TD 3: Bipartite Graphs

1 Turan

What is the maximum number of edges of a bipartite graph with n vertices?
Solution:

If n is even, the answer is $\frac{n^2}{4}$, which is adhieved by a complete bipartite graph $K_{n/2,n/2}$. Let us prove that no bipartite graph on n vertices can have more edges. Consider the bipartite graph $G = (A, B, E)$ with maximum number of edges such that $|A| + |B| = n$. We first notice that G must be complete bipartite, as if there exist $a \in A, b \in B$ with $ab \notin E$, adding ab to G preserves bipartiteness and increases the number of edges. We therefore need to prove that $|A| = |B|$. Suppose without loss of generality that this is not the case and that we have $|A| > |B|$, which implies that $|A| \geq |B| + 2$ (since n is odd). Remove one vertex a from A and add a new vertex b to B. We have removed $|B|$ edges from the graph and then added $|A| - 1$ edges from the graph. But $|A| - 1 > |B|$, so we now have a graph with n vertices and strictly more edges than before. Repeating this gives $|A| = |B|$. If n is odd a similar argument shows that the bipartite graph with maximum number of edges is $K_{\frac{n-1}{2},\frac{n+1}{2}}$, which gives $\frac{n^2-1}{4}$ edges.

2 Regularity Makes Perfect

Recall that a graph is k-regular if all vertices have degree exactly k. Show that for all $k > 0$, if a bipartite graph is k-regular, then it has a perfect matching. (Note: first convince yourselves that if a bipartite graph is k-regular, then its two parts have the same size.)
Solution:

Suppose that $G = (A, B, E)$ is k-regular. First, we observe that $|E| = \sum_{v \in A} \deg(v) = \sum_{v \in B} \deg(v)$, therefore $|A| = |B|$.

To prove that a perfect matching exists, we use Hall't theorem and claim that for all $S \subseteq A$ we have $|N(S)| \geq |S|$. Indeed, the number of edges incident on S is $k|S|$. If $|N(S)| < |S|$, then the number of edges with one endpoint in $N(S)$ and the other in S is at most $k|N(S)| < k|S|$, contradiction.

3 Maximal Matchings

A *maximal* matching M is a matching such that $M + e$, where e is an edge not in M, is no longer a matching. Prove that if M is a maximal matching and M' is a maximum matching, then $|M|$ is at least $|M'|/2$.
Solution:

Suppose that $|M| < \frac{|M'|}{2}$, therefore $|M'| > 2|M|$. We claim that in this case there exists $e \in M'$ such that $M + e$ is still a matching, contradicting the maximality of M. To see this, observe that each $f \in M$ intersects at most two edges of M' (one for each endpoint of f). Hence, if we remove from M' all edges that intersect an edge of M, we will remove at most $2|M|$ edges. If $|M'| > 2|M|$, there will be an edge left in M' which does not intersect any edge of M and hence can be added to it without destroying the matching.

4 Dominating Set

We saw in class that the MINIMUM VERTEX COVER problem is easier on bipartite graphs than it is on general graphs. For this exercise we look at a problem which is as hard on general graphs as it is on bipartite graphs.

Recall that a *dominating set* of a graph $G = (V, E)$ is a set $S \subseteq V$ such that all vertices of $V \setminus S$ have a neighbor in S. In the MINIMUM DOMINATING SET problem we are given G, k and are asked if G has a dominating set of size at most k. Show that if we had an efficient algorithm for MINIMUM DOMINATING SET on bipartite graphs, we would have such an algorithm for the same problem on general graphs. (Hint: Given an arbitrary graph G you must modify it so that you construct a bipartite graph G' but preserve the solution.)

Solution:

Given an arbitrary graph $G = (V, E)$ we construct a bipartite graph G' by taking two copies of V, call them V_1, V_2. For $u \in V_1, v \in V_2$ we construct the edge uv if and only if $uv \in E$ or $u = v$. Furthermore, we add a new vertex x and connect it to all vertices of V_1; and we add a new vertex y and connect it only to x. We claim that G' has a dominating set of size at most $k + 1$ if and only if G has a dominating set of size at most k.

To see this, first suppose G has such a dominating set S. We select in G' the set $S \subseteq V_1$ as well as x, which has size at most $k + 1$. The vertex x dominates V_1, x, and y, so we need to prove that we dominate V_2. However, this follows because S is a dominating set of G.

For the converse direction, suppose G' has a dominating set S' of size $k + 1$. S' must contains x or y to dominate y, so we can assume without loss of generality that $x \in S'$ (otherwise we exchange y with x). x dominates V_1, so we can also assume without loss of generality that all other vertices of S' are contained in V_1 (so that they dominate something in V_2); indeed, if $u \in V_2$ belongs in S', we can replace it with $u \in V_1$. Therefore, S' contains k vertices of V_1. We claim these vertices are a dominating set of G, which follows from the fact that they dominate V_2.

Now, we observe that G' is bipartite, so if we could solve MINIMUM DOMINATING SET in polynomial time on bipartite graphs, we could use the procedure above to solve it also on general graphs.

5 Perfect Matchings on Trees

Show that a tree has a perfect matching if and only if for all v, o(G-v)=1, where o(G) is the number of odd-order components.

Solution:

Perfect matching $\Rightarrow$ for all v, $o(G - v) = 1$:

Fix a perfect matching M. Let $T_1, \ldots, T_k$ be the trees of $G - v$ and suppose that v is matched in M with a vertex of T_1. Then, $M \cap T_i$ for all $i \geq 2$ is a perfect matching, so all trees excepth T_1 have even order. Furthermore, $M \cap (T_1 \cup \{v\})$ is also a perfect matching, so T_1 must have odd order.

Perfect matching $\Leftarrow$ for all v, $o(G - v) = 1$:

We prove this by induction on the size of the given tree. The statement is true for trees with up to three vertices. Take now a tree T, such that for all v we have $o(G - v) = 1$. Observe that this implies that T has an even number of vertices, as can be seen if we set v to be a leaf. Consider now any non-leaf vertex v. Let $T_1, \ldots, T_k$ be the trees of $T - v$, with T_1 being the unique tree of odd order. We claim that $T_1 \cup \{v\}$ has a perfect matching and for all $i \geq 2$, T_i has a perfect matching and will prove this via the inductive hypothesis.

- $T_1 \cup \{v\}$ has a perfect matching: We need to show that $T_1 \cup \{v\}$ satisfies the property. Let v' be a vertex of T_1 and let $T_{1,1}, T_{1,2}, \ldots, T_{1,r}$ be the trees of $T_1 \cup \{v\} - v'$. Suppose without loss of generality that $v \in T_{1,1}$. Then, if we consider the original tree T, the forest $T - v'$ also contains the trees $T_{1,2}, \ldots, T_{1,r}$. Furthermore, the remaining tree of $T - v'$ is just $T_{1,1}$ together with the trees $T_2, \ldots, T_k$, all of which have even order. Hence, if exactly one of the trees of $T - v'$ has odd order, the same is true for the trees of $T_1 \cup \{v\} - v'$.

- For all $i \geq 2$, T_i has a perfect matching: Again, we need to show that T_i satisfies the property. Let v' be a vertex of T_i and let $T_{i,1}, T_{i,2}, \ldots, T_{i,r}$ be the trees of $T_i - v'$. Suppose without loss of generality that v is a neighbor of $T_{i,1}$. Then, if we consider the original tree T, the forest $T - v'$ also contains the trees $T_{i,2}, \ldots, T_{i,r}$. Furthermore, $T - v'$ contains a tree that contains $T_{i,1}$, v, and T_1, but since $|T_1 \cup \{v\}|$ is even, this tree has a size that has the same parity as the size of $T_{i,1}$. Hence, if exactly one of the trees of $T - v'$ has odd order, the same is true for the trees of $T_i - v'$.

Note that the above gives a polynomial-time algorithm for MAXIMUM MATCHING on trees. However, we already knew that such an algorithm exists, as trees are bipartite. Furthermore, a simpler greedy algorithm is also optimal: while there is a leaf, match the leaf to its unique neighbor.