TD 6: More Matchings and Cuts

1 Two Matchings Make One

Let $G = (A, B, E)$ be a bipartite graph. Let M_1 be a matching that touches all vertices of $X \subseteq A$. Let M_2 be a matching that touches all vertices of $Y \subseteq B$. Prove that there always exists a matching M_3 that touches all vertices of $X \cup Y$.

Solution:

Consider the graph formed by $M_1 \cup M_2$, which is a subgraph of G of maximum degree 2. Every non-trivial connected component is either (i) an even cycle (ii) a path with an even number of vertices (iii) a path with an odd number of vertices. (We cannot have odd cycles, as the graph is bipartite.) Furthermore, all vertices of $X \cup Y$ are contained in some non-trivial connected component of this graph.

We form a matching M_3 by selecting for each component of type (i) or (ii) a perfect matching inside this component.

For each component P_{2k+1} of type (iii) we claim that one of the two endpoints of the path is not in $X \cup Y$. To see this, suppose without loss of generality that the first vertex p_1 of the path belongs in X, therefore, if the last vertex p_{2k+1} belongs in $X \cup Y$, it must belong in X as well (since their distance is even). It must therefore be the case that $p_1p_2 \in M_1$ ($p_1 \in X$ so it is matched by M_1) and $p_{2k}p_{2k+1} \in M_1$ (similarly). For every internal vertex of the path, exactly one of its incident edges is in each of M_1, M_2. So, we have an even number of edges; the first and last edge are in M_1; edges alternate between M_1 and M_2. It is not hard to see that this gives a contradiction.

Given that for a component of type (iii) one endpoint is not in $X \cup Y$ we select a matching in this component that matches all other vertices except this endpoint and add it to M_3. Taking the union of all matching we have selected so far guarantees that we touch all of $X \cup Y$.

2 Kőnig and Maximum Degree

Show that any bipartite graph G with m edges and maximum degree Δ has a matching of size at least $\frac{m}{\Delta}$. Is the statement true for non-bipartite graphs?

Solution:

We will equivalently show that the size of a minimum vertex cover of G is at least $\frac{m}{\Delta}$. Since the maximum matching size is equal to the minimum vertex cover size on bipartite graphs, the claim will follow. Suppose then that we have a vertex cover of size $k < \frac{m}{\Delta}$. Each vertex of this set covers at most Δ edges, so in total we would cover at most $k\Delta < m$ edges, contradiction.

The statement is false for odd cycles: C_{2n+1} has $m = 2n + 1$ edges, $\Delta = 2$, but the maximum matching size is $n < \frac{m}{\Delta}$.

3 Connectivy and Cycles

For each $k \geq 2$, show that if G is k-vertex connected and has at least $2k$ vertices, then G contains a cycle of length at least $2k$.

Solution:

For contradiction, suppose that the longest cycle C in G has length at most $2k - 1$. Consider the vertices of C that have a neighbor outside of C. There are at least k such vertices, otherwise deleting these vertices would disconnect the graph while deleting only at most $k-1$ vertices, contradicting the assumption on the k-connectivity of G. In any set of k or more vertices on a cycle of length $2k-1$ or less, there are two vertices x, y which are consecutive in the cycle. If x, y have a common neighbor outside C, we are done as we can insert this neighbor inside the cycle, contradicting the assumption that C is longest. Let then x' be a neighbor of x outside the cycle and y' be a neighbor of y outside the cycle.

Suppose now that $G-C$ has a path P from x' to y'. We can construct a cycle longer than C in G as follows: remove the edge xy and instead add the edges xx', yy', and the path P. This again contradicts the assumption that C is longest.

Finally, suppose that $G-C$ has no path from x' to y', therefore all paths from x' to y' pass through C. By Menger's theorem, there are k disjoint paths from x' to y', so there are k disjoint paths from x' to C. Let $z_1, \ldots, z_k$ be the first vertex of C from each such path. Two of these must be consecutive in the cycle C, say z_1, z_2. Then, C, minus the edge $z_1 z_2$, plus the paths from z_1, z_2 to z' form a longer cycle, contradiction.

4 Latin Rectangles and Squares

In combinatorics, a Latin rectangle with dimensions $n \times m$, for $n \leq m$, is a matrix with n lines, m columns, such that every element is an integer in $\{1, \ldots, m\}$, and no element appears twice in the same row or in the same column. A Latin square is a Latin rectangle where the number of rows is equal to the number of columns.

Prove that any Latin rectangle can be extended to a Latin square by adding $m-n$ new rows.

Example of a Latin rectangle:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 5 & 3 \\ 3 & 5 & 2 & 1 & 4 \end{pmatrix}$$

Example of a Latin square we can obtain from the previous rectangle by adding two rows:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 5 & 3 \\ 3 & 5 & 2 & 1 & 4 \\ 4 & 3 & 5 & 2 & 1 \\ 5 & 1 & 4 & 3 & 2 \end{pmatrix}$$

Solution:

We prove that whenever we have a Latin rectangle with dimensions $n \times m$, with $n < m$, we can always add a row to it to obtain a Latin rectangle with dimensions $(n+1) \times m$. Repeating this will eventually produce a Latin square.

Construct a bipartite graph $G = (A, B, E)$ with $|A| = |B| = m$. The vertices of A represent the positions of the new row, while the vertices of B represent the values $\{1, \ldots, m\}$. We construct the edge $a_i b_j$ if it is possible to place value j in position i of the new row, that is, if column i of the current rectangle does not contain the number j.

We claim that this bipartite graph is $(m-n)$-regular. If we prove this, we are done, because regular bipartite graphs have a perfect matching. If we have such a matching, for each edge $a_i b_j$ in the matching we write the number j in column i of the new row and this ensures that we have correctly extended the rectangle because: (i) since we have a matching, we have used each value exactly once in the new row (ii) in each column we have only used values which did not already appear.

Consider then a vertex a_i, representing position i in the new row. Column i contains n elements in our current square, so the remaining $m-n$ elements can be written in this position. Hence, a_i has degree $m-n$. Consider a vertex b_j. Value j appears exactly once in each row of the current rectangle, hence it appears in n distinct columns. Hence, j can be written in $m-n$ distinct columns of the new row, hence the degree of b_j is also $m-n$. We conclude that the graph is regular.