TD 11: Chordal, Split, and Interval Graphs

1 Interval Trees

A caterpillar is a tree T such that all vertices of degree strictly more than 1 lie on a single path P. Prove that for all graphs G, G is a caterpillar if and only G is a tree and an interval graph.

2 Interval Graphs and Vertex Orderings

Recall that a graph is chordal if and only if there exists an ordering of the vertices $v_1, v_2, \ldots, v_n$, such that for each v_i the set of neighbors of v_i with indices $j > i$ (i.e. coming later in the ordering) induces a clique. This is called a Perfect Elimination Ordering.

Show that a graph is an interval graph if and only if there exists an ordering of its vertices $v_1, v_2, \ldots, v_n$, such that for each $i < j < k$, if $v_i v_k \in E$, then $v_j v_k \in E$.

3 Maximal Cliques in Chordal Graphs

Show that in a connected chordal graph on n vertices, with $n \geq 2$, there exist at most $n - 1$ distinct maximal cliques. A clique C is maximal if it is impossible to increase it by adding a vertex, i.e. each $v \in V \setminus C$ has a non-neighbor in C.

Show that there exists a non-chordal graph with $2n$ vertices and 2^n distinct maximal cliques.

4 Split Graphs and Degree Sequences

Let $(d_1, d_2, \ldots, d_n)$ be the degree sequence of a graph G, with $d_i \geq d_{i+1}$ for all i. Prove the following: G is a split graph if and only if $\sum_{i=1}^{k} d_i = k(k-1) + \sum_{i=k+1}^{n} d_i$, where k is the maximum index i such that $d_i \geq i - 1$.

5 Short cycles in chordal graphs

Show that if G is chordal and an edge $e \in E(G)$ is part of a cycle, then there exists a K_3 in G that contains e.