Graph Theory Midterm Exam – 8/11/2024

Family name:

First name:

Student Number:

Guidelines

1. For all questions, dedicated answer boxes are provided. **Provide your answer inside these boxes**. Anything written outside the boxes will not be graded.
2. The sizes of the boxes give you a hint for the length of the answer expected.
3. **Write clearly!**
4. Extra pages are provided in the end to allow you to work out your answers before filling them in. Copies of the given graphs also appear there, to allow you to experiment with them, if necessary.
5. The maximum (theoretical) score in this test is 21/20.
6. Exam duration: 90 minutes.
7. When you are finished, raise your hand and wait for your copy to be collected before (quietly) leaving the room.
8. **Don't panic!**
9. **Good luck!**

1 Lightning Round (4 points)

For each of the following statements, indicate whether the statement is true or false. Provide a justification for your answer (proof, example, counter-example, as appropriate).

1. There exists a tree of average degree 2.

   False. The average degree is $\frac{\sum_{v\in V}\deg(v)}{n} = \frac{2n-2}{n} < 2$, where we used that trees have $n-1$ edges and the sum of the degrees is $2m$.

2. There exists a 3-vertex-connected graph G (i.e. $\kappa(G) \geq 3$), such that G has average vertex degree 2.

   False. If a graph has average degree 2, then at least one vertex has degree at most 2, so deleting its neighbors gives a cut of size less than three.

3. There exists a graph G, where $\mathrm{mm}(G) < \mathrm{vc}(G)$, where mm, vc are the sizes of the maximum matching and minimum vertex cover respectively.

   True. For example K_3 has max matching of size 1, vertex cover of size 2.

4. There exists a bipartite graph G that satisfies the condition of the previous question.

   False, according to Konig's theorem, the two values are equal for bipartite graphs.

5. Every tree is bipartite.

   True, trees contain no cycles, hence no odd cycles.

6. Every subgraph of a tree is a tree.

   False, the subgraph may be disconnected.

7. There exists a graph G where $\kappa(G), \kappa'(G), \delta(G)$ have three distinct values, where κ, κ', δ are respectively the vertex connectivity, edge connectivity, and minimum degree.

   True. Take two copies of a K_5. Place two edges from one vertex of the first copy to two vertices of the other copy. We have $\delta = 4$, $\kappa = 1$, and $\kappa' = 2$.

8. In all bipartite graphs with at least 3 vertices, $\alpha(G) \geq \omega(G)$, where α, ω are respectively the size of the largest independent set and largest clique.

   True. Because $n \geq 3$ and $\alpha(G) \geq n/2$ we have $\alpha(G) \geq 2$. But $\omega(G) \leq 2$ on bipartite graphs, as otherwise we would have an odd cycle.

2 Complements and Connectivity (3 points)

Suppose that a graph G with $n \geq 4$ vertices contains no vertices of degree 0. Prove that if G is disconnected, then $\kappa(\overline{G}) \geq 2$, that is, $\overline{G}$ is at least 2-vertex-connected.

Proof: Since no vertex has degree 0, every connected component of G has at least two vertices. We prove that for any two vertices x, y, there exist at least two vertex-disjoint paths $x \to y$ in $\overline{G}$. We have two cases:

- x, y are in the same component C of G. Then, there exists another component C' (since G is disconnected). C' has at least two vertices a, b. The edges xa, xb, ya, yb all exist in $\overline{G}$, as G has no edge from C to C'. So we have the paths $x \to a \to y$ and $x \to b \to y$.

- x, y are in distinct components of G. There must exist another vertex x' in the same component of G as x, and another vertex y' in the same component as y. The edges $xy, xy', x'y, x'y'$ all exist in $\overline{G}$. We have the path $x \to y' \to x' \to y$ and the edge xy, giving two vertex-disjoint paths.

3 Edge Cuts (4 points)

Suppose a graph $G = (V, E)$ is connected and all vertices have even degree. Show that in this case G has no cut edge, that is, for all $e \in E$ we have that $G - e$ is connected.

Proof: suppose for contradiction that $G - e$ is disconnected for some $e = xy$. Therefore, x, y are in distinct components of $G - e$. Let X be the component of $G - e$ that contains x and consider the graph $G[X]$. The degrees of all vertices other than x are unchanged compared to G, while the degree of x is now odd (we have removed one of its incident edges). Therefore, the sum of all degrees in this graph is odd, contradiction.

4 Degree Sequences (3 points)

Suppose a graph G has $n = 6$ vertices, $x_1, x_2, \ldots, x_6$. You are given the following information regarding their degrees: $\deg(x_1) = 5$, $\deg(x_2) = 4$, $\deg(x_3) = 3$, $\deg(x_4) = 2$, $\deg(x_5) = a$, $\deg(x_6) = b$, where a, b are unknown values.

Among all the values of (a, b) for which such a graph exists, determine the minimum possible value of $a + b$.

Proof: We make the following observations, assuming without loss of generality that $a \geq b$.

- Because $\deg(x_1) = 5$, x_1 is adjacent to all vertices, so $a, b > 0$. Hence $a + b \geq 2$.
- Because $\deg(x_2) = 4$, x_2 is adjacent to at least one of x_5, x_6, so $a \geq 2$, which implies that $a + b \geq 3$.
- Because the sum of all degrees must be even, we have $a + b \geq 4$.

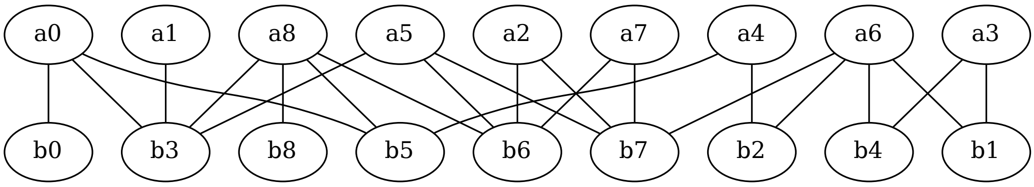
Consider now for example $a = b = 2$, which gives the sequence $(5, 4, 3, 2, 2, 2)$. Running the Havel-Hakimi algorithm gives: $(5, 4, 3, 2, 2, 2) \to (3, 2, 1, 1, 1, 0) \to (3, 2, 1, 1, 1) \to (1, 0, 0, 1) \to (1, 1)$. Since the last sequence is graphic, we conclude that the original is graphic. Hence, the minimum value of $a + b$ is 4.

Comments:

- Setting $a = 3, b = 1$ also gives a valid solution.
- Instead of running the Havel-Hakimi algorithm, one could also supply a graph with the prescribed sequence to prove that $a = b = 2$ is feasible.
- However, proving that $a + b \leq 3$ is not feasible is still necessary.
- Running Havel-Hakimi on the sequence $(5, 4, 3, 2, a, b)$ is ill-advised, as we cannot be sure that this sequence is sorted without knowing the values of a, b (and the same problem arises again for intermediate steps of the algorithm).

5 Maximum Matching (4 points)

Consider the bipartite graph given below.



Give a matching of maximum size in the graph above. That is, give the list of edges that form your matching. (To ease presentation, you may also highlight the selected edges in the graph above. However, the list below will be considered your official answer.)

Answer: Take the vertical edges, $a_0b_0, a_1b_3, a_8b_8, a_2b_6, a_7b_7, a_4b_2, a_6b_4, a_3b_1$. Size: 8

Prove that the graph does not contain a matching larger than the one you supplied.

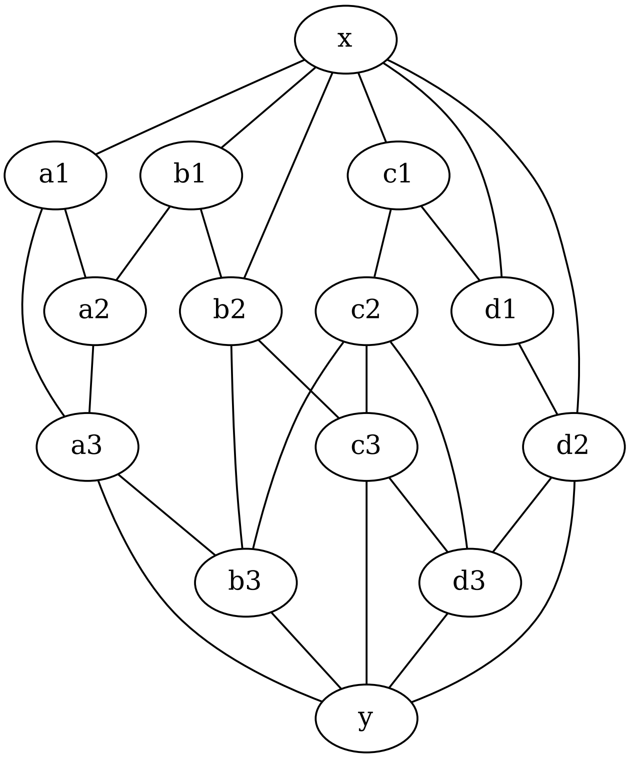
Answer: A larger matching would be perfect. However, the set $\{a_1, a_5, a_2, a_7\}$ has size 4 but a neighborhood of size 3 ($\{b_3, b_6, b_7\}$).

Give a minimum vertex cover of the given graph and explain why it is minimum.

Answer: We take the set $\{b_3, b_6, b_7, a_0, a_3, a_4, a_6, a_8\}$ which has size 8. No cover can be smaller, as we have a matching of size 8.

6 Cuts and Menger (3 points)

Consider the graph G given below.



Find a minimum xy vertex cut in G. Explain why your cut is minimum. (1 point)

Answer: $\{a_3, b_2, c_2, d_2\}$. No smaller cut exists, as we have four disjoint paths ($xa_1a_2a_3y$, $xb_1b_2b_3y$, $xc_1c_2c_3y$, $xd_1d_2d_3y$).

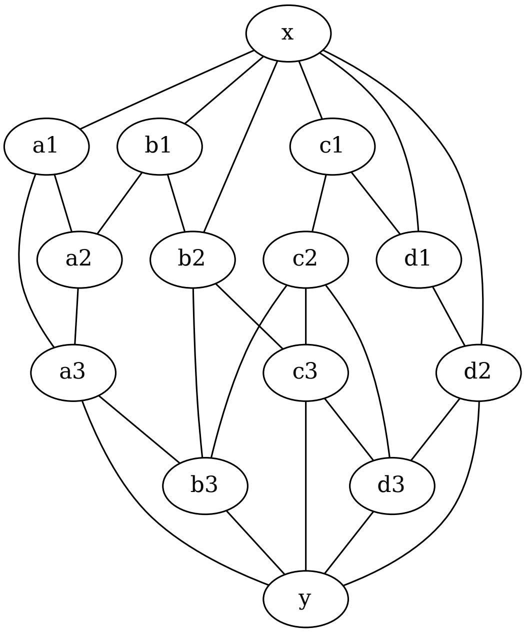
Suppose we add to G the edge $(a2, b3)$. Find a minimum xy vertex cut in the new graph. Explain why your cut is minimum. (2 points)

Answer: We add to the previous set the vertex a_2. To prove that the new set is minimum, we show that the graph now has 5 disjoint paths: $xa_1a_3y, xb_1a_2b_3y, xb_2c_3y, xc_1c_2d_3y, xd_1d_2y$.

Comment: The optimality of a set of size 5 **cannot** be proved using Menger's theorem. Menger's theorem can be used to show that "if the separator of size 5 is minimum, then there exist 5 disjoint paths". However, this implication is in the wrong direction, as what we want to prove is that the separator is minimum.

Extra Space

Use this space for notes or to play (this part will not be read). For convenience, we also give you two copies of each of the graphs supplied in previous exercises.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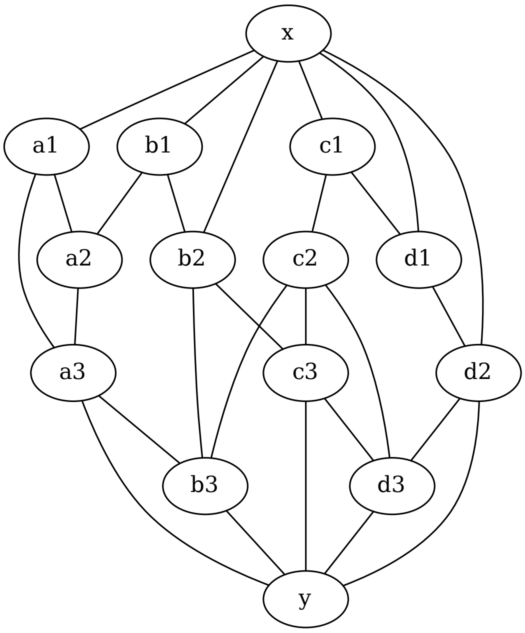
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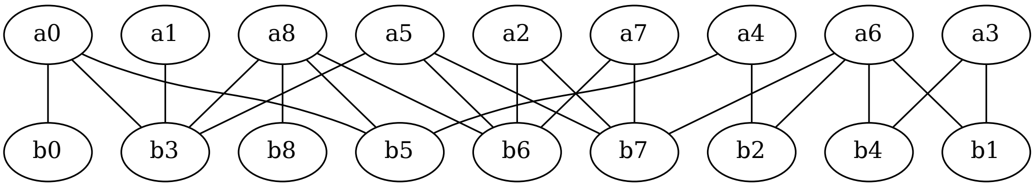
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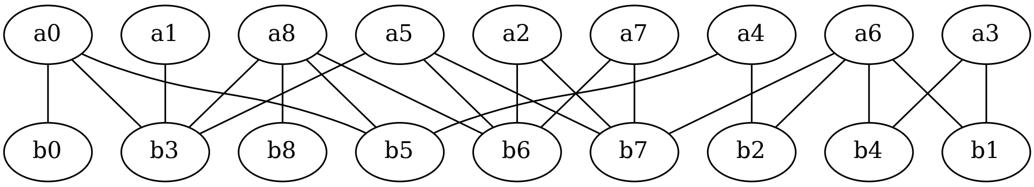
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