Graph Theory: Lecture 6

Planar Graphs

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Definition

A graph is **planar** if it can be embedded (drawn) on the plane without edge crossings.

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Reminder:

- We said we usually don't care about how a graph is drawn.
- Today we make a slight exception, because planar graphs are important.

Planar Graphs

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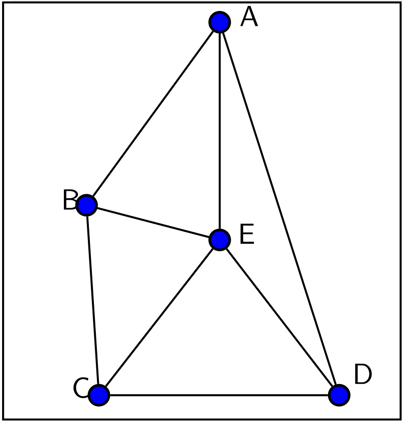

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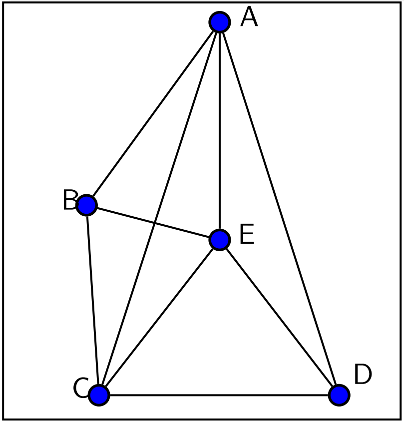

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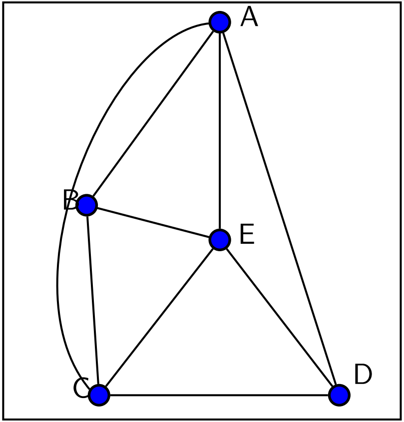

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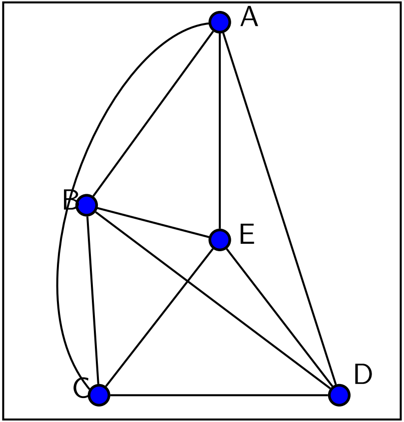

Planar Graphs

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A graph is **planar** if it can be embedded (drawn) on the plane without edge crossings.

- "Is this graph planar?" is in NP (certificate?)
- We will see that it is also in coNP and in fact in P (without proofs).

Planar Graphs

Definition

A graph is **planar** if it can be embedded (drawn) on the plane without edge crossings.

Examples:

- Trees are planar
- Cycles are planar
- (Bi-)Cliques are (usually) not planar

Cliques are not planar

Theorem

K_5 is not planar.

Proof.

(By picture) □

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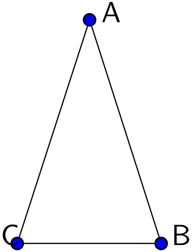

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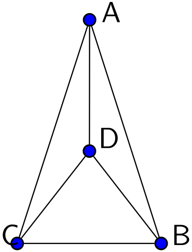

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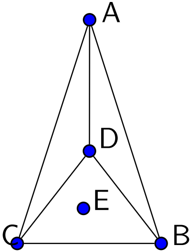

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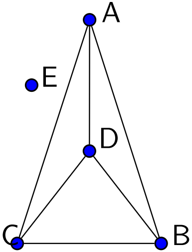

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Cliques are not planar

Theorem

K_5 is not planar.

Proof.

(By picture)
A few details missing:

- Is it OK to only use straight lines? (Yes)
- Actually, doesn't matter: cycles are Jordan curves
- Outside face symmetric to inside face. . .

□

Faces

Definition

A **face** of a plane drawing of a planar graph is a maximal connected region not intersecting any edge.

Intuitively: the border of a face is a cycle, such that one side of the cycle has no vertex.

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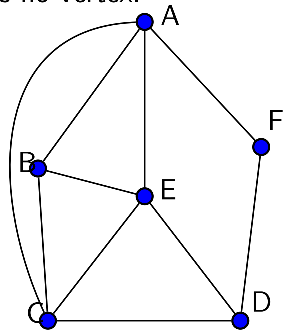

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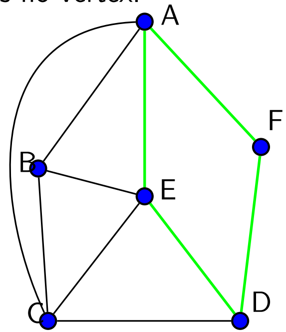

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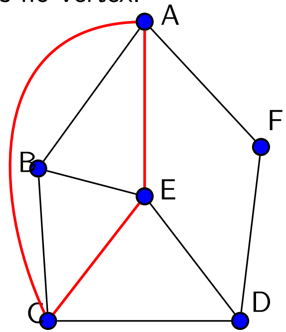

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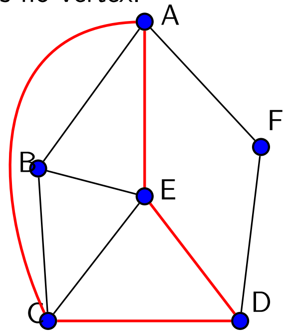

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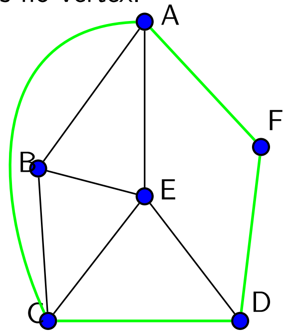

Faces

Definition

A **face** of a plane drawing of a planar graph is a maximal connected region not intersecting any edge.

Intuitively: the border of a face is a cycle, such that one side of the cycle has no vertex.
In other words:

- A face is defined by a cycle (walk) that is **not** a separator of the graph.

Euler's formula

Theorem

For all planar drawings with f faces of a connected *planar graph with n vertices and m edges we have:*

$$n + f = m + 2$$

Proof.

□

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Proof.

By induction on m

- If $m = 1$, since G is connected, G is a K_2
- $\Rightarrow n = 2, f = 1$, good.

□

Euler's formula

Theorem

For all planar drawings with f faces of a connected *planar graph with n vertices and m edges we have:*

$$n + f = m + 2$$

Proof.

By induction on m

- If G is a tree, then $m = n - 1, f = 1$ good.

□

Euler's formula

Theorem

For all planar drawings with f faces of a connected *planar graph with n vertices and m edges we have:*

$$n + f = m + 2$$

Proof.

By induction on m

- Suppose G contains a cycle, m edges, statement true for connected graphs with $m - 1$ edges.
- Remove an edge e of a cycle, $G - e$ has:
  - $n' = n$, $m' = m - 1$, $f' = f - 1$
  - (IH) $n' + f' = m' + 2$
  - $\Rightarrow n + f - 1 = m - 1 + 2$, good.

□

Euler's formula

Theorem

For all planar drawings with f *faces of a* connected *planar graph with* n *vertices and* m *edges we have:*

$$n + f = m + 2$$

Proof.

By induction on m
Key step:
Removing an edge merges two faces into one.

□

Euler's formula: Applications

Theorem

All planar embeddings of a planar graph G have the same number of faces.

Theorem

For all planar graphs $m \leq 3n - 6$

Corollary

For all planar graphs $\delta \leq 5$

Corollary

K_5 *is not planar*

Planar graphs are sparse

Theorem

For all planar graphs $m \leq 3n - 6$

Proof.

- Suppose G is planar, has maximum number of edges.

Planar graphs are sparse

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Proof.

- Suppose G is planar, has maximum number of edges.
- Then, every face is a C_3.

Planar graphs are sparse

Theorem

For all planar graphs $m \leq 3n - 6$

Proof.

- Suppose G is planar, has maximum number of edges.
- Then, every face is a C_3.
- $\Rightarrow 3f = 2m$, because every edge appears in two faces.
- $n + f = m + 2 \Rightarrow n = \frac{m}{3} + 2 \Rightarrow m = 3n - 6$

□

Characterization of Planar Graphs

Forbidden Subgraphs?

- Reminder: G is bipartite if and only if G has no odd cycle subgraph.
- Would be nice to have a similar theorem for planar graphs!
  - Among other reasons: recognition in NP$\cap$coNP.
- Example: G is planar if and only if G has no K_5 subgraph.

Forbidden Subgraphs?

- Reminder: G is bipartite if and only if G has no odd cycle subgraph.
- Would be nice to have a similar theorem for planar graphs!
  - Among other reasons: recognition in NP∩coNP.
- Example: G is planar if and only if G has no K_5 subgraph.
- This is false because:
  - K_5 is not the only minimal non-planar graph.
  - Subgraphs are too restricted an operation for planarity.

Can we “fix” this?

Minimal Non-Planar Graphs I

Theorem

$K_{3,3}$ is non-planar.

Proof.

(Proof by picture)

□

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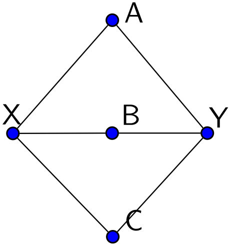

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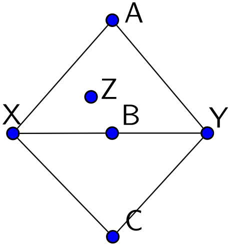

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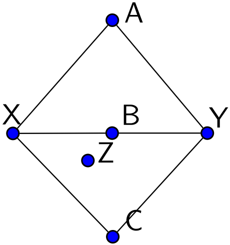

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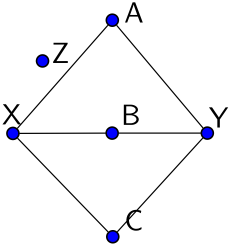

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Minimal Non-Planar Graphs II

Definition

A **sub-division** of an edge $e = xy$ is the operation that removes from the graph e and replaces it with a new vertex z and the edges xz, yz.

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If G is planar, its sub-divisions are planar.
If G is non-planar, its sub-divisions are non-planar.

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If G is planar, its sub-divisions are planar.
If G is non-planar, its sub-divisions are non-planar.

Idea:

- Forbidding a subgraph of G cannot precisely characterize planarity: sub-dividing edges destroys most subgraphs.
- What if we try to forbid a **sub-division** instead of a subgraph?

Kuratowski's Theorem

Theorem

G is planar if and only if G does not contain a subgraph that is a sub-division of K_5 or $K_{3,3}$.

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- K_5 and $K_{3,3}$ are the only minimal non-planar graphs for the sub-division operation!
- Planarity is in NP∩coNP
  - Counter-certificate (which always exists): a sub-divided copy of K_5 or $K_{3,3}$.

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- K_5 and $K_{3,3}$ are the only minimal non-planar graphs for the sub-division operation!
- Planarity is in NP∩coNP
  - Counter-certificate (which always exists): a sub-divided copy of K_5 or $K_{3,3}$.
- Actually, Planarity is in P (but algorithm too complicated for this course).

Coloring of Planar Graphs

6-color Theorem

Minimum number of colors that is sufficient to color any planar graph?

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Proof.

$\delta(G) \leq 5$, run First-Fit with this vertex last. □

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Minimum number of colors that is sufficient to color any planar graph?

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Theorem

There exists a planar graph G with $\chi(G) \geq 4$.

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K_4 is planar. □

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Minimum number of colors that is sufficient to color any planar graph?

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Proof.

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K_4 is planar. □

Correct answer is 4, 5, or 6. . .

The 5-color Theorem

Theorem

If G is planar, then $\chi(G) \leq 5$.

Proof.

- G has a vertex v of degree at most 5.
- By induction $G - v$ can be 5-colored.

□

The 5-color Theorem

Theorem

If G is planar, then $\chi(G) \leq 5$.

Proof.

- G has a vertex v of degree at most 5.
- By induction $G - v$ can be 5-colored.
- If 5-coloring of $G - v$ uses ≤ 4 colors in neighbors of v, done!

□

The 5-color Theorem

Theorem

If G is planar, then $\chi(G) \leq 5$.

Proof.

- G has a vertex v of degree at most 5.
- By induction $G - v$ can be 5-colored.
- Suppose G has neighbors $x_1, \ldots, x_5$ (in clockwise order) with distinct respective colors $\{1, \ldots, 5\}$ in $G - v$.
- Let $G_{1,3}$ be the graph induced by colors $1, 3$.
  - If x_1, x_3 in distinct components of $G_{1,3}$, flip colors $1, 3$ in component of x_1, done!
  - Otherwise, $x_1 \to x_3$ path in $G_{1,3}$ plus v form a cycle that separates x_2 from x_4. Flip colors $2, 4$ in component of $G_{2,4}$ that contains x_2, done!

□

The 4-color Theorem

Theorem

If G is planar, then $\chi(G) \leq 4$.

The 4-color Theorem

Theorem

If G is planar, then $\chi(G) \leq 4$.

- Conjectured already in 19th century.
- Several incorrect proofs published!
- First "real" proof: Appel and Haken 1976
  - Controversially, first computer-assisted proof.
  - Was later found to contain small (fixable) errors.
- Simplified proof: Robertson, Sanders, Seymour, and Thomas, 1996
  - Still computer-assisted!
  - Gives $O(n^2)$ algorithm for producing 4-coloring.
- Computer-assisted proofs have now also been computer-verified.

What about 3 colors?

Theorem

Deciding if a planar graph can be colored with 3 colors is NP-complete.

What about 3 colors?

Theorem

Deciding if a planar graph can be colored with 3 colors is NP-complete.

- Deciding if $\chi(G) \leq 2$ is easy (bipartiteness).
- Deciding if $\chi(G) \leq 4$ is easy (always Yes).
- Deciding if $\chi(G) \leq 3$ is hard!

What about 3 colors?

Theorem

Deciding if a planar graph can be colored with 3 colors is NP-complete.

- Deciding if $\chi(G) \leq 2$ is easy (bipartiteness).
- Deciding if $\chi(G) \leq 4$ is easy (always Yes).
- Deciding if $\chi(G) \leq 3$ is hard!

- Actually, the vast majority of interesting problems are (unfortunately) still hard on planar graphs.