TD 1: Introduction

1 Enumeration

List all non-isomorphic graphs on 2,3,4, and 5 vertices.
Solution:

- $n = 2$: K_2 and $2K_1$
- $n = 3$: K_3, P_3, $P_2 + K_1$, $3K_1$
- $n = 4$: We list graphs with at most 3 edges (we can then take their complements for the full list). $4K_1$, $K_2 + 2K_1$, $2K_2$, $P_3 + K_1$, $K_3 + K_1$, P_4, $K_{1,3}$
- $n = 5$: As before, we only list graph with up to 5 edges, as we can take the complements of these graphs to obtain the full list.
  - 0 edges: $5K_1$
  - 1 edge: $K_2 + 3K_1$
  - 2 edges: $2K_2 + K_1$, $P_3 + 2K_1$
  - 3 edges: $K_3 + 2K_1$, $K_2 + P_3$, $P_4 + K_1$
  - 4 edges:
    * 1 component: All trees on 5 vertices: P_5, $K_{1,4}$, $K_{1,3}$ with one edge sub-divided.
    * 2 components: $K_2 + K_3$, $K_1 + C_4$, $K_1 + (K_3 \oplus \ell)$
  - 5 edges:
    * 1 component: C_5, Bull (C_3 with two leaves attached), Kite (C_3 with a P_2 attached), C_4 with a leaf attached
    * 2 components: one component must be K_1, so we can use the list of all 4-vertex, 5-edge graphs, obtained above.
    * 3 components: impossible

2 Connected Complements

Prove that for all graphs $G = (V, E)$, at least one of $G, \overline{G}$ is connected.
Solution:

Suppose that G is not connected and let C be a connected component of G. We show that $\overline{G}$ is connected as follows: take a vertex $x \in C$ and we will show that there is a path from x to every other vertex in $\overline{G}$. For a vertex $y \in V \setminus C$, we have $xy \notin E$ (otherwise y would be in the same connected component), therefore $xy \in E(\overline{G})$. For a vertex $z \in C$, let y be an arbitrary vertex of $V \setminus C$ and by the same reasoning as before $xy, zy \in E(\overline{G})$, so there is an $x - z$ path in $\overline{G}$. Therefore, x is in the same connected component as every vertex of $\overline{G}$, hence $\overline{G}$ is connected.

3 Connected Complements II

Show that if for a graph $G = (V, E)$ we have $\mathrm{diam}(G) \geq 3$, then $\mathrm{diam}(\overline{G}) \leq 3$.
Solution:
Suppose that x, y are two vertices which are at maximum distance in G. We observe the following:

- $xy \notin E$, therefore $xy \in E(\overline{G})$
- For all other $z \in V$ we have $xz \notin E$ or $yz \notin E$. Indeed, otherwise we would have a path of length 2 from x to y.

From the above we obtain the bound on the diameter of $\overline{G}$ as follows: the vertices x, y dominate $\overline{G}$ and are adjacent, so any two other vertices z, w have in the worst case a path of the form z, x, y, w connecting them.

4 Many edges connect the graph

Show that for any n-vertex graph $G = (V, E)$ with m edges, if $m > \binom{n-1}{2}$, then G is connected. Is this bound **sharp**? (meaning, is the claim false if we decrease the right-hand-side by 1?)
Solution:
We will prove that among all disconnected graphs on n vertices the one that has the largest number of edges is $K_{n-1} + K_1$. Since this graph has $\binom{n-1}{2}$ edges, this will prove the claim.
Consider then a disconnected graph on n vertices and suppose that it contains at least three components. Then, adding an edge between two components gives a disconnected graph with more edges. Suppose then that there are exactly two components C_1, C_2 with $|C_1| \geq |C_2|$ (wlog). If C_2 contains no edges, clearly that graph that has the maximum number of edges and has this form is indeed $K_{n-1} + K_1$. Suppose then that C_2 contains an edge xy. We remove from the graph all edges incident on x and add all edges with one endpoint x and the other in C_1. This increases the number of edges of the graph, as we added $|C_1|$ edges and removed at most $|C_2| - 1$ edges. Continuing like this we obtain the graph we described.

5 Walks and Adjacency Matrices

A **walk** is a path which is allowed to repeat vertices. Show that if A is the adjacency matrix of a graph G, then for all positive integers k we have that $A^k[i, j]$ is equal to the number of distinct walks of length exactly k from i to j in G.
Solution:
Proof by induction: for $k = 1$, $A[i, j]$ contains only $0/1$ values and has a 1 if and only if there is an edge ij, that is, a walk of length 1.
Inductive step: suppose the statement is true for A^k and consider A^{k+1}.

$$A^{k+1}[i, j] = \sum_{z \in [n]} A^k[i, z] A[z, j]$$

Observe now that we can count the number of $i - j$ walks of length exactly $k + 1$ by considering the penultimate vertex z, counting for each neighbor z of j how many length k walks there are from i to z and taking the sum. This is exactly what is calculated by the formula above.

6 Min degree to Path

Show that if all vertices of G have degree at least k, then G contains a path of length at least k.
Solution:
We execute a simple greedy algorithm: start at an arbitrary vertex v and maintain a path that is initially just (v). At each step we select an arbitrary neighbor of the current vertex and add it to the path if possible.

Suppose now that this algorithm terminates and the last vertex appended to the path was x. Since the algorithm stopped, all neighbors of x are already in the path. But x has at least k neighbors, so the path contains at least $k + 1$ vertices, so has length at least k.

7 Odd degrees

Prove that if a graph G contains exactly two vertices of odd degree, then they are connected by a path.
Solution:

Suppose that x, y are the only odd-degree vertices, but a connected component C contains only x and not y. Then $G[C]$ (the graph induced by C) is a graph where all vertices of C have the same degree as in G, hence x is the only vertex of odd degree. But $G[C]$ cannot have an odd number of odd-degree vertices, contradiction.

8 Ramsey

Prove that in any group of 6 people, there are either 3 people who all know each other or 3 people who do not know each other. Show that this is false for groups of 5 people.

Generalization: prove that for all k, in any group of 4^k people, there are either at least k who all know each other, or at least k who do not know each other.
Solution:

6 people: we model this with a graph on 6 vertices and prove that there exists a clique or an independent set of size at least 3. Let a be the vertex of highest degree. If the degree of a is at most 2, then the graph is a union of paths and cycles, so there is an independent set of size 3. If not, we check to see if $N(a)$ induces any edges. If yes, we have a triangle; otherwise we have an independent set of size at least 3. For 5 people, it suffices to consider a C_5.

4^k people: we prove that for positive integers s, c any graph with at least 2^{s+c} vertices contains an independent set of size s or a clique of size c. By setting $s = c = k$ we obtain the result.

To prove the claim we use induction on $s + c$. For $s + c = 2$ (which is the minimum value) the statement holds. Consider now two fixed values s, c and suppose the statement is shown for any smaller pair. Take a graph $G = (V, E)$ with at least 2^{s+c} vertices and take an arbitrary vertex x. If $|N(x)| \geq 2^{s+c-1}$, then $G[N(x)]$ contains either a clique of size $c - 1$ or an independent set of size s; in the latter case we are done, in the former case we form a clique of size c by adding x. Otherwise, $|N(x)| \leq 2^{s+c-1} - 1$, therefore, $|V \setminus N(x)| \geq 2^{s+c-1} + 1$. Consider then the graph induced by $V \setminus (N(x) \cup \{x\})$, which has at least 2^{s+c-1} vertices. By inductive hypothesis this graph has at least a clique of size c (in which case we are done) or an independent set of size $s - 1$, to which we can add x to obtain an independent set of size s.