TD 7: Coloring

1 Edges and Colors

Show that if a graph G has $m > 0$ edges and chromatic number k, then $m \geq \binom{k}{2}$.
Solution:

Consider a k-coloring of G, $V_1, V_2, \ldots, V_k$. We observe that for each $i, j \in [k], i \neq j$ there must exist at least one edge with one endpoint in V_i and the other in V_j. Indeed, otherwise, $V_i \cup V_j$ is an independent set, and we can color G with $k-1$ colors, contradicting the assumption that G has chromatic number k. Therefore, G has at least $\binom{k}{2}$ edges.

2 Chromatic Number and Average Degree

Prove or disprove: if G is connected and has average degree d, then G can be colored with at most $\lceil 1+d \rceil$ colors.
Solution:

The statement is false: consider a graph made up of a K_5, a path on 95 vertices, and an edge connecting them. This graph has $n = 100$, $m = 105$, therefore the average degree is $2m/n < 3$. According to the claim, the graph can be colored with 4 colors, but this is impossible, as it contains K_5 as a subgraph.

NB: It is tempting to try to prove that the statement is true algorithmically as follows: G must contain a vertex v of degree at most d, so we can first color $G - v$ and then insert v; in the worst case v has d neighbors with distinct colors, so it will receive color $d+1$. This proof is **false**! The error here is that $G - v$ could (and probably will) have higher average degree than G, because d is a low-degree vertex, so the inductive hypothesis may already be using too many colors in $G - v$. In our example, by repeatedly deleting a low-degree vertex we end up with K_5 which has average degree 4, while the orginal graph had average degree < 3.

3 Blanche Descartes Construction

We saw in class a construction due to Mycielsky that gives for each $k \geq 2$ a graph with chromatic number k that does not contain any K_3 as a subgraph. We consider now a different construction, due to Blanche Descartes. Define the sequence of graphs D_i inductively as follows: $D_1 = K_1$; if D_i has n_i vertices, then D_{i+1} starts with a set S_{i+1} of $i(n_i - 1) + 1$ vertices and for each $S' \subseteq S_{i+1}$ with $|S'| = n_i$ we construct a distinct copy of D_i and place a perfect matching between S' and this new copy.

1. Which construction is more efficient (has smaller n_i), this one or the one by Mycielski? Why?
2. Prove that D_i can be colored with i colors.
3. Prove that D_i cannot be colored with $i-1$ colors.
4. Prove that D_i does not contain any C_3, C_4, or C_5 as induced subgraphs.

Solution:

1. Mycielski's construction will, at each step roughly double the size of the graph, so give $n_i = 2^{O(i)}$. In contrast, in this construction we have that n_{i+1} is **exponential** in n_i. More specifically, n_{i+1} contains at least $\binom{in_i}{n_i}$ vertices, which is at least i^{n_i}. Hence, this construction has graphs whose size increases as a **tower of exponentials**, and is therefore much less efficient.

2. The vertices of S_i are an independent set, so we can assigne one color to them. The rest of D_i constists of disjoint copies of D_{i-1}, which by inductive hypothesis can be colored with $i-1$ colors.

3. Suppose that D_i can be colored with $i-1$ colors and i is minimum (that is, D_{i-1} needs $i-1$ colors to be colored properly). Suppose that the color that is used the largest number of times in S_i is color 1 and that it appears at least $\frac{(i-1)(n_{i-1}-1)+1}{i-1} \geq n_{i-1} - 1 + \frac{1}{i-1}$ times. Since the number of appearances of color 1 in S_i is an integer, color 1 appears at least n_{i-1} times in S_i. Let $S' \subseteq S_i$ be a set of size n_{i-1} where all vertices have color 1. Color 1 cannot be used in the copy of D_{i-1} which has a perfect matching to S', so D_{i-1} must be using colors $\{2, \ldots, i-1\}$, which would give a $(i-2)$-coloring of D_{i-1}, contradiction.

4. First, let us see that if D_i contains no C_3, then D_{i+1} contains no C_3. In this case a C_3 in D_{i+1} would need to contain at least one new vertex from S_{i+1}. It cannot contain two such vertices, as such vertices are independent. Therefore, it must contain exactly one such vertex v. In each copy of D_{i-1} we have constructed, v has at most one neighbor, so the two remaining vertices of a supposed C_3 cannot be from the same copy. However, they also cannot be from distinct copies, as there are no edges between distinct copies.

   Now, suppose that D_{i+1} contains a C_4 or C_5. The supposed cycle cannot contain more than two vertices of S_{i+1}, as such vertices are independent. It cannot contain exactly one vertex $v \in S_{i+1}$, for reasons similar to the previous paragraph, namely: if the rest of the cycle comes from a single copy of D_i, then v has degree 1 in the cycle; and if it comes from two copies and uses no other vertex of S_{i+1}, then v is a cut-vertex of the cycle (contradiction). If the cycle contains no vertex of S_{i+1}, then it can be found in D_i, contradiction. Therefore, the cycle contains two vertices $u, v \in S_{i+1}$. If the cycle has length at most 5, then it must have a common neighbor x of u, v. However, every vertex of D_{i+1} is adjacent to at most one of these two vertices, since we add perfect matchings for each copy of D_i we construct.

4 Colorings and Complements

Prove that for all G on n vertices we have $\chi(G)\chi(\overline{G}) \geq n$. Conclude that for all G on n vertices, $\chi(G) + \chi(\overline{G}) \geq 2\sqrt{n}$. Give a tight example.

Solution:

We have $\chi(\overline{G}) \geq \omega(\overline{G}) = \alpha(G)$. Therefore, $\chi(G)\chi(\overline{G}) \geq \alpha(G)\chi(G)$ and we have seen in class that $\chi(G) \geq \frac{n}{\alpha(G)}$.

For the second part, if we had $\chi(G) + \chi(\overline{G}) < 2\sqrt{n}$ this would imply that $\chi(G)\chi(\overline{G}) < \chi(G)(2\sqrt{n} - \chi(G))$. If we consider the right-hand side as a function of $\chi(G)$, this is maximized when $\chi(G) = \sqrt{n}$ so we would have $\chi(G)\chi(\overline{G}) < n$, contradiction.

A tight example can be formed by taking a union of n cliques K_n, forming a graph with n^2 vertices. Clearly, $\chi(G) = n$. The complement of this graph is a graph with n parts, each part being an independent set of size n, so $\chi(\overline{G}) = n$.

5 Colorings and Kőnig

Suppose that G has $\chi(G) > k$ but $V(G)$ can be partitioned into two sets X, Y such that $G[X], G[Y]$ are both k-colorable. Then, there are at least k edges with one endpoint in X and the other in Y.

Solution:

Let $X_1, \ldots, X_k$ and $Y_1, \ldots, Y_k$ be the k-colorings of $G[X], G[Y]$. We form a bipartite graph with k vertices on each side, where the vertices of the left side represent the sets X_i and the vertices on the right side represent the sets Y_i. We place an edge between two vertices if the sets X_i, Y_j have no edge connecting them, that is, if $X_i \cup Y_j$ is independent.

What we want to prove now is that this bipartite graph has a perfect matching whenever there are less than k edges linking X to Y. This will lead to a contradiction as follows: each edge of the matching gives an independent set, so we can partition all of $V(G)$ into k color classes, contradicting the hypothesis that $\chi(G) > k$.

Let us then prove that if $< k$ edges connect X to Y in G, then the bipartite graph has a perfect matching. Equivalently, by Kőnig's theorem, we will show that the bipartite graph does not have a vertex cover of size $k-1$. Indeed, the bipartite graph has at most k^2 possible edges and each edge of G connecting X to Y eliminates at most one edge of the bipartite graph. If G has $< k$ edges connecting X to Y, then the bipartite graph has $\geq k(k-1)+1$ edges. However, each vertex of the bipartite graph has degree at most k, so can cover at most k edges. Hence, a supposed vertex cover of size $(k-1)$ can only cover at most $k(k-1)$ edges, contradiction.