TD 5: Cuts, Disjoint Paths, Line Graphs

1 Vertex vs Edge Connectivity on Cubic Graphs

Show that if G is 3-regular then $\kappa(G) = \kappa'(G)$. Recall that $\kappa(G)$ is the size of the smallest vertex cut-set and $\kappa'(G)$ the size of the smallest edge cut-set of G.

Solution:

We recall that $\kappa'(G) \geq \kappa(G)$ for all G, so we want to prove that $\kappa'(G) \leq \kappa(G)$ for 3-regular graphs. Suppose that we have a vertex set S of size $\kappa(G)$ such that $G - S$ is disconnected and let C_1, C_2 be two distinct connected components of $G - S$. If a vertex $x \in S$ has no neighbor in C_1, then $S \setminus \{x\}$ is still a vertex separator, contradicting the assumption that $|S| = \kappa(G)$. Therefore, all $x \in S$ have at least one edge connecting them to C_1, and similarly at least one edge connecting them to C_2. We construct an edge separator S' as follows: for each $x \in S$ that has a unique edge e connecting x to C_1, we place e in S'; for each remaining $x \in S$, there exist two edges connecting x to C_1, therefore a unique e connecting x to C_2, so we place e in S'.

It is clear that $|S'| = \kappa$, as we have included in S' one edge for each vertex of S. To see that in $G - S'$ there is no path from C_1 to C_2, observe that any such path must at some point exit C_1 and enter S. Suppose $x \in S$ is the vertex of such a path immediately following the last vertex of C_1. Then, since the edge connecting x to C_1 is still in $G - S'$, we know that x has another edge to C_1 and degree exactly 2 if $G - S'$. Therefore, the path must continue to C_1, contradicting the selection of x.

Note that in the above we may assume without loss of generality that $\kappa(G) \leq 2$. Indeed, if $\kappa(G) \geq 3$, since we have $\kappa \leq \kappa' \leq \delta = 3$, we immediately get $\kappa' = 3$. So even though we did not make any explicit assumptions about the size of S above, actually we only need to consider two cases: G has a cut vertex and G has a vertex cut of size 2.

2 Connectivity, Diameter, Graph size

Suppose that a graph G has diameter d and vertex-connectivity κ. Show that $n \geq \kappa(d - 1) + 2$.

Solution:

Take two vertices s, t which are at distance d in G. By Menger's theorem, there exists κ vertex disjoint paths from s to t. All such paths have length at least d, therefore contain at least $d - 1$ internal vertices. Hence, the internal vertices of these paths are at least $\kappa(d - 1)$ and adding s, t gives the bound on n.

3 Minimum Degree and Connectivity

Show that if in a graph G we have that all vertices have degree at least $\delta \geq \frac{n-1}{2}$, then G is connected. Furthermore, for all $k \geq 1$, if $\delta \geq \frac{n+k-2}{2}$, then G is k-vertex-connected.

Solution:

The proof is essentially the same in both cases, so we prove the stronger statement. Let x, y be two vertices of G which are not adjacent. Then $|N(x) \cap N(y)| = |N(x)| + |N(y)| - |N(x) \cup N(y)| \geq n + k - 2 - (n - 2) \geq k$. Hence, since any vertex xy-separator must contain all common neighbors of x, y, any such separator must have size at least k.

4 Fans and Cycles

Let $G = (V, E)$ be a graph, k an integer, $x \in V$ a vertex and $U \subseteq V$ a set of vertices of size at least k. We say that G has a k-**fan** from x to U if there exist k paths from x to U which are vertex-disjoint except for x. Observe that, without loss of generality, we may assume that each such path has one endpoint in x, the other in U, and all other vertices in $V \setminus (U \cup \{x\})$.

1. Show that if a graph $G = (V, E)$ is k-vertex connected, then for all $x \in V$ and $U \subseteq V \setminus \{x\}$ with $|U| \geq k$ there exists a k-fan from x to U.

2. Show that if $G = (V, E)$ is k-vertex connected (with $k \geq 2$), then for all $v_1, v_2, \ldots, v_k$ there exists a simple cycle that passes through all v_i, $i \in [k]$ (in some order).

Solution:

For the first item, we add to the graph a new vertex x' and make it adjacent to all of U. We claim that the new graph is still k-vertex connected and in particular, it is impossible to separate x from x' by deleting $k - 1$ vertices. If this is true, by Menger's theorem, there are k vertex-disjoint $x \to x'$ paths, which means there is a k-fan from x to U. To see that in the new graph we cannot disconnect x from x' by deleting $k - 1$ vertices, observe that after we remove at most $k - 1$ vertices, the graph will always still contain some vertex $u \in U$ (because $|U| \geq k$), which will be adjacent to x'. Hence, to disconnect x from x' we would need to disconnect x from $u \in U$ while removing $k - 1$ vertices, which would contradict the hypothesis that G is k-connected.

For the second item, we proceed by induction on k. For $k = 2$ the claim follows from Menger's theorem: for any x_1, x_2, there are two vertex-disjoint paths $x_1 \to x_2$, so their union is a cycle containing x_1, x_2.

Suppose the statement holds for $k - 1$ and we want to prove it for k. Since G is k-connected, it is also $(k - 1)$-connected, therefore by inductive hypothesis there is a cycle going through $x_1, \ldots, x_{k-1}$. Let U be the vertices of this cycle. If $|U| \geq k$, then by the previous statement, there is a k-fan from x_k to U in G. We can partition the cycle contained in U that passes through $x_1, \ldots, x_{k-1}$ into $k - 1$ intervals, each interval being a maximal sub-path of the cycle whose endpoints are from $\{x_1, \ldots, x_{k-1}\}$ and whose internal vertices (if any) are not. The k-fan guarantees that there is such an interval I with the property that there are 2 vertex-disjoint paths from x_k to I. We can therefore "insert" x_k in the cycle inside the interval I to obtain a cycle passing through all k vertices. If $|U| < k$, then U contains exactly the vertices $x_1, \ldots, x_{k-1}$ and nothing else. We now observe that there is a $(k - 1)$-fan from x_k to U, therefore, there are vertex-disjoint paths from x_k to all the $k - 1$ vertices of U. This again allows us to insert x_k between two consecutive vertices of the cycle.

5 Line Graphs

Recall that for a graph $G = (V, E)$, the line graph $L(G)$ is defined as follows: the set of vertices of $L(G)$ is E (that is, $L(G)$ has a vertex for each edge of G), and for each $e_1, e_2 \in E$ we have that e_1, e_2 are adjacent in $L(G)$ if and only if the edges e_1, e_2 share an endpoint in G.

1. What is $L(P_n)$ and $L(C_n)$?

2. Show that if G_1, G_2 are isomorphic, then $L(G_1), L(G_2)$ are isomorphic.

3. Show that the converse is not true, by demonstrating two non-isomorphic four-vertex graphs G_1, G_2 such that $L(G_1)$ is isomorphic to $L(G_2)$.

4. Show that the converse is, however, true, for all pairs of connected graphs except the specific example you found in the previous question.

Solution:

1. $L(P_n) = P_{n-1}$ and $L(C_n) = C_n$.

2. Suppose there is a bijective function $\phi : V_1 \to V_2$ such that $uv \in E_1 \Leftrightarrow \phi(u)\phi(v) \in E_2$. We construct a function $\phi' : E_1 \to E_2$ as follows: for each $e = uv \in E_1$ we set $\phi'(uv) = \phi(u)\phi(v)$. We observe that:

   - The image of ϕ' is indeed E_2, that is, when $uv \in E_1$, then $\phi'(uv) \in E_2$. This follows from the properties of ϕ.
   - ϕ' is one-to-one. Indeed, for two distinct edges $e_1 = v_1v_2 \in E_1$ and $e_2 = u_1u_2 \in E_1$, suppose without loss of generality that $v_1 \neq u_1$. We can then see that $\phi'(e_1) = \phi(v_1)\phi(v_2) \neq \phi'(e_2) = \phi(u_1)\phi(u_2)$, because ϕ is one-to-one, so $v_1 \neq u_1 \Rightarrow \phi(v_1) \neq \phi(u_1)$.
   - Because ϕ' is one-to-one, and $|E_1| = |E_2|$, we have that ϕ' is onto, that is, ϕ' is a bijection.
   - Finally, suppose that $e_1, e_2 \in E_1$ share an endpoint, say u. Then $\phi'(e_1), \phi'(e_2)$ will also share an endpoint, namely $\phi(u)$. Conversely, suppose that $e'_1, e'_2 \in E_2$ share an endpoint, say $e'_1 = v_1v_2$ and $e'_2 = v_1v_3$ and we have $\phi'(e_1) = e'_1$ and $\phi'(e_2) = e'_2$ for some $e_1, e_2 \in E_1$. We claim that in this case e_1, e_2 also share an endpoint. Indeed, if this were not the case, then e'_1, e'_2 would not share v_1 as an endpoint, because ϕ is one-to-one, so it cannot map four distinct vertices to the set $\{v_1, v_2, v_3\}$.

3. K_3 and $K_{1,3}$ are non-isomorphic, but their line graphs are both K_3.

4. Suppose we have an isomorphism ϕ' from $L(G_1)$ to $L(G_2)$, G_1, G_2 are connected and have at least 5 vertices (the case of at most 4 vertices can easily be handled by considering all graphs). Furthermore, suppose that both G_1, G_2 contains at least one vertex of degree at least 3 (because otherwise, one of the two graphs is a path or a cycle, and the claim is easy to see). We want to construct an isomorphism ϕ from G_1 to G_2.

   The key idea now is that the edges of $K_{1,3}$ subgraphs of G_1 must be mapped to the edges of $K_{1,3}$ subgraphs of G_2 (in particular, a $K_{1,3}$ cannot be mapped to a K_3). Suppose that we have three edges $e_1, e_2, e_3 \in E_1$ which share an endpoint, v, and let $e'_1 = \phi'(e_1), e'_2 = \phi'(e_2), e'_3 = \phi'(e_3)$. We want to claim that e'_1, e'_2, e'_3 also share an endpoint, so suppose for contradiction that this is not the case. However, since e'_1, e'_2 share an endpoint, if e'_3 does not also share this endpoint it must be touching the other two endpoints of e'_1, e'_2, that is, e'_1, e'_2, e'_3 must form a K_3. Now, consider a fourth edge $e_4 \in E_1$ which shares an endpoint with at least one of e_1, e_2, e_3. Such an edge must exist, because G_1 has at least 5 vertices and is connected. We observe that e_4 either shares an endpoint with all three of e_1, e_2, e_3 or with exactly one. However, $\phi'(e_4)$ can only share an endpoint with exactly 2 of e'_1, e'_2, e'_3, contradiction.

   Given the previous observation, we now see that, if we denote for $v \in V(G_1)$ by $E(v)$ the set of edges of G_1 incident on v, then ϕ' bijectively maps $E(v)$ to $E(v')$ for a unique vertex $v' \in V(G_2)$. Indeed, if v has degree at least 3, suppose its incident edges are $e_1, e_2, \ldots, e_k$. By the claim of the previous paragraph, ϕ' maps e_1, e_2, e_3 to three edges with a common endpoint, say v'. Furthermore, it maps e_2, e_3, e_4 to three edges with a common endpoint, which must therefore still be v'. Continuing in this way, all edges of $E(v)$ are injectively mapped to the edges incident on $E(v')$. Furthermore, if there is an edge $e' \in E(v')$ that is not the image of an edge in $E(v)$, we have a contradiction, as e' shares an endpoint with all edges of $E(v')$, therefore its pre-image must share an endpoint with all edges of $E(v)$, therefore the pre-image of e' must be incident on v. For vertices of degree 2, it is clear that the two edges e_1, e_2 incident on v must be mapped on two edges with a common endpoint v'. Finally, for vertices v with degree 1, the neighbor u of v must have degree at least 2, and since the edges of $E(v)$ are mapped to some $E(v')$, the edge vu is mapped to an edge incident on a leaf of G_2. We can therefore extract a one-to-one mapping of vertices of G_1 to vertices of G_2 by defining that $\phi(v) = v'$ if and only if ϕ' maps $E(v)$ to $E(v')$.