbert basis* if each integer vector in their cone can

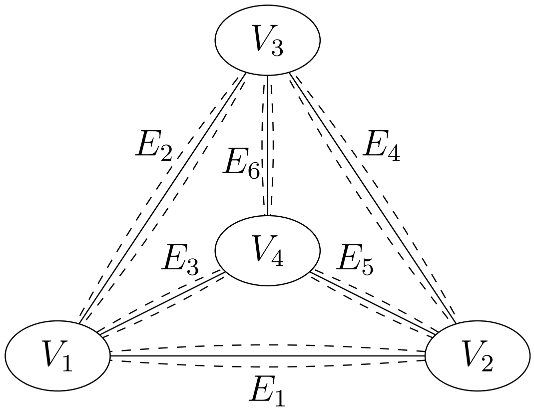

Fig. 1. Edges in the figure represent sets of edges of G having endpoints in distinct V_i's. Solid lines depict $e_1, \ldots, e_6$ given in the proof of Theorem 2.1.

be expressed as a nonnegative integer combination of $\mathbf{v}_1, \ldots, \mathbf{v}_k$. A Hilbert basis is *integer* if it is composed of integer vectors, and it is a *minimal integer Hilbert basis* if it has the smallest number of vectors among all integer Hilbert basis generating the same cone. Each pointed rational cone has a unique minimal integer Hilbert basis [15, Theorems 16.4]. The link between Hilbert basis and TDIness is in the following result.

Theorem 1.1 (*Corollary 22.5a of [16]*)**.** *A system $Ax \geq \mathbf{0}$ is TDI if and only if the rows of A form a Hilbert basis.*

2. When do multicuts form a Hilbert basis?

2.1. Characterization

The following result characterizes the graphs for which the multicuts form a Hilbert basis.

Theorem 2.1. *The multicuts of a graph form a Hilbert basis if and only if the graph is series–parallel.*

Proof. First, let us show that the incidence vectors of the multicuts of a non series–parallel graph do not form a Hilbert basis. Suppose that $G = (V, E)$ has K_4 as a minor. Without loss of generality, we may assume G connected. Then, V can be partitioned into four sets $\{V_1, \ldots, V_4\}$ such that V_i induces a connected subgraph and at least one edge connects each pair V_i, V_j for $i, j = 1, \ldots, 4$. We subdivide $\delta(V_1, V_2, V_3, V_4)$ into $E_1, \ldots, E_6$ as in Fig. 1.

Let $\hat{E} = \{e_1, \ldots, e_6\}$ where $e_i \in E_i$ for all $i = 1, \ldots, 6$, and let $\mathbf{w} \in \mathbb{Z}^E$ be as follows:

$$\mathbf{w}_e = \begin{cases} 2 & \text{if } e \in E_1, \\ 1 & \text{if } e \in E_2, \ldots, E_6, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\mathbf{w} = \frac{1}{2}\chi^{\delta(V_1)} + \frac{1}{2}\chi^{\delta(V_2)} + \frac{1}{2}\chi^{\delta(V_1 \cup V_3)} + \frac{1}{2}\chi^{\delta(V_1 \cup V_4)}$, it belongs to the cut cone of G. Moreover, $\mathbf{w}^\top \chi^{\hat{E}} = 7$. Any conic combination of multicuts yielding $\mathbf{w}$ involves only multicuts contained in $\delta(V_1, \ldots, V_4)$. Each of these multicuts contains between 3 and 6 edges of $\hat{E}$. Hence, if $\mathbf{w}$ is an integer combination of such multicuts, it is the sum of two multicuts containing 3 and 4 edges of $\hat{E}$, respectively. This means that $\mathbf{w}$ is the sum of $\chi^{\delta(V_i)}$ and $\chi^{\delta(V_i, V_j)}$ for some $i \neq j$. Since $\mathbf{w}_{e_1} = 2$, we have $i \in \{1, 2\}$ and $j \in \{3, 4\}$. But then $\delta(V_i) \cap \delta(V_i, V_j)$ contains an edge among $e_2, \ldots, e_5$, a contradiction with $\mathbf{w}_{e_2} = \mathbf{w}_{e_3} = \mathbf{w}_{e_4} = \mathbf{w}_{e_5} = 1$.

Therefore, $\mathbf{w}$ is not an integer combination of multicuts, implying that the set of multicuts of G is not a Hilbert basis.

For the other direction, remark that each multicut of a series–parallel graph is the disjoint union of multicuts of its 2-connected components. Since they belong to disjoint spaces, if the set of multicuts of each 2-connected component forms a Hilbert basis, then so does their union. Hence, it is enough to prove that the multicuts of a 2-connected series–parallel graph form a Hilbert basis. From now on, assume the graph to be 2-connected.

We prove the result by induction on the number of edges of G. When $G = (\{u, v\}, \{e, f\})$ is the circuit of length two, the only nonempty multicut is $\{e, f\}$, and its incidence vector forms a Hilbert basis. Similarly, when G consists of a single edge, its incidence vector forms a Hilbert basis.

Now, let $\tilde{G} = (\tilde{V}, \tilde{E})$ be obtained from a 2-connected series–parallel graph $G = (V, E)$ by either adding a parallel edge or subdividing an edge. By the induction hypothesis, $\mathcal{M}_G$ is a Hilbert basis.

Suppose first that $\tilde{G}$ is obtained from G by adding an edge f parallel to an edge e of E. A subset of edges M of G containing (respectively not containing) e is a multicut if and only if $M \cup f$ (respectively M) is a multicut of $\tilde{G}$. Thus, the

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incidence vector of each multicut of $\tilde{G}$ is obtained by copying the component associated with e in the component of f. Since the incidence vectors of the multicuts of G are a Hilbert basis, so are the incidence vectors of the multicuts of $\tilde{G}$.

Suppose now that $\tilde{G}$ is obtained from G by subdividing an edge $\bar{e} \in E$. We denote by u the new vertex and by f and g the edges adjacent to it. A multicut M of $\tilde{G}$ can be expressed as the half-sum of the bonds of $\tilde{G}$. Moreover, as each bond is a multicut, bonds and multicuts of $\tilde{G}$ generate the same cone: the cut cone. Since System (1) is TDI in series–parallel graphs [3, end of Section 6.4], the set of vectors $\{\frac{1}{2}\chi^B : B \in \mathcal{B}_{\tilde{G}}\}$ forms a Hilbert basis.

Let $\mathbf{v}$ be an integer vector in the cut cone. There exist $\lambda_B \in \frac{1}{2}\mathbb{Z}_+$ for all $B \in \mathcal{B}_{\tilde{G}}$ such that $\mathbf{v} = \sum_{B\in\mathcal{B}_{\tilde{G}}} \lambda_B \chi^B$. The vector $\mathbf{v}$ is an integer combination of multicuts of $\tilde{G}$ if and only if $\mathbf{v} - \lfloor\lambda_{\delta(u)}\rfloor\chi^{\delta(u)}$ is, thus we may assume that $\lambda_{\delta(u)} \in \{0, \frac{1}{2}\}$. Define $\mathbf{w} \in \mathbb{Z}^E$ by:

$$\mathbf{w}_e = \begin{cases} \mathbf{v}_f + \mathbf{v}_g - 2\lambda_{\delta(u)} & \text{if } e = \bar{e}, \\ \mathbf{v}_e & \text{otherwise.} \end{cases}$$

Remark that $(B \setminus \bar{e}) \cup f$ and $(B \setminus \bar{e}) \cup g$ are bonds of $\tilde{G}$ whenever B is a bond of G containing $\bar{e}$. Moreover, a bond B of G which does not contain $\bar{e}$ is a bond of $\tilde{G}$. Since $\delta(u)$ is the unique bond of $\tilde{G}$ containing both f and g, we have:

$$\mathbf{w} = \sum_{B\in\mathcal{B}_G:\bar{e}\in B} (\lambda_{(B\setminus\bar{e})\cup f} + \lambda_{(B\setminus\bar{e})\cup g})\chi^B + \sum_{B\in\mathcal{B}_G:\bar{e}\notin B} \lambda_B\chi^B.$$

Thus, $\mathbf{w}$ belongs to the cut cone of G. Moreover, as $\lambda_{\delta(u)}$ is half-integer, $\mathbf{w}$ is integer. By the induction hypothesis, $\mathcal{M}_G$ is a Hilbert basis, hence there exist $\mu_M \in \mathbb{Z}_+$ for all $M \in \mathcal{M}_G$ such that $\mathbf{w} = \sum_{M\in\mathcal{M}_G} \mu_M\chi^M$. Consider the family $\mathcal{N}$ of multicuts of G where each multicut M of G appears μ_M times.

Suppose first that $\lambda_{\delta(u)} = 0$. Then, $\mathbf{v}_f + \mathbf{v}_g$ multicuts of $\mathcal{N}$ contain $\bar{e}$. Let $\mathcal{P}$ be a family of $\mathbf{v}_f$ multicuts of $\mathcal{N}$ containing $\bar{e}$ and $\mathcal{Q} = \{M \in \mathcal{N} : \bar{e} \in M\} \setminus \mathcal{P}$. Then, we have

$$\mathbf{v} = \sum_{M\in\mathcal{N}:\bar{e}\notin M} \chi^M + \sum_{M\in\mathcal{P}} \chi^{(M\setminus\bar{e})\cup f} + \sum_{M\in\mathcal{Q}} \chi^{(M\setminus\bar{e})\cup g},$$

hence $\mathbf{v}$ is a nonnegative integer combination of multicuts of $\tilde{G}$.

Suppose now that $\lambda_{\delta(u)} = \frac{1}{2}$. Then, $\mathbf{v}_f + \mathbf{v}_g - 1$ multicuts of $\mathcal{N}$ contain $\bar{e}$. Let $\mathcal{P}$ be a family of $\mathbf{v}_f - 1$ multicuts of $\mathcal{N}$ containing $\bar{e}$, let $\mathcal{Q}$ be a family of $\mathbf{v}_g - 1$ multicuts in $\{M \in \mathcal{N} : \bar{e} \in M\} \setminus \mathcal{P}$, and denote by N the unique multicut of $\mathcal{N}$ containing $\bar{e}$ which is not in $\mathcal{P} \cup \mathcal{Q}$. Then, we have

$$\mathbf{v} = \sum_{M\in\mathcal{N}:\bar{e}\notin M} \chi^M + \sum_{M\in\mathcal{P}} \chi^{(M\setminus\bar{e})\cup f} + \sum_{M\in\mathcal{Q}} \chi^{(M\setminus\bar{e})\cup g} + \chi^{N\setminus\bar{e}\cup\{f,g\}}.$$

Hence $\mathbf{v}$ is a nonnegative integer combination of multicuts of $\tilde{G}$. This proves that $\mathcal{M}_{\tilde{G}}$ is a Hilbert basis. □

2.2. An integer box-TDI system for the flow cone in series–parallel graphs

Combining the box-TDIness of the flow cone and Theorems 1.1 and 2.1 yields a box-TDI system for the flow cone of a series–parallel graph with only integer coefficients. This provides a first answer to the question of [3].

Corollary 2.2. *The following statements are equivalent:*

i. G is a series–parallel graph,
ii. System (2) is TDI,
iii. System (2) is box-TDI.

Proof (*i.*⇔*ii.*)**.** This equivalence follows by combining Theorems 1.1 and 2.1.

(*ii.*⇔*iii.*) If G is series–parallel, then System (1) is box-TDI [3, end of Section 6.4]. Hence, the flow cone of G is box-TDI. Since a TDI system describing a box-TDI polyhedron is a box-TDI system [5], point *ii.* implies point *iii.*. A box-TDI system being TDI by definition, point *iii.* implies point *ii.*. □

3. Which multicuts form Hilbert basis?

3.1. A minimal integer Hilbert basis

Theorem 2.1 provides the set of graphs whose multicuts form a Hilbert basis. The following theorem refines this result by characterizing the multicuts which form the minimal Hilbert basis.

A multicut is *chordal* when its reduced graph is 2-connected and chordal. Note that bonds are chordal multicuts.

Theorem 3.1. *The chordal multicuts of a series–parallel graph form a minimal integer Hilbert basis.*

Proof. Let $G = (V, E)$ be a series–parallel graph. By Theorem 2.1, the multicuts of G form an integer Hilbert basis. Hence, the minimal integer Hilbert basis is composed of the multicuts which are not disjoint union of other multicuts. These multicuts are characterized in the following lemma, from which stems the desired theorem.

Lemma 3.2. *A multicut of a series–parallel graph G is chordal if and only if it cannot be expressed as the disjoint union of other nonempty multicuts.*

Proof. Let M be a multicut of G. Recall that every multicut of G_M is a multicut of G. Besides, since the disjoint union of multicuts is a multicut, a disjoint union of nonempty multicuts is actually the disjoint union of two nonempty multicuts.

We first prove that, if G_M is 2-connected and chordal, then M is not the disjoint union of two nonempty multicuts. By contradiction, suppose that G_M is 2-connected and chordal, and $M = M_1 \cup M_2$ where M_1, M_2 are disjoint multicuts of G_M. If C is a circuit of length at most three in G_M, then $C \subseteq M_i$ for some $i = 1, 2$. Indeed, the edges of C are partitioned by M_1 and M_2, and a multicut and a circuit intersect in either none or at least two edges.

Since G_M is 2-connected and M_i is nonempty for $i = 1, 2$, there exists at least a circuit containing edges of both M_1 and M_2. Let C be such a circuit, of smallest length. Then, C has length at least 4, as otherwise it would be contained in one of M_1 and M_2. Since G_M is chordal, there exists a chord c of C. Denote by P_1 and P_2 the two paths of C between the endpoints of c. For $i = 1, 2$, the circuit $P_i \cup \{c\}$ is strictly shorter than C. Since C is the shortest circuit intersecting both M_1 and M_2, we get that $P_i \cup \{c\} \subseteq M_i$ for $i = 1, 2$. But then $c \in M_1 \cap M_2$, a contradiction.

To prove the other direction, first suppose that G_M is not 2-connected. Then, the set of edges of each 2-connected component of G_M is a multicut of G, and M is the disjoint union of these multicuts. Now, suppose that G_M is not chordal, that is, G_M contains a chordless circuit C of length at least 4. We will apply the following.

Claim 3.3. *Let C be a circuit of length at least 4 in a series–parallel graph G. Then, there exists a pair of vertices nonadjacent in $G[V(C)]$ whose removal disconnects G.*

Proof. We can assume that there are two nonadjacent vertices u and v of $G[V(C)]$ such that there exists a path P between u and v that has no internal vertex in C. Indeed, otherwise, removing any two nonadjacent vertices of $G[V(C)]$ would disconnect G.

Let us show that removing u and v disconnects G. Denote by Q and R the two paths of C between u and v. By contradiction, suppose that $G \setminus \{u, v\}$ is connected. Then, there exists a path containing neither u nor v between an internal vertex of R and an internal vertex of either P or Q. Let S be a minimal path of this kind. Then, no internal vertex of S belongs to P, Q, or R, and the subgraph composed of P, Q, R and S is a subdivision of K_4. This contradicts the hypothesis that G is series–parallel. □

By Claim 3.3 there exist two vertices u and v of C, nonadjacent in $G[V(C)]$, whose removal disconnects G. Denote by $V_1, \ldots, V_k$ the sets of vertices of the connected components of $G \setminus \{u, v\}$. Let $G_i = G[V_i \cup \{u, v\}]$ and denote by $E(G_i)$ the set of edges of G_i, for $i = 1, \ldots, k$. Note that, since u and v are not adjacent, $E(G_i) \cap E(G_j) = \emptyset$ for all distinct i and j. Thus, M is the disjoint union of $E(G_1), \ldots, E(G_k)$.

Let us prove that $E(G_i)$ is a multicut of G_M, for $i = 1, \ldots, k$. Consider a circuit D of G_M. If D is contained in one of the G_i's, then $|D \cap E(G_j)| \neq 1$ for $j = 1, \ldots, k$. Otherwise, D is the union of two paths from u to v, these paths being contained in two different G_i's. Without loss of generality, let these paths be $P_1 \in G_1$ and $P_2 \in G_2$. Then, we have $D \cap E(G_i) = P_i$ if $i = 1, 2$, and $\emptyset$ otherwise. Since u and v are not adjacent, the shortest path from u to v in each G_i is of length at least two, hence $|P_i| \geq 2$. Therefore $|D \cap E(G_i)| \neq 1$ for $i = 1, \ldots, k$.

Therefore, $E(G_i)$ is a multicut of G_M, and hence of G, for $i = 1, \ldots, k$. Hence, M is the disjoint union of multicuts of G. □

□

3.2. The Schrijver system of the flow cone in series–parallel graphs

Corollary 2.2 provides an integer box-TDI description of the flow cone in series–parallel graphs. However, this box-TDI description is not minimal: there are redundant inequalities whose removal preserves box-TDIness. Here, we provide the minimal integer box-TDI system for this cone. This completely answers the question of [3, end of Section 6.4].

Corollary 3.4. *The Schrijver system for the flow cone of a series–parallel graph G is the following:*

$$x(M) \geq 0 \quad \text{for all chordal multicuts } M \text{ of } G. \tag{3}$$

Moreover, this system is box-TDI.

Proof. By Theorems 1.1 and 3.1, System (3) is a minimal integer TDI system. Since every bond is a chordal multicut, this system describes the flow cone for series–parallel graphs. Therefore, by [5, Corollary 2.5] and by the flow cone being box-TDI for series–parallel graphs, System (3) is box-TDI. □

We mention that, by planar duality, Corollary 3.4 provides the Schrijver system for the cone of conservative functions [17, Corollary 29.2h] in series–parallel graphs.

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Box-Total Dual Integrality and Edge-Connectivity

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Abstract Given a graph $G = (V, E)$ and an integer $k \geq 1$, the graph $H = (V, F)$, where F is a family of elements (with repetitions allowed) of E, is a k-edge-connected spanning subgraph of G if H cannot be disconnected by deleting any $k - 1$ elements of F. The convex hull of incidence vectors of the k-edge-connected subgraphs of a graph G forms the k-edge-connected subgraph polyhedron of G. We prove that this polyhedron is box-totally dual integral if and only if G is series-parallel. In this case, we also provide an integer box-totally dual integral system describing this polyhedron.

Keywords Box-total dual integrality · k-edge connected subgraph · Polyhedron · Series-parallel graph

1 Introduction

Totally dual integral systems, introduced in the late 70's, are strongly connected to min-max relations in combinatorial optimization [34]. A rational system of linear inequalities $Ax \geq b$ is *totally dual integral (TDI)* if the maximization problem in the linear programming duality

$$\min\{c^\top x : Ax \geq b\} = \max\{b^\top y : A^\top y = c, y \geq \mathbf{0}\}$$

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admits an integer optimal solution for each integer vector c such that the optimum is finite. Every rational polyhedron can be described by a TDI system [28]. For instance, the polyhedron $\{x : Ax \geq b\}$ can be described by TDI systems of the form $\frac{1}{q}Ax \geq \frac{1}{q}b$ for certain positive q. However, a polyhedron is integer if and only if it can be described by a TDI system with only integer coefficients [23, 28]. Integer TDI systems yield min-max results that may have combinatorial interpretation.

A stronger property is box-total dual integrality: a system $Ax \geq b$ is *box-totally dual integral (box-TDI)* if $Ax \geq b, \ell \leq x \leq u$ is TDI for all rational vectors ℓ and u (possibly with infinite components). General properties of such systems can be found in Cook [12] and Chapter 22.4 of Schrijver [34]. Note that, although every rational polyhedron can be described by a TDI system, not every polyhedron can be described by a box-TDI system. A polyhedron which can be described by a box-TDI system is called a *box-TDI polyhedron*. As proved by Cook [12], every TDI system describing such a polyhedron is actually box-TDI.

Recently, several new box-TDI systems have been exhibited. Chen, Ding, and Zang [6] characterized box-Mengerian matroid ports. Ding, Tan, and Zang [18] characterized the graphs for which the Edmonds' system defining the matching polytope [21] is box-TDI. Ding, Zang, and Zhao [19] exhibited new subclasses of box-perfect graphs. Cornaz, Grappe, and Lacroix [14] provided several box-TDI systems in series-parallel graphs. Barbato, Grappe, Lacroix, Lancini, and Wolfler Calvo [3] gave the minimal box-TDI system with integer coefficients for the flow cone for series-parallel graphs. For these graphs, Chen, Ding, and Zang [7] provided a box-TDI system describing the 2-edge-connected spanning subgraph polyhedron.

In this paper, we are interested in integrality properties of systems related to k-edge-connected spanning subgraphs. A *k-edge-connected spanning subgraph* of a graph $G = (V, E)$ is a graph $H = (V, F)$, with F being a collection of elements of E where each element can appear several times, that remains connected after the removal of any $k - 1$ edges.

These objects model a kind of failure resistance of telecommunication networks. More precisely, they represent networks which remain connected when $k - 1$ links fail. The underlying network design problem is the *k-edge-connected spanning subgraph problem* (*k-ECSSP*): given a graph G and positive edge costs, find a k-edge-connected spanning subgraph of G of minimum cost. Special cases of this problem are related to classical combinatorial optimization problems. The 2-ECSSP is a well-studied relaxation of the traveling salesman problem [24] and the 1-ECSSP is nothing but the well-known minimum spanning tree problem. While this latter is polynomial-time solvable, the k-ECSSP is **NP**-hard for every fixed $k \geq 2$ [27].

Different algorithms have been devised in order to deal with the k-ECSSP, such as branch-and-cut procedures [4][15], approximation algorithms [8][26], cutting plane algorithms [30], and heuristics [11]. In [36], Winter introduced a linear-time algorithm solving the 2-ECSSP on series-parallel graphs. Most of these algorithms rely on polyhedral considerations.

Given a graph $G = (V, E)$, the convex hull of incidence vectors of all the families of E inducing a k-edge-connected spanning subgraph of G forms a polyhedron, hereafter called the *k-edge-connected spanning subgraph polyhedron* of G and denoted by $P_k(G)$. Cornuéjols, Fonlupt, and Naddef [16] gave a system describing $P_2(G)$ when G is series-parallel. Vandenbussche and Nemhauser [35] characterized in terms of forbidden minors the graphs for which this system describes $P_2(G)$. Chopra [10] described $P_k(G)$ for outerplanar graphs when k is odd. Didi Biha and Mahjoub [17] extended these results to series-parallel graphs for all $k \geq 2$. By a result of Baïou, Barahona, and Mahjoub [1], the inequalities in these descriptions can be separated in polynomial time, which implies that the k-ECSSP is solvable in polynomial time for series-parallel graphs.

When studying k-edge-connected spanning subgraphs of a graph G, we can add the constraint that each edge of G can be taken at most once. We denote the corresponding polyhedron by $Q_k(G)$. Barahona and Mahjoub [2] described $Q_2(G)$ for Halin graphs. Further polyhedral results for the case $k = 2$ have been obtained by Boyd and Hao [5] and Mahjoub [32][33]. Grötschel and Monma [29] described several classes of facets of $Q_k(G)$. Moreover, Fonlupt and Mahjoub [25] extensively studied the extremal points of $Q_k(G)$ and characterized the class of graphs for which this polytope is described by cut inequalities and $\mathbf{0} \leq x \leq \mathbf{1}$.

The polyhedron $P_1(G)$ is known to be box-TDI for all graphs [31]. For series-parallel graphs, the system given in [16] describing $P_2(G)$ is not TDI. Chen, Ding, and Zang [7] showed that dividing it by 2 yields a TDI system for such graphs. Actually, they proved that this system is box-TDI if and only if the graph is series-parallel.

Contributions. Our starting point is the result of Chen, Ding, and Zang [7]. First, their result implies that $P_2(G)$ is a box-TDI polyhedron for series-parallel graphs. However, this leaves open the question of the box-TDIness of $P_2(G)$ for non series-parallel graphs. More generally, for which integers k and graphs G is $P_k(G)$ a box-TDI polyhedron?

We answer this question by proving that, for $k \geq 2$, $P_k(G)$ is a box-TDI polyhedron if and only if G is series-parallel. Note that this work is one of the first ones that proves the box-TDIness of a polyhedron without giving a box-TDI system describing it. Instead, our proof is based on the recent matricial characterization of box-TDI polyhedra given by Chervet, Grappe, and Robert [9].

By [34, Theorem 22.6], there exists a TDI system with integer coefficients describing $P_k(G)$. For series-parallel graphs, the system provided by Chen, Ding, and Zang [7] has noninteger coefficients. Moreover, the system given by Didi Biha and Mahjoub [17] describing $P_k(G)$ when k is even is not TDI. When $k \geq 2$ and G is series-parallel, which combinatorial objects yield an integer TDI system describing $P_k(G)$?

We answer this question by exhibiting integer TDI systems based on multicuts. When k is even, we use multicuts to provide an integer TDI system for $P_k(G)$ when G is series-parallel. Our proof relies on the standard constructive characterization of series-parallel graphs. When k is odd, we prove that the description of $P_k(G)$ given by Didi Biha and Mahjoub [17] based on multicuts is TDI if and only if the graph is series-parallel. For this case, our proof relies on new properties of the set of degree 2 vertices in simple series-parallel graphs stated in Lemma 2.3.

The box-totally dual integral characterization of $P_k(G)$ implies that these systems are actually box-TDI if and only if G is series-parallel. By definition of box-TDIness, adding $x \leq \mathbf{1}$ to these systems yields box-TDI systems for $Q_k(G)$ for series-parallel graphs.

Outline. In Section 2, we give the definitions and preliminary results used throughout the paper. In Section 3, we prove that, for $k \geq 2$, $P_k(G)$ is a box-TDI polyhedron if and only if G is series-parallel. In Section 4, we provide a TDI system with integer coefficients describing $P_k(G)$ when G is series-parallel and $k \geq 2$ is even. In Section 5, we show the TDIness of the system given by Didi Biha and Mahjoub [17] that describes $P_k(G)$ for G series-parallel and $k \geq 3$ odd.

2 Definitions and Preliminary Results

This section is devoted to the definitions, notation, and preliminary results used throughout the paper.

2.1 Graphs and Combinatorial Objects

Given a set E, a *family* of E is a collection of elements of E where each element can appear multiple times. The incidence vector of a family F of E is the vector χ^F of $\mathbb{Z}_+^E$ such that e's coordinate is the multiplicity of e in F for all e in E. Since there is a bijection between families and their incidence vectors, we will often use the same terminology for both.

Given a graph $G = (V, E)$ and the incidence vector $z \in \mathbb{Z}_+^E$ of a family F of E, $G(z)$ denotes the graph (V, F).

Let $G = (V, E)$ be a loopless undirected graph. Two edges of G are *parallel* if they share the same endpoints, and G is *simple* if it does not have parallel edges. A graph is 2-*connected* if it cannot be disconnected by removing a vertex. The graph obtained from two disjoint graphs by identifying two vertices, one of each graph, is called a *1-sum.* A 2-connected graph is *trivial* if it is composed of a single edge. We denote by K_n the complete graph on n vertices, that is the simple graph with n vertices and one edge between each pair of vertices. Given an edge e of G, we denote by $G \setminus e$ (respectively G/e) the graph obtained by removing (respectively contracting) the edge e, where *contracting* an edge uv consists in removing it and identifying u and v. Similarly, we denote by $G \setminus v$ the graph obtained form G by removing the vertex v, and by $G[W]$ the *graph induced by* W, that is, the graph obtained by removing all vertices not in the vertex subset W. Given a vector $x \in \mathbb{R}^E$ and a subgraph H of G, we denote by $x_{|H}$ the vector obtained by restricting x to the components associated with the edges of H.

A subset of edges of G is called a *circuit* if it induces a connected graph in which every vertex has degree 2. Given a subset U of V, the *cut* $\delta(U)$ is the set of edges having exactly one endpoint in U. A *bond* is a minimal nonempty cut. Given a partition $\{V_1, \ldots, V_n\}$ of V, the set of edges having endpoints in two distinct V_i's is called a *multicut* and is denoted by $\delta(V_1, \ldots, V_n)$. We denote respectively by $\mathcal{M}_G$ and $\mathcal{B}_G$ the set of multicuts and the set of bonds of G. For every multicut M, there exists a unique partition $\{V_1, \ldots, V_{d_M}\}$ of vertices of V such that $M = \delta(V_1, \ldots, V_{d_M})$, and $G[V_i]$ is connected for all $i = 1, \ldots, d_M$. We say that d_M is the *order* of M and $V_1, \ldots, V_{d_M}$ are the *classes* of M. Multicuts are characterized in terms of circuits, as stated in the following.

Lemma 2.1 ([13]) *A set of edges M is a multicut if and only if $|M \cap C| \neq 1$ for all circuits C of G.*

We denote the symmetric difference of two sets S and T by $S \triangle T$. It is well-known that the symmetric difference of two cuts is a cut. Moreover, the following result holds.

Observation 2.2 *Let G be a graph, v be a degree 2 vertex of G, and M be a multicut such that $|M \cap \delta(v)| = 1$. Then, $M \cup \delta(v)$ and $M \triangle \delta(v)$ are multicuts. Moreover, $d_{M \cup \delta(v)} = d_M + 1$, and $d_{M \triangle \delta(v)} = d_M$.*

A graph is *series-parallel* if its nontrivial 2-connected components can be constructed from a circuit of length 2 by repeatedly adding edges parallel to an existing one, and subdividing edges, that is, replacing an edge by a path of length two. Equivalently, series-parallel graphs are those having no K_4-minor [20].

By construction, simple nontrivial 2-connected series-parallel graphs have at least one degree 2 vertex. Moreover, these vertices satisfy the following.

Lemma 2.3 *For a simple nontrivial 2-connected series-parallel graph, at least one of the following holds:*

(i) two degree 2 vertices are adjacent,
(ii) a degree 2 vertex belongs to a circuit of length 3,
(iii) two degree 2 vertices belong to the same circuit of length 4.

Proof We proceed by induction, the base case is K_3 for which (i) holds.

Let G be a simple 2-connected series-parallel graph. Since G is simple, it can be built from a series-parallel graph H by subdividing an edge e into a path f, g. Let v be the degree 2 vertex added with this operation. By the induction hypothesis, either H is not simple, or one among (i), (ii), and (iii) holds for H. Hence, there are four cases.

Case 1: H is not simple. Since G is simple, e is parallel to exactly one edge h. Hence, f, g, h is a circuit of G length 3 containing v, thus (ii) holds for G.
Case 2: (i) holds for H. Then, it holds for G.
Case 3: (ii) holds for H. Let C be a circuit of H of length 3 containing a degree 2 vertex, say w. If $e \notin C$, then (ii) holds for G. Otherwise, by subdividing e, we obtain a circuit of length 4 containing v and w, and hence (iii) holds for G.
Case 4: (iii) holds for H. Let C be a circuit of H of length 4 containing two degree 2 vertices. If $e \notin C$, then (iii) holds for G. Otherwise, by subdividing e, we obtain a circuit of length 5 containing three degree 2 vertices. Then, at least two of them are adjacent, and so (i) holds for G. □

2.2 Box-Total Dual Integrality

Let $A \in \mathbb{R}^{m \times n}$ be a full-row rank matrix. This matrix is *equimodular* if all its $m \times m$ non-zero determinants have the same absolute value. The matrix A is *face-defining for* a face F of a polyhedron $P \subseteq \mathbb{R}^n$ if $\mathrm{aff}(F) = \{x \in \mathbb{R}^n : Ax = b\}$ for some $b \in \mathbb{R}^m$, where $\mathrm{aff}(F)$ denotes the affine hull of F. Such matrices are the *face-defining matrices of* P.

Theorem 2.4 ([9, Theorem 1.4]) *Let P be a polyhedron. Then, the following statements are equivalent:*

(i) P is box-TDI.

(ii) *Every face-defining matrix of P is equimodular.*
(iii) *Each face of P has an equimodular face-defining matrix.*

In Theorem 2.4, the equivalence of conditions (*ii*) and (*iii*) follows from the following observation.

Observation 2.5 ([9, Observation 4.10]) *Let F be a face of a polyhedron. If a face-defining matrix for F is equimodular, then so are all the face-defining matrices for F.*

We will also use the following.

Observation 2.6 *Let $A \in \mathbb{R}^{I \times J}$ be a full row rank matrix and $j \in J$. If A is equimodular, then so are following two matrices:*

(i) $\begin{bmatrix} A \\ \pm\chi^j \end{bmatrix}$ *if it is full row-rank,*
(ii) $\begin{bmatrix} A & \mathbf{0} \\ \pm\chi^j & \pm 1 \end{bmatrix}$.

Observation 2.7 ([9, Observation 4.11]) *Let $P \subseteq \mathbb{R}^n$ be a polyhedron and let $F = \{x \in P : Bx = b\}$ be a face of P. If B has full-row rank and $n - \dim(F)$ rows, then B is face-defining for F.*

2.3 The k-Edge-Connected Spanning Subgraph Polyhedron

Note that $P_k(G)$ is the dominant of the convex hull of incidence vectors of all the families of E containing at most k copies of each edge and inducing a k-edge-connected spanning subgraph of G. Since the dominant of a polyhedron is a polyhedron, $P_k(G)$ is a full-dimensional polyhedron even though it is the convex hull of an infinite number of points.

From now on, we assume that $k \geq 2$. Didi Biha and Mahjoub [17] gave a complete description of $P_k(G)$ for all k, when G is series-parallel.

Theorem 2.8 ([17]) *Let G be a series-parallel graph and h be a positive integer. Then, $P_{2h}(G)$ is described by:*

$$(1) \begin{cases} x(D) \geq 2h \quad \text{for all cuts } D \text{ of } G, & \text{(1a)} \\ x \geq \mathbf{0}, & \text{(1b)} \end{cases}$$

and $P_{2h+1}(G)$ is described by:

$$(2) \begin{cases} x(M) \geq (h+1)d_M - 1 \quad \text{for all multicuts } M \text{ of } G, & \text{(2a)} \\ x \geq \mathbf{0}. & \text{(2b)} \end{cases}$$

Since the incidence vector of a multicut $\delta(V_1, \ldots, V_\ell)$ of order ℓ is the half-sum of the incidence vectors of the bonds $\delta(V_1), \ldots, \delta(V_\ell)$, we can deduce another description of $P_{2h}(G)$.

Corollary 2.9 *Let G be a series-parallel graph and h be a positive integer. Then, $P_{2h}(G)$ is described by:*

$$(3)\begin{cases} x(M) \geq hd_M & \textit{for all multicuts } M \textit{ of } G, \quad (3a)\\ x \geq \mathbf{0}. & (3b)\end{cases}$$

We call constraints (2a) and (3a) *partition constraints*. A multicut M is *tight for a point* of $P_k(G)$ if this point satisfies with equality the partition constraint (2a) (respectively (3a)) associated with M when k is odd (respectively even). Moreover, M is *tight for a face* F of $P_k(G)$ if it is tight for all the points of F.

The following results give some insights on the structure of tight multicuts.

Theorem 2.10 ([17, Theorem 2.3 and Lemma 3.1]) *Let x be a point of $P_{2h+1}(G)$, and let $M = \delta(V_1, \ldots, V_{d_M})$ be a multicut tight for x. Then, the following hold:*

(i) if $d_M \geq 3$, then $x\,(\delta(V_i) \cap \delta(V_j)) \leq h+1$ for all $i \neq j \in \{1, \ldots, d_M\}$.
(ii) $G \setminus V_i$ is connected for all $i = 1, \ldots, d_M$.

Lemma 2.11 *Let v be a degree 2 vertex of G and M be a multicut of G strictly containing $\delta(v) = \{uv, vw\}$. If M is tight for a point of $P_k(G)$ with $k \geq 2$, then both $M \setminus uv$ and $M \setminus vw$ are multicuts of G of order $d_M - 1$.*

Proof It suffices to show that u and w belong to different classes of $M = \delta(v, V_2, \ldots, V_{d_M})$. Suppose that $u, w \in V_2$. Then M is the union of the two multicuts $\delta(v)$ and $M' = \delta(v \cup V_2, \ldots, V_{d_M})$. Since $d_{\delta(v)} + d_{M'} = d_M + 1$, the sum of the partition inequalities associated with $\delta(v)$ and M' implies that the partition inequality associated with M is tight for no point of $P_k(G)$ for every $k \geq 2$. □

Chopra [10] gave sufficient conditions for an inequality to be facet-defining for $P_k(G)$. The following proposition is a direct consequence of Theorems 2.4 and 2.6 of [10].

Lemma 2.12 *Let G be a connected graph having a K_4-minor. Then, there exist two disjoint nonempty subsets of edges of G, E' and E'', and a rational b such that*

$$x(E') + 2x(E'') \geq b, \tag{4}$$

is a facet-defining inequality of $P_{2h+1}(G)$.

Chen, Ding, and Zang [7] provided a box-TDI system for $P_2(G)$ for series-parallel graphs.

Theorem 2.13 ([7, Theorem 1.1]) *The system:*

$$\begin{cases} \frac{1}{2}x(D) \geq 1 & \textit{for all cuts } D \textit{ of } G,\\ x \geq \mathbf{0} \end{cases} \tag{5}$$

is box-TDI if and only if G is a series-parallel graph.

This result proves that the polyhedron $P_2(G)$ is box-TDI for all series-parallel graphs, and gives a TDI system describing this polyhedron in this case. However, Theorem 2.13 is not sufficient to state that $P_2(G)$ is a box-TDI polyhedron if and only if G is series-parallel.

3 Box-TDIness of $P_k(G)$

In this section we show that, for $k \geq 2$, $P_k(G)$ is a box-TDI polyhedron for a connected graph G if and only if G is series-parallel. Since $P_k(G) = \emptyset$ when G is not connected, we assume from now on that G is connected.

When $k \geq 2$, $P_k(G)$ is not always box-TDI, as stated in Lemma 3.1. Indeed, by Theorem 2.4, if a polyhedron has a nonequimodular face-defining matrix, then it is not box-TDI. The proof of Lemma 3.1 exhibits such a matrix when G has a K_4-minor. This follows from the existence of a particular facet-defining inequality when k is odd, as shown by Chopra [10]. When k is even, we build a nonequimodular face-defining matrix based on the structure of cuts in a K_4-minor.

Lemma 3.1 *For $k \geq 2$, if $G = (V, E)$ has a K_4-minor, then $P_k(G)$ is not box-TDI.*

Proof When $k = 2h + 1$ is odd, Lemma 2.12 shows that there exists a facet-defining inequality that is described by a nonequimodular matrix as $P_k(G)$ is full-dimensional. Thus, $P_k(G)$ is not box-TDI by Statement (ii) of Theorem 2.4.

We now prove the case when k is even. Since G has a K_4-minor, there exists a partition $\{V_1, \ldots, V_4\}$ of V such that $G[V_i]$ is connected and $\delta(V_i, V_j) \neq \emptyset$ for all $i < j \in \{1, \ldots, 4\}$. We now prove that the matrix A whose three rows are $\chi^{\delta(V_i)}$ for $i = 1, 2, 3$ is a face-defining matrix of $P_k(G)$ which is not equimodular. This will end the proof by Statement (ii) of Theorem 2.4.

Let e_{ij} be an edge in $\delta(V_i, V_j)$ for all $i < j \in \{1, \ldots, 4\}$. The submatrix of A formed by the columns associated with edges e_{ij} is the following:

$$\begin{array}{c c}
 & \begin{array}{cccccc} e_{12} & e_{13} & e_{23} & e_{14} & e_{24} & e_{34} \end{array} \\
\begin{array}{c} \chi^{\delta(V_1)} \\ \chi^{\delta(V_2)} \\ \chi^{\delta(V_3)} \end{array} & \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}
\end{array}$$

The matrix A is not equimodular as the first three columns form a matrix of determinant -2 whereas the last three ones give a matrix of determinant 1.

By Observation 2.7, to show that A is face-defining, it is enough to exhibit $|E| - 2$ affinely independent points of $P_k(G)$ satisfying $x(\delta(V_i)) = k$ for $i = 1, 2, 3$.

Let $D_1 = \{e_{12}, e_{14}, e_{23}, e_{34}\}$, $D_2 = \{e_{12}, e_{13}, e_{24}, e_{34}\}$, $D_3 = \{e_{13}, e_{14}, e_{23}, e_{24}\}$ and $D_4 = \{e_{14}, e_{24}, e_{34}\}$. First, we define the points $S_j = \sum_{i=1}^{4} k\chi^{E[V_i]} + \frac{k}{2}\chi^{D_j}$, for $j = 1, 2, 3$, and $S_4 = \sum_{i=1}^{4} k\chi^{E[V_i]} + k\chi^{D_4}$. Note that they are affinely independent.

Now, for each edge $e \notin \{e_{12}, e_{13}, e_{14}, e_{23}, e_{24}, e_{34}\}$, we construct the point S_e as follows. When $e \in E[V_i]$ for some $i = 1, \ldots, 4$, we define $S_e = S_4 + \chi^e$. Adding the point S_e maintains affine independence as S_e is the only point not satisfying $x_e = k$. When $e \in \delta(V_i, V_j)$ for some i, j, we define $S_e = S_\ell - \chi^{e_{ij}} + \chi^e$, where S_ℓ is S_1 if $e \in \delta(V_1, V_4) \cup \delta(V_2, V_3)$ and S_2 otherwise. Affine independence comes because S_e is the only point involving e.

In total, we built $4 + |E| - 6 = |E| - 2$ affinely independent points. $\square$

The following theorem characterizes the class of graphs for which $P_k(G)$ is box-TDI. The case k even is obtained using the box-TDIness for $k = 2$ and the fact that integer dilations maintain box-TDIness. For the case k odd, on the contrary to what is generally done, the proof does not exhibit a box-TDI system describing $P_k(G)$. For this case, the proof is by induction on the number of edges of G. We prove that series-parallel operations preserve the box-TDIness of the polyhedron. The most technical part of the proof is the subdivision of an edge uw into two edges uv and vw. We proceed by contradiction: by Theorem 2.4, we suppose that there exists a face F of $P_k(G)$ defined by a nonequimodular matrix. We study the structure of the inequalities corresponding to this matrix. In particular, we show that they are all associated with multicuts, and that these multicuts contain either both uv and vw, or none of them—see Claims 3.1, 3.2, and 3.3. These last results allow us to build a nonequimodular face-defining matrix for the smaller graph, which contradicts the induction hypothesis.

Theorem 3.2 *For $k \geq 2$, $P_k(G)$ is a box-TDI polyhedron if and only if G is series-parallel.*

Proof Necessity follows from Lemma 3.1. Let us now prove sufficiency. When $k = 2$, the box-TDIness of System (5) has been shown by Chen, Ding, and Zang [7]. This implies box-TDIness for all even k: multiplying the right-hand side of a box-TDI system by a positive rational preserves its box-TDIness [34, Section 22.5]. The system obtained by multiplying by $\frac{k}{2}$ the right-hand side of System (5) describes $P_k(G)$ when k is even. Hence, the latter is a box-TDI polyhedron.

The rest of the proof is devoted to the case where $k = 2h + 1$ for some $h \geq 1$. To this end, we prove that for every face of $P_{2h+1}(G)$ there exists an equimodular face-defining matrix. The characterization of box-TDIness given in Theorem 2.4 concludes. We proceed by induction on the number of edges of G.

If G is trivial, then $P_{2h+1}(G) = \{x \in \mathbb{R}_+ : x \geq 2h + 1\}$ is box-TDI. If G is the circuit $\{e, f\}$, then $P_{2h+1}(G) = \{x_e, x_f \in \mathbb{R}_+ : x_e + x_f \geq 2h+1\}$ is also box-TDI.

(1-sum) Let G be the 1-sum of two series-parallel graphs $G' = (W', E')$ and $G'' = (W'', E'')$. By induction, there exist two box-TDI systems $A'y \geq b'$ and $A''z \geq b''$ describing respectively $P_{2h+1}(G')$ and $P_{2h+1}(G'')$. If v is the vertex of G obtained by the identification, $G \setminus v$ is not connected, hence, by Statement (ii) of Theorem 2.10, a multicut M of G is tight for a face of $P_{2h+1}(G)$ only if $M \subseteq E'$ or $M \subseteq E''$. It follows that for every face F of $P_{2h+1}(G)$ there exist faces F' and F'' of $P_{2h+1}(G')$ and $P_{2h+1}(G'')$ respectively, such that $F = F' \times F''$. Then $P_{2h+1}(G) = \{(y, z) \in \mathbb{R}_+^{E'} \times \mathbb{R}_+^{E''} : A'y \geq b', A''z \geq b''\}$ and so it is box-TDI.

(Parallelization) Let $G = (V, E)$ be obtained from a series-parallel graph G' by adding an edge g parallel to an edge f of G' and suppose that $P_{2h+1}(G')$ is box-TDI. Let $A'x \geq b$ be a box-TDI system describing $P_{2h+1}(G')$. Note that $P_{2h+1}(G)$ is described by $Ax \geq b, x_f \geq 0, x_g \geq 0$, where A is the matrix obtained by duplicating f's column. By Theorem 22.10 of [34], the system $Ax \geq b$ is box-TDI, hence so is $Ax \geq b, x_f \geq 0, x_g \geq 0$. Thus, $P_{2h+1}(G)$ is a box-TDI polyhedron.

(Subdivision) Let $G = (V, E)$ be obtained by subdividing an edge uw of a series-parallel graph $G' = (V', E')$ into a path of length two uv, vw. By contradiction, suppose there exists a nonempty face $F = \{x \in P_{2h+1}(G) : A_F x = b_F\}$ such that A_F is a face-defining matrix for F which is not equimodular. Take such a face with maximum dimension. Then, every submatrix of A_F which is face-defining for a face of $P_{2h+1}(G)$ is equimodular. We may assume that A_F is defined by

the partition constraints (2a) associated with the set of multicuts $\mathcal{M}_F$ and the nonnegativity constraints associated with the set of edges $\mathcal{E}_F$.

Claim 3.1 $\mathcal{E}_F = \emptyset$.

Proof Suppose there exists an edge $e \in \mathcal{E}_F$. Let $H = G \setminus e$ and let $A_{F_H}x = b_{F_H}$ be the system obtained from $A_F x = b_F$ by removing the column and the nonnegativity constraint associated with e. Since the matrix A_F is of full row rank, so is A_{F_H}. Since $e \in \mathcal{E}_F$, for all multicuts M tight for F not containing e, $M \cup e$ is not a multicut. Hence $M \setminus e$ is a multicut of H of order d_M, for all M in $\mathcal{M}_F$. Hence, the set $F_H = \{x \in P_{2h+1}(H) : A_{F_H}x = b_{F_H}\}$ is a face of $P_{2h+1}(H)$. Moreover, deleting e's coordinate of $\mathrm{aff}(F)$ gives $\mathrm{aff}(F_H)$ so A_{F_H} is face-defining for F_H. By the induction hypothesis, A_{F_H} is equimodular. Since maximal invertible square submatrices of A_F are in bijection with those of A_{F_H} and have the same determinant in absolute value, A_F is equimodular, a contradiction. ☐

Claim 3.2 *For $e \in \{uv, vw\}$, at least one multicut of $\mathcal{M}_F$ different from $\delta(v)$ contains e.*

Proof By contradiction, suppose for instance that uv belongs to no multicut of $\mathcal{M}_F$ different from $\delta(v)$.

First, suppose that $\delta(v)$ does not belong to $\mathcal{M}_F$. Then, the column of A_F associated with uv is zero. Let A'_F be the matrix obtained from A_F by removing this column. Every multicut of G not containing uv is a multicut of G' (relabelling vw by uw), so the rows of A'_F are associated with multicuts of G'. Thus, $F' = \{x \in P_k(G') : A'_F x = b_F\}$ is a face of $P_{2h+1}(G')$. Removing uv's coordinate from the points of F gives a set of points of F' of affine dimension at least $\dim(F) - 1$. Since A'_F has the same rank as A_F and has one column fewer than A_F, then A'_F is face-defining for F' by Observation 2.7. By the induction hypothesis, A'_F is equimodular. Since adding a column of zeros preserves equimodularity, A_F is also equimodular.

Suppose now that $\delta(v)$ belongs to $\mathcal{M}_F$. Then, the column of A_F associated with uv has zeros in each row but $\chi^{\delta(v)}$. Let $A^\star_F x = b^\star_F$ be the system obtained from $A_F x = b_F$ by removing the equation associated with $\delta(v)$. Then $F^\star = \{x \in P_k(G) : A^\star_F x = b^\star_F\}$ is a face of $P_k(G)$ of dimension $\dim(F)+1$. Indeed, it contains F and $z + \alpha\chi^{uv} \notin F$ for every point z of F and $\alpha > 0$. Hence, $A^\star_F$ is face-defining for $F^\star$. This matrix is equimodular by the maximality assumption on F, and so is A_F by Statement (ii) of Observation 2.6. ☐

Claim 3.3 $|M \cap \delta(v)| \neq 1$ *for every multicut $M \in \mathcal{M}_F$.*

Proof Suppose there exists a multicut M tight for F such that $|M \cap \delta(v)| = 1$. Without loss of generality, suppose that M contains uv but not vw. Then, $F \subseteq \{x \in P_{2h+1}(G) : x_{vw} \geq x_{uv}\}$ because of the partition inequality (2a) associated with the multicut $M \triangle \delta(v)$. Moreover, the partition inequality associated with $\delta(v)$ and the integrality of $P_{2h+1}(G)$ imply $F \subseteq \{x \in P_{2h+1}(G) : x_{vw} \geq h + 1\}$. The proof is divided into two cases.

Case 1: $F \subseteq \{x \in P_{2h+1}(G) : x_{vw} = h + 1\}$. We prove this case by exhibiting an equimodular face-defining matrix for F. By Observation 2.5, this implies that A_F is equimodular, which contradicts the assumption on F.

Equality $x_{vw} = h + 1$ can be expressed as a linear combination of equations of $A_F x = b_F$. Let $A'_F x = b'_F$ denote the system obtained by replacing an equation of $A_F x = b_F$ by $x_{vw} = h + 1$ in such a way that the underlying affine space remains unchanged. Denote by $\mathcal{N}$ the set of multicuts of $\mathcal{M}_F$ containing vw but not uv. If $\mathcal{N} \neq \emptyset$, then let N be in $\mathcal{N}$. We now modify the system $A'_F x = b'_F$ by performing the following operations.

1. For all $M \in \mathcal{M}_F$ strictly containing $\delta(v)$, replace the equation associated with M by the partition constraint (2a) associated with $M \setminus vw$ set to equality, that is, $x(M \setminus vw) = (h + 1)d_{M \setminus vw} - 1$.
2. If $\delta(v) \in \mathcal{M}_F$, then replace the equation associated with $\delta(v)$ by the box constraint $x_{uv} = h$.
3. If $\mathcal{N} \neq \emptyset$, then replace the equation associated with N by the box constraint $x_{uv} = h + 1$.
4. For all $M \in \mathcal{N} \setminus N$, replace the equation associated with M by the partition constraint (2a) associated with $M \triangle \delta(v)$ set to equality, that is, $x(M \triangle \delta(v)) = (h + 1)d_{M \triangle \delta(v)} - 1$.

These operations do not change the underlying affine space. Indeed, for every multicut M strictly containing $\delta(v)$ and tight for F, the set $M \setminus vw$ is a multicut tight for F by Lemma 2.11 and $F \subseteq \{x \in P_{2h+1}(G) : x_{vw} = h+1\}$. If $\delta(v)$ is tight for F, then $F \subseteq \{x \in P_{2h+1}(G) : x_{uv} = h\}$ because $F \subseteq \{x \in P_{2h+1}(G) : x_{vw} = h + 1\}$. For $M \in \mathcal{N}$, by Observation 2.2, the set $M \triangle \delta(v)$ is a multicut of order d_M. The tightness of the constraint (2a) associated with N and the constraint (2a) associated with $M \triangle \delta(v)$ imply that $F \subseteq \{x \in P_{2h+1}(G) : x_{vw} \leq x_{uv}\}$. Since $F \subseteq \{x \in P_{2h+1}(G) : x_{vw} \geq x_{uv}\}$, we have $F \subseteq \{x \in P_{2h+1}(G) : x_{uv} = h + 1\}$ and $M \triangle \delta(v)$ is tight for F. It follows that, if $\delta(v) \in \mathcal{M}_F$, then $\mathcal{N} = \emptyset$. Therefore, at most one among Operations 2 and 3 is applied so the rank of the matrix remains unchanged.

Let $A''_F x = b''_F$ be the system obtained by removing the equation $x_{vw} = h + 1$ from $A'_F x = b'_F$. By construction, $A''_F x = b''_F$ is composed of constraints (2a) set to equality and possibly $x_{uv} = h$ or $x_{uv} = h + 1$. Moreover, the column of A''_F associated with vw is zero. Let $F'' = \{x \in P_{2h+1}(G) : A''_F x = b''_F\}$. For every point z of F and $\alpha \geq 0$, $z + \alpha\chi^{vw}$ belongs to F'' because the column of A''_F associated with vw is zero, and $z + \alpha\chi^{vw} \in P_{2h+1}(G)$. This implies that $\dim(F'') \geq \dim(F) + 1$.

If F'' is a face of $P_{2h+1}(G)$, then A''_F is face-defining for F'' by Observation 2.7 and because A'_F is face-defining for F. By the maximality assumption on F, A''_F is equimodular, and hence so is A'_F by Statement (i) of Observation 2.6.

Otherwise, by construction, $F'' = F^\star \cap \{x \in \mathbb{R}^E : x_{uv} = t\}$ where $F^\star$ is a face of $P_{2h+1}(G)$ strictly containing F and $t \in \{h, h + 1\}$. Therefore, there exists a face-defining matrix for F'' given by a face-defining matrix for $F^\star$ and the row χ^{uv}. Such a matrix is equimodular by the maximality assumption of F and Statement (i) of Observation 2.6. Hence, A''_F is equimodular by Observation 2.5, and so is A'_F by Statement (i) of Observation 2.6.

Case 2: $F \not\subseteq \{x \in P_{2h+1}(G) : x_{vw} = h + 1\}$. Thus, there exists $z \in F$ such that $z_{vw} > h + 1$. By Claim 3.2, there exists a multicut $N \neq \delta(v)$ containing vw which is tight for F. By Statement (i) of Theorem 2.10, the existence of z implies that N is a bond, hence it does not contain uv. The set $L = N \triangle \delta(v)$ is a bond of G. The partition inequality (2a) associated with L implies that $F \subseteq$

$\{x \in P_{2h+1}(G) : x_{vw} = x_{uv}\}$ and L is tight for F. Moreover, N is the unique multicut tight for F containing vw. Suppose indeed that there exists a multicut B containing vw tight for F. Then, B is a bond by Statement (i) of Theorem 2.10 and the existence of z. Moreover, $B \triangle N$ is a multicut not containing vw. This implies that no point x of F satisfies the partition constraint associated with $B \triangle N$ because $x(B \triangle N) = x(B) + x(N) - 2x(B \cap N) = 2(2h+1) - 2x(B \cap N) \leq 4h + 2 - 2x_{vw} \leq 2h$, a contradiction.

Consider the matrix $A_F^\star$ obtained from A_F by removing the row associated with N. Matrix $A_F^\star$ is a face-defining matrix for a face $F^\star \supseteq F$ of $P_{2h+1}(G)$ because $F^\star$ contains F and $z + \alpha\chi^{uv}$ for every point z of F and $\alpha > 0$. By the maximality assumption, the matrix $A_F^\star$ is equimodular. Let B_F be the matrix obtained from A_F by replacing the row χ^N by the row $\chi^N - \chi^L$. Then, B_F is face-defining for F. Moreover, B_F is equimodular by Statement (ii) of Observation 2.6—a contradiction. $\square$

Let $A'_F x = b'_F$ be the system obtained from $A_F x = b_F$ by removing uv's column from A_F and subtracting $h+1$ times this column to b_F. We now show that $\{x \in P_{2h+1}(G') : A'_F x = b'_F\}$ is a face of $P_{2h+1}(G')$ if $\delta(v) \notin \mathcal{M}_F$, and of $P_{2h+1}(G') \cap \{x : x_{uw} = h\}$ otherwise. Indeed, consider a multicut M in $\mathcal{M}_F$. If $M = \delta(v)$, then the equation of $A'_F x = b'_F$ induced by M is nothing but $x_{uw} = h$. Otherwise, by Lemma 2.11 and Claim 3.3, the set $M \setminus uv$ is a multicut of G' (relabelling vw by uw) of order d_M if $uv \notin M$ and $d_M - 1$ otherwise. Thus, the equation of $A'_F x = b'_F$ induced by M is the partition constraint (2a) associated with $M \setminus uv$ set to equality.

By construction and Claim 3.3, A'_F has full row rank and one column less than A_F. We prove that A'_F is face-defining by exhibiting $\dim(F)$ affinely independent points of $P_{2h+1}(G')$ satisfying $A'_F x = b'_F$. Because of the integrality of $P_{2h+1}(G)$, there exist $n = \dim(F) + 1$ affinely independent integer points $z^1, \ldots, z^n$ of F. By Claims 3.2 and 3.3, there exists a multicut strictly containing $\delta(v)$. Then, Statement (i) of Theorem 2.10 implies that $F \subseteq \{x \in \mathbb{R}^E : x_{uv} \leq h+1, x_{vw} \leq h+1\}$. Combined with the partition inequality $x_{uv} + x_{vw} \geq 2h+1$ associated with $\delta(v)$, this implies that at least one of z^i_{uv} and z^i_{vw} is equal to $h+1$ for $i = 1, \ldots, n$. Since exchanging the uv and vw coordinates of any point of F gives a point of F by Claim 3.3, the hypotheses on $z^1, \ldots, z^n$ are preserved under the assumption that $z^i_{uv} = h+1$ for $i = 1, \ldots, n-1$. Let $y^1, \ldots, y^{n-1}$ be the points obtained from $z^1, \ldots, z^{n-1}$ by removing uv's coordinate. Since every multicut of G' is a multicut of G with the same order, $y^1, \ldots, y^{n-1}$ belong to $P_{2h+1}(G')$. By construction, they satisfy $A'_F x = b'_F$ so they belong to a face of $P_{2h+1}(G')$ or $P_{2h+1}(G') \cap \{x : x_{uw} = h\}$. This implies that A'_F is a face-defining matrix of $P_{2h+1}(G')$ if $\delta(v) \notin \mathcal{M}_F$, and of $P_{2h+1}(G') \cap \{x : x_{uw} = h\}$ otherwise.

By induction, $P_{2h+1}(G')$ is a box-TDI polyhedron and hence so is $P_{2h+1}(G') \cap \{x : x_{uw} = h\}$. Hence, A'_F is equimodular by Theorem 2.4. Since A_F is obtained from A'_F by copying a column, then also A_F is equimodular—a contradiction. $\square$

By definition of box-TDIness and $Q_k(G)$, Theorem 3.2 implies that $Q_k(G)$ is box-TDI when G is series-parallel. The converse does not hold. Indeed, for instance, when $G = (V, E)$ is a minimal k-edge-connected graph, $Q_k(G)$ is nothing but the single point χ^E so it is a box-TDI polyhedron.

4 An Integer TDI System for $P_{2h}(G)$

Let G be a series-parallel graph. In this section we provide an integer TDI system for $P_{2h}(G)$ with h positive and integer.

The proof of the main result of the section is based on the characterization of TDIness by means of Hilbert bases. A set of vectors $\{v^1, \ldots, v^k\}$ is a *Hilbert basis* if each integer vector that is a nonnegative combination of $v^1, \ldots, v^k$ can be expressed as a nonnegative integer combination of them. The link between Hilbert basis and TDIness is stated in the following theorem.

Theorem 4.1 (Theorem 22.5 of [34]) *A system $Ax \geq b$ is TDI if and only if for every face F of $P = \{x : Ax \geq b\}$, the rows of A associated with tight constraints for F form a Hilbert basis.*

In the previous theorem, we could restrict to minimal faces: indeed, the cone generated by the constraints tight for a face F is a face of the cone generated by the constraints active for a face $F' \subseteq F$ [34].

Remark 4.2 A system $Ax \geq b$ is TDI if and only if, for each minimal face F of $P = \{x : Ax \geq b\}$, the rows of A associated with constraints tight for F form a Hilbert basis.

The rest of the section is devoted to prove that the system given by the partition constraints and the nonnegativity constraints, which describes $P_k(G)$ when k is even, is TDI when G is series-parallel.

The proof is based on the TDIness of System (5) and the structure of inequalities (3a). Their right-hand sides are proportional to k, hence it is enough to prove the case $k = 2$. This allows us to use Theorem 2.13 to obtain a TDI system for $P_2(G)$. In terms of Hilbert bases, the TDIness of this system implies that, given a face F of $P_2(G)$, the integer points of the associated cone are the half sum of the cuts tight for F. The technical part of the proof is to show that each integer point of this cone is also the sum of incidence vectors of the multicuts tight for F.

Theorem 4.3 *For a series-parallel graph G and a positive integer h, System* (3) *is TDI.*

Proof We only prove the case $h = 1$ since multiplying the right hand side of a system by a positive constant preserves its TDIness [34, Section 22.5].

The proof is done by induction on the number of edges of the graph $G = (V, E)$. When G consists of two vertices connected by a single edge ℓ, System (3) is $x_\ell \geq 2, x_\ell \geq 0$ and is TDI. If G is the circuit $\{e, f\}$, System (3) is $x_e + x_f \geq 2, x \geq \mathbf{0}$ and is TDI.

(Parallelization) Let now G be obtained from a series-parallel graph H by adding an edge g parallel to an edge f of H. System (3) associated with G is obtained from that associated with H by duplicating f's column in constraints (3a) and adding the nonnegativity constraint $x_g \geq 0$. By Lemma 3.1 of [7], System (3) is TDI.

For the other cases, we prove the TDIness of System (3) associated with G using Remark 4.2. More precisely, we prove that for any extreme point z of $P_2(G)$, the set of vectors $\{\chi^M : M \in \mathcal{T}_z\} \cup \{\chi^e : e \in E, z_e = 0\}$ is a Hilbert basis, where $\mathcal{T}_z$ is the set of multicuts tight for z.

(1-sum) Let G be the 1-sum of two series-parallel graphs $G^1 = (W^1, E^1)$ and $G^2 = (W^2, E^2)$ and let z be an extreme point of $P_2(G)$. By construction, we have $z = (z^1, z^2)$ where $z^i \in P_2(G^i)$ for $i = 1, 2$. Moreover, for each multicut $M \in \mathcal{T}_z$, the graph obtained from $G(z)$ by contracting the edges of $E \setminus M$ is a circuit. Indeed, it is 2-edge-connected since $G(z)$ is, and it has $z(M) = d_M$ edges and d_M vertices. Therefore M is either a multicut of G^1 tight for z^1 or one of G^2 tight for z^2.

By induction, Systems (3) associated with G^1 and G^2 are TDI. Thus, $\{\chi^M : M \in \mathcal{T}_z \cap \mathcal{M}(G^i)\} \cup \{\chi^e : e \in E^i, z_e = 0\}$ is a Hilbert basis for $i = 1, 2$ by Theorem 4.1. Since they belong to disjoint spaces, their union is a Hilbert basis. By Theorem 4.1, System (3) is TDI.

(Subdivision) Let $G = (V, E)$ be obtained by subdividing an edge uw of a series-parallel graph $G' = (V', E')$ into a path of length two uv, vw, and let z be an extreme point of $P_2(G)$.

Without loss of generality, suppose $z_{uv} \geq z_{vw}$. Define $z' \in \mathbb{Z}^{E'}$ by $z'_{uw} = z_{vw}$ and $z'_e = z_e$ for all edges e in $E' \setminus uw$. Note that z' belongs to $P_2(G')$ since $G'(z')$ is obtained by contracting the edge uv in $G(z)$, and this contraction preserves 2-edge-connectivity.

Note that for all $e \in E$, $z_e \in \{0, 1, 2\}$. Indeed, since z is an extreme point of $P_2(G)$ which is also described by System (1), if $z_e > 0$, then e belongs to a cut D tight for z. Moreover, as $z_{uv} \geq z_{vw}$, the partition constraint (3a) associated with $\delta(v)$ implies that $z_{uv} \in \{1, 2\}$. We now consider two different cases depending on the value of z_{uv}.

Case 1: $z_{uv} = 2$.

We first show that every multicut of $\mathcal{T}_z$ containing uv is a bond. Indeed, note that every multicut M with $d_M = 2$ is a bond. If a multicut $M = \delta(V_1, \dots, V_{d_M}) \in \mathcal{T}_z$ satisfies $d_M \geq 3$ and $uv \in \delta(V_1, V_2)$, then $M' = \delta(V_1 \cup V_2, V_3, \dots, V_{d_M})$ is a multicut and satisfies

$$z(M') \leq z(M) - 2 < d_M - 1 = d_{M'}.$$

Hence, the partition constraint (3a) associated with M' is violated, a contradiction.

Moreover, there exists at most one bond of $\mathcal{T}_z$, say N, containing uv. As otherwise suppose there exist two bonds B_1 and B_2 in $\mathcal{T}_z$ containing uv. Then, $z(B_1 \triangle B_2) \leq z(B_1) + z(B_2) - 2z_{uv} = 0$, which contradicts the constraint (3a) associated with the multicut $B_1 \triangle B_2$. For a multicut M not containing uv, $M \in \mathcal{T}_z$ if and only if $M \in \mathcal{T}_{z'}$. This implies that $\mathcal{T}_z = \mathcal{T}_{z'} \cup N$. By induction and Theorem 4.1, $\mathcal{T}_{z'} \cup \mathcal{E}_{z'}$ is a Hilbert basis. As $\mathcal{E}_z = \mathcal{E}_{z'}$ (identifying uv and vw) and N is the only member of $\mathcal{T}_z \cup \mathcal{E}_z$ containing uv, $\mathcal{T}_z \cup \mathcal{E}_z$ is also a Hilbert basis.

Case 2: $z_{uv} = 1$.

Let $\mathbf{v}$ be an integer point of the cone generated by $\mathcal{T}_z \cup \mathcal{E}_z$. We prove that $\mathbf{v}$ can be expressed as an integer nonnegative combination of the vectors of $\mathcal{T}_z \cup \mathcal{E}_z$. This implies that $\mathcal{T}_z \cup \mathcal{E}_z$ is a Hilbert basis.

Let $\mathcal{B}_z$ be the set of bonds of $\mathcal{T}_z$. Since System (5) is a TDI system describing $P_2(G)$ in series-parallel graphs, the set of vectors $\{\frac{1}{2}\chi^B : B \in \mathcal{B}_z\} \cup \mathcal{E}_z$ forms a Hilbert basis by Theorem 4.1. Then, there exist $\lambda_B \in \frac{1}{2}\mathbb{Z}_+$ for all $B \in \mathcal{B}_z$ and $\mu_e \in \mathbb{Z}_+$ for all $e \in \mathcal{E}_z$ such that $\mathbf{v} = \sum_{B\in\mathcal{B}_z} \lambda_B \chi^B + \sum_{e\in\mathcal{E}_z} \mu_e \chi^e$.

Since $z_{uv} \geq z_{vw}$, the partition inequality (3a) associated with $\delta(v)$ implies that $z_{vw} = 1$ and $\delta(v) \in \mathcal{T}_z$. In particular, $vw \notin \mathcal{E}_z$. The vector $\mathbf{v}$ is an integer combination of vectors of $\mathcal{T}_z \cup \mathcal{E}_z$ if and only if $\mathbf{v} - \lfloor \lambda_{\delta(v)} \rfloor \chi^{\delta(v)}$ is, thus we may assume that $\lambda_{\delta(v)} \in \{0, \frac{1}{2}\}$. Define $\mathbf{w} \in \mathbb{Z}^{E'}$ by:

$$\mathbf{w}_e = \begin{cases} \mathbf{v}_{uv} + \mathbf{v}_{vw} - 2\lambda_{\delta(v)} & \text{if } e = uw, \\ \mathbf{v}_e & \text{otherwise.} \end{cases}$$

Note that $(B \setminus uw) \cup uv$ and $(B \setminus uw) \cup vw$ are bonds of $\mathcal{T}_z$ whenever B is a bond of $\mathcal{T}_{z'}$ containing uw because $z'_{uw} = z_{uv} = z_{vw} = 1$. Moreover, a bond B of $\mathcal{T}_{z'}$ which does not contain uw is a bond of $\mathcal{T}_z$. Since $\delta(v)$ is the unique bond of G containing both uv and vw and $\mathcal{E}_z = \mathcal{E}_{z'}$, we have:

$$\mathbf{w} = \sum_{B\in\mathcal{B}_{z'}:uw\in B} (\lambda_{(B\setminus uw)\cup uv} + \lambda_{(B\setminus uw)\cup vw})\chi^B + \sum_{B\in\mathcal{B}_{z'}:uw\notin B} \lambda_B \chi^B + \sum_{e\in\mathcal{E}_{z'}} \mu_e \chi^e.$$

Thus, $\mathbf{w}$ belongs to the cone generated by $\mathcal{T}_{z'} \cup \mathcal{E}_{z'}$. By the induction hypothesis, $\mathcal{T}_{z'} \cup \mathcal{E}_{z'}$ is a Hilbert basis, hence there exist $\lambda'_M \in \mathbb{Z}_+$ for all $M \in \mathcal{T}_{z'}$ and $\mu'_e \in \mathbb{Z}_+$ for all $e \in \mathcal{E}_{z'}$ such that $\mathbf{w} = \sum_{M\in\mathcal{T}_{z'}} \lambda'_M \chi^M + \sum_{e\in\mathcal{E}_{z'}} \mu'_e \chi^e$.

Consider the family $\mathcal{N}$ of multicuts of $\mathcal{T}_{z'}$ where each multicut M of $\mathcal{T}_{z'}$ appears λ'_M times. Suppose first that $\lambda_{\delta(v)} = 0$. Then, $\mathbf{v}_{uv}+\mathbf{v}_{vw}$ multicuts of $\mathcal{N}$ contain uw. Let $\mathcal{P}$ be a family of $\mathbf{v}_{uv}$ multicuts of $\mathcal{N}$ containing uw and $\mathcal{Q} = \{F \in \mathcal{N} : uw \in F\} \setminus \mathcal{P}$. Then, we have

$$\mathbf{v} = \sum_{M\in\mathcal{N}:uw\notin M} \chi^M + \sum_{M\in\mathcal{P}} \chi^{(M\setminus uw)\cup uv} + \sum_{M\in\mathcal{Q}} \chi^{(M\setminus uw)\cup vw} + \sum_{e\in\mathcal{E}_{z'}} \mu'_e \chi^e. \quad (6)$$

Suppose now that $\lambda_{\delta(v)} = \frac{1}{2}$. Then, $\mathbf{w}_{uw} = \mathbf{v}_{uv} + \mathbf{v}_{vw} - 1$ multicuts of $\mathcal{N}$ contain uw. Let $\mathcal{P}$ be a family of $\mathbf{v}_{uv} - 1$ multicuts of $\mathcal{N}$ containing uw, let $\mathcal{Q}$ be a family of $\mathbf{v}_{vw} - 1$ multicuts in $\{F \in \mathcal{N} : uw \in F\} \setminus \mathcal{P}$, and denote by N the unique multicut of $\mathcal{N}$ containing uw which is not in $\mathcal{P} \cup \mathcal{Q}$. Then, we have

$$\mathbf{v} = \sum_{M\in\mathcal{N}:uw\notin M} \chi^M + \sum_{M\in\mathcal{P}} \chi^{(M\setminus uw)\cup uv} + \sum_{M\in\mathcal{Q}} \chi^{(M\setminus uw)\cup vw} + \chi^{(N\setminus uw)\cup\delta(v)} + \sum_{e\in\mathcal{E}_{z'}} \mu'_e \chi^e. \quad (7)$$

Every $M \in \mathcal{T}_{z'}$ not containing uw is in $\mathcal{T}_z$. For every $M \in \mathcal{T}_{z'}$ containing uw, $(M \setminus uw) \cup uv$, $(M \setminus uw) \cup vw$ and $(M \setminus uw) \cup \delta(v)$ belong to $\mathcal{T}_z$ since $z'_{uw} = z_{uv} = z_{vw} = 1$. Since $\mathcal{E}_z = \mathcal{E}_{z'}$, then $\mathbf{v}$ is a nonnegative integer combination of vectors of $\mathcal{T}_z \cup \mathcal{E}_z$ in both (6) and (7). This proves that $\mathcal{T}_z \cup \mathcal{E}_z$ is a Hilbert basis. Therefore by Remark 4.2, System (3) is TDI. □

The box-TDIness of $P_k(G)$ and the TDIness of System (3) give the following result.

Corollary 4.4 *System* (3) *is box-TDI if and only if* G *is series-parallel.*

Proof If G is series-parallel, then System (3) is box-TDI by Theorems 3.2 and 4.3, since a TDI system describing a box-TDI polyhedron is box-TDI [12]. If G is not series-parallel, Theorem 3.2 ensures that $P_k(G)$ is not box-TDI, therefore System (3) is not box-TDI. □

Theorem 4.3 leaves open the following problem:

Open Problem 4.5 *Characterize the classes of graphs such that System* (3) *is TDI.*

5 An Integer TDI System for $P_{2h+1}(G)$

In this section, we prove that System (2) is TDI if and only if G is a series-parallel graph—see Theorem 5.1. Proving the TDIness for k odd is considerably more involved than for k even. The first difference with the even case is the lack of a known TDI system describing $P_k(G)$ when k is odd, even a noninteger one. Thus, no property of the Hilbert bases associated with $P_k(G)$ is known, and the approach used to prove Theorem 4.3 cannot be applied. Instead, following the definition of TDIness, we prove the existence of an integer optimal solution to each feasible dual problem.

Another difference with the case k even follows from the structure of the partition inequalities (2a). In particular, the presence of the constant "-1" in the right-hand sides perturbs the structure of tight multicuts. Indeed, when k is odd, the tightness of $\delta(V_1, \dots, V_n)$ does not imply that of $\delta(V_1), \dots, \delta(V_n)$. Consequently, it is not clear how the contraction of an edge impacts the tightness of a multicut $\delta(V_1, \dots, V_n)$: merging adjacent V_i's is not sufficient to obtain new tight multicuts. Due to the link between tight multicuts and positive dual variables, the structure of the optimal solutions to the dual problem is completely modified when subdividing an edge. Proving directly that subdivision preserves TDIness turned out to be challenging, and we overcome this difficulty by deriving new properties of series-parallel graphs—see Lemma 2.3.

The proof of Theorem 5.1 focuses on properties of vertices of degree 2 in a minimal counterexample to the TDIness of System (2). In particular, we prove that no two vertices of degree 2 are adjacent (Claim 5.9), or in the same circuit of length 4 (Claim 5.11). Moreover, no triangle contains vertices of degree 2 (Claim 5.10). By Lemma 2.3, this implies that the graph is not series-parallel. To derive these properties, we study the interplay between cuts associated with degree 2 vertices and dual optimal solutions—see Claims 5.3-5.8.

Theorem 5.1 *For h positive and integer, System* (2) *is TDI if and only if G is series-parallel.*

Proof If G is not series-parallel, then System (2) is not TDI because every TDI system with integer right-hand side describes an integer polyhedron [22], but when G has a K_4-minor, System (2) describes a noninteger polyhedron [10].

We now prove that, if G is series-parallel, then System (2) is TDI. We prove the result by contradiction. Let $G = (V, E)$ be a series-parallel graph such that

System (2) is not TDI. By definition of TDIness, there exists $c \in \mathbb{Z}^E$ such that $\mathcal{D}_{(G,c)}$:

$$\max \sum_{M \in \mathcal{M}_G} b_M y_M$$

$$\text{s.t.} \begin{cases} \displaystyle\sum_{M \in \mathcal{M}_G : e \in M} y_M \leq c_e & \text{for all } e \in E, \quad (8a) \\ y_M \geq 0 & \text{for all } M \in \mathcal{M}_G, \quad (8b) \end{cases}$$

is feasible, bounded, but admits no integer optimal solution, where $b_M = (h+1)d_M - 1$ for all $M \in \mathcal{M}_G$. Without loss of generality, we assume that:

(i) G has a minimum number of edges,
(ii) $\sum_{e \in E} c_e$ is minimum with respect to (i).

By definition, $\mathcal{D}_{(G,c)}$ is feasible if and only if $c \geq \mathbf{0}$. Hence, by minimality assumption (ii), $\mathcal{D}_{(G,c')}$ has an optimal integer solution for every integer $c' \neq c$ such that $0 \leq c' \leq c$.

Let M be a multicut of G. We denote by ξ_M the vector of $\{0,1\}^{\mathcal{M}_G}$ whose only nonzero coordinate is the one associated with M. We say that M is *active* for a solution y to $\mathcal{D}_{(G,c)}$ if $y_M > 0$. Note that, by complementary slackness, a multicut is active for an optimal solution to $\mathcal{D}_{(G,c)}$ only if it is tight for an optimal solution to the primal problem. In particular, if a multicut is tight for no point of $P_{2h+1}(G)$, then it is active for no optimal solution to $\mathcal{D}_{(G,c)}$. Thus, we will use Lemma 2.11 and Theorem 2.10 to deduce properties on the optimal solutions to $\mathcal{D}_{(G,c)}$.

Claim 5.1 *G is simple, 2-connected, and nontrivial.*

Proof Suppose for a contradiction that there exist two parallel edges e_1 and e_2 and $c_{e_1} \leq c_{e_2}$. Since a multicut contains either both e_1 and e_2 or none of them, the inequality (8a) associated with e_2 is redundant because $c_{e_1} \leq c_{e_2}$. This contradicts minimality assumption (i), so G is simple.

Assume for a contradiction that G is not 2-connected. Then G is the 1-sum of two distinct graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. By Statement (ii) of Theorem 2.10, the multicuts of G that intersect both E_1 and E_2 are not tight for the points of $P_{2h+1}(G)$, by complementary slackness, these multicuts are not active for the optimal solutions to $\mathcal{D}_{(G,c)}$. Hence, every optimal solution y to $\mathcal{D}_{(G,c)}$ is of the form:

$$y_M = \begin{cases} y^1_M & \text{if } M \in \mathcal{M}_{G_1}, \\ y^2_M & \text{if } M \in \mathcal{M}_{G_2}, \\ 0 & \text{otherwise}, \end{cases} \qquad \text{for all } M \in \mathcal{M}_G,$$

where y^i is an optimal solution to $\mathcal{D}_{(G_i, c_{|E_i})}$ for $i = 1,2$. By minimality assumption (i), there exists an integer optimal solution $\bar{y}^i$ to $\mathcal{D}_{(G_i, c_{|E_i})}$ for $i = 1,2$, implying that $(\bar{y}^1, \bar{y}^2)$ is an integer optimal solution to $\mathcal{D}_{(G,c)}$, a contradiction.

Finally, if $G = K_2$, $\mathcal{M}_G$ contains only one multicut, say $\{e\}$, and the optimal solution to $\mathcal{D}_{(G,c)}$ is $y^*_{\{e\}} = c_e$ which is integer. □

Claim 5.2 *For all edges $e \in E$, $c_e \geq 1$.*

Proof By hypothesis, $c \geq \mathbf{0}$ is integer and $\mathcal{D}_{(G,c)}$ has an optimal solution, say y^*. Suppose for a contradiction that there exists an edge $e \in E$ with $c_e = 0$. Set $G' = G/e$ and $c' = c_{|(E \setminus e)}$. The active multicuts for y^* do not contain the edge e so they are multicuts of G' since $\mathcal{M}_{G'} = \{M \in \mathcal{M}_G | e \notin M\}$. Hence, the point $y' \in \mathbb{R}^{\mathcal{M}_{G'}}$ defined by $y'_M = y^*_M$ for all $M \in \mathcal{M}_{G'}$ is a solution to $\mathcal{D}_{(G',c')}$.

By minimality assumption (i), there exists an integer optimal solution $\tilde{y}$ to $\mathcal{D}_{(G',c')}$. Extending $\tilde{y}$ to a point of $\mathbb{Z}^{\mathcal{M}_G}$ by setting to 0 the missing component gives an integer solution to $\mathcal{D}_{(G,c)}$ with cost $b^\top \tilde{y} \geq b^\top y' = b^\top y^*$. This is an integer optimal solution to $\mathcal{D}_{(G,c)}$ since y^* is optimal, a contradiction to the hypothesis that $\mathcal{D}_{(G,c)}$ has no integer optimal solution. □

Claim 5.3 *Every optimal solution y to $\mathcal{D}_{(G,c)}$ satisfies $0 \leq y_M < 1$ for all $M \in \mathcal{M}_G$.*

Proof By contradiction, suppose that y^* is an optimal solution to $\mathcal{D}_{(G,c)}$ such that there exists a multicut M such that $y^*_M \geq 1$. Therefore, the point y' defined by $y' = y^* - \xi_M$ is a solution to $\mathcal{D}_{(G,c')}$ where $c' = c - \chi^M$. By minimality assumption (ii), $\mathcal{D}_{(G,c')}$ admits an integer optimal solution y''. The point $\tilde{y}$ defined by $\tilde{y} = y'' + \xi_M$ is an integer solution to $\mathcal{D}_{(G,c)}$ and we have:

$$b^\top \tilde{y} = b^\top y'' + b_M \geq b^\top y' + b_M = b^\top y^*.$$

Therefore $\tilde{y}$ is an integer optimal solution to $\mathcal{D}_{(G,c)}$, a contradiction. □

From the definition of series-parallel graphs, Claim 5.1 implies that G contains at least one degree 2 vertex. Let $\widehat{V}$ be the set of degree 2 vertices of G.

Claim 5.4 *Let $v \in \widehat{V}$, $\delta(v) = \{e_1, e_2\}$, y be an optimal solution to $\mathcal{D}_{(G,c)}$, and M_1 be an active multicut for y such that $M_1 \cap \delta(v) = e_1$. If $\delta(v)$ is active for y, then no multicut whose intersection with $\delta(v)$ is e_2 is active for y.*

Proof We prove the result by contradiction. Assume that M_1 and $\delta(v)$ are active for y and that there exists a M_2 active for y with $M_2 \cap \delta(v) = e_2$. By Observation 2.2, $M'_i = M_i \cup \delta(v)$ is a multicut of G such that $d_{M'_i} = d_{M_i} + 1$ for $i = 1, 2$. Let $0 < \varepsilon \leq \min\{y_{M_1}, y_{M_2}, y_{\delta(v)}\}$. Then, the point:

$$y' = y - \varepsilon \left(\chi^{M_1} + \chi^{M_2} + \chi^{\delta(v)} \right) + \varepsilon \left(\chi^{M'_1} + \chi^{M'_2} \right)$$

is a solution to $\mathcal{D}_{(G,c)}$, and we have $b^\top y' = b^\top y + \varepsilon$, implying that y is not optimal, a contradiction. □

Claim 5.5 *For every optimal solution to $\mathcal{D}_{(G,c)}$, the constraints* (8a) *associated with the edges incident to a degree 2 vertex are tight.*

Proof We prove the result by contradiction. Suppose that there exist an optimal solution y^* to $\mathcal{D}_{(G,c)}$ and a vertex v with $\delta(v) = \{e_1, e_2\}$ such that the inequality (8a) associated with e_1 is not tight. For $i = 1, 2$, let s_i be the slack of the constraint associated with e_i, that is,

$$s_i = c_{e_i} - \sum_{M \in \mathcal{M}_G : e_i \in M} y^*_M.$$

Inequality (8a) associated with e_2 is tight, as otherwise there exists $0 < \eta \leq \min\{s_1, s_2\}$, such that $y^* + \eta\xi_{\delta(v)}$ is a solution to $\mathcal{D}_{(G,c)}$, a contradiction to the optimality of y^*. Hence, Claims 5.2 and 5.3 imply that there are at least two distinct multicuts M_1 and M_2 active for y^* and containing e_2. Let $0 < \varepsilon \leq \min\{y^*_{M_1}, y^*_{M_2}, s_1\}$. For $i = 1, 2$, $e_1 \in M_i$, as otherwise $y' = y^* + \varepsilon(\xi_{M_i \cup e_1} - \xi_{M_i})$ is a solution to $\mathcal{D}_{(G,c)}$. This solution is such that $b^\top y' = b^\top y^* + \varepsilon(h+1) > b^\top y^*$, a contradiction to the optimality of y^*. Thus, both M_1 and M_2 contain $\delta(v)$. Since they are distinct, at least one of them, say M_1, strictly contains $\delta(v)$. Then, $y'' = y^* + \varepsilon(-\xi_{M_1} + \xi_{M_1 \setminus e_2} + \xi_{\delta(v)})$ is a solution to $\mathcal{D}_{(G,c)}$ because $M_1 \setminus e_2$ belongs to $\mathcal{M}_G$ by Lemma 2.11. Then, $b^\top y'' = b^\top y^* + \varepsilon(-b_{M_1} + b_{M_1} - (h+1) + 2h+1) > b^\top y^*$, a contradiction. $\square$

Given a solution y to $\mathcal{D}_{(G,c)}$, we define for each vertex $v \in \widehat{V}$ the set $\mathcal{A}^y_v$ as the set of multicuts active for y that strictly contain $\delta(v)$. Moreover we define the value α^y_v as:

$$\alpha^y_v = \sum_{M \in \mathcal{A}^y_v} y_M. \tag{9}$$

Claim 5.6 *Every optimal solution y to $\mathcal{D}_{(G,c)}$ satisfies $0 < \alpha^y_v < 1$ for all $v \in \widehat{V}$.*

Proof Suppose for a contradiction that there exist an optimal solution y^* to $\mathcal{D}_{(G,c)}$ and a vertex v of $\widehat{V}$ such that either $\alpha^{y^*}_v \geq 1$ or $\alpha^{y^*}_v = 0$. Denote the two edges incident to v by e_1 and e_2 in such a way that $c_{e_1} \leq c_{e_2}$.

Suppose first that $\alpha^{y^*}_v \geq 1$. By Claim 5.3, there exist at least two multicuts in $\mathcal{A}^{y^*}_v$. Let $\mathcal{A}^{y^*}_v = \{M_1, \ldots, M_n\}$. By Lemma 2.11, for all $i = 1, \ldots, n$, $M'_i = M_i \setminus e_1$ is a multicut of G with $d_{M'_i} = d_{M_i} - 1$. Let $c' = c - \chi^{e_1}$. By $\alpha^{y^*}_v \geq 1$, there exist ϵ_i for all $i = 1, \ldots, n$, such that $0 \leq \epsilon_i \leq y^*_{M_i}$ and $\sum_{i=1}^n \epsilon_i = 1$. The point y^1 defined by:

$$y^1 = y^* + \sum_{i=1}^n \left(-\epsilon_i \xi_{M_i} + \epsilon_i \xi_{M'_i}\right)$$

is a solution to $\mathcal{D}_{(G,c')}$. By definition of b, we have:

$$b^\top y^1 = b^\top y^* - h - 1. \tag{10}$$

By minimality assumption (ii), $\mathcal{D}_{(G,c')}$ admits an integer optimal solution, say y^2. This solution satisfies with equality the constraint (8a) associated with e_2 as otherwise $y^2 + \xi_{\delta(v)}$ would be a solution to $\mathcal{D}_{(G,c)}$ with cost $b^\top y^2 + b_{\delta(v)} \geq b^\top y^1 + 2h + 1$, contradicting the assumption that y^* is optimal by (10) and $h \geq 1$. Hence, there exists a multicut $\bar{M}$ active for y^2 containing e_2 but not e_1 since $c'_{e_1} + 1 \leq c'_{e_2}$. By definition, $\bar{M} \cup e_1$ is a multicut of G of order $d_{\bar{M}} + 1$. Define $y^3 \in \mathbb{Z}^{\mathcal{M}_G}$ by:

$$y^3_M = y^2 - \chi^{\bar{M}} + \chi^{\bar{M} \cup e_1}$$

By definition of c' and y^2, the point y^3 is an integer solution to $\mathcal{D}_{(G,c)}$. Therefore, by (10), since y^2 is optimal in $\mathcal{D}_{(G,c')}$ and by definition of y^3, we have:

$$b^\top y^* = b^\top y^1 + h + 1 \leq b^\top y^2 + h + 1 \leq b^\top y^3.$$

Thus, y^3 is an integer optimal solution to $\mathcal{D}_{(G,c)}$, a contradiction.

Suppose now that $\alpha_v^{y^*} = 0$. First, note that $\delta(v)$ is not an active multicut for y^*. Otherwise by Claims 5.2, 5.3 and 5.5, there would be a multicut containing e_1 and not e_2, say N_1, and a multicut containing e_2 and not e_1, say N_2, which are both active for y^*. This contradicts Claim 5.4. Hence, by definition of $\alpha_v^{y^*}$, no active multicut contains $\delta(v)$.

By Observation 2.2, if a multicut M contains e_2 but not e_1, then $M \triangle \delta(v)$ is a multicut with the same order and $b_M = b_{M\triangle\delta(v)}$. Hence, we can define the point $y^4 \in \mathbb{Q}^{\mathcal{M}_G}$:

$$y^4_M = \begin{cases} 0 & \text{if } e_1 \in M, \\ y^*_M + y^*_{M\triangle\delta(v)} & \text{if } e_1 \notin M \text{ and } e_2 \in M, \\ y^*_M & \text{otherwise,} \end{cases} \qquad \text{for all } M \in \mathcal{M}_G,$$

which is a solution to $\mathcal{D}_{(G,\hat{c})}$, where $\hat{c}$ is defined by:

$$\hat{c}_e = \begin{cases} c_{e_1} + c_{e_2} & \text{if } e = e_2, \\ 0 & \text{if } e = e_1, \\ c_e & \text{otherwise,} \end{cases} \qquad \text{for all } e \in E.$$

By construction, we have:

$$b^\top y^4 = b^\top y^*. \tag{11}$$

Using the argument given in the proof of Claim 5.2, we deduce that $\mathcal{D}_{(G,\hat{c})}$ admits an integer optimal solution, say y^5. Let $\mathcal{S}$ be the family of active multicuts for y^5 containing e_2, where each multicut M appears y^5_M times in $\mathcal{S}$. We have $|\mathcal{S}| > c_{e_2}$ as otherwise y^5 would be an integer optimal solution to $\mathcal{D}_{(G,c)}$, a contradiction.

We now construct from y^5 an integer solution y^6 to $\mathcal{D}_{(G,c)}$ with the same cost by replacing e_2 by e_1 in some active multicuts for y^5. More formally, since $|\mathcal{S}| \geq c_{e_1}$, there exists $\mathcal{S}' \subseteq \mathcal{S}$ with $|\mathcal{S}'| = c_{e_1}$. By Observation 2.2, $M\triangle\delta(v)$ is a multicut of G for all $M \in \mathcal{S}'$ and $b_M = b_{M\triangle\delta(v)}$. Let $y^6 \in \mathbb{Z}^{\mathcal{M}_G}$ be the point defined by:

$$y^6 = y^5 + \sum_{M\in\mathcal{S}'} \left(\xi_{M\triangle\delta(v)} - \xi_M\right) \tag{12}$$

By construction, we have:

$$b^\top y^6 = b^\top y^5. \tag{13}$$

Note that for each $M \in \mathcal{S}'$, adding $\xi_{M\triangle\delta(v)} - \xi_M$ to a point of $\mathbb{R}^{\mathcal{M}_G}$ increases (resp. decreases) by 1 the left-hand side of the inequality (8a) associated with e_1 (resp. e_2) while not changing the left-hand side of the inequalities (8a) associated with the edges of $E \setminus \{e_1, e_2\}$. Therefore, by definition of $\hat{c}$, y^6 is a solution to $\mathcal{D}_{(G,c)}$. By (13), the optimality of y^5, and (11), we have:

$$b^\top y^6 = b^\top y^5 \geq b^\top y^4 = b^\top y^*.$$

Therefore y^6 is an integer optimal solution to $\mathcal{D}_{(G,c)}$, a contradiction. $\square$

Claim 5.6 implies that for each optimal solution y and for each $v \in \widehat{V}$ there exists at least one multicut strictly containing $\delta(v)$ that is active for y. For the following claims we need to define a subset of optimal solutions to $\mathcal{D}_{(G,c)}$: let $\mathfrak{D}_v$ be the set of optimal solutions to $\mathcal{D}_{(G,c)}$ for which $\delta(v)$ is not active. Note that if $\mathfrak{D}_v$ is not empty, then there exists a solution y in $\mathfrak{D}_v$ maximizing α_v^z over all $z \in \mathfrak{D}_v$.

The following claim presents the structure of a specific optimal solution to $\mathcal{D}_{(G,c)}$ for which $\delta(v)$ is not active.

Claim 5.7 *Let $v \in \widehat{V}$ with $\delta(v) = \{e_1, e_2\}$ and let $y^* \in \mathfrak{D}_v$ maximize α_v^z over all $z \in \mathfrak{D}_v$. Then, there are exactly 3 multicuts active for y^* intersecting $\delta(v)$: two bonds $F \cup e_1$ and $F \cup e_2$ and a multicut $F \cup \{e_1, e_2\}$ of order 3, for some $F \subseteq E$.*

Proof By Claim 5.6, there exists at least one multicut strictly containing $\delta(v)$ which is active for y^*, say M_0. By definition of $\mathfrak{D}_v$, $\delta(v)$ is not active for y^*. Hence, by Claim 5.5, there exists at least one multicut active for y^* which contains e_i and not $\delta(v) \setminus e_i$, for $i = 1, 2$. Let M_i be such a multicut with maximum order.

First, we prove that $d_{M_0} = 3$. By definition, $M_0 = \delta(v, V_2, V_3, \ldots, V_{d_{M_0}})$. Moreover, by Lemma 2.11 and complementary slackness, the two vertices adjacent to v belong to two different classes, say V_2 and V_3. By contradiction, suppose that $d_{M_0} \geq 4$. Then, $M_0' = \delta(v \cup V_2 \cup V_3, \ldots, V_{d_{M_0}})$ is a multicut of order $d_{M_0} - 2$. For $i = 1, 2$, $M_i' = M_i \cup \delta(v)$ is a multicut of order $d_{M_i} + 1$. Let $0 < \varepsilon \leq \min\{y^*_{M_0}, y^*_{M_1}, y^*_{M_2}\}$. Then, let $y' \in \mathbb{R}^{\mathcal{M}_G}$ be the point defined by:

$$y' = y^* - \varepsilon\xi_{M_0} + \varepsilon\xi_{M_0'} + \varepsilon \sum_{i=1,2} \left(-\xi_{M_i} + \xi_{M_i'}\right).$$

By construction, y' is a solution to $\mathcal{D}_{(G,c)}$ with $b^\top y^* = b^\top y'$. Hence y' is an optimal solution, but we have $\alpha_v^{y'} = \alpha_v^{y^*} + \varepsilon$ because $\delta(v) \subsetneq M_i'$ for $i = 1, 2$. This contradicts the maximality of $\alpha_v^{y^*}$. Therefore $d_{M_0} = 3$.

Now, we show that M_1 is a bond. The result for M_2 holds by symmetry. By contradiction, suppose that $M_1 = \delta(V_1, \ldots, V_{d_{M_1}})$ with $d_{M_1} \geq 3$. Without loss of generality, we suppose that $e \in \delta(V_1) \cap \delta(V_2)$. Then, $M_1' = \delta(V_1 \cup V_2, \ldots, V_{d_{M_1}})$ is a multicut of order $d_{M_1} - 1$. Moreover, $M_2' = M_2 \cup \delta(v)$ is a multicut of order $d_{M_2} + 1$. Let $0 < \varepsilon \leq \min\{y^*_{M_1}, y^*_{M_2}\}$ and $y' \in \mathbb{R}^{\mathcal{M}_G}$ be the point defined by:

$$y' = y^* - \varepsilon\xi_{M_1} + \varepsilon\xi_{M_1'} - \varepsilon\xi_{M_2} + \varepsilon\xi_{M_2'}.$$

By construction, y' is a solution to $\mathcal{D}_{(G,c)}$ with $b^\top y^* = b^\top y'$. Hence y' is an optimal solution, but we have $\alpha_v^{y'} = \alpha_v^{y^*} + \varepsilon$ because $\delta(v) \subsetneq M_2'$. This contradicts the maximality of $\alpha_v^{y^*}$. Therefore, $d_{M_1} = d_{M_2} = 2$.

We now prove that there exists a set F such that $M_0 = F \cup \delta(v)$, and $M_i = F \cup e_i$ for $i = 1, 2$. Note that $M_1 \cup M_2$ is a multicut so $y'' = y^* + \varepsilon(\xi_{M_1 \cup M_2} - \xi_{M_1} - \xi_{M_2})$ is a solution to $\mathcal{D}_{(G,c)}$. The optimality of y^* implies $d_{M_1 \cup M_2} \leq 3$. Since M_1 and M_2 are distinct bonds, there exists $F \subseteq E \setminus \delta(v)$ such that $M_i = F \cup e_i$, for $i = 1, 2$. Finally, let $N_0 = M_0 \setminus e_2$ and $N_1 = M_1 \cup e_2$. Note that $\tilde{y} = y^* + \varepsilon(\xi_{N_0} - \xi_{M_0} + \xi_{N_1} - \xi_{M_1})$ is an optimal solution to $\mathcal{D}_{(G,c)}$ for which $\delta(v)$ is not active. Moreover, N_0 and M_2 are bonds active for $\tilde{y}$ since $d_{M_0} = 3$. This implies that $N_0 = F \cup e_1$, and hence $M_0 = F \cup \delta(v)$.

This implies that M_0, M_1, and M_2 are the only multicuts active for y^* intersecting $\delta(v)$. Indeed, if M is a multicut active for y^* strictly containing $\delta(v)$, then repeating the proof above with M, M_1, and M_2 shows that there exists F' such that $M = F' \cup \delta(v)$, and $M_i = F' \cup e_i$ for $i = 1, 2$. Since $M_i = F \cup e_i$ for $i = 1, 2$, we have $F' = F$ and hence $M = M_0$. A similar argument holds for any multicut active for y^* and intersecting $\delta(v)$. □

Claim 5.8 *Let $v \in \widehat{V}$ and y^* be an optimal solution to $\mathcal{D}_{(G,c)}$. Then,*

(i) if $y^*_{\delta(v)} = 0$, then $c_e = 1$ for all $e \in \delta(v)$,

(ii) if $y^*_{\delta(v)} > 0$, then $\alpha_v^{y^*} + y^*_{\delta(v)} = 1$, and there exists $e \in \delta(v)$ such that $c_e = 1$.

Proof (*i.*) First suppose that $y^*_{\delta(v)} = 0$, then $\mathfrak{D}_v \neq \emptyset$. Let $y' \in \mathfrak{D}_v$ maximize α_v^z over all $z \in \mathfrak{D}_v$. Then, by Claim 5.7, there exist exactly two active multicuts for y' containing e_i for $i = 1, 2$. Combining Claims 5.3 and 5.5, and the integrality of c, we obtain that $c_{e_i} = 1$ for $i = 1, 2$.

(*ii.*) Let now $y^*_{\delta(v)} > 0$. By Claim 5.4, there exists an edge $e \in \delta(v)$ such that all multicuts containing e that are active for y contain $\delta(v)$. Hence, the constraint (8a) associated with e is:

$$c_e \geq \sum_{M:e\in M} y^*_M = y^*_{\delta(v)} + \sum_{M\in\mathcal{A}_v^{y^*}} y^*_M = y^*_{\delta(v)} + \alpha_v^{y^*}. \tag{14}$$

By Claim 5.5, the constraint (8a) associated with e is tight. Thus, $y^*_{\delta(v)} + \alpha_v^{y^*} = c_e$. By Claims 5.3 and 5.6 and since c_e is integer, we have $c_e = 1$. □

The last three claims of the proof give some structural property of the graph G. In particular we focus our attention on the vertices of $\widehat{V}$.

Claim 5.9 *Vertices of degree 2 are pairwise nonadjacent.*

Proof Assume for a contradiction that there exist two adjacent vertices v_1 and v_2 in $\widehat{V}$, and denote $\delta(v_i) = \{e_0, e_i\}$ for $i = 1, 2$.

We prove that $\delta(v_1)$ is active for all optimal solutions to $\mathcal{D}_{(G,c)}$, the result for $\delta(v_2)$ is obtained by symmetry. By contradiction, suppose that $\mathfrak{D}_{v_1} \neq \emptyset$. Among all the solutions $y \in \mathfrak{D}_{v_1}$, let y^1 be one having $\alpha_{v_1}^y$ maximum. Then, by Claim 5.7, the three multicuts active for y^1 intersecting $\delta(v_1)$ are $M_0 = F \cup \delta(v_1)$, $B_0 = F \cup e_0$, and $B_1 = F \cup e_1$, where B_i are bonds for $i = 0, 1$, and $F \subseteq E \setminus \delta(v_1)$ contains no nonempty multicut. By Claim 5.6 on v_2, there exists a multicut M active for y^1 strictly containing $\delta(v_2)$. By $\delta(v_1) \cap \delta(v_2) = e_0$, M intersects $\delta(v_1)$. Since $d_M \geq 3$, Claim 5.7 for v_1 implies $M = M_0$, $F = \{e_2\}$, and $B_0 = \delta(v_2)$.

As $y^1_{\delta(v_1)} = 0$, by Statement (i) of Claim 5.8, $c_{e_0} = c_{e_1} = 1$. By Claim 5.5, the constraints associated with e_0 and e_1 are tight. Since $\mathcal{A}_{v_1}^{y^1} = \{M_0\}$ by Claim 5.7, we have:

$$c_{e_i} = y^1_{M_0} + y^1_{B_i} = 1 \quad \text{for } i = 0, 1. \tag{15}$$

Let $\{M_1, \ldots, M_n\}$ be the set of active multicuts for y^1 such that $M_i \cap \{e_0, e_1, e_2\} = e_2$, for $i = 1, \ldots, n$. By Claim 5.5, the constraint (8a) associated with e_2 is tight, hence, using (15):

$$c_{e_2} = y^1_{M_0} + y^1_{B_0} + y^1_{B_1} + \sum_{i=1}^{n} y^1_{M_i} = 1 + y^1_{B_0} + \sum_{i=1}^{n} y^1_{M_i}. \tag{16}$$

By Claim 5.3, B_0 active for y^1, and $c_{e_2} \in \mathbb{Z}$, we have $\{M_1, \dots, M_n\} \neq \emptyset$ and $c_{e_2} \geq 2$. Thus, combining (15) and (16), we have:

$$\sum_{i=1}^{n} y^1_{M_i} = c_{e_2} - 1 - y^1_{B_0} \geq y^1_{M_0}. \tag{17}$$

Then, there exist $\epsilon_1, \dots, \epsilon_n$ such that $0 \leq \epsilon_i \leq y^1_{M_i}$ for $i = 1, \dots, n$, and

$$\sum_{i=1}^{n} \epsilon_i = y^1_{M_0}.$$

For $i = 1, \dots, n$, $M_i \cup e_0$ is a multicut with order $d_{M_i} + 1$, hence we can consider the following solution to $\mathcal{D}_{(G,c)}$:

$$y^2 = y^1 - \left(y^1_{M_0} \xi_{M_0} + \sum_{i=1}^{n} \epsilon_i \xi_{M_i} \right) + \left(y^1_{M_0} \xi_{M_0 \setminus e_0} + \sum_{i=1}^{n} \epsilon_i \xi_{M_i \cup e_0} \right). \tag{18}$$

We have $b^\top y^1 = b^\top y^2$, but $\alpha^{y^2}_{v_1} = 0$, a contradiction to Claim 5.6. Therefore $\mathfrak{D}_v \neq \emptyset$, and by symmetry we deduce that both $\delta(v_1)$ and $\delta(v_2)$ are active for all optimal solutions to $\mathcal{D}_{(G,c)}$.

By Claim 5.4, for every optimal solution y to $\mathcal{D}_{(G,c)}$ and every multicut M of G, if M is active for y and contains e_i for some $i \in \{1, 2\}$, then $e_0 \in M$.

Let y^* be the optimal solution to $\mathcal{D}_{(G,c)}$ maximizing $\alpha^y_{v_1}$ over all y solutions to $\mathcal{D}_{(G,c)}$. We have $\mathcal{A}^{y^*}_{v_2} \subseteq \mathcal{A}^{y^*}_{v_1}$ and all the multicuts in $\mathcal{A}^{y^*}_{v_2}$ have order at most 3. Otherwise, let $M \in \mathcal{A}^{y^*}_{v_2} \setminus \mathcal{A}^{y^*}_{v_1}$ (resp. $M \in \mathcal{A}^{y^*}_{v_2}$ such that $d_M \geq 4$), and $0 < \varepsilon \leq \min\{y^*_M, y^*_{\delta(v_1)}\}$. The solution

$$y^3 = y^* - \varepsilon(\xi_M + \xi_{\delta(v_1)}) + \varepsilon(\xi_{M \setminus e_2} + \xi_{\delta(v_1) \cup e_2})$$

is optimal, but $\alpha^{y^3}_{v_1} = \alpha^{y^*}_{v_1} + \varepsilon$ by the choice of M, a contradiction to the maximality of $\alpha^{y^*}_{v_1}$. Thus, $\bar{M} = \{e_0, e_1, e_2\}$ is the only multicut in $\mathcal{A}^{y^*}_{v_2}$.

Let $\{N_1, \dots, N_m\}$ be the set of active multicuts for y^* such that $N_i \cap \{e_0, e_1, e_2\} = e_0$ for $i = 1, \dots, m$. The constraint associated with e_0 is tight by Claim 5.5, hence, by $\mathcal{A}^{y^*}_{v_2} \subseteq \mathcal{A}^{y^*}_{v_1}$, we have:

$$c_{e_0} = \alpha^{y^*}_{v_1} + y^*_{\delta(v_1)} + y^*_{\delta(v_2)} + \sum_{i=1}^{m} y^*_{N_i}. \tag{19}$$

By Statement (ii) of Claim 5.8 applied to v_1, we have $y^*_{\delta(v_1)} + \alpha^{y^*}_{v_1} = 1$, and so:

$$c_{e_0} = 1 + y^*_{\delta(v_2)} + \sum_{i=1}^{m} y^*_{N_i}. \tag{20}$$

By $\mathcal{A}^{y^*}_{v_2} = \{\bar{M}\}$ and Statement (ii) of Claim 5.8 applied to v_2, we have $y^*_{\delta(v_2)} + y^*_{\bar{M}} = 1$, hence:

$$c_{e_0} = 2 - y^*_{\bar{M}} + \sum_{i=1}^{m} y^*_{N_i}. \tag{21}$$

Since c_{e_0} is integer and since $y^*_{\bar{M}} < 1$ by Claim 5.3, by (21), we have:

$$\sum_{i=1}^{m} y^*_{N_i} \geq y^*_{\bar{M}}. \tag{22}$$

Hence, let $\lambda_1, \ldots, \lambda_m$ be such that $0 \leq \lambda_i \leq y^*_{N_i}$ for $i = 1, \ldots, m$, and $\sum_{i=1}^{m} \lambda_i = y^*_{\bar{M}}$. Note that $\delta(v_2) = \bar{M} \setminus e_1$. Then, the point

$$y^5 = y^* - \left(y^*_{\bar{M}} \xi_{\bar{M}} + \sum_{i=1}^{m} \lambda_i \xi_{N_i} \right) + \left(y^*_{\bar{M}} \xi_{\delta(v_2)} + \sum_{i=1}^{m} \lambda_i \xi_{N_i \cup e_1} \right)$$

is a solution to $\mathcal{D}_{(G,c)}$, and it is optimal by definition of b. Moreover,

$$y^5_{\delta(v_2)} = y^*_{\bar{M}} + y^*_{\delta(v_2)} = 1,$$

a contradiction to Claim 5.3. □

The following claim forbids a circuit of length 3 to contain a vertex of $\widehat{V}$.

Claim 5.10 *No circuit of length 3 contains a vertex of degree 2.*

Proof Assume for a contradiction that in G there exist a vertex $v \in \widehat{V}$ and a circuit $\{e_1, e_2, e_3\}$ such that $\delta(v) = \{e_1, e_2\}$. By Lemma 2.1, a multicut contains e_3 only if it intersects $\delta(v)$. On the other hand, by Lemma 2.11 and complementary slackness, each multicut strictly containing $\delta(v)$ and active for an optimal solution contains e_3. Thus, for every optimal solution y to $\mathcal{D}_{(G,c)}$, we have:

$$\sum_{M: e_3 \in M} y_M = \sum_{M: e_1 \in M, M \neq \delta(v)} y_M + \sum_{M: e_2 \in M, M \neq \delta(v)} y_M - \alpha^y_v. \tag{23}$$

Let y^* be an optimal solution to $\mathcal{D}_{(G,c)}$. By the constraint (8a) associated with e_3, (23), and Claim 5.5, we have:

$$c_{e_3} \geq \sum_{M: e_3 \in M} y^*_M = c_{e_1} + c_{e_2} - 2y^*_{\delta(v)} - \alpha^{y^*}_v. \tag{24}$$

By Claim 5.6 and Statement (ii) of Claim 5.8, we have $2y^*_{\delta(v)} + \alpha^{y^*}_v < 2$. Thus, by (24) and $c_{e_3} \in \mathbb{Z}$, we have $c_{e_3} \geq c_{e_1} + c_{e_2} - 1$.

Define $G' = G \setminus e_3$ and $c' = c_{|(E \setminus e_3)}$. Note that for each multicut $M \in \mathcal{M}_G$, $M \setminus e_3$ is a multicut of G' with order at least d_M. Hence, y^* induces a solution to $\mathcal{D}_{(G',c')}$ of cost at least $b^\top y^*$. By minimality assumption (i), there exists an integer optimal solution y' to $\mathcal{D}_{(G',c')}$, and we have $b^\top y' \geq b^\top y^*$.

Let $\mathcal{M}_1$ (resp. $\mathcal{M}_2$) be the set of multicuts $M = \delta(V_1, \ldots, V_{d_M})$ of G' active for y' such that the endpoints of e_3 belong (resp. do not belong) to a same V_i for some $i \in \{1, \ldots, d_M\}$. For each $M \in \mathcal{M}_1$ (resp. $M \in \mathcal{M}_2$), M (resp. $M \cup e_3$) is a multicut of G with the same order. Hence,

$$y'' = \sum_{M \in \mathcal{M}_1} y'_M \xi_M + \sum_{M \in \mathcal{M}_2} y'_M \xi_{M \cup e_3}$$

is a point of $\mathbb{Z}_+^{\mathcal{M}_G}$ with $b^\top y'' = b^\top y'$. Thus, $b^\top y'' \geq b^\top y^*$, and y'' is not a solution to $\mathcal{D}_{(G,c)}$. By definition, y'' respects every constraint of $\mathcal{D}_{(G,c)}$ but the constraint (8a) associated with e_3.

By definition of y'', we have:

$$\sum_{M:e_3\in M} y''_M = \sum_{M:e_1\in M, M\neq\delta(v)} y''_M + \sum_{M:e_2\in M, M\neq\delta(v)} y''_M - \alpha_v^{y''}. \tag{25}$$

Therefore, by y'' violating the constraint (8a) associated with e_3, (25), Statement (*ii*) of Claim 5.8, and the inequalities (8a) associated with e_1 and e_2, we have:

$$c_{e_3} < \sum_{M:e_3\in M} y''_M = \sum_{M:e_1\in M} y''_M + \sum_{M:e_2\in M} y''_M - \alpha_v^{y''} - 2y''_{\delta(v)} \leq c_{e_1} + c_{e_2} - \alpha_v^{y''} - 2y''_{\delta(v)}. \tag{26}$$

Thus, by (24), we have $\alpha_v^{y''} + 2y''_{\delta(v)} < \alpha_v^{y^*} + 2y^*_{\delta(v)} < 2$. By $c_{e_3} \geq c_{e_2} + c_{e_1} - 1$, the integrality of y'', and (26), we have $\alpha_v^{y''} = y''_{\delta(v)} = 0$, and so $c_{e_3} = c_{e_1} + c_{e_2} - 1$. Hence, by the integrality of y'' and equation (25):

$$c_{e_3} + 1 = \sum_{M:e_3\in M} y''_M = \sum_{M:e_1\in M} y''_M + \sum_{M:e_2\in M} y''_M = c_{e_1} + c_{e_2}. \tag{27}$$

For $i = 1, 2$, since $c_{e_i} \geq 1$, there exists a multicut M_i active for y'' such that $M_i \cap \delta(v) = e_i$.

We claim that the constraint (8a) associated with e_3 is not tight for y^*. By $c_{e_3} = c_{e_1} + c_{e_2} - 1$, (24), and Claim 5.6, $\delta(v)$ is active for y^*. Hence, by Statement (ii) of Claim 5.8, we have:

$$\alpha_v^{y^*} + y^*_{\delta(v)} = 1. \tag{28}$$

Hence, by (23) and Claim 5.5, (28), (27), and $\delta(v)$ active for y^*, we have:

$$\sum_{M:e_3\in M} y^*_M = c_{e_1} + c_{e_2} - 1 - y^*_{\delta(v)} = c_{e_3} - y^*_{\delta(v)} < c_{e_3}. \tag{29}$$

The point y'' respects all the constraints of $\mathcal{D}_{(G,c)}$ except the one associated with e_3, and this constraint is not tight for y^*. Therefore, there exists $0 < \lambda < 1$ such that

$$\tilde{y} = \lambda y^* + (1-\lambda) y''$$

is a solution to $\mathcal{D}_{(G,c)}$. Moreover, $\tilde{y}$ is optimal because $b^\top y^* \leq b^\top y''$.

All multicuts active for at least one between y^* and y'' are active for $\tilde{y}$. Since $\delta(v)$ is active for y^* and M_1, M_2 are active for y'', the three multicuts M_1, M_2, and $\delta(v)$ are active for $\tilde{y}$, a contradiction to Claim 5.4. □

Claim 5.11 *Each circuit of length 4 contains at most one vertex of degree 2.*

Proof Assume for a contradiction that there exists a circuit $C = \{e_1, \ldots, e_4\}$ in G covering two vertices of $\widehat{V}$, say v_1, v_2. By Claim 5.9, v_1 and v_2 are not adjacent, hence we assume that $\delta(v_1) = \{e_1, e_2\}$ and $\delta(v_2) = \{e_3, e_4\}$. Let v_3 and v_4 be the remaining vertices of C.

We prove that $\delta(v_1)$ is active for all optimal solutions to $\mathcal{D}_{(G,c)}$. Indeed, if $\mathfrak{D}_{v_1} \neq \emptyset$, then let $y' \in \mathfrak{D}_{v_1}$ maximize $\alpha_{v_1}^z$ over all $z \in \mathfrak{D}_{v_1}$. By Statement (ii) of

Theorem 2.10, for every multicut M in $\mathcal{A}_{v_2}^{y'}$, we have $M = \delta\{v_2, V_2, \dots, V_{d_M}\}$, with v_3 and v_4 belonging to different V_i's, hence $M \cap \delta(v_1) \neq \emptyset$. However, $M \setminus \delta(v_1)$ contains $\delta(v_2)$, a contradiction to Claim 5.7 applied to v_1. Exchanging the role of v_1 and v_2, we deduce that $\delta(v_2)$ is active for all optimal solutions to $\mathcal{D}_{(G,c)}$.

Without loss of generality, there exists an optimal solution y such that $\alpha_{v_1}^y \geq \alpha_{v_2}^y$. Then, we can build from y an optimal solution y^* to $\mathcal{D}_{(G,c)}$ such that $\mathcal{A}_{v_2}^{y^*} \subseteq \mathcal{A}_{v_1}^{y^*}$. Indeed, suppose $\mathcal{A}_{v_2}^y \setminus \mathcal{A}_{v_1}^y = \{M_1, \dots, M_n\}$. Then, since $\alpha_{v_1}^y \geq \alpha_{v_2}^y$, there exist $N_1, \dots, N_m \in \mathcal{A}_{v_1}^y \setminus \mathcal{A}_{v_2}^y$ such that:

$$\sum_{i=1}^{n} y_{M_i} \leq \sum_{j=1}^{m} y_{N_j}. \tag{30}$$

Hence, there exist $\epsilon_1, \dots, \epsilon_m$ such that $0 \leq \epsilon_j \leq y_{N_j}$, for $j = 1, \dots, m$, and

$$\sum_{j=1}^{m} \epsilon_j = \sum_{i=1}^{n} y_{M_i}. \tag{31}$$

By Statement (ii) of Theorem 2.10 and complementary slackness, v_3 and v_4 belong to different classes of N_j for each $j = 1, \dots, m$, implying that $N_j \cap \delta(v_2) \neq \emptyset$. Moreover, since $N_j \notin \mathcal{A}_{v_2}^y$, we have $|N_j \cap \delta(v_2)| = 1$, for all $j = 1, \dots, m$. Furthermore, since $\delta(v_2)$ is active for y, by Claim 5.4, there exists an edge in $\delta(v_2)$, say e_3, such that $N_j \cap \delta(v_2) = e_3$ for all $j = 1, \dots, m$. Therefore, the point

$$y^* = y - \left(\sum_{i=1}^{n} y_{M_i} \xi_{M_i} - \sum_{i=1}^{n} y_{M_i} \xi_{M_i \setminus e_4} \right) + \left(\sum_{j=1}^{m} \epsilon_j \xi_{N_j \cup e_4} - \sum_{j=1}^{m} \epsilon_j \xi_{N_j} \right) \tag{32}$$

is a solution to $\mathcal{D}_{(G,c)}$ with $b^\top y^* = b^\top y$ and $\mathcal{A}_{v_2}^{y^*} \subseteq \mathcal{A}_{v_1}^{y^*}$. Let $\mathcal{A}_{v_2}^{y^*} = \{M'_1 \dots, M'_p\}$. For $i = 1, \dots, p$, since $M'_i \in \mathcal{A}_{v_1}^{y^*}$, Statement (ii) of Theorem 2.10 implies $M'_i = \delta(v_1, v_2, V_3^i, V_4^i, \dots, V_{d_{M'_i}}^i)$, where V_3^i and V_4^i contain respectively v_3 and v_4. Then, $M''_i = \delta(v_1, v_2 \cup V_3^i \cup V_4^i, \dots, V_{d_{M'_i}}^i)$ is a multicut of order $d_{M'_i} - 2$ for $i = 1, \dots, p$. Since $\delta(v_2)$ is active for y^*, by Statement (ii) of Claim 5.8, we have $\alpha_{v_2}^{y^*} + y^*_{\delta(v_2)} = 1$. Then, the point $y^1 \in \mathbb{Q}^{\mathcal{M}_G}$ defined by:

$$y^1 = y^* - \left(y^*_{\delta(v_2)} \xi_{\delta(v_2)} + \sum_{i=1}^{p} y^*_{M'_i} \xi_{M'_i} \right) + \left(\sum_{i=1}^{p} y^*_{M'_i} \xi_{M''_i} \right),$$

is a solution to $\mathcal{D}_{(G,c')}$, where $c' = c - \chi^{\delta(v_2)}$.

By $d_{M''_i} = d_{M'_i} - 2$ for all $i = 1, \dots, p$, and $\alpha_{v_2}^{y^*} + y^*_{\delta(v_2)} = 1$, we have:

$$b^\top y^1 = b^\top y^* - \alpha_{v_2}^{y^*}(2h+2) - y^*_{\delta(v_2)}(2h+1) = b^\top y^* - (2h+1) - \alpha_{v_2}^{y^*}. \tag{33}$$

By minimality assumption (ii), $\mathcal{D}_{(G,c')}$ admits an integer optimal solution, say y^2. The point $y^3 \in \mathbb{Z}^{\mathcal{M}_G}$ defined by $y^3 = y^2 + \xi_{\delta(v_2)}$ is a solution to $\mathcal{D}_{(G,c)}$ such that:

$$b^\top y^3 = b^\top y^2 + 2h + 1. \tag{34}$$

Therefore, by (33), the optimality of y^2, and (34), we have:

$$b^\top y^* = b^\top y^1 + 2h + 1 + \alpha_{v_2}^{y^*} \leq b^\top y^2 + 2h + 1 + \alpha_{v_2}^{y^*} = b^\top y^3 + \alpha_{v_2}^{y^*}. \tag{35}$$

By integrality of $P_{2h+1}(G)$ and duality, we have $b^\top y^* \in \mathbb{Z}$. Furthermore, y^3 is integer by construction, so $b^\top y^3 \in \mathbb{Z}$. Then, by (35) and Claim 5.6, we have $b^\top y^* \leq b^\top y^3$, and so y^3 is an integer optimal solution to $\mathcal{D}_{(G,c)}$, a contradiction. □

Claims 5.1, 5.9, 5.10, 5.11 and Lemma 2.3 imply that G is not series-parallel, a contradiction. □

The box-TDIness of $P_k(G)$ and the TDIness of System (2) give the following result.

Corollary 5.2 *System* (2) *is box-TDI if and only if G is series-parallel.*

Proof By Theorem 5.1, when G is not series-parallel, System (2) is not TDI. Whenever G is series-parallel, $P_k(G)$ is box-TDI by Theorem 3.2 and System (2) is TDI by Theorem 5.1. □

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