TD 5: Cuts, Disjoint Paths, Line Graphs

1 Vertex vs Edge Connectivity on Cubic Graphs

Show that if G is 3-regular then $\kappa(G) = \kappa'(G)$. Recall that $\kappa(G)$ is the size of the smallest vertex cut-set and $\kappa'(G)$ the size of the smallest edge cut-set of G.

2 Connectivity, Diameter, Graph size

Suppose that a graph G has diameter d and vertex-connectivity κ. Show that $n \geq \kappa(d-1)+2$.

3 Minimum Degree and Connectivity

Show that if in a graph G we have that all vertices have degree at least $\delta \geq \frac{n-1}{2}$, then G is connected. Furthermore, for all $k \geq 1$, if $\delta \geq \frac{n+k-2}{2}$, then G is k-vertex-connected.

4 Fans and Cycles

Let $G = (V, E)$ be a graph, k an integer, $x \in V$ a vertex and $U \subseteq V$ a set of vertices of size at least k. We say that G has a k-**fan** from x to U if there exist k paths from x to U which are vertex-disjoint except for x. Observe that, without loss of generality, we may assume that each such path has one endpoint in x, the other in U, and all other vertices in $V \setminus (U \cup \{x\})$.

1. Show that if a graph $G = (V, E)$ is k-vertex connected, then for all $x \in V$ and $U \subseteq V \setminus \{x\}$ with $|U| \geq k$ there exists a k-fan from x to U.
2. Show that if $G = (V, E)$ is k-vertex connected (with $k \geq 2$), then for all $v_1, v_2, \ldots, v_k$ there exists a simple cycle that passes through all v_i, $i \in [k]$ (in some order).

5 Line Graphs

Recall that for a graph $G = (V, E)$, the line graph $L(G)$ is defined as follows: the set of vertices of $L(G)$ is E (that is, $L(G)$ has a vertex for each edge of G), and for each $e_1, e_2 \in E$ we have that e_1, e_2 are adjacent in $L(G)$ if and only if the edges e_1, e_2 share an endpoint in G.

1. What is $L(P_n)$ and $L(C_n)$?
2. Show that if G_1, G_2 are isomorphic, then $L(G_1), L(G_2)$ are isomorphic.
3. Show that the converse is not true, by demonstrating two non-isomorphic four-vertex graphs G_1, G_2 such that $L(G_1)$ is isomorphic to $L(G_2)$.
4. Show that the converse is, however, true, for all pairs of connected graphs except the specific example you found in the previous question.