TD 8: More Coloring

1 Colorings and Complements – again!

In a previous exercise we saw that for all G on n vertices we have $\chi(G) + \chi(\overline{G}) \geq 2\sqrt{n}$. Here we want to prove an upper bound on the same quantity. Show that $\chi(G) + \chi(\overline{G}) \leq n + 1$. Give a tight example.
Solution:

We prove the statement by induction on n. For $n = 1$ or $n = 2$ the statement is easy to see. Suppose the statement is true for graphs on n vertices and we have a graph G on $n + 1$ vertices. We want to show that $\chi(G) + \chi(\overline{G}) \leq n + 2$. Pick an arbitrary vertex x of G and suppose its degree is d. The inductive hypothesis holds for $G-x$ (which has n vertices) so $\chi(G-x)+\chi(\overline{G-x}) \leq n+1$. We also have $\chi(G) \leq \chi(G-x)+1$ and $\chi(\overline{G}) \leq \chi(\overline{G-x}) + 1$, because a new color can always be used for x. Unfortunately, adding these inequalities gives $\chi(G) + \chi(\overline{G}) \leq n + 3$, which is not enough. We therefore consider two cases:

1. One of the inequalities $\chi(G) \leq \chi(G - x) + 1$ and $\chi(\overline{G}) \leq \chi(\overline{G-x}) + 1$ is strict. In that case $\chi(G) + \chi(\overline{G}) \leq n + 2$ as desired.

2. Both of the above inequalities are actually equalities. Then, deleting x from G decreases the chromatic number by 1. It must be the case that the degree d of x in G is at least $\chi(G - x)$, otherwise a coloring of $G - x$ with $\chi(G - x)$ colors can always be extended to G without using a new color for x. Similarly, the degree of x in $\overline{G}$ (which is $n-d$) must be at least $\chi(\overline{G-x})$. Summing up, we get $\chi(G-x)+\chi(\overline{G-x}) \leq n$, which is better than what we had from the inductive hypothesis, so adding the inequalities as before yields the desired result.

A tight example is a clique on n vertices.

2 First-Fit and Trees

Recall the First-Fit coloring algorithm we saw in class: consider the vertices of G in some (arbitrary) order, and for each vertex assign to it the minimum color that is still available. This algorithm is not guaranteed to produce an optimal coloring. In this exercise we focus on how far this algorithm can be from giving an optimal coloring.

1. Show that there exists, for each k, a tree T_k such that First-Fit uses k colors to color T_k. (Recall that trees can be colored with 2 colors, so this is bad. . .)

2. Show that if G is a tree on n vertices, then First-Fit will never use more than $1 + \log n$ colors (no matter the order considered).

3. Show that there exists, for each k, a bipartite graph on $2k$ vertices for which First-Fit uses $k + 1$ colors.

4. Show that in any bipartite graph on at most $2n + 1$ vertices, First-Fit will use at most $n + 1$ colors.

Solution:

1. We show by induction that for each k there exists a tree T_k, a special vertex $u \in T_k$ (which we call the root) and an ordering of the vertices of T_k such that First-Fit assigns color k to u and u is last. For $k = 1$ and $k = 2$ the claim is easy to see. Suppose the claim is true for k and we want to prove it for $k+1$. Take two copies of T_k, call them T_k^1, T_k^2 and place an edge between the two roots. Call one of them, say the one in T_k^2 the new root. Order the vertices of the new graph by first giving the vertices of T_k^1 (in the order assumed in the inductive hypothesis) and then those of T_k^2. We claim that First-Fit colors the new root with $k+1$. Indeed, by inductive hypothesis the root of T_k^1 will receive color k. Every non-root vertex of T_k^2 is non-adjacent to T_k^1, so will receive the same colors as if we were coloring T_k^2 alone. Finally, u, which is last, has a neighbor of color k (the root of T_k^2) and neighbors with colors $\{1, \ldots, k-1\}$ (because its neighbors in T_k^2 receive the same colors as in a coloring of T_k^2 and in such a coloring u receives color k). Hence, $k+1$ is the lowest available color.

2. We prove by induction that if First-Fit uses k colors for a tree T, then T has at least 2^{k-1} vertices. To see that this implies the original statement, consider a tree with n vertices, where First-Fit uses k colors. If $k > 1 + \log n$, then (assuming we prove the statement of the first sentence) $n \geq 2^{k-1} > 2^{\log n} = n$, contradiction. So we get that $k \leq 1 + \log n$, as desired.

   The statement is true for $k = 1$ and $k = 2$. Suppose that we are using $k+1$ colors for a tree T. We then want to prove that T has at least 2^k vertices. Let u be a vertex that received color $k+1$. The vertex u must have at least k neighbors $x_1, \ldots, x_k$ with each x_i having received color i. Consider now the forest $T - u$, where each x_i belongs in a distinct sub-tree T_i. By inductive hypothesis T_i has at least 2^{i-1} vertices. So the whole graph has at least $1 + \sum_{i=1}^{k} 2^{i-1}$ vertices (where we have also counted u). But $1 + (1 + 2 + 4 + \ldots + 2^{k-1}) = 2^k$, as desired.

3. Consider a graph $K_{n,n}$ with the vertices of the two sets $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$. Edit this graph by removing a matching of size $n-1$, say we remove the edges $a_1b_1, a_2b_2, \ldots, a_{n-1}b_{n-1}$. Consider now the order $a_1, b_1, a_2, b_2, \ldots, a_n, b_n$. We claim that First-Fit will assign color $n+1$ to b_n. Indeed, for $i \leq n-1$, First-Fit will assign color i to a_i and b_i, which can easily be shown by induction. Then, a_n will receive color n. Then, b_n will have neighbors with all colors in $\{1, \ldots, n\}$, so will receive $n+1$.

4. Suppose that in a bipartite graph $G = (A, B, E)$ with at most $2n+1$ vertices First-Fit uses color $n+2$ for $u \in B$. Then, u has neighbors with colors $\{1, \ldots, n+1\}$, which are distinct vertices of A, so $|A| \geq n+1$. Furthermore, let v be the neighbor of u that received color $n+1$. This vertex has neighbors with colors $\{1, \ldots, n\}$, which are distinct vertices of $B \setminus \{u\}$. So, $|B| \geq n+1$. Therefore, G has at least $2n+2$ vertices, contradiction.

3 Edge Coloring and Bipartite Graphs

An edge coloring of a graph $G = (V, E)$ with k colors is an assignment of colors from $\{1, \ldots, k\}$ to E so that any two edges e_1, e_2 that share an endpoint receive distinct colors. In other words an edge coloring of G is a vertex coloring of $L(G)$, the line graph of G. We use $\chi'(G)$ to denote the minimum number of colors needed to color the edges of G.

1. Show that $\chi'(G) \geq \Delta(G)$ for all G.
2. Show that if G is bipartite, then $\chi'(G) = \Delta(G)$.
3. Show that there exists a (non-bipartite) G with $\chi'(G) > \Delta(G)$.

Solution:

The first part is easy to see as the edges incident on a vertex of degree Δ must take distinct colors (that is, they form a clique of size Δ in the line graph).

We prove the second part by induction on the number of edges of G. For G with at most 3 edges, the statement is easy to see. Suppose we are given a bipartite graph G of maximum degree Δ and we remove one edge $e = xy$. $G - e$ is still bipartite and has maximum degree at most Δ, so we can color all edges of $G - e$ with at most Δ colors. We want to insert e into this coloring.

Observe that x, y have degree at most $\Delta - 1$ in $G - e$, so there is a color $i \in \{1, \dots, \Delta\}$ which is not used for any edge of $G - e$ incident on x, and there is a color j which is not used for any edge incident on y. If $i = j$ we just color e with color i and we are done. So, suppose $i \neq j$. Consider now the subgraph of $G - e$ formed by edges colored i or j. This subgraph has maximum degree 2, so each component is a path or a cycle. Furthermore, the component C that contains y must be a path, as y has no edge colored j. We also note that C cannot contain x, because every vertex of C that is on the same side as x is incident on an edge of color i. Inside the component C we exchange colors i and j, so now i is not used on any edge incident on y. Since x is not in C, i is still available to use in x and we use i to color e.

An example for the last statement is any odd cycle.

4 Cograph Coloring

Prove that if G contains no induced copy of P_4 (the path on 4 vertices), then First-Fit always produces an optimal coloring of G. (Note: graphs that contain no induced P_4 are called cographs.)

Solution:

We prove that if First-Fit coloring uses k colors to color a P_4-free graph G, then G contains a clique of size k, hence needs at least k colors.

We will construct for each $i \in \{1, \dots, k\}$ a clique containing one vertex from each of the i highest colors used in G. For $i = 1$ this is trivial, as we can just take a vertex of color k. Suppose then that we have a clique C with one vertex for each of the colors $\{k, k-1, \dots, k-i\}$ and we want to show that we can always extend this to a clique by adding a vertex of color $k - i - 1$. If we show this we are done.

Let x_j for $j \in \{k - i, \dots, k\}$ be the element of C with color j. Let S_j be the set of neighbors of x_j that have color $k - i - 1$. S_j must be non-empty, as otherwise x_j would have received color $k - i - 1$ itself. Now we claim that for $j \neq j'$, we have either $S_j \subseteq S_{j'}$ or $S_{j'} \subseteq S_j$. Indeed, suppose for contradiction that neither holds, so $S_j \setminus S_{j'}$ and $S_{j'} \setminus S_j$ are both non-empty and contain vertices a, b respectively. The vertices a, b are non-adjacent (they have the same color), a is adjacent to x_j, x_j is adjacent to $x_{j'}$ (C is a clique), $x_{j'}$ is adjacent to b, $x_{j'}$ is non-adjacent to a and x_j is non-adjacent to b. We have therefore found an induced P_4 ($ax_jx_{j'}b$) contradiction.

Now, if for all $j \neq j'$ we have either $S_j \subseteq S_{j'}$ or $S_{j'} \subseteq S_j$ we can order the x_j according to the size of S_j. Let x_k be such that $|S_k|$ is minimum and we have that $S_k \subseteq S_j$ for all other $x_j \in C$. Therefore, a vertex $a \in S_k$ is a vertex of color $k - i - 1$ which is adjacent to all of C, so we have managed to extend C as promised.