Graph Theory: Lecture 9

Cographs and Friends

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December 5, 2024

Forbidden Subgraph Characterizations

Wider question: how does **local** structure lead to **global** structure?

- A graph is a forest if and only if it has no C_k (induced) subgraph.
- A graph is bipartite if and only if it has no C_{2k+1} (induced) subgraph.
- A graph is planar if and only if it has not $K_{3,3}, K_5$ topological minor.
- A graph is chordal if it contains no induced C_k subgraph, for $k \geq 4$.
- A graph is split if it contains no induced $2K_2$, C_4, or C_5.
- A graph is interval if it is chordal and contains no Asteroidal Triple

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We examined what happens if we forbid long or odd induced cycles. What if we forbid paths?

Cographs

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Recall: for all G', at least one of $G', \overline{G'}$ is connected, so G is a cograph if **exactly** one of the two is connected for each induced subgraph.

Cographs

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Examples:

- C_4 is a cograph
- C_k, $k \geq 5$ is not a cograph
- P_k, $k \geq 4$ is not a cograph

Cographs – Characterization

Theorem

The following are equivalent:

1. G *is a cograph*
2. G *can be constructed from* K_1*s using* **Join** *and* **Union** *operations*
3. G *can be constructed from* K_1*s using* **Union** *and* **Complement** *operations*
4. G *contains no induced* P_4

Cographs – Characterization

Theorem

The following are equivalent:

1. *G is a cograph*
2. *G can be constructed from K_1s using* **Join** *and* **Union** *operations*
3. *G can be constructed from K_1s using* **Union** *and* **Complement** *operations*
4. *G contains no induced P_4*

Note: Implies that cograph recognition is in NP$\cap$coNP and in fact in P. (why?)

Cographs and Cotrees

Definition

A cotree of a cograph G is a rooted tree where:

- Each leaf is a vertex of G.
- Each internal node is labeled 1 (Join) or 0 (Union)

The cotree shows how to construct G from individual vertices using the two operations Join and Union.

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Examples:
Join (Union (a,b)) (Union (a,b)) $\rightarrow C_4$

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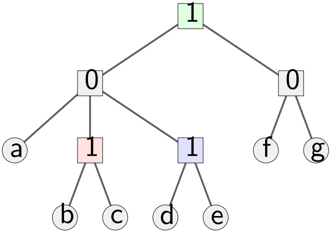
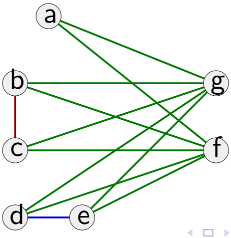
Examples:





Cographs – Characterization continued

Lemma

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Cographs – Characterization continued

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G is a cograph if and only if G has a cotree.

Proof.

Proof by induction:

- G is cograph $\Rightarrow$ G has a cotree
  - G is cograph $\Rightarrow$ G is disconnected or $\overline{G}$ is disconnected into components $C_1, \ldots, C_k$.
  - By inductive hypothesis, we have a cotree for each C_i
  - If G disconnected, take Union of cotrees; if not, take Join of cotrees.
- G is cograph $\Leftarrow$ G has a cotree
  - If root of tree is 0, G is disconnected into components $C_1, \ldots, C_k$.
  - Any induced subgraph contained in a C_i is good by IH.
  - Any subgraph with vertices from two components is disconnected.
  - Proof is symmetric if root is 1.

Cographs – Characterization continued

Lemma

G is a cograph if and only if G has no induced P_4.

Proof.

□

Cographs – Characterization continued

Lemma

G is a cograph if and only if G has no induced P_4.

Proof.

G is cograph $\Rightarrow$ no induced P_4:
Easy: $P_4 = \overline{P_4}$, so if G contains P_4, G contains an induced subgraph that proves that it is not a cograph.

□

Cographs – Characterization continued

Lemma

G is a cograph if and only if G has no induced P_4.

Proof.

G is cograph $\Leftarrow$ no induced P_4:
Proof by induction on the size of G

- Let $x \in V(G)$ and consider $G - x$, apply IH, $G - x$ is cograph.
- Suppose wlog that $G - x$ is disconnected into $C_1, C_2, \ldots, C_k$ (otherwise take its complement)
- If x is universal:
  - All subgraphs that contain x have disconnected complements.
  - All other subgraphs are OK by IH.

□

Cographs – Characterization continued

Lemma

G is a cograph if and only if G has no induced P_4.

Proof.

G is cograph $\Leftarrow$ no induced P_4:
Proof by induction on the size of G

- Then, x is not universal.
- If x has no neighbor in a component C_i:
    - Let $a \in V(C_i)$
    - Subgraphs without $x \Rightarrow$ Good! (IH)
    - Subgraphs without $a \Rightarrow$ Good! (IH)
    - Subgraphs with a and $x \Rightarrow$ disconnected, Good!

□

Cographs – Characterization continued

Lemma

G is a cograph if and only if G has no induced P_4.

Proof.

G is cograph $\Leftarrow$ no induced P_4:

Proof by induction on the size of G

- Then, x is not universal and x has a neighbor is each component.
- Let $ax \notin E$, $bx \in E$, $a, b \in C_1$
- Let $cx \in E$, $c \in C_2$
- Then, $a \rightarrow b, x, c$ induces a P_k, $k \geq 4$, contradiction!

□

Algorithmic Questions

Theorem

The following are polynomial-time solvable:

- *Deciding if G is a cograph.*
- *Computing the max independent set of a cograph.*
- *Computing the max clique of a cograph.*
- *Computing the chromatic number of a cograph.*

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Proof.

Construct a cotree recursively

□

Algorithmic Questions

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- *Deciding if G is a cograph.*
- *Computing the max independent set of a cograph.*
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- *Computing the chromatic number of a cograph.*

Proof.

- If $G = G_1 \cup G_2$, return $\alpha(G_1) + \alpha(G_2)$.
- If $G = G_1 \times G_2$, return $\max\{\alpha(G_1), \alpha(G_2)\}$.

□

Algorithmic Questions

Theorem

The following are polynomial-time solvable:

- *Deciding if G is a cograph.*
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- *Computing the chromatic number of a cograph.*

Proof.

Run previous algorithm on complement of G.

□

Algorithmic Questions

Theorem

The following are polynomial-time solvable:

- *Deciding if G is a cograph.*
- *Computing the max independent set of a cograph.*
- *Computing the max clique of a cograph.*
- *Computing the chromatic number of a cograph.*

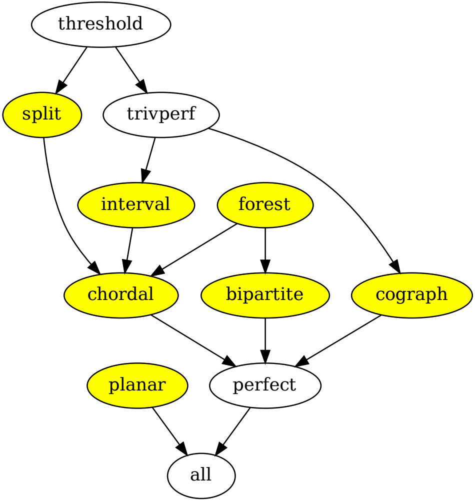
Proof.

- If $G = G_1 \cup G_2$, return $\max\{\chi(G_1), \chi(G_2)\}$.
- If $G = G_1 \times G_2$, return $\chi(G_1) + \chi(G_2)$.

□

More graph classes!

Where we are


threshold
split
trivperf
interval
forest
chordal
bipartite
cograph
planar
perfect
all

Perfect Graphs

Definition

A graph G is perfect if for every induced subgraph G' we have $\chi(G') = \omega(G')$.

Perfect Graphs

Definition

A graph G is perfect if for every induced subgraph G' we have $\chi(G') = \omega(G')$.

- Defined by Berge in the 1960's
- Closure under complement open for 10 years (Lovasz 1970's)
- Forbidden subgraph characterization open for 40 years (Chudnovsky et al. 2006)
- Generalize many poly-time solvable cases of independent set, clique, coloring.

Perfect Graphs

Definition

A graph G is perfect if for every induced subgraph G' we have $\chi(G') = \omega(G')$.

Theorem (Weak Perfect Graph Theorem)

G is perfect if and only if $\overline{G}$ is perfect.

Theorem (Strong Perfect Graph Theorem)

G is perfect if and only if G has no C_{2k+1} or $\overline{C}_{2k+1}$ induced subgraph, for $k \geq 2$ (no odd holes or anti-holes).

Bipartite Graphs are Perfect

Theorem

If G is bipartite, then G is perfect.

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Proof.

Straight from definition: G' non-empty induced subgraph of $G \Rightarrow G'$ bipartite $\Rightarrow \omega(G') = 2$ and $\chi(G') = 2$. □

Bipartite Graphs are Perfect

Theorem

If G is bipartite, then G is perfect.

Proof.

Straight from definition: G' non-empty induced subgraph of $G \Rightarrow G'$ bipartite $\Rightarrow \omega(G') = 2$ and $\chi(G') = 2$. □

Proof.

(Using Strong PG theorem) G bipartite, so G has no odd holes. $\overline{C}_5 = C_5$ is also not in G. $\overline{C}_{2k+1}$, for $2k+1 \geq 7$ contains a K_3, so also not in G. □

Cographs are Perfect

Theorem

If G is a cograph, then G is perfect.

Cographs are Perfect

Theorem

If G is a cograph, then G is perfect.

Proof.

(Using Strong PG theorem)

- G is cograph $\Rightarrow$ all induced subgraphs G' which are connected have $\overline{G}'$ disconnected.
- If G had a $G' = C_{2k+1}$ (or $G' = \overline{C}_{2k+1}$), for $k \geq 2$ as an induced subgraph, then $G', \overline{G}'$ are both connected, contradiction.

□

Cographs are Perfect

Theorem

If G is a cograph, then G is perfect.

Proof.

Direct application of definition and induction:

- If G is disconnected, $\omega(G)$ is **max** over all components, $\chi(G)$ is max over all components, by IH in each component C, $\omega(C) = \chi(C)$.
- If G is connected, $\omega(G)$ is **sum** over all components, $\chi(G)$ is sum over all components, by IH in each component C, $\omega(C) = \chi(C)$.

□

Chordal Graphs are Perfect

Theorem

If G is chordal, then G is perfect.

Chordal Graphs are Perfect

Theorem

If G is chordal, then G is perfect.

Proof.

Direct application of definition and induction:

- Let x be a simplicial vertex. Two cases:
  - $\omega(G) = \omega(G - x) + 1$. By IH $\omega(G - x) = \chi(G - x) \geq \chi(G) - 1$ so $\omega(G) \geq \chi(G) \Rightarrow \omega(G) = \chi(G)$.
  - $\omega(G) = \omega(G - x) = \chi(G - x)$. In this case, $\chi(G - x) \geq \deg(x) + 1$, because $\omega(G) \geq \deg(x) + 1$. So, after coloring $G - x$ there is always an available color for x.

□

Chordal Graphs are Perfect

Theorem

If G is chordal, then G is perfect.

Proof.

Using Strong PG theorem

- G is chordal $\Rightarrow$ no odd holes or $\overline{C}_5$
- If G has a $\overline{C}_{2k+1}$ for $2k+1 \geq 7$ as induced subgraph, call its vertices $x_1, x_2, \dots, x_{2k+1}$.
- Observe that x_1, x_3, x_{2k+1}, x_4 induces a C_4 contradiction.

□

An Application

Line Graphs of Bipartite Graphs are Perfect

Theorem

If G is bipartite, then $L(G)$ is perfect.

Line Graphs of Bipartite Graphs are Perfect

Theorem

If G is bipartite, then $L(G)$ is perfect.

Proof.

Using Strong PG theorem

- G is bipartite, contains no odd holes, so $L(G)$ contains no odd holes.
- If $L(G)$ has a $\overline{C}_{2k+1}$ for $2k+1 \geq 7$ as induced subgraph, call its vertices $x_1, x_2, \ldots, x_{2k+1}$.
- Consider x_1, x_3, x_4, x_5, x_6, each corresponding to an edge $a_i b_i$ of G
  - x_1 is adjacent to all others, say $a_3 = a_1$ so $b_1 \neq b_3$
  - Because x_3, x_4 non-adjacent, $a_4 \neq a_1$, $b_4 = b_1$
  - Because x_4, x_5 non-adjacent, $b_5 \neq b_4$, $a_5 = a_1 = a_3$
  - Because x_5, x_6 non-adjacent, $a_6 \neq a_1$, $b_6 = b_4 = b_1$
  - But x_3, x_6 adjacent, while $b_3 \neq b_6$ and $a_3 \neq a_6$!!

□

An application

Theorem (Again?)

If G is bipartite, then its maximum matching equals its minimum vertex cover.

Proof.

- $L(G)$ is perfect $\Rightarrow \overline{L(G)}$ is perfect
- $\alpha(L(G)) = \overline{\chi}(L(G))$
    - $\alpha(L(G))$ is just max matching of G
    - $\overline{\chi}(L(G))$ is minimum clique cover
    - Cliques of $L(G)$ are vertices of G
    - $\Rightarrow \overline{\chi}(L(G))$ is minimum vertex cover of G

□