TD 7: Coloring

1 Edges and Colors

Show that if a graph G has $m > 0$ edges and chromatic number k, then $m \geq \binom{k}{2}$.

2 Chromatic Number and Average Degree

Prove or disprove: if G is connected and has average degree d, then G can be colored with at most $\lceil 1 + d \rceil$ colors.

3 Blanche Descartes Construction

We saw in class a construction due to Mycielsky that gives for each $k \geq 2$ a graph with chromatic number k that does not contain any K_3 as a subgraph. We consider now a different construction, due to Blanche Descartes. Define the sequence of graphs D_i inductively as follows: $D_1 = K_1$; if D_i has n_i vertices, then D_{i+1} starts with a set S_{i+1} of $i(n_i - 1) + 1$ vertices and for each $S' \subseteq S_{i+1}$ with $|S'| = n_i$ we construct a distinct copy of D_i and place a perfect matching between S' and this new copy.

1. Which construction is more efficient (has smaller n_i), this one or the one by Mycielski? Why?
2. Prove that D_i can be colored with i colors.
3. Prove that D_i cannot be colored with $i - 1$ colors.
4. Prove that D_i does not contain any C_3, C_4, or C_5 as induced subgraphs.

4 Colorings and Complements

Prove that for all G on n vertices we have $\chi(G)\chi(\overline{G}) \geq n$. Conclude that for all G on n vertices, $\chi(G) + \chi(\overline{G}) \geq 2\sqrt{n}$. Give a tight example.

5 Colorings and Kőnig

Suppose that G has $\chi(G) > k$ but $V(G)$ can be partitioned into two sets X, Y such that $G[X], G[Y]$ are both k-colorable. Then, there are at least k edges with one endpoint in X and the other in Y.