TD 12: Revision

1 Outerplanar Graphs and Kuratowski

Show that a graph G is outerplanar if and only if G contains no subgraph that is a subdivision of K_4 or $K_{2,3}$. (For the definition of outerplanar graphs see the TD on planar graphs).

Solution:

One direction is easy: we have already proved that outerplanar graphs have at most $2n-3$ edges. For K_4 we have 6 edges but $2n-3=5$. Similarly, bipartite outerplanar graphs have at most $\frac{3n}{2}-2$ edges, as all cycles have length at least 4, but $K_{2,3}$ has 6 edges and $\frac{3n}{2}-2=5.5$. Since these graphs are not outerplanar, their sub-divisions are also not outerplanar, so no outerplanar graph can contain them.

The converse direction is more interesting. Suppose that G is connected and contains neither K_4 nor $K_{2,3}$ but is not outerplanar and among all such graphs take a minimal counter-example. We can assume that G has no cut vertex. Indeed, if $G-v$ is disconnected, let $C_1, C_2, \ldots, C_k$ be the components of $G-v$ and $G'_1, \ldots, G'_k$ be $G[C_i \cup \{v\}]$. Each such graph (i) contains no K_4 or $K_{2,3}$ (ii) is therefore outerplanar by the minimality of G (iii) therefore we can obtain an outerplanar drawing of G by gluing the outerplanar drawings of each G'_i, contradiction.

We also conclude that G is planar, by Kuratowski's theorem, as G contains neither K_5 nor $K_{3,3}$ (if it did, it would contain a K_4 or $K_{2,3}$ respectively). Take a planar drawing of G that has a maximum number of vertices on the outer face. Let C be a cycle on the outer face obtained by repeatedly removing degree 1 vertices lying on the outer face.

Let v be a vertex not on the outer face. If there are three vertex-disjoint paths from v to C, the graph contains a sub-divided $K_{1,4}$ as a subgraph: let x_1, x_2, x_3 be the endpoints of the paths in C and we keep in the graph the edges of the three paths plus the edges of C. We therefore assume in the remainder that all vertices in the inside have at most two vertex-disjoint paths to C. Since the graph has no cut vertex, all vertices in the inside have exactly 2 disjoint paths to C.

If v has exactly 2 vertex-disjoint paths to C we distinguish two cases: the endpoints x_1, x_2 of these paths are non-consecutive, in which case the graph contains a sub-divided $K_{2,3}$ (with x_1, x_2 the vertices of the small part, and v plus two vertices of $C \setminus \{x_1, x_2\}$ the larger part); or the endpoints x_1, x_2 are consecutive. Let P be the path that goes from x_1 to x_2 through v inside the inner face. Suppose there is a vertex y in this path such that there is a path $y \to C$ which avoids P. This vertex would have three disjoint paths to C, contradiction. Therefore, we can draw the component of $G-\{x_1, x_2\}$ that contains P on the outside, obtaining a drawing where strictly more vertices are on the outside face, contradiction.

2 Menger from Kőnig

Show that Kőnig's theorem implies Menger's theorem. In particular, show how a polynomial-time algorithm that decides if a bipartite graph has a matching of size at least k can be used to obtain a polynomial-time algorithm that decides for two vertices s, t of a graph G whether there exist at least k disjoint paths from s to t. (Reminder: in class we saw the opposite direction, namely, how Menger's theorem implies Kőnig's theorem.)

Solution:

We are given a graph $G=(V,E)$ and vertices $s, t \in V$ with $st \notin E$ and are asked whether there are at least k vertex-disjoint paths from s to t in V. Let $n = |V \setminus \{s,t\}|$. Construct a bipartite graph $G'=(A,B,E')$ as follows:

1. Place in A n copies of the vertex s, call them $s_1, \dots, s_n$ and place in B n copies of t, call them $t_1, \dots, t_n$.
2. For each $v \in V \setminus \{s,t\}$, add a vertex v_1 in A and a vertex v_2 in B. Add the edge $v_1v_2 \in E'$.
3. For each $uv \in E$ add the edges u_1v_2 and v_1u_2 in E'.
4. For each $v \in N(s)$ add the edges s_iv_2 for $i \in \{1, \dots, n\}$
5. For each $v \in N(t)$ add the edges v_1t_i for $i \in \{1, \dots, n\}$

We now make two claims:

- If G' has a matching of size $n+k$, then G has at least k vertex-disjoint $s \to t$ paths.
- If G' has a vertex cover of size $n+k$, then G has an st-separator of size at most k.

Observe that the two claims together imply Menger's theorem: because G' is bipartite, its maximum matching size is equal to its minimum vertex cover size. Let k be such that G' has a matching and a vertex cover of size $n+k$. Then, G has k disjoint $s \to t$ paths and an st-separator of size k. Every separator must have size at least k (because of the k disjoint paths), so this separator is minimum; it is impossible to find $k+1$ disjoint paths (because of the separator), so this collection of paths is maximum. Hence, in G we obtain Menger's theorem. Of course, since we can decide the maximum matching size in G' in polynomial time (Hungarian method seen in class), we therefore also obtain an algorithm for computing the minimum size of an st-separator in G.

For the first claim, suppose we have a matching M of size $n+k$ in G'. If for some $u \in V$ exactly one of u_1, u_2 is incident on an edge of M, place the edge u_1u_2 in the matching are remove the edge of M incident on u_1 or u_2, maintaining a matching of the same size. We now have the property that in M, for each $u \in V$, either both u_1, u_2 are matched or neither is.

Observe that at least k of the vertices s_i are matched and at least k of the vertices t_i are matched, otherwise M could not have size $n+k$. Select in G every edge $e = uv$ such that either u_1v_2 or v_1u_2 is in the matching, as well as the edges incident on s, t which appear in the matching. This gives a graph where s, t have degree at least k, and all other vertices have degree 0 or 2. We remove from this graph every connected component that contains neither s nor t, and obtain a subgraph H of G. We claim that H has k disjoint $s \to t$ paths.

Consider $H - \{s,t\}$. Every component C of this graph is a path with both endpoints adjacent to $\{s,t\}$, because all vertices of H except s, t have degree 0 or 2. We claim that it cannot be the case that both endpoints of C are adjacent to s. Indeed, if there are $|C|$ internal vertices in C, we have $|C|+1$ edges of M which gave us the component C (counting the two supposed edges connecting the endpoints to s). However, in G' this gives a bipartite graph which has $|C|$ vertices on one side and $|C|+2$ vertices on the other (as the two copies of s are on the same side). Clearly, forming a matching of size $|C|+1$ is impossible on this graph. Hence, each component C of $H - \{s,t\}$ is in fact a path from s to t. Since s has degree at least k, there exist at least k such paths.

For the second claim, we assume without loss of generality that $n > k$ (if $k \geq n$, then G clearly has an st-separator of size k). Suppose we have a vertex cover S of G' of size $n+k$. We claim that S must contain, for each $v \in N(s)$, the vertex v_2. Indeed, if for some $v \in N(s)$ we have $v_2 \notin S$, this would force all n copies of s to be in S. For each $u \in V$ we have at least one of u_1, u_2 in S, so we would have $|S| \geq 2n > n+k$ contradiction. Similarly, for all $v \in N(t)$ we have $v_1 \in S$. As a result, S does not contain any of the copies of s, t in G'.

We therefore have a vertex cover which for each $v \in V$ selects at least one of v_1, v_2 (to cover the edge v_1v_2). Since the size of the cover is $n+k$, we conclude that there are exactly k vertices $v \in V$ such that $v_1, v_2 \in S$ and for all other vertices S contains exactly one of v_1, v_2. We claim that these k vertices form an st-separator S' in G.

To see that S' is an st-separator in G, suppose there exists an $s \to t$ path that avoids S'. The same path becomes a path in G' from a copy of s to a copy of t with an odd length, that is, with an even number of vertices. Any vertex cover of such a path must either include an endpoint (which is not the case), or both endpoints of an edge v_1v_2 (which is also not the case, as we assumed the path avoids such vertices). We therefore reach a contradiction and conclude that S' is an st-separator of the desired size.

3 Rates of growth

Asymptotically, how many graphs on n vertices are there in the following classes? For classes marked with (*), give an upper bound (because a lower bound is harder to show).

1. All graphs
2. Forests(*)
3. Split graphs
4. Bipartite graphs
5. Chordal graphs
6. Interval graphs(*)
7. Planar graphs(*)

Solution:

All graphs: at most $2^{\binom{n}{2}}$. Since there are at most $n! = n^{O(n)}$ possible isomorphisms, the correct number is $2^{\Theta(n^2)}$.

Forests: at most $\binom{\binom{n}{2}}{n-1}$, because forest have at most $n-1$ edges. The correct asymptotic estimation is actually $2^{\Theta(n)}$, but this is harder to show.

Split and bipartite graphs: at least $2^{\Theta(n^2)}$. Take two sides of size $n/2$, we have $2^{n^2/4}$ possible choices for the edges.

Chordal graphs: contain split graphs, so the answer is the same if we don't care about constants in the exponent.

Interval graphs: $n^{\Theta(n)}$. Upper bound: each graph can be described by n intervals with numbers in $1, \ldots, 2n$. Lower bound is harder to show.

Planar graphs: $n^{O(n)}$ from upper bound on edges. The correct asymptotic estimation is actually $2^{\Theta(n)}$, but this is harder to show.

4 Brooks and bipartiteness

Let G be a connected graph with n vertices, m edges, and maximum degree 3 that is not a K_4. Show that G contains a bipartite subgraph with at least $m - \frac{n}{3}$ edges.

Solution:

By Brooks' theorem, G can be colored using 3 colors, let V_1, V_2, V_3 be the three classes. We will remove some of the edges of G so that it becomes bipartite. One of the three classes contains at most $n/3$ vertices, say V_3. For each $v \in V_3$ if v has at least two neighbors in V_1 delete its (at most one) edge to V_2; otherwise delete its (at most one) edge to V_1. Now, each vertex of V_3 either has no neighbors in V_1 (so can be placed in V_1) or in V_2. We therefore have a bipartite graph and in the process deleted one edge for each vertex of V_3, meaning the graph has at least $m - \frac{n}{3}$ edges remaining.

5 Cobipartite graphs are perfect

Prove that for all G, if $\overline{G}$ is bipartite, then G is perfect. Do not use the perfect graph theorem! (otherwise this is too easy)

Solution:

We want to prove that $\chi(G) = \omega(G)$ when G is co-bipartite. Note that since co-bipartiteness is preserved by taking induced subgraphs, we only need to prove this for G itself.

Observe that if $\chi(G) = k$ then $\overline{G}$ has clique-cover number k, that is, there exist k cliques in $\overline{G}$ whose union is V, but there do not exist $k-1$ such cliques. Since $\overline{G}$ is bipartite, the cliques in question are K_1 or K_2 and we can assume that the number of K_2's is equal to the maximum matching of G. Therefore, $\chi(G) = n - mm(\overline{G})$, where mm is the maximum matching size. But, since $\overline{G}$ is bipartite, $\chi(G) = n - vc(\overline{G}) = \alpha(\overline{G}) = \omega(G)$, as desired.