# TD 11: Chordal, Split, and Interval Graphs

## 1 Interval Trees

A caterpillar is a tree $T$ such that all vertices of degree strictly more than 1 lie on a single path $P$. Prove that for all graphs $G$, $G$ is a caterpillar if and only $G$ is a tree and an interval graph.

**Solution:**

If a tree $T$ is a caterpillar $\Rightarrow T$ is an interval graph:

Let $P = x_1, x_2, \dots, x_p$ be the path that contains all vertices of degree at least 2 in $T$, that is, all vertices of $T$ outside of $P$ have degree 1. We assign to each $x_i$ an interval of length $\deg(x) + 1$ such that the right endpoint of the interval of $x_i$ is the left endpoint of the interval of $x_{i+1}$. In particular, $x_1$ is assigned the interval $[0, \deg(x_1) + 1]$, $x_2$ is assigned $[\deg(x_1) + 1, \deg(x_2) + \deg(x_1) + 2]$, $x_3$ is assigned $[\deg(x_1) + \deg(x_2) + 2, \deg(x_1) + \deg(x_2) + \deg(x_3) + 3]$ and so on. In other words, the right endpoint of the interval assigned to $x_i$ is $i + \sum_{j=1}^{i} \deg(x_i)$, which is also the left endpoint of the interval of $x_{i+1}$. Observe that the interval we have constructed so far are a representation of the path $P$, so what remains is to insert the remaining vertices of $T$ in this representation.

Consider now the neighbors of $x_i$ which lie outside of $P$. For each such neighbor $y$ we select a distinct integer $j$ lying in the interior of the interval assigned to $x_i$ (i.e. not the endpoints of the interval) and assign to $y$ the interval $[j, j]$. Because we have $\deg(x_i)$ integers to choose from, but $x_i$ has at most $\deg(x_i)$ leaves attached to it, we can assign distinct integers to each leaf neighbor of $x_i$, ensuring that the intervals indeed represent vertices of degree 1.

If a tree $T$ is an interval graph $\Rightarrow T$ is a caterpiller:

Suppose that $T$ is not a caterpillar. We will then show that the graph contains an asteroidal triple. Let $P$ be the path of the tree that contains as many non-leaf vertices as possible. We may assume that $P$ contains no leaves, as a leaf cannot be an internal vertex of $P$ and if an endpoint of $P$ is a leaf we can shorten $P$ without decreasing the number of non-leaf vertices it contains.

There is a vertex of degree at least 2 not contained in $P$, call it $v$. Remove $v$ from $T$ and we obtain a component $C_1$ that contains all of $P$ and at least one component $C_2$ which contains no vertex of $P$. Let $x$ be a leaf of $G[C_2]$.

Take a path $P'$ from $v$ to $P$ and let $v'$ be the first vertex of this path that belongs in $P$. It must be the case that $v'$ is an internal vertex of $P$, otherwise $P \cup P'$ would be a path that contains more non-leaf vertices than $P$. Therefore, $P$ contains at least three vertices. Let $y, z$ be the endpoints of $P$, which have degree at least 2 in $T$. Let $y', z'$ be the neighbors of $y, z$ respectively which do not lie in $P$. Then $x, y', z'$ is an asteroidal triple. Indeed, removing $x$ and its neighbors does not affect $P$, so there is still a path from $y'$ to $z'$; and removing $y'$ and its neighbors leaves $P - y$ intact, which ensures that $P' \cup P \cup \{v\} \cup C_2$ contains a path from $x$ to $z'$ (similarly for removing $z'$).

## 2 Interval Graphs and Vertex Orderings

Recall that a graph is chordal if and only if there exists an ordering of the vertices $v_1, v_2, \dots, v_n$, such that for each $v_i$ the set of neighbors of $v_i$ with indices $j > i$ (i.e. coming later in the ordering) induces a clique. This is called a Perfect Elimination Ordering.

Show that a graph is an interval graph if and only if there exists an ordering of its vertices $v_1, v_2, \dots, v_n$, such that for each $i < j < k$, if $v_i v_k \in E$, then $v_j v_k \in E$.

**Solution:**

An ordering exists $\Rightarrow$ the graph is an interval graph:

We produce from the ordering $v_1, \dots, v_n$ an interval representation, assigning to each $v_k$ the interval $[f(k), k]$, where $f(k)$ is defined as the minimum $i \in [1, k-1]$ such that $v_i v_k \in E$ and if no such $i$ exists, we set $f(k) = k$. We claim that the resulting interval graph is exactly $G$.

First, we show that for all $j < k$, if $[f(j), j] \cap [f(k), k] \neq \emptyset$, then $v_j v_k \in E$. Indeed, if the intersection of the two intervals is non-empty, then $j \geq f(k)$, so $v_i v_k \in E$ for some $i \leq j$. By the properties of the ordering, $v_i v_k \in E$ implies that $v_j v_k \in E$, as desired.

Second, we show that for all $j < k$, if $[f(j), j] \cap [f(k), k] = \emptyset$, then $v_j v_k \notin E$. Indeed, we have $j < f(k)$, so the minimum $i$ such that $v_i v_k \in E$ is strictly larger than $j$, hence $v_j v_k \notin E$.

An ordering exists $\Leftarrow$ the graph is an interval graph:

Suppose that $G = (V, E)$ is an interval graph and for each $v \in V$ we have an interval $[s_v, t_v]$ such that $vu \in E$ if and only if $[s_v, t_v] \cap [s_u, t_u] \neq \emptyset$. Order the vertices $v_1, \dots, v_n$ in non-decreasing order of their right endpoint, that is, so that whenever $i < j$, then $t_{v_i} \leq t_{v_j}$. We claim that this ordering satisfies the desired property. Consider $v_i, v_j, v_k$, with $i < j < k$ and $v_i v_k \in E$. Therefore, $t_{v_i} \geq s_{v_k}$. However, $t_{v_j} \geq t_{v_i}$ (since $j > i$ and we ordered by right endpoint) and $t_{v_j} \leq t_{v_k}$ (for the same reason), so $t_{v_j} \in [s_{v_k}, t_{v_k}]$, therefore $v_j v_k \in E$ as desired.

## 3 Maximal Cliques in Chordal Graphs

Show that in a connected chordal graph on $n$ vertices, with $n \geq 2$, there exist at most $n - 1$ distinct maximal cliques. A clique $C$ is maximal if it is impossible to increase it by adding a vertex, i.e. each $v \in V \setminus C$ has a non-neighbor in $C$.

Show that there exists a non-chordal graph with $2n$ vertices and $2^n$ distinct maximal cliques.

**Solution:**

For the first part, we do induction on $n$ and observe that the statement is true for $n = 2$. Let $n \geq 3$ and consider a simplicial vertex $v$ of $G$. We observe that the maximal cliques of $G$ can be partitioned into two classes: those that contain $v$; and those that contain a non-neighbor of $v$ and are therefore maximal cliques of $G - v$. By inductive hypothesis, $G - v$ contains at most $n - 2$ maximal cliques. On the other hand, there exists exactly one maximal clique in $G$ that contains $v$, namely $\{v\} \cup N(v)$, since $v$ is simplicial. Therefore, $G$ has at most $n - 1$ maximal cliques.

For the second part, consider a graph made up of $n$ independent sets of size 2 where we add all possible edges between parts (i.e. $\overline{nK_2}$). Any set that contains exactly one vertex from each part is a maximal clique.

## 4 Split Graphs and Degree Sequences

Let $(d_1, d_2, \dots, d_n)$ be the degree sequence of a graph $G$, with $d_i \geq d_{i+1}$ for all $i$. Prove the following: $G$ is a split graph if and only if $\sum_{i=1}^{k} d_i = k(k-1) + \sum_{i=k+1}^{n} d_i$, where $k$ is the maximum index $i$ such that $d_i \geq i - 1$.

**Solution:**

Suppose that $G$ is a split graph with clique $C$ and independent set $I$ and suppose that $|C| = \omega(G)$, that is, all vertices of $I$ have a non-neighbor in $C$. We observe that for any vertex $x \in C$ and $y \in I$ we have $\deg(x) \geq \deg(y)$, therefore in the sorted degree sequence the degrees of vertices of $C$ appear before the degrees of vertices of $I$. Let $k = |C|$. Then, the $k$-th vertex in the sequence is a vertex of $C$ of minimum degree. We have $\sum_{i=1}^{k} d_i = 2|E(C)| + |E(C, I)| = k(k-1) + |E(C, I)|$ and $\sum_{i=k+1}^{n} d_i = |E(C, I)|$, so the equality is verified.

Suppose now that the degree sequence satisfies the property, let $C$ be the set of the $k$ highest-degree vertices of the graph and $I = V \setminus C$. We want to show that $C$ is a clique and $I$ an independent set. Indeed, suppose that $C$ is not a clique. Then $\sum_{v \in C} \deg(v) = \sum_{i=1}^{k} d_i = 2|E(C)| + |E(C, I)| < k(k-1) + \sum_{i=k+1}^{n} d_i$, where we used the fact that $|E(C, I)| \leq \sum_{v \in I} \deg(v)$ and that $|E(C)| < \frac{k(k-1)}{2}$, as $C$ is not a clique. We therefore have a contradiction, since we assumed that $\sum_{i=1}^{k} d_i = k(k-1) + \sum_{i=k+1}^{n} d_i$. Similarly, suppose that $I$ induces at least one edge. Then $\sum_{v \in I} \deg(v) = \sum_{i=k+1}^{n} d_i = 2|E(I)| + |E(C, I)| > \sum_{i=1}^{k} d_i - k(k-1)$, where we used that $|E(I)| > 0$ and $\sum_{i=1}^{k} d_i = k(k-1) + |E(C, I)|$, which follows because $C$ is a clique. We again have a contradiction, as we had assumed $\sum_{i=k+1}^{n} d_i = \sum_{i=1}^{k} d_i - k(k-1)$.

# 5 Short cycles in chordal graphs

Show that if $G$ is chordal and an edge $e \in E(G)$ is part of a cycle, then there exists a $K_3$ in $G$ that contains $e$.
**Solution:**

Let $C$ be the shortest cycle of $G$ that contains $e$. If $C$ has length 3, we are done. Otherwise, since $G$ is chordal, $C$ must induce an edge $e'$, which partitions the cycle $C$ into two paths $P_1, P_2$, such that $P_1 + e'$ and $P_2 + e'$ are both cycles shorter than $C$. However, the endpoints of $e$ are contained in either $P_1$ or $P_2$, so $e$ is contained in a cycle shorter than $C$, contradiction.