TD 9: Planar Graphs

1 Planar Graphs without Short Cycles

Show that if in a planar (non-acyclic) graph G on n vertices and m edges, all cycles have length at least g, then $m \leq (n-2)\frac{g}{g-2}$. Conclude that $K_{3,3}$ is non-planar, using the fact that it is bipartite.

2 Planarity and Complements

Show that if G is planar and has $n \geq 11$ vertices, then $\overline{G}$ is non-planar.

3 Outerplanarity

A graph is outerplanar if it has a planar drawing where all the vertices lie on a single face. Prove the following:

1. In an outerplanar graph with n vertices and m edges we have $m \leq 2n-3$.
2. More strongly, in an outerplanar graph where all cycles have length at least g we have $m \leq \frac{g-1}{g-2}n - \frac{g}{g-2}$.
3. Conclude that K_4 and $K_{2,3}$ are not outerplanar.
4. Prove that every outerplanar graph contains a vertex of degree at most 2. Observe that this implies the first point.
5. Conclude that outerplanar graphs can always be colored with 3 colors.
6. Conclude a second time that outerplanar graphs can always be colored with 3 colors by invoking the 4-color theorem.

4 Euler's formula for disconnected graphs

We saw that if a planar graph G is connected, then $n + f = m + 2$. Show that for (possibly) disconnected planar graphs with c connected components we have $n + f = m + c + 1$.

5 Kuratowski

Prove that if a graph G has at most 8 edges, then G is planar.