TD 4: More Trees, Bipartite Graphs, Connectivity

1 Vertices, Edges, Components

Prove that all graphs with n vertices, m edges, and c connected components satisfy the inequality $n \leq m + c$.
Solution:

We first prove the statement for connected graphs, that is, for $c = 1$. In this case we must show that $n \leq m + 1$. We do this by induction on $n + m$. If $n \leq 2, m \leq 1$, the inequality is easy to see. Suppose we have proved the inequality for smaller values of $n + m$ and we are given a connected graph G on n vertices and m edges. We have two cases:

1. G contains a cycle. Let e be an edge of this cycle. Then, $G - e$ is connected and has $m - 1$ edges. By inductive hypothesis on $G - e$ we have $n \leq (m - 1) + 1$ so $n \leq m + 1$ as desired.

2. G does not contain any cycle. Find the longest path P on G and suppose that a, b are its endpoints. We observe that a is adjacent to exactly one vertex of P (otherwise we would have a cycle). Furthermore, a cannot have any neighbor x which is not contained in P, otherwise the path from x to b going through a would be longer than P. We conclude that a has degree 1. Remove from the graph a and its incident edge, to obtain G', which has $n-1$ vertices and $m-1$ edges. By inductive hypothesis we have $(n-1) \leq (m - 1) + 1 \Rightarrow n \leq m + 1$ as desired.

We have now established the statement for $c = 1$. Consider then a disconnected graph G with $c > 1$ components $C_1, C_2, \dots, C_c$, where each component C_i has n_i vertices and m_i edges. By the statement for connected graphs we have $n_i \leq m_i + 1$ for all $i \in \{1, \dots, c\}$. So $\sum_{i\in\{1,\dots,c\}} n_i \leq \sum_{i\in\{1,\dots,c\}}(m_i + 1) \Rightarrow n \leq m + c$.

2 Average degrees and Trees

Prove that the average degree of a connected graph G is strictly less than 2, if and only if G is a tree.
Solution:

G is a tree $\Rightarrow$ G is connected and the average degree is strictly less than 2:

If G is a tree and G has n vertices, then G has $m = n - 1$ edges. The average degree is $\frac{\sum_{v\in V} \deg(v)}{n} = \frac{2m}{n} = \frac{2n-2}{n} < 2$.

G is a tree $\Leftarrow$ G is connected and the average degree is strictly less than 2:

If G is acyclic, since G is connected, then G is a tree and there is nothing to prove. Suppose then that G has a cycle and let e be an edge of the cycle, so $G - e$ is also connected. Therefore, by the previous exercise, $(m - 1) \geq n - 1$, because $G - e$ has $m - 1$ edges and is connected. But then, $m \geq n$, which implies than $\frac{\sum_{v\in V} \deg(v)}{n} = \frac{2m}{n} \geq 2$, contradicting the assumption on the average degree of G.

3 Degrees and Bipartite Graphs

Let $\delta(G)$ denote the minimum degree of a graph G and $\Delta(G)$ denote the maximum degree of G. Does there exist a bipartite graph with $\delta(G) + \Delta(G) > n$? Does there exist a bipartite graph with $\delta(G) + \Delta(G) = n$? What is the maximum value of $\delta(G) + \Delta(G)$ for non-bipartite graphs?
Solution:

Suppose there exists bipartite $G = (A, B, E)$ with $|A| + |B| = n$ such that $\delta(G) + \Delta(G) > n$. Assume without loss of generality that $|A| \le |B|$. We now observe that $\Delta \le |B|$, because for all $v \in V$ we have either $N(v) \subseteq A$ or $N(v) \subseteq B$, so $\deg(v) \le |N(v)| \le |B|$. Furthermore, $\delta \le |A|$, because for a vertex $v \in B$ we have $N(v) \subseteq A$. Hence, $\delta + \Delta \le |A| + |B| = n$.

The bipartite graph $K_{n,m}$ has $\delta + \Delta = n + m = |V(K_{n,m}|$.

In general graphs that maximum value of $\delta + \Delta$ is $2n - 2$. This is achieved by a clique K_n. That $\delta + \Delta$ cannot attain a higher value is obvious, as the maximum degree is always at most $n - 1$.

4 Undirected Geography

Two people, Alice and Bob, play the following game on a graph G. Starting with Alice, the players alternate and at each round, the current player selects a vertex that has not been selected before and that is adjacent to the last selected vertex. The first player who is unable to find such a vertex loses.

Show that Alice has a winning strategy in this game if and only if G has no perfect matching.

Note: this game is called Geography, because it (supposedly) derives from the following children's game: Alice names a city (e.g. Athens) and then Bob is supposed to respond with a city whose name begins with the **last** letter of Alice's city and has not been mentioned before (e.g. Sparta). Alice then continues with another city that obeys the same restriction (e.g. Amsterdam), and the first player unable to come up with a new legal city loses. Why is the game above not a faithful model for the children's game?

Solution:

G has no perfect matching $\Rightarrow$ Alice wins:

Let M be a maximum matching of G and v_1 be a vertex unmatched by M. Alice begins with vertex v_1. We observe that Bob must respond with a vertex v_2 that is matched by M, because he is forced to have $v_1v_2 \in E$ and if v_2 is unmatched, then $M \cup \{v_1v_2\}$ would be a larger matching. Since v_2 is matched, Alice responds with vertex v_3 such that $v_2v_3 \in M$.

We now claim the following invariant: at move $2i$, Bob is forced to play a vertex v_{2i} that is matched by M, following an edge $v_{2i-1}v_{2i} \notin M$, and such that the neighbor of v_{2i} in M has not yet been played. If this holds, then Alice can respond with the vertex v_{2i+1} such that $v_{2i}v_{2i+1} \in M$. This maintains the invariant because: (i) Bob is then forced to follow another edge incident on v_{2i+1}, hence an edge e not in the matching (ii) if the other endpoint of e is unmatched, the path from the original vertex v_1 to this new vertex would be an augmenting path, contradicting the assumption that M is maximum (iii) Alice's strategy ensures that as soon as a matched vertex is played by Bob she immediately responds with its match, so when Bob plays each edge of M has either both of its endpoints available or none. Hence, Alice wins, because for each move of Bob she has a response.

G has no perfect matching $\Leftarrow$ Alice wins:

We prove equivalently that if G has a perfect matching, then Bob has a winning strategy. Let M be the perfect matching and suppose that Alice first plays v_1. Bob responds with the neighbor of v_1 in the matching M, which must exist as M is perfect. Maintaining this strategy, whenever it is Bob's turn to play at round $2i$ we know that in the first $2i - 2$ round we have played both endpoints of $(i - 1)$ edges of the matching, and Alice has just played a vertex v_{2i-1}; Bob then responds by playing the neighbor of v_{2i-1} in M. Since Bob always has a valid response, the game must end with Alice losing.

This game is not a faithful representation of the original Geography, because the graph is undirected, while in the original game Athens$\to$Seoul is valid and Seoul$\to$Athens is not. To more accurately capture the game we would need a directed graph. It is important to note that this would have a huge impact on the problem's complexity. On undirected graphs, it is polynomial-time solvable to decide which player has a winning strategy (by deciding if G has a perfect matching). For directed graphs, the problem is PSPACE-complete (this means, not solvable in polynomial time, unless very strange things happen, including P=NP).

5 Tree Degree Sequences

Show that a sequence $(d_1, \dots, d_n)$ of positive integers is the degree sequence of a tree if and only if $\sum_{i \in \{1,\dots,n\}} d_i = 2(n - 1)$.

Solution:
(Note: in this exercise we will assume, to ease presentation, that the degree sequences are not necessarily sorted.)

$(d_1, \ldots, d_n)$ is the degree sequence of a tree $\Rightarrow \sum_i d_i = 2(n-1)$: this follows from the fact that for a tree $m = n-1$ and the sum of the degrees is $2m$.

$(d_1, \ldots, d_n)$ is the degree sequence of a tree $\Leftarrow \sum_i d_i = 2(n-1)$:

We prove the claim by induction on n. For $n = 1$ the claim is vacuous, and for $n = 2$ the only possible sequence is $(1, 1)$, which is indeed the sequence of a tree (K_2). Suppose that the statement holds for smaller n and we have a sequence $(d_1, \ldots, d_n)$. We claim that there must exist d_i in this sequence such that $d_i = 1$. Indeed, since all elements are positive, if for all i we have $d_i \geq 2$, then $\sum_i d_i \geq 2n$, contradiction. Suppose then, without loss of generality, that $d_n = 1$. We also claim that there must exist $d_j \geq 2$ as otherwise, all elements are equal to 1 and $\sum_i d_i = n < 2(n-1)$ for $n \geq 3$. Suppose without loss of generality that $d_{n-1} \geq 2$. Consider then the sequence $(d_1, \ldots, d_{n-2}, d_{n-1} - 1)$. All its elements are positive and its sum is equal to $(\sum_{i \in \{1,\ldots,n-1\}} d_i) - 1 = (\sum_{i \in \{1,\ldots,n\}} d_i) - 2 = 2(n-1) - 2 = 2(n-2)$. By inductive hypothesis, there exists a tree with the new degree sequence, one of whose vertices has degree $d_{n-1} - 1$. Attach to this vertex a new leaf, to obtain a tree with the original degree sequence.