Graph Theory: Lecture 4

Cuts, Connectivity, Flows

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Vertex and Edge Cuts

Definition

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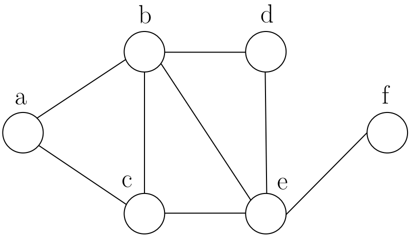
Variants:

- Edge separator $\rightarrow$ edge set whose removal disconnects G
- XY-separator $\rightarrow$ separator that disconnects vertex set X from vertex set Y (without intersecting X, Y)
- xy-separator $\rightarrow$ $\{x\}\{y\}$-separator

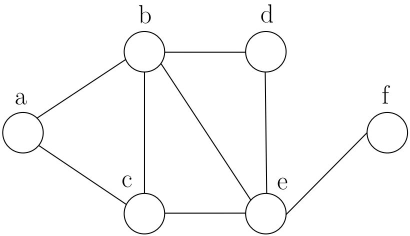
Special cases:

- If a separator $S = \{x\}$, then x is a **cut vertex**
- If an edge cut $H = \{e\}$, then e is a **cut edge** or **bridge**

Understanding the definitions

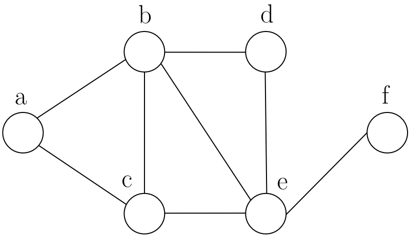


Understanding the definitions



- $\{b, e\}$ is a vertex cut (indeed an af-cut, and an $\{a, c\}\{d, f\}$-cut)
- e is a cut vertex
- $\{bd, be, ce\}$ is an edge cut
- ef is a bridge

Understanding the definitions



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Theorem

In a tree, every edge is a bridge, every non-leaf vertex is a cut vertex.

A basic inequality

Definition

- $\kappa(G)$ is the size of the smallest **vertex** separator of G.
- $\kappa'(G)$ is the size of the smallest **edge** separator of G.
- $\delta(G)$ is the minimum degree of G.

A basic inequality

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Proof.

- $\kappa(G) \leq \kappa'(G)$ because I can select one endpoint of each edge of a min edge cut
- $\kappa'(G) \leq \delta(G)$ because I can select all edges incident on a vertex v of min degree

□

A basic min/max relation

Definition

- $\kappa_{x,y}(G)$: min xy-vertex cut
- $\mathrm{vdp}_{x,y}(G)$: max number of vertex disjoint $x \to y$ paths

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Proof.

- Every separator must intersect all paths from x to y.
- If we have k disjoint such paths $\Rightarrow$ need $\geq k$ vertices to hit them.

□

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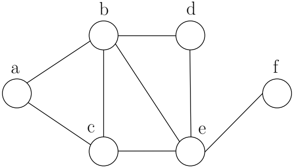

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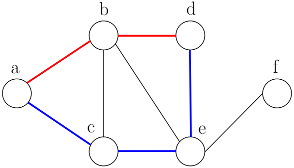

Main Message

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For all G, x, y *we have* $\kappa_{x,y}(G) \geq \mathrm{vdp}_{x,y}(G)$

Theorem

For all G, x, y *we have* $\kappa_{x,y}(G) = \mathrm{vdp}_{x,y}(G)$

- Result known as Menger's theorem
- Implies: decide min-cut value is in NP$\cap$coNP
- Also holds for edge cuts/edge-disjoint paths

The case $k = 2$

Whitney's theorem

Theorem

$G = (V, E)$ is 2-connected if and only if for each $u, v \in V$ there exist two disjoint $u \to v$ paths in G.

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Note: 2-connected $\Leftrightarrow$ min vertex cut has size ≥ 2

Whitney's theorem

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Proof.

$\Leftarrow$:

- If there exists two disjoint $u \to v$ paths, then all uv-separators have size at least 2.
- This is true for all u, v, so all separators have size at least 2.

□

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Theorem

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Proof.

$\Rightarrow$: all separators have size ≥ 2, $\Rightarrow \forall u, v$ we have two disjoint $u \to v$ paths
Proof by induction:

- Induction on distance of u, v
- If $d(u, v) = 1$ (i.e. $uv \in E$) $\Rightarrow$
- $\kappa' \geq \kappa \geq 2$, therefore $G - uv$ is connected
- $\Rightarrow$ there are two disjoint paths $u \to v$ in G

□

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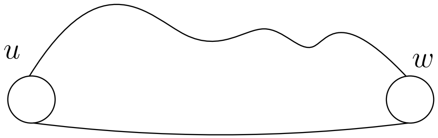
Proof.

$\Rightarrow$: all separators have size ≥ 2, $\Rightarrow \forall u, v$ we have two disjoint $u \to v$ paths
Inductive step:

- $d(u, v) = k$, let w be last vertex of $u \to v$ path of length k
- $d(u, w) = k - 1$
- Inductive hypothesis: exist two disjoint $u \to w$ paths

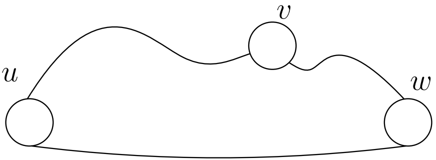
□

Whitney's theorem (proof)



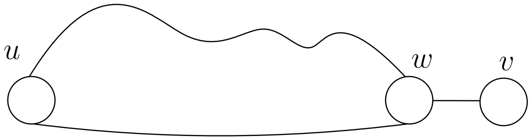
- We have two disjoint paths $u \to w$
- Need two disjoint paths $u \to v$

Whitney's theorem (proof)



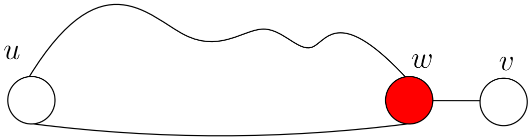
- If v is in one of the two paths, DONE!

Whitney's theorem (proof)



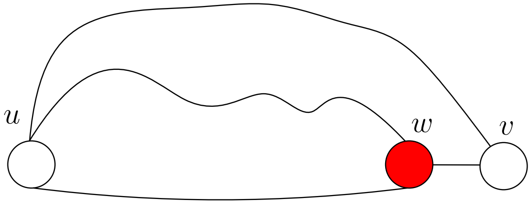
- Otherwise, recall that v is adjacent to w

Whitney's theorem (proof)



- $G - w$ is connected (as $\kappa(G) \geq 2$)
- There exists $u \to v$ path in $G - w$

Whitney's theorem (proof)



- If this path is disjoint from other two, DONE!

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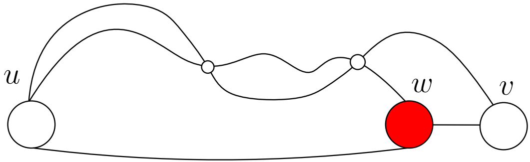

- If this path is disjoint from **one of the** other two, DONE!

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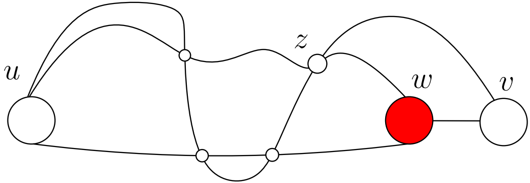

Otherwise:

- z is last common vertex of new path and two old paths
- Construct two $u \to v$ paths:
  - Take first old path up to z, then continue with new path to v
  - Take second old path up to w, continue to v

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(Menger 1927) If x, y are vertices of $G = (V, E)$ with $xy \notin E$, then the minimum size of an xy-cut is equal to the maximum number of vertex-disjoint $x \to y$-paths.

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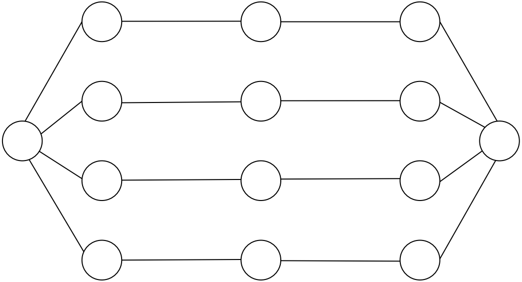

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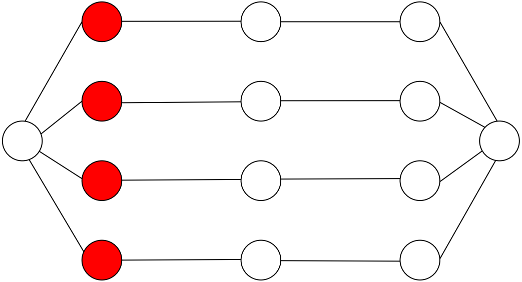

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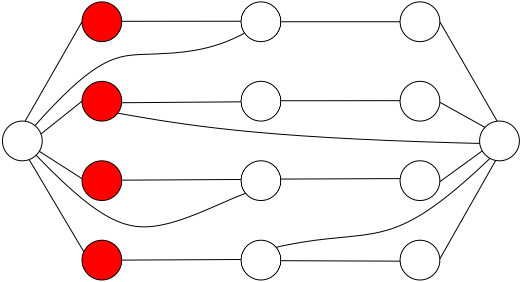

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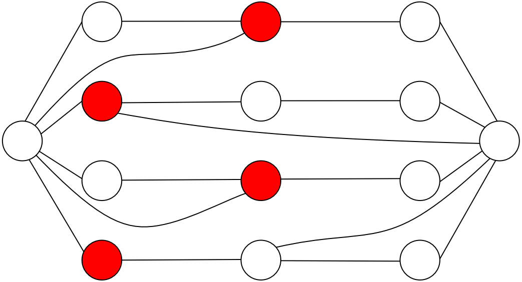

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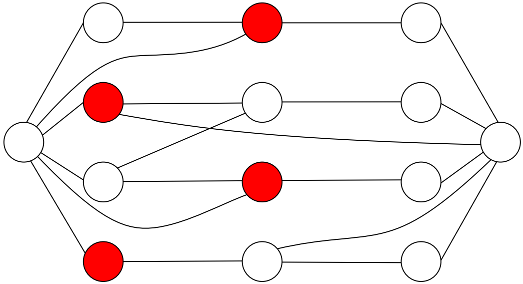

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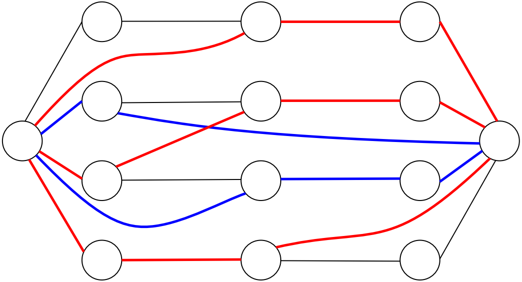
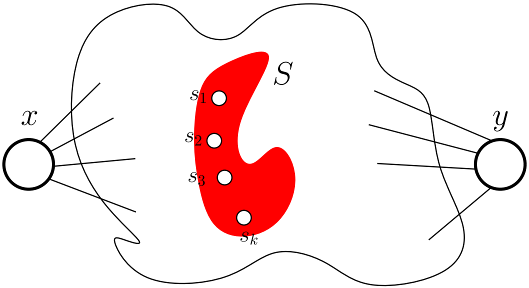
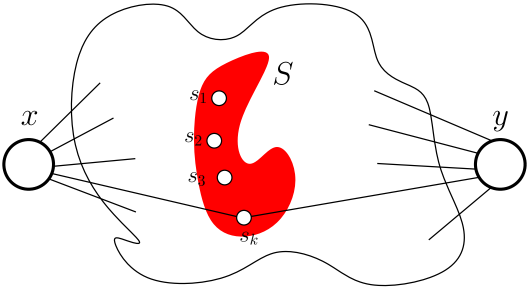


Menger's Theorem (proof)



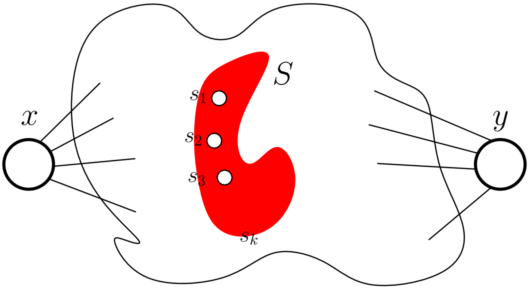
- S is a min xy-separator of size k
- Will prove by induction that there are k $x \to y$ disjoint paths

Menger's Theorem (proof)



- If S contains a common neighbor of x, y, say s_k, remove it

Menger's Theorem (proof)



- $S - s_k$ is a min-cut of the new graph of size $k - 1$
- Inductive hypothesis $\Rightarrow$ exist $k - 1$ disjoint paths

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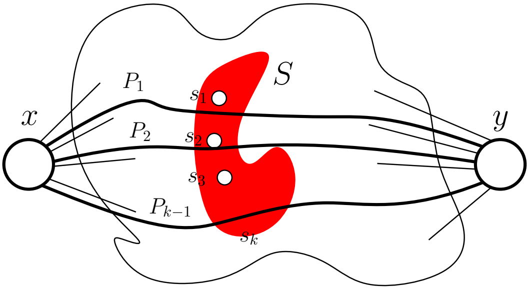

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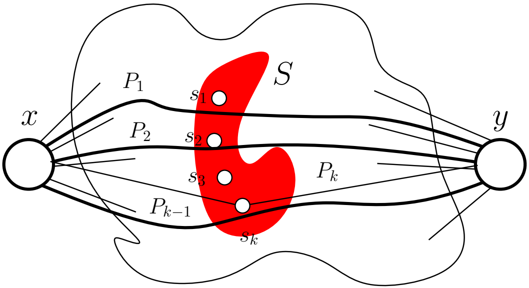
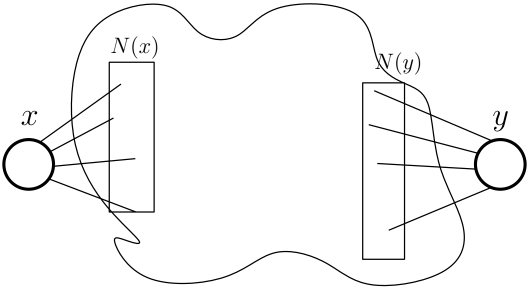


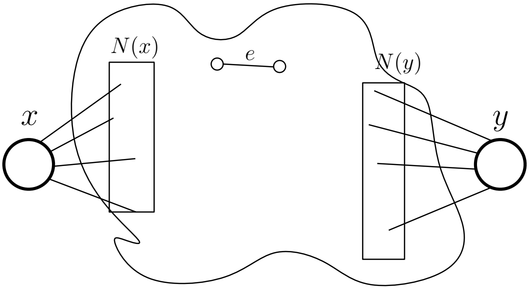
- Inductive hypothesis $\Rightarrow$ exist $k-1$ disjoint paths
- Together with $x \to s_k \to y$ we have k disjoint paths

Menger's Theorem (proof)



- Suppose then that $N(x) \cap N(y) = \emptyset$
- Let e be an edge not incident on x, y
- Remove it to get $G - e$

Menger's Theorem (proof)



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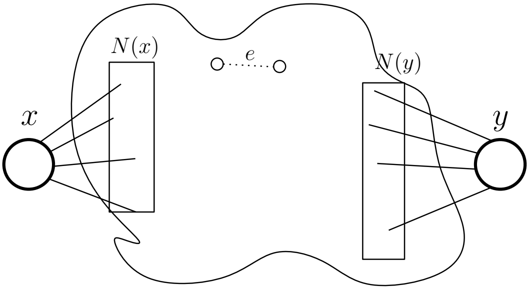

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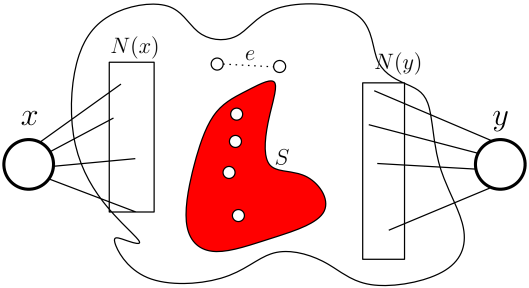
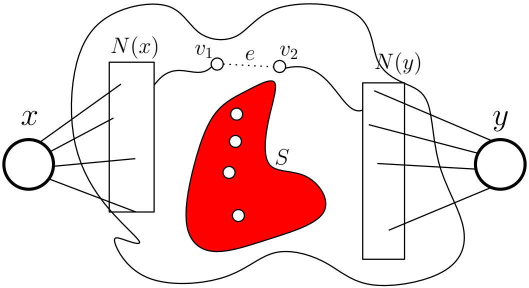


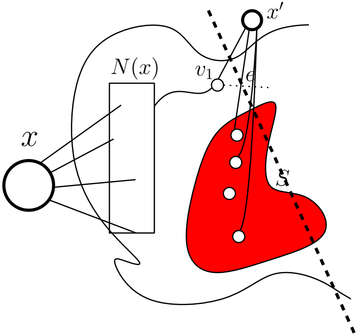
- S is a min xy-separator of $G - e$
  - $|S| \geq k$?
  - $|S| = k - 1$?
  - $|S| \leq k - 2$?

Menger's Theorem (proof)



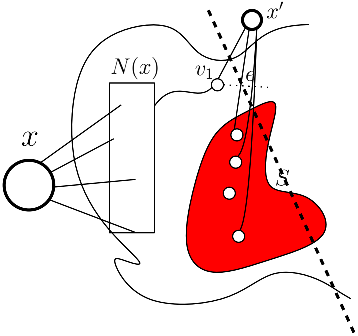
- Interesting case: $|S| = k - 1$
- $\Rightarrow$ $|S|$ is not an xy-separator in G
- $\Rightarrow$ there is an $x \to y$ path in $G - S$ which must use e

Menger's Theorem (proof)



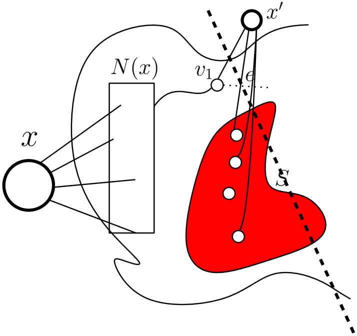
- Consider "left" part and add new vertex x' connected to $S \cup \{v_1\}$
- This graph is smaller than $G \Rightarrow$ inductive hypothesis
- Min xx'-separator has size $\geq k$

Menger's Theorem (proof)



- Min xx'-separator has size $\geq k$
  - If there exists xx'-separator of size $k-1$, this would also be an xy-separator in G

Menger's Theorem (proof)



- Inductive hypothesis $\Rightarrow$ k disjoint paths from $x \to x'$
- $\Rightarrow$ k disjoint paths from $x \to S \cup \{v_1\}$
- Symmetrically... $\Rightarrow$ k disjoint paths from $y \to S \cup \{v_2\}$
- $\Rightarrow$ k disjoint $x \to y$ paths

Déjà vu ?

Reminder: Kőnig's theorem

Theorem

If G is bipartite, then $\mathrm{mm}(G) = \mathrm{vc}(G)$.

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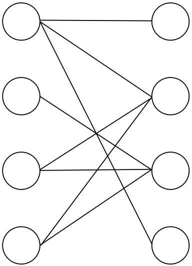

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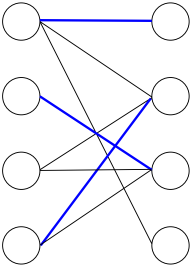

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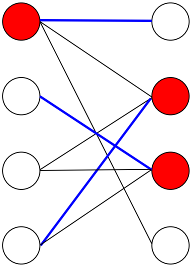

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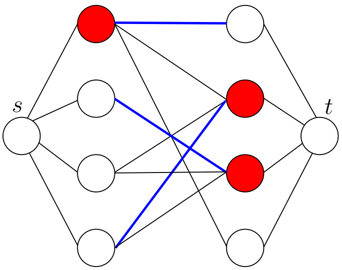

- Edges of matching $\rightarrow$ vertex-disjoint $s \rightarrow t$ paths
- Vertices of vertex cover $\rightarrow$ st separator

Reminder: Kőnig's theorem

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Conclusion:

Menger's theorem $\Rightarrow$ Kőnig's theorem

Edge Cuts

Line Graphs

Definition

For a graph $G = (V, E)$, the **line graph** $L(G)$ is defined as follows:

- $V(L(G)) = E$
- For $e_1, e_2 \in E$ we have $e_1 e_2 \in E(L(G))$ iff $e_1 e_2$ is a path of length 2.

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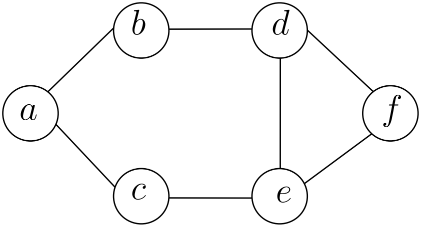

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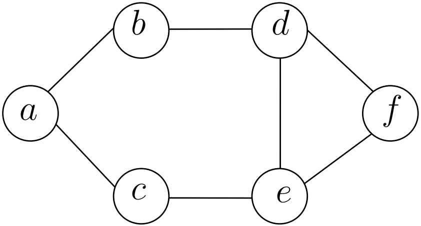

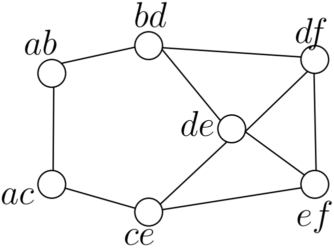

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Line graphs of:

- Paths? Cycles?
- Trees? Bipartite graphs?

Menger's Theorem for Edges

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*(Menger 1927) If x, y are vertices of $G = (V, E)$ with $x \neq y$, then the minimum size of an xy-***edge cut** *is equal to the maximum number of* **edge***-disjoint $x \to y$-paths.*

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Proof.

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Menger's Theorem for Edges

Theorem

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- Edge xy-cuts in $G' \Leftrightarrow$ Vertex $(xx')(yy')$-cuts in $L(G')$
- Apply Menger's theorem to $L(G')$

□

Max-Flow/Min-Cut

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  - Both vertex and edge versions...
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  - If edge has capacity c, there can be at most c paths using it.
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Theorem

In a directed weighted digraph $G = (V, E)$ for $s, t \in V$ we have that the maximum **flow** *from s to t is equal to the capacity of the minimum st-cut.*