# TD 4: More Trees, Bipartite Graphs, Connectivity

## 1 Vertices, Edges, Components

Prove that all graphs with $n$ vertices, $m$ edges, and $c$ connected components satisfy the inequality $n \leq m + c$.

## 2 Average degrees and Trees

Prove that the average degree of a connected graph $G$ is strictly less than 2, if and only if $G$ is a tree.

## 3 Degrees and Bipartite Graphs

Let $\delta(G)$ denote the minimum degree of a graph $G$ and $\Delta(G)$ denote the maximum degree of $G$. Does there exist a bipartite graph with $\delta(G) + \Delta(G) > n$? Does there exist a bipartite graph with $\delta(G) + \Delta(G) = n$? What is the maximum value of $\delta(G) + \Delta(G)$ for non-bipartite graphs?

## 4 Undirected Geography

Two people, Alice and Bob, play the following game on a graph $G$. Starting with Alice, the players alternate and at each round, the current player selects a vertex that has not been selected before and that is adjacent to the last selected vertex. The first player who is unable to find such a vertex loses.

Show that Alice has a winning strategy in this game if and only if $G$ has no perfect matching.

Note: this game is called Geography, because it (supposedly) derives from the following children's game: Alice names a city (e.g. Athens) and then Bob is supposed to respond with a city whose name begins with the **last** letter of Alice's city and has not been mentioned before (e.g. Sparta). Alice then continues with another city that obeys the same restriction (e.g. Amsterdam), and the first player unable to come up with a new legal city loses. Why is the game above not a faithful model for the children's game?

## 5 Tree Degree Sequences

Show that a sequence $(d_1, \ldots, d_n)$ of positive integers is the degree sequence of a tree if and only if $\sum_{i \in \{1,\ldots,n\}} d_i = 2(n-1)$.