TD 8: More Coloring

1 Colorings and Complements – again!

In a previous exercise we saw that for all G on n vertices we have $\chi(G) + \chi(\overline{G}) \geq 2\sqrt{n}$. Here we want to prove an upper bound on the same quantity. Show that $\chi(G) + \chi(\overline{G}) \leq n + 1$. Give a tight example.

2 First-Fit and Trees

Recall the First-Fit coloring algorithm we saw in class: consider the vertices of G in some (arbitrary) order, and for each vertex assign to it the minimum color that is still available. This algorithm is not guaranteed to produce an optimal coloring. In this exercise we focus on how far this algorithm can be from giving an optimal coloring.

1. Show that there exists, for each k, a tree T_k such that First-Fit uses k colors to color T_k. (Recall that trees can be colored with 2 colors, so this is bad...)
2. Show that if G is a tree on n vertices, then First-Fit will never use more than $\log n$ colors (no matter the order considered).
3. Show that there exists, for each k, a bipartite graph on $2k$ vertices for which First-Fit uses $k + 1$ colors.
4. Show that in any bipartite graph on at most $2n + 1$ vertices, First-Fit will use at most $n + 1$ colors.

3 Edge Coloring and Bipartite Graphs

An edge coloring of a graph $G = (V, E)$ with k colors is an assignment of colors from $\{1, \ldots, k\}$ to E so that any two edges e_1, e_2 that share an endpoint receive distinct colors. In other words an edge coloring of G is a vertex coloring of $L(G)$, the line graph of G. We use $\chi'(G)$ to denote the minimum number of colors needed to color the edges of G.

1. Show that $\chi'(G) \geq \Delta(G)$ for all G.
2. Show that if G is bipartite, then $\chi'(G) = \Delta(G)$.
3. Show that there exists a (non-bipartite) G with $\chi'(G) > \Delta(G)$.

4 Cograph Coloring

Prove that if G contains no induced copy of P_4 (the path on 4 vertices), then First-Fit always produces an optimal coloring of G. (Note: graphs that contain no induced P_4 are called cographs.)