TD 6: More Matchings and Cuts

1 Two Matchings Make One

Let $G = (A, B, E)$ be a bipartite graph. Let M_1 be a matching that touches all vertices of $X \subseteq A$. Let M_2 be a matching that touches all vertices of $Y \subseteq B$. Prove that there always exists a matching M_3 that touches all vertices of $X \cup Y$.

2 Kőnig and Maximum Degree

Show that any bipartite graph G with m edges and maximum degree Δ has a matching of size at least $\frac{m}{\Delta}$. Is the statement true for non-bipartite graphs?

3 Connectivy and Cycles

For each $k \geq 2$, show that if G is k-vertex connected and has at least $2k$ vertices, then G contains a cycle of length at least $2k$.

4 Latin Rectangles and Squares

In combinatorics, a Latin rectangle with dimensions $n \times m$, for $n \leq m$, is a matrix with n lines, m columns, such that every element is an integer in $\{1, \ldots, m\}$, and no element appears twice in the same row or in the same column. A Latin square is a Latin rectangle where the number of rows is equal to the number of columns.

Prove that any Latin rectangle can be extended to a Latin square by adding $m - n$ new rows.

Example of a Latin rectangle:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 5 & 3 \\ 3 & 5 & 2 & 1 & 4 \end{pmatrix}$$

Example of a Latin square we can obtain from the previous rectangle by adding two rows:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 5 & 3 \\ 3 & 5 & 2 & 1 & 4 \\ 4 & 3 & 5 & 2 & 1 \\ 5 & 1 & 4 & 3 & 2 \end{pmatrix}$$