Graph Theory: Lecture 8

Split and Interval Graphs

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November 15, 2024

Forbidden Subgraph Characterizations

Wider question: how does **local** structure lead to **global** structure?

- A graph is a forest if and only if it has no C_k (induced) subgraph.
- A graph is bipartite if and only if it has no C_{2k+1} (induced) subgraph.
- A graph is planar if and only if it has not $K_{3,3}, K_5$ topological minor.

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- A graph is planar if and only if it has not $K_{3,3}, K_5$ topological minor.
- Previous lecture: a graph is chordal if it contains no induced C_k subgraph, for $k \geq 4$.

We show that chordality is useful. But where is it coming from?

Split Graphs

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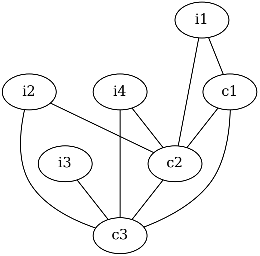

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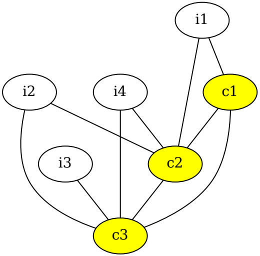

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Relation between split and ...:

- Forest?
- Bipartite?
- Planar?
- Chordal?

Basic Facts

Lemma

G is split if and only if $\overline{G}$ is split.

Lemma

If G is split, then G is chordal.

Lemma

If G is split, then we have one of the three following conditions:

1. $|C| = \omega(G)$ *and* $|I| = \alpha(G)$, *and the partition into* C, I *is unique.*
2. $|C| = \omega(G) - 1$ *and* $|I| = \alpha(G)$
3. $|C| = \omega(G)$ *and* $|I| = \alpha(G) - 1$

Closure under complement

Lemma

G is split if and only if $\overline{G}$ is split.

Proof.

If $G = (V, E)$ and $V = C \cup I$, where C is a clique and I an independent set, then in $\overline{G}$, C is an independent set and I is a clique, so $\overline{G}$ is split.

□

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Split graphs are the bizarro cousins of bipartite graphs. . .

Partition into clique and independent set

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How can the partition into C, I not be unique? (Compare with bipartite graphs)

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Proof.

We have:

- $|C| \leq \omega(G)$ and $|I| \leq \alpha(G)$
- $|C| + |I| = n$
- $\omega(G) + \alpha(G) \leq n + 1$ (why?)
- $\Rightarrow n \leq \alpha(G) + \omega(G) \leq n + 1$

□

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3. $|C| = \omega(G)$ *and* $|I| = \alpha(G) - 1$

Proof.

- If $|C| \leq \omega(G) - 2 \ldots$
- Or $|I| \leq \alpha(G) - 2 \ldots$
- Or $(|C| \leq \omega(G) - 1$ and $|I| \leq \alpha(G) - 1) \ldots$
- $\Rightarrow n = |C| + |I| \leq \alpha(G) + \omega(G) - 2 \leq n - 1$ contradiction!

□

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3. $|C| = \omega(G)$ *and* $|I| = \alpha(G) - 1$

Proof.

If $|C| = \omega(G)$ and $|I| = \alpha(G)$, then the partition is unique

- Suppose (C', I') is alternative partition
  - We must have $|C'| = |C|$ and $|I'| = |I|$
- C' contains one vertex of I, so $C' = C \setminus \{x\} \cup \{y\}$, with $x \in C, y \in I$.
- If $xy \in E$, $C \cup \{y\}$ is larger clique (!), otherwise $I \cup \{x\}$ is larger independent set, contradiction!

Split Graphs are Chordal

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Proof.

We can form a PEO by starting with the vertices of I and continuing with the vertices of C.

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Proof.

If G is split, then G contains no induced C_k, with $k \geq 4$ (why?).

□

Split Graphs Characterization

Theorem

For all G, the following are equivalent:

1. *G is split*
2. *G is chordal and $\overline{G}$ is chordal*
3. *G has none of the following as induced subgraphs: $C_4, C_5, 2K_2$.*

Split Graphs Characterization

Theorem

For all G, the following are equivalent:

1. *G is split*
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3. *G has none of the following as induced subgraphs: $C_4, C_5, 2K_2$.*

Proof.

1. $1 \Rightarrow 2$
   - G is split $\Rightarrow \overline{G}$ is split, $\Rightarrow$ both are chordal.
2. $2 \Rightarrow 3$
   - G chordal $\Rightarrow$ no C_4, C_5
   - $\overline{G}$ chordal $\Rightarrow G$ has no $2K_2$ (otherwise $\overline{G}$ has $\overline{2K_2} = C_4$ induced subgraph)

□

Something easier

Lemma

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If G has none of $2K_2, C_4, C_5$ as induced subgraphs, then G and $\overline{G}$ are chordal.

Proof.

- No $2K_2 \Rightarrow$ no $C_6, C_7, \ldots$ as induced subgraphs, so G is chordal
- $\overline{G}$ contains no $\overline{2K_2}, \overline{C_4}, \overline{C_5} = C_4, 2K_2, C_5$, so $\overline{G}$ is chordal.

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Still missing: (2,3) $\Rightarrow$ (1)

Forbidden subgraphs to split partition

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Proof.

High-level idea:

- Start with "best possible" partition into C, I
  - Assume $|C| = \omega(G)$ and $G[I]$ has minimum number of edges.
- Assume that I is not an independent set
- Argue locally to find a forbidden induced subgraph $\Rightarrow$ contradiction.

□

Part 1: an induced path

Suppose that C is a maximum clique and $I = V \setminus C$ induces minimum number of edges but contains an edge $i_1 i_2 \in E$.

- Each of i_1, i_2 has a non-neighbor in C
- In fact, they have distinct non-neighbors c_1, c_2
- We know that: $i_1 i_2 \in E, c_1 c_2 \in E, i_1 c_1 \notin E, i_2 c_2 \notin E$
- Exactly one of $i_1 c_2, i_2 c_1$ is an edge
- Without loss of generality $i_2 \to i_1 \to c_2 \to c_1$ is an induced P_4.
- Plan: show that there is some way to obtain from this a C_4 or C_5.

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  - Otherwise, there exists unique $c \in C$ that is non-neighbor of i_1, i_2, so $C - c + \{i_1, i_2\}$ is a larger clique.
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  - None are edges $\Rightarrow 2K_2$
  - Both are edges $\Rightarrow C_4$
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Part 2: another independent vertex

So far:

- C is maximum clique, $I = V \setminus C$ induces minimum number of edges
- $i_2 \to i_1 \to c_2 \to c_1$ is an induced P_4, with $i_1, i_2 \in I$ and $c_1, c_2 \in C$

We observe:

- $\{i_1, i_2\}$ dominates $C \setminus \{c_1, c_2\}$
- In fact, i_1 dominates $C \setminus \{c_1, c_2\}$
- Therefore, $C' = C - c_1 + i_1$ is also a maximum clique, so $I' = V \setminus C'$ must induce at least as many edges as I
- Therefore, there exists i_3 with $i_1 i_3 \notin E$ and $i_3 c_1 \in E$.

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Part 3: using the new vertex

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- $i_2 \to i_1 \to c_2 \to c_1$ is an induced P_4, with $i_1, i_2 \in I$ and $c_1, c_2 \in C$
- $i_3 \in I$ with $i_3 c_1 \in E$ and $i_1 i_3 \notin E$

To wrap this up:

- If $i_3 i_2 \notin E$, we have a $2K_2$ $(i_3 c_1, i_1 i_2)$
- So, $i_3 i_2 \in E$ and $i_3 \to i_2 \to i_1 \to c_2 \to c_1 \to i_3$ is a C_5
- If the C_5 is induced, done!
- If it has a chord:
  - It cannot be $i_1 i_3$, nor $i_1 c_1$, so it is not incident on i_1
  - It cannot be $i_2 c_2$, nor $i_2 c_1$, so it is not incident on i_2
  - Therefore, it must be $i_3 c_2$, but then $i_3 \to i_2 \to i_1 \to c_2 \to i_3$ is an induced C_4, done!

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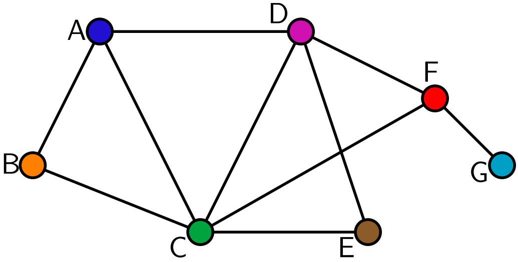
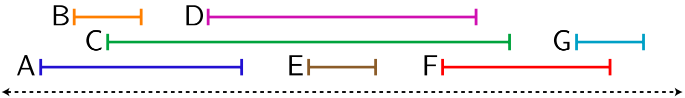
- Intersection graph: more general notion where E represents which pairs have non-empty intersection from a ground set.
- Interval graphs arise naturally in scheduling applications.

Interval Graphs – Example





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Proof.

We produce a PEO:

- Let $x \in V(G)$ such that $[s_x, t_x]$ has minimum **right endpoint** t_x.
- Claim: x is simplicial.
- Remove x from G, repeat.

□

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Illuminating exercise: can also prove this by showing that C_k, for $k \geq 4$ is not an interval graph.

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Clearly: x, y, z must be an independent set

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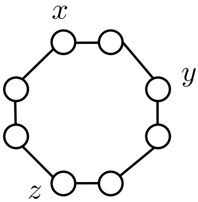

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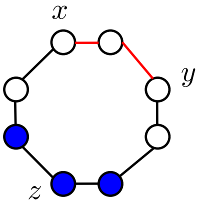

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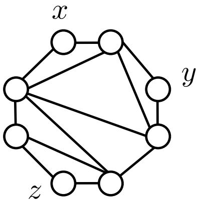

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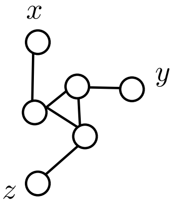

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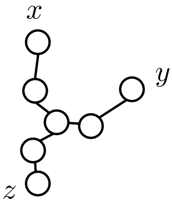

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Proof.

- x, y, z are disjoint intervals.
- Wlog $t_x < s_y < t_y < s_z$.
- Then, $N[y]$ separates x from z!

□

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Theorem

G is an interval graph if and only if G is chordal and AT-free.

Relationships between classes

Subclasses of chordal graphs:

- Split
- Interval
- Trees

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Subclasses of chordal graphs:

- Split
- Interval
- Trees

Classes are incomparable:

- $\exists$ split graph which is not interval, nor tree
- $\exists$ interval graph which is not split, nor tree
- $\exists$ tree which is not split, nor interval

Relationships between classes

Subclasses of chordal graphs:

- Split
- Interval
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Classes are incomparable:

- $\exists$ split graph which is not interval, nor tree
- $\exists$ interval graph which is not split, nor tree
- $\exists$ tree which is not split, nor interval

All three recognizable in polynomial time.

Algorithmic Example

Dominating Set

Definition

A **Dominating Set** of a graph $G = (V, E)$ is a set $S \subseteq V$ such that all $v \in V \setminus S$ have a neighbor in S.

Definition

Algorithmic problem: decide if a given graph has a dominating set of size at most k.

Note: this problem is NP-complete in general.

Dominating Set – Interval Graphs

Greedy algorithm:

1. Initially $S := \emptyset$
2. Order intervals by their right endpoint
3. As long as there exists a non-dominated interval, pick non-dominated x with minimum t_x
   - Select y such that y dominates x and t_y is maximum
   - Set $S := S \cup \{y\}$

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Correctness:

- Clearly polynomial-time
- Optimality:
  - Consider an optimal solution, sorted by the right endpoint, and let $[s_k, t_k]$ be the first interval we select that is not in this optimal solution.
  - We selected $[s_k, t_k]$ to dominate x, while the optimal solution used $[s_{k'}, t_{k'}]$, with $t_{k'} \leq t_k$.
  - Replace $[s_{k'}, t_{k'}]$ with $[s_k, t_k]$ in the optimal, we still have a valid solution of the same size.

Dominating Set – Split Graphs

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Dominating Set is NP-complete on split graphs.

Proof.

We show that we can transform **any** graph G into a split graph G' such that G' has a dominating set of size k if and only if G does.

- G' has two copies of $V(G)$, V_1, V_2
- V_1 is a clique, V_2 is an independent set
- $u \in V_1$ is adjacent to $v \in V_2$ iff $uv \in E(G)$ or $u = v$.

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Simplicial vertices are not enough for this problem!