Graph Theory: Lecture 5

Coloring

Michael Lampis

October 18, 2024

Coloring

Definition

For a graph $G = (V, E)$ a **proper coloring** of G with k colors is a partition of V into k **independent** sets $V_1, \ldots, V_k$.

Definition

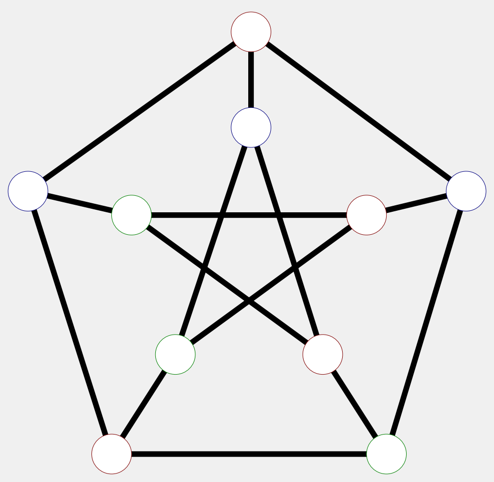
The **chromatic number** of G, denoted $\chi(G)$ is the smallest k for which G admits a proper k-coloring.

Definition

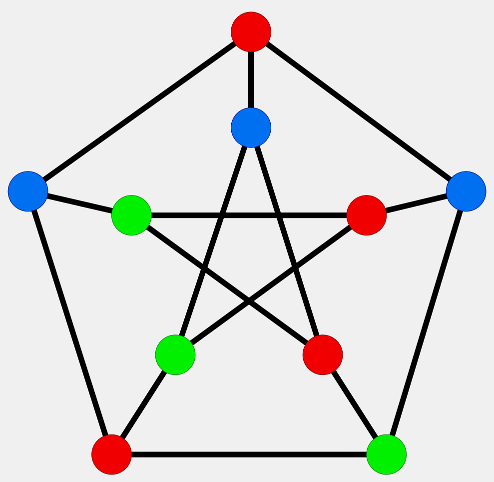
In the GRAPH COLORING problem we are given a graph G and are asked to determine $\chi(G)$.

Note: $\chi(G) \leq 2$ if and only if G is bipartite.

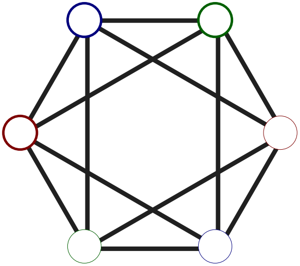
Examples



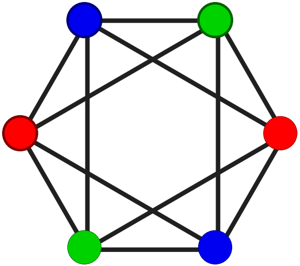
Examples



Examples



Examples



Colorings and Cliques

Theorem

For all graphs G, $\chi(G) \geq \omega(G)$.

(Reminder: $\omega(G)$: size of maximum clique)

Colorings and Cliques

Theorem

For all graphs G, $\chi(G) \geq \omega(G)$.

(Reminder: $\omega(G)$: size of maximum clique)
This is **not** an equivalence!

- Construct a graph with $\chi(G) \geq \omega(G) + 1$

Colorings and Cliques

Theorem

For all graphs G, $\chi(G) \geq \omega(G)$.

(Reminder: $\omega(G)$: size of maximum clique)

This is **not** an equivalence!

- Construct a graph with $\chi(G) \geq \omega(G) + 1$
  - C_5
- Construct a graph with $\chi(G) \gg \omega(G)$

Colorings and Cliques

Theorem

For all graphs G, $\chi(G) \geq \omega(G)$.

(Reminder: $\omega(G)$: size of maximum clique)

This is **not** an equivalence!

- Construct a graph with $\chi(G) \geq \omega(G) + 1$
  - C_5
- Construct a graph with $\chi(G) \gg \omega(G)$
  - Will see a construction later...

Colorings and Cliques

Theorem

For all graphs G, $\chi(G) \geq \omega(G)$.

(Reminder: $\omega(G)$: size of maximum clique)

This is **not** an equivalence!

- Construct a graph with $\chi(G) \geq \omega(G) + 1$
  - C_5
- Construct a graph with $\chi(G) \gg \omega(G)$
  - Will see a construction later...
- Graph Coloring is in NP

Colorings and Cliques

Theorem

For all graphs G, $\chi(G) \geq \omega(G)$*.*

(Reminder: $\omega(G)$: size of maximum clique)

This is **not** an equivalence!

- Construct a graph with $\chi(G) \geq \omega(G) + 1$
  - C_5
- Construct a graph with $\chi(G) \gg \omega(G)$
  - Will see a construction later...
- Graph Coloring is in NP
  - Certificate is the coloring
- ...but not in coNP (unless NP=coNP)

Colorings and Independent Sets

Theorem

For all graphs G, $\chi(G) \geq n/\alpha(G)$.

(Reminder: $\alpha(G)$: size of maximum independent set)

Colorings and Independent Sets

Theorem

For all graphs G, $\chi(G) \geq n/\alpha(G)$.

(Reminder: $\alpha(G)$: size of maximum independent set)

Proof.

- Suppose that $\chi < \frac{n}{\alpha}$ and that the color classes are $V_1, V_2, \ldots, V_\chi$.
- Since each V_i is an independent set, $|V_i| \leq \alpha$.
- Then $|V| = \sum_{i\in[\chi]} |V_i| \leq \chi\alpha < n$, contradiction!

□

Colorings and Degrees

Theorem

For all graphs G, $\chi(G) \leq \Delta(G) + 1$.

(Reminder: $\Delta(G)$: maximum degree)

Colorings and Degrees

Theorem

For all graphs G, $\chi(G) \leq \Delta(G) + 1$.

(Reminder: $\Delta(G)$: maximum degree)

Proof.

First-Fit algorithm:

- Consider vertices in some order $v_1, v_2, \ldots, v_n$
- For each v_i assign to it the minimum color in $\{1, 2, \ldots\}$ that is not yet used by its neighbors.

Colorings and Degrees

Theorem

For all graphs G, $\chi(G) \leq \Delta(G) + 1$.

(Reminder: $\Delta(G)$: maximum degree)

Proof.

First-Fit algorithm:

- Consider vertices in some order $v_1, v_2, \ldots, v_n$
- For each v_i assign to it the minimum color in $\{1, 2, \ldots\}$ that is not yet used by its neighbors.
- **Worst case:** the (at most Δ) neighbors of v_i use all colors in $\{1, \ldots, \Delta\}$, so v_i gets color $\Delta + 1$.

□

Colorings and Degrees

Theorem

For all graphs G*,* $\chi(G) \leq \Delta(G) + 1$*.*

(Reminder: $\Delta(G)$: maximum degree)

Proof.

First-Fit algorithm:

- Consider vertices in some order $v_1, v_2, \ldots, v_n$
- For each v_i assign to it the minimum color in $\{1, 2, \ldots\}$ that is not yet used by its neighbors.
- **Worst case:** the (at most Δ) neighbors of v_i use all colors in $\{1, \ldots, \Delta\}$, so v_i gets color $\Delta + 1$.

□

Can this be improved?

The First-Fit Algorithm

Lemma

There exists a graph G and an ordering of $V(G)$ such that **First-Fit** *uses* **strictly more** *than $\chi(G)$ colors.*

The First-Fit Algorithm

Lemma

There exists a graph G and an ordering of $V(G)$ such that **First-Fit** *uses* **strictly more** *than $\chi(G)$ colors.*

NB: If the above were false, then we would have a P-time algorithm for GRAPH COLORING!

The First-Fit Algorithm

Lemma

There exists a graph G and an ordering of $V(G)$ such that **First-Fit** *uses* **strictly more** *than $\chi(G)$ colors.*

Example: P_4, with ordering $1, 4, 2, 3$.

The First-Fit Algorithm

Lemma

There exists a graph G and an ordering of $V(G)$ such that **First-Fit** *uses* **strictly more** *than $\chi(G)$ colors.*

Lemma

For all G, there exists an ordering of $V(G)$ such that **First-Fit** *uses $\chi(G)$ colors.*

The First-Fit Algorithm

Lemma

There exists a graph G and an ordering of $V(G)$ such that **First-Fit** *uses* **strictly more** *than $\chi(G)$ colors.*

Lemma

For all G, there exists an ordering of $V(G)$ such that **First-Fit** *uses $\chi(G)$ colors.*

Proof.

Let $V_1, V_2, \ldots, V_k$ be a proper coloring of G with k colors. We can use an ordering $V_1 \prec V_2 \prec \ldots V_k$.

□

Coloring and Degeneracy

Definition

The **degeneracy** of G is the minimum δ^* such that all subgraphs of G contain a vertex of degree at most δ^*.

Theorem

For all G we have $\chi(G) \leq \delta^*(G) + 1$.

Coloring and Degeneracy

Definition

The **degeneracy** of G is the minimum δ^* such that all subgraphs of G contain a vertex of degree at most δ^*.

Theorem

For all G we have $\chi(G) \leq \delta^*(G) + 1$.

Note that $\delta^* \leq \Delta$, because all subgraphs contain a vertex of degree Δ, so this is **better** than previous theorem.

Coloring and Degeneracy

Definition

The **degeneracy** of G is the minimum δ^* such that all subgraphs of G contain a vertex of degree at most δ^*.

Theorem

For all G we have $\chi(G) \leq \delta^*(G) + 1$.

Proof.

By induction:

- Suppose statement true for G with $\leq n - 1$ vertices.
- G contains a vertex of degree $\leq \delta^*$, call it v.
- $\delta^*(G - v) \leq \delta^*(G)$, so by IH $G - v$ can be colored with δ^* colors.
- Use the smallest available color for v to extend this coloring to G.

□

Brooks' Theorem

Upper bounds on chromatic number

Theorem

For all G we have $\chi(G) \leq \delta^*(G) + 1$.

Theorem

For all graphs G, $\chi(G) \leq \Delta(G) + 1$.

Upper bounds on chromatic number

Theorem

For all G we have $\chi(G) \leq \delta^*(G) + 1$.

Theorem

For all graphs G, $\chi(G) \leq \Delta(G) + 1$.

Because $\delta^* \leq \Delta$, the first theorem implies the second.

Upper bounds on chromatic number

Theorem

For all G we have $\chi(G) \leq \delta^*(G) + 1$.

Theorem

For all graphs G, $\chi(G) \leq \Delta(G) + 1$.

Are these theorems tight?

Upper bounds on chromatic number

Theorem

For all G we have $\chi(G) \leq \delta^*(G) + 1$.

Theorem

For all graphs G, $\chi(G) \leq \Delta(G) + 1$.

Are these theorems tight?

- Cliques K_n have $\Delta = \delta^* = n - 1$, $\chi = n$

Upper bounds on chromatic number

Theorem

For all G we have $\chi(G) \leq \delta^*(G) + 1$.

Theorem

For all graphs G, $\chi(G) \leq \Delta(G) + 1$.

Are these theorems tight?

- Cliques K_n have $\Delta = \delta^* = n - 1$, $\chi = n$
- Stars $K_{1,n}$ have $\Delta = n$, $\delta^* = 1$, $\chi = 2$

Upper bounds on chromatic number

Theorem

For all G we have $\chi(G) \leq \delta^*(G) + 1$.

Theorem

For all graphs G, $\chi(G) \leq \Delta(G) + 1$.

Are these theorems tight?

- Cliques K_n have $\Delta = \delta^* = n - 1$, $\chi = n$
- Stars $K_{1,n}$ have $\Delta = n$, $\delta^* = 1$, $\chi = 2$
- Cycles C_{2n+1} have $\Delta = 2$, $\delta^* = 2$, $\chi = 3$

Upper bounds on chromatic number

Theorem

For all G we have $\chi(G) \leq \delta^*(G) + 1$.

Theorem

For all graphs G, $\chi(G) \leq \Delta(G) + 1$.

Are these theorems tight?

- Cliques K_n have $\Delta = \delta^* = n - 1$, $\chi = n$
- Stars $K_{1,n}$ have $\Delta = n$, $\delta^* = 1$, $\chi = 2$
- Cycles C_{2n+1} have $\Delta = 2$, $\delta^* = 2$, $\chi = 3$

Actually, cliques and odd cycles are **the only** cases where the second theorem is tight!

Brooks' Theorem

Theorem

For all G such that G is not a clique or an odd cycle, $\chi(G) \leq \Delta(G)$.

Brooks' Theorem

Theorem

For all G such that G is not a clique or an odd cycle, $\chi(G) \leq \Delta(G)$.

Proof.

Proof by minimal counter-example:

- Suppose G is the smallest (non-clique, non-odd-cycle) graph for which $\chi(G) \geq \Delta(G) + 1$.
- We will reach a contradiction, assuming that the theorem is true for all graphs with fewer vertices.
- 3 cases:
  - G has a cut vertex
  - G has a vertex cut of size 2
  - G is 3-connected
- Assume throughout that $\Delta \geq 3$ and G is Δ-regular (why?)

Cut Vertex Case

Assumption: G has $\chi(G) \geq \Delta(G) + 1$ and G has a cut vertex x.

Proof.

- Let $G_1, \ldots, G_k$ be the components of $G - v$
- Let $G_i' = G_i + v$ (where we keep all edges of G incident on v in G_i).
- G_i' is Δ-colorable, wlog v has color 1

Cut Vertex Case

Assumption: G has $\chi(G) \geq \Delta(G) + 1$ and G has a cut vertex x.

Proof.

- Let $G_1, \ldots, G_k$ be the components of $G - v$
- Let $G_i' = G_i + v$ (where we keep all edges of G incident on v in G_i).
- G_i' is Δ-colorable, wlog v has color 1
  - v has degree at most $\Delta - 1$ in G_i'
  - If G_i' is a clique, then $\chi(G_i') \leq \Delta$
  - If G_i' is an odd cycle, $\chi(G_i') = 3 \leq \Delta$
  - Otherwise G_i' is not a counter-example, so $\chi(G_i') \leq \Delta$.

Cut Vertex Case

Assumption: G has $\chi(G) \geq \Delta(G) + 1$ and G has a cut vertex x.

Proof.

- Let $G_1, \ldots, G_k$ be the components of $G - v$
- Let $G_i' = G_i + v$ (where we keep all edges of G incident on v in G_i).
- G_i' is Δ-colorable, wlog v has color 1
  - v has degree at most $\Delta - 1$ in G_i'
  - If G_i' is a clique, then $\chi(G_i') \leq \Delta$
  - If G_i' is an odd cycle, $\chi(G_i') = 3 \leq \Delta$
  - Otherwise G_i' is not a counter-example, so $\chi(G_i') \leq \Delta$.
- Gluing colorings together we get a Δ-coloring of G, contradiction.

□

Cut of Size 2

Assumption: G has $\chi(G) \geq \Delta(G) + 1$ and G has a cut set $\{x, y\}$.

Proof.

- Let $G_1, \ldots, G_k$ be the components of $G - \{x, y\}$
- Let $G_i' = G_i + \{x, y\}$ (where we keep all edges of G incident on x, y in G_i).
- Furthermore, add to G_i' the edge xy (if it is not already there).
- G_i' is Δ-colorable, wlog x, y have colors $1, 2$

Cut of Size 2

Assumption: G has $\chi(G) \geq \Delta(G) + 1$ and G has a cut set $\{x, y\}$.

Proof.

- Let $G_1, \ldots, G_k$ be the components of $G - \{x, y\}$
- Let $G_i' = G_i + \{x, y\}$ (where we keep all edges of G incident on x, y in G_i).
- Furthermore, add to G_i' the edge xy (if it is not already there).
- G_i' is Δ-colorable, wlog x, y have colors $1, 2$
  - x, y have degree at most $\Delta - 1$ in G_i'
  - Adding the edge xy makes their degrees at most Δ
  - If G_i' is a clique, then $\chi(G_i') \leq \Delta + 1$ (!!!)
  - If G_i' is an odd cycle, $\chi(G_i') = 3 \leq \Delta$
  - Otherwise G_i' is not a counter-example, so $\chi(G_i') \leq \Delta$.

Cut of Size 2

Assumption: G has $\chi(G) \geq \Delta(G) + 1$ and G has a cut set $\{x, y\}$.

Proof.

- Let $G_1, \ldots, G_k$ be the components of $G - \{x, y\}$
- Let $G_i' = G_i + \{x, y\}$ (where we keep all edges of G incident on x, y in G_i).
- Furthermore, add to G_i' the edge xy (if it is not already there).
- G_i' is Δ-colorable, wlog x, y have colors $1, 2$
  - x, y have degree at most $\Delta - 1$ in G_i'
  - Adding the edge xy makes their degrees at most Δ
  - If G_i' is a clique, then $\chi(G_i') \leq \Delta + 1$ (!!!)
  - If G_i' is an odd cycle, $\chi(G_i') = 3 \leq \Delta$
  - Otherwise G_i' is not a counter-example, so $\chi(G_i') \leq \Delta$.
- Gluing colorings together we get a Δ-coloring of G, contradiction.

□

Cut of Size 2 – Missing case

Assumption: G has $\chi(G) \geq \Delta(G) + 1$ and G has a cut set $\{x, y\}$.

Proof.

- Let $G_1, \ldots, G_k$ be the components of $G - \{x, y\}$
- Sticky case: G_1 is a clique of size $\Delta - 1$, x, y are adjacent to all of G_1.

Cut of Size 2 – Missing case

Assumption: G has $\chi(G) \geq \Delta(G) + 1$ and G has a cut set $\{x, y\}$.

Proof.

- Let $G_1, \ldots, G_k$ be the components of $G - \{x, y\}$
- Sticky case: G_1 is a clique of size $\Delta - 1$, x, y are adjacent to all of G_1.
    - There exists only one other component G_2, x, y have degree 1 in G_2.
    - Since $\Delta \geq 3$, there is a coloring of $G_2 + \{x, y\}$ where x, y receive the same color.
    - This coloring can be extended to a Δ-coloring of G.

□

All Cuts of size at least 3

Assumption: G has $\chi(G) \geq \Delta(G) + 1$ and G is 3-connected.

Proof.

- Since G is not a clique, there exist $x, y \in V$ with $xy \notin E$.
- In fact, there exist such x, y with distance 2 (common neighbor z)

All Cuts of size at least 3

Assumption: G has $\chi(G) \geq \Delta(G) + 1$ and G is 3-connected.

Proof.

- Since G is not a clique, there exist $x, y \in V$ with $xy \notin E$.
- In fact, there exist such x, y with distance 2 (common neighbor z)
  - Consider the pair x, y with minimum distance. If the shortest path has length ≥ 3, x with the third vertex of the path make a better pair.

All Cuts of size at least 3

Assumption: G has $\chi(G) \geq \Delta(G) + 1$ and G is 3-connected.

Proof.

- Since G is not a clique, there exist $x, y \in V$ with $xy \notin E$.
- In fact, there exist such x, y with distance 2 (common neighbor z)
  - Consider the pair x, y with minimum distance. If the shortest path has length ≥ 3, x with the third vertex of the path make a better pair.
- $\{x, y\}$ is not a separator. If G' is G where we remove all edges incident on x, y, except xz, yz, G' is connected.
- Run First-Fit on G for ordering $x, y, V \setminus \{x, y, z\}, z$, where $V \setminus \{x, y, z\}$ is ordered in decreasing distance from z in G'.

All Cuts of size at least 3

Assumption: G has $\chi(G) \geq \Delta(G) + 1$ and G is 3-connected.

Proof.

- Since G is not a clique, there exist $x, y \in V$ with $xy \notin E$.
- In fact, there exist such x, y with distance 2 (common neighbor z)
  - Consider the pair x, y with minimum distance. If the shortest path has length ≥ 3, x with the third vertex of the path make a better pair.
- $\{x, y\}$ is not a separator. If G' is G where we remove all edges incident on x, y, except xz, yz, G' is connected.
- Run First-Fit on G for ordering $x, y, V \setminus \{x, y, z\}, z$, where $V \setminus \{x, y, z\}$ is ordered in decreasing distance from z in G'.
  - x, y receive color 1
  - All vertices of $V \setminus \{x, y, z\}$ have an uncolored neighbor when considered $\Rightarrow$ at most Δ colors used in this part
  - z has two neighbors with identical color $\Rightarrow$ receives color $\leq \Delta$.

Mycielski

Colorings and Cliques (again)

Theorem

For all graphs G, $\chi(G) \geq \omega(G)$.

Colorings and Cliques (again)

Theorem

For all graphs G, $\chi(G) \geq \omega(G)$.

This inequality is **NOT** tight in general!

- Otherwise we would have $\textsc{Coloring} \in \mathsf{NP} \cap \mathsf{coNP}$

We will construct a triangle-free graph with arbitrarily large chromatic number.

Mycielski Construction

Definition

If $G = (V, E)$ is a graph with $V = \{v_1, \ldots, v_n\}$, then G^* is the graph obtained by:

- $V(G^*) = V \cup U \cup \{w\}$, where $U = \{u_1, \ldots, u_n\}$
- $E(G^*) = E \cup \{v_i u_j, u_i v_j \mid v_i v_j \in E\} \cup \{wu_i \mid i \in [n]\}$

Mycielski Construction

Definition

If $G = (V, E)$ is a graph with $V = \{v_1, \ldots, v_n\}$, then G^* is the graph obtained by:

- $V(G^*) = V \cup U \cup \{w\}$, where $U = \{u_1, \ldots, u_n\}$
- $E(G^*) = E \cup \{v_i u_j, u_i v_j \mid v_i v_j \in E\} \cup \{w u_i \mid i \in [n]\}$

In words:

- For each v_i we add a new "copy" u_i adjacent to the neighbors of v_i.
- However, the u_i's are an independent set.
- We add a new vertex w adjacent to all other new vertices.

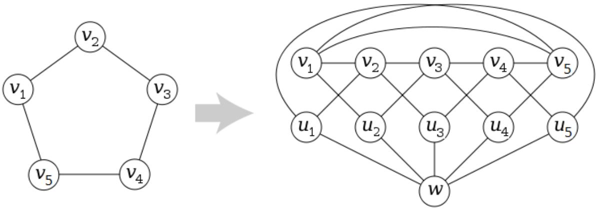
Mycielski Construction

Definition

If $G = (V, E)$ is a graph with $V = \{v_1, \ldots, v_n\}$, then G^* is the graph obtained by:

- $V(G^*) = V \cup U \cup \{w\}$, where $U = \{u_1, \ldots, u_n\}$
- $E(G^*) = E \cup \{v_i u_j, u_i v_j \mid v_i v_j \in E\} \cup \{wu_i \mid i \in [n]\}$

Example:



Mycielski Construction Works

Theorem

$\chi(G^*) = \chi(G) + 1.$

Theorem

If G has no triangle, then G^* has no triangle.

Mycielski Construction Works

Theorem

$\chi(G^*) = \chi(G) + 1.$

Theorem

If G has no triangle, then G^* has no triangle.

Proof.

- w cannot be in a triangle, as its neighbors are independent.
- u_i, u_j cannot be together in a triangle.
- If v_i, v_j, u_k is a triangle, v_i, v_j, v_k is also a triangle.

□

Mycielski Construction Works

Theorem

$\chi(G^*) = \chi(G) + 1$.

Theorem

If G has no triangle, then G^* has no triangle.

Proof.

- $\chi(G^*) \leq \chi(G) + 1$ is easy
- $\chi(G) \leq \chi(G^*) - 1$:
  - In an optimal coloring U is using $\chi(G^*) - 1$ colors
  - For $v_i \in V$ with color $\chi(G^*)$, assign it the color of u_i; keep the other colors of V intact.

□