TD 2: Trees

1 Degrees in Trees

For the following statements, decide if they are true or false and give a proof or counter-example.

1. There exists a tree where at least half the vertices have degree 1.
2. There exists a tree where at least half the vertices have degree 2.
3. There exists a tree where at least half the vertices have degree 3.
4. Suppose that in a tree T no vertex has degree 2. Then, at least half of its vertices have degree 1.
5. Suppose that a tree has a vertex of degree k. Then, the tree has at least k leaves.
6. Every tree with exactly two leaves is isomorphic to a path P_n.

2 Balanced Edge Separators

Recall that every tree has a balanced **vertex** separator, that is, a vertex whose removal leaves a collection of trees of size at most $\frac{n}{2}$. The question in this exercise is whether a similar claim can be made for **edge** separators.

1. Is it true that for all trees $T = (V, E)$ on n vertices there exists $e \in E$ such that both components of $T - e$ have at most $\frac{n}{2}$ vertices? How about $\frac{2n}{3}$ vertices? $\frac{3n}{4}$?
2. Show that in any tree T on n vertices of maximum degree Δ there exists an edge e such that both components of $T - e$ have at most $\frac{(\Delta-1)n+1}{\Delta}$ vertices.

3 Polynomial-Time Algorithms

Give polynomial-time algorithms which take as input a tree T and calculate:

1. An independent set of T of maximum size
2. A clique of T of maximum size
3. The longest simple path contained in T
4. A dominating set of T of minimum size

4 Edges and Cycles

Show that every connected graph G on n vertices and m edges contains at least $m - n + 1$ distinct cycles.

5 Helly Property

Let T be a tree and $T_1, T_2, \ldots, T_k$ be k sub-trees of T. Suppose that for all $i, j \in [k]$ we have that $T_i \cap T_j \neq \emptyset$, that is, any two of the sub-trees share a vertex. Show that in this case there exists $v \in \cap_{i\in[k]} T_i$, that is, there exists a vertex that is common to all trees.