TD 9: Planar Graphs

1 Planar Graphs without Short Cycles

Show that if in a planar (non-acyclic) graph G on n vertices and m edges, all cycles have length at least g, then $m \leq (n-2)\frac{g}{g-2}$. Conclude that $K_{3,3}$ is non-planar, using the fact that it is bipartite.

Solution:

Consider a planar G that satisfies the conditions. Then, each face has a border with length at least g. For a face ϕ, let $b(\phi)$ be the number of edges on the border of ϕ. We have $fg \leq \sum b(\phi) \leq 2m$, where f is the number of faces of a drawing of G. The first inequality follows because each face has at least g edges on its border; the second because each edge is counted at most twice in the sum.

We now recall that $n+f = m+2$, so if $f \leq \frac{2m}{g}$ then $n+f \leq \frac{2m}{g}+n \Rightarrow m+2 \leq \frac{2m}{g}+n \Rightarrow m \leq (n-2)\frac{g}{g-2}$

Since $K_{3,3}$ is bipartite, all its cycles have length at least 4. If it were planar, it would need to have at most $(6-2) \cdot \frac{4}{2} = 8$ edges, but it has 9 edges.

2 Planarity and Complements

Show that if G is planar and has $n \geq 11$ vertices, then $\overline{G}$ is non-planar.

Solution:

G has at most $m \leq 3n-6$ edges, therefore $\overline{G}$ has at least $\binom{n}{2} - m \geq \frac{n(n-1)}{2} - 3n + 6$ edges. We claim that $\frac{n(n-1)}{2} - 3n + 6 > 3n - 6$, which would imply that $\overline{G}$ is non-planar. Indeed, we equivalently get $n(n-1) > 2(6n-12) \Leftrightarrow n^2 > 13n - 24$. It is not hard to verify that this inequality holds for $n \geq 11$.

Note: the statement remains true for $n \geq 9$, but the proof is tedious and involves many cases. There does, however, exist a planar graph with $n = 8$ vertices whose complement is still planar.

3 Outerplanarity

A graph is outerplanar if it has a planar drawing where all the vertices lie on a single face. Prove the following:

1. In an outerplanar graph with n vertices and m edges we have $m \leq 2n-3$.
2. More strongly, in an outerplanar graph where all cycles have length at least g we have $m \leq \frac{g-1}{g-2}n - \frac{g}{g-2}$.
3. Conclude that K_4 and $K_{2,3}$ are not outerplanar.
4. Prove that every outerplanar graph contains a vertex of degree at most 2. Observe that this implies the first point.
5. Conclude that outerplanar graphs can always be colored with 3 colors.
6. Conclude a second time that outerplanar graphs can always be colored with 3 colors by invoking the 4-color theorem.

Solution:

1. The outer face has at its border n vertices and therefore at least n edges. The inside faces have at least 3 edges each. If we define $b(\phi)$ as in the first exercise we have $3(f-1)+n \leq \sum b(\phi) \leq 2m$ which gives $2m \geq 3f+n-3$. However, from Euler's formula we have $3f = 3m-3n+6$, so $2m \geq 3m-2n+3 \Rightarrow m \leq 2n-3$.

2. We repeat the previous calculation, but now inner faces have at least g edges, so $2m \geq g(f-1)+n = gf+n-g$. However, from Euler's formula $gf = gm-gn+2g$, so $2m \geq gm-gn+n+g \Rightarrow (g-2)m \leq (g-1)n-g \Rightarrow m \leq \frac{g-1}{g-2}n - \frac{g}{g-2}$.

3. K_4 and $K_{2,3}$ would need to have at most 5 and 5.5 edges respectively, but they both have 6 edges (we used that for $K_{2,3}$ the shortest cycles has length 4).

4. Suppose that we have a maximal outerplanar graph G (that is, we add as many edges as possible preserving outerplanarity). The outer face contains all n vertices connected in a cycle. Consider now the **dual** graph, which has a vertex for each inner face and two vertices are adjacent if the corresponding faces share an edge. This dual graph restricted to inner faces must be acyclic, because otherwise a vertex would not lie on the outside face. Every acyclic graph has a vertex of degree 1. In the corresponding face, one vertex has degree at most 2.

5. Run the First-Fit algorithm placing the vertex of degree 2 last, recurse...

6. Take an outerplanar graph G and add a universal vertex v adjacent to all previous vertices. The new graph is planar, as we can take a drawing of G and place v in the outer face. Then $G+v$ can be colored with 4 colors, by the 4-color theorem. Hence, G can be colored with 3 colors, since the color of v does not appear in any other vertex.

4 Euler's formula for disconnected graphs

We saw that if a planar graph G is connected, then $n+f = m+2$. Show that for (possibly) disconnected planar graphs with c connected components we have $n+f = m+c+1$.

Solution:

Let $C_1, C_2, \ldots, C_c$ be the components of a planar graph G, where each component has n_i vertices and m_i edges. We have that in a planar drawing of $G[C_i]$ we have $f_i = m_i - n_i + 2$ faces. One of these faces is the outer face, so we have $f_i' = m_i - n_i + 1$ inner faces. The total number of faces of a drawing of G is then $f = 1 + \sum_i f_i' = 1 + \sum_i (m_i - n_i + 1) = 1 + m - n + c$ so $f + n = m + c + 1$.

5 Kuratowski

Prove that if a graph G has at most 8 edges, then G is planar.

Solution:

For contradiction, suppose G is non-planar. Then, it must contain a subgraph that is a sub-division of either K_5 or $K_{3,3}$, therefor it must contain at least as many edges as one of these graphs. However, both these graphs have at least 9 edges, contradiction.