Graph Theory: Lecture 3

Bipartite Graphs

Michael Lampis

September 17, 2024

Bipartite Graphs and Matchings

Bipartite Graphs

Definition

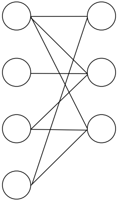
A graph $G = (V, E)$ is **bipartite** if V can be partitioned into two independent sets A, B.

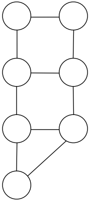
Bipartite Graphs

Definition

A graph $G = (V, E)$ is **bipartite** if V can be partitioned into two independent sets A, B.

Examples:



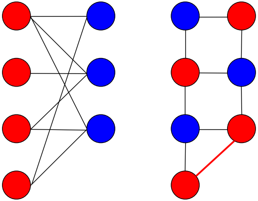


Bipartite Graphs

Definition

A graph $G = (V, E)$ is **bipartite** if V can be partitioned into two independent sets A, B.

Examples:



Bipartite Graphs

Definition

A graph $G = (V, E)$ is **bipartite** if V can be partitioned into two independent sets A, B.

Relation with:

- Paths?
- Cycles?
- Trees?
- Cliques?

Bipartite Graphs

Definition

A graph $G = (V, E)$ is **bipartite** if V can be partitioned into two independent sets A, B.

Definition

A graph $G = (V, E)$ is k-colorable if V can be partitioned into k independent sets.

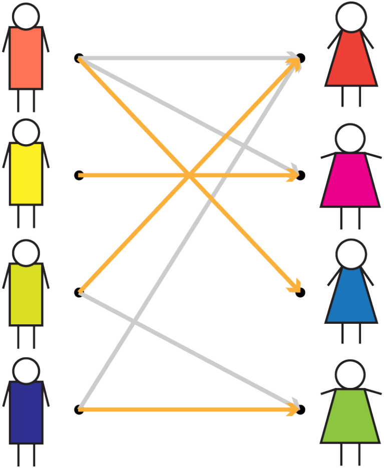
- Graph Coloring is a notorious graph problem.
- Deciding if a graph is bipartite is the special case for $k = 2$.

Motivation

Why care about bipartite graphs?

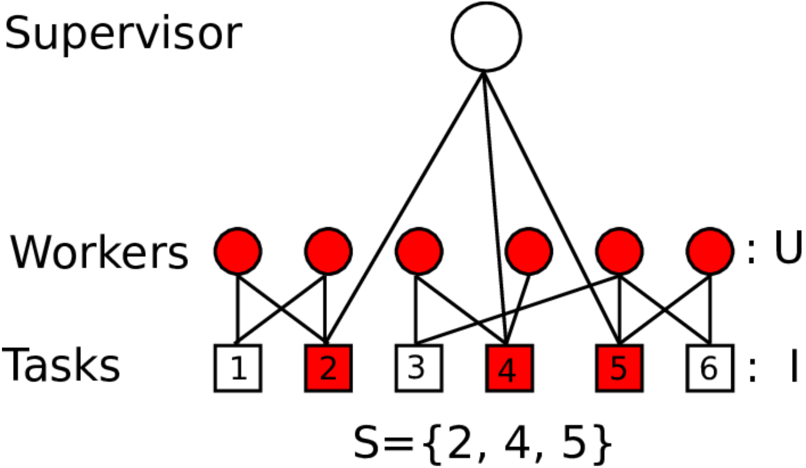
Motivation

Why care about bipartite graphs?



Motivation

Why care about bipartite graphs?



Motivation

Why care about bipartite graphs?

- Come up naturally when we have two groups of elements and only care about relations from one group to the other.
- What structure arises from this restriction?
- Can we use it algorithmically?

Basic Facts

Characterization

Theorem

A graph G is bipartite if and only if G contains no odd cycles as subgraphs.

Characterization

Theorem

A graph G is bipartite if and only if G contains no odd cycles as subgraphs.

Proof.

Bipartite $\Rightarrow$ No odd cycle:

- Easy: C_{2k+1} is **not** bipartite, bipartiteness is preserved by subgraphs, so if $C_{2k+1} \subseteq G$, then G is not bipartite.

□

Characterization

Theorem

A graph G is bipartite if and only if G contains no odd cycles as subgraphs.

Proof.

Bipartite $\Leftarrow$ No odd cycle:

- Let x be a vertex of G, V_1 vertices at odd distance from x, $V_2 = V \setminus V$, distances at even distance from x.
- Claim: V_1, V_2 are independent sets.
  - Take $y, z \in V_1$, shortest $x \to y, x \to z$ paths.
  - Let x' be the last common vertex of these paths.
  - $x' \to y, x' \to z$ paths have the same parity.
  - If $yz \in E$ we have an odd cycle, contradiction!

□

Recognition Complexity

Problem

Given G, decide if G is bipartite.

Recognition Complexity

Problem

Given G, decide if G is bipartite.

- Is in NP

Recognition Complexity

Problem

Given G, decide if G is bipartite.

- Is in NP
  - Certificate is the bipartition.
- Is in coNP

Recognition Complexity

Problem

Given G, decide if G is bipartite.

- Is in NP
  - Certificate is the bipartition.
- Is in coNP
  - Counter-certificate is an odd cycle.

Recognition Complexity

Problem

Given G, decide if G is bipartite.

- Is in NP
  - Certificate is the bipartition.
- Is in coNP
  - Counter-certificate is an odd cycle.
- $\Rightarrow$ is in NP$\cap$coNP
- In fact is in P

Proof.

Algorithm (for connected graph):

- Initially, pick a vertex and place it in A
- While $\exists$ undecided v with decided neighbor, color v

Correctness? □

Matchings

Matchings

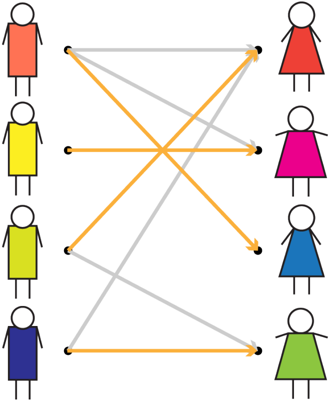
Definition

A **matching** in a graph $G = (V, E)$ is a set $M \subseteq E$ such that no two elements of M share a vertex.

Matchings

Definition

A **matching** in a graph $G = (V, E)$ is a set $M \subseteq E$ such that no two elements of M share a vertex.



Matchings

Definition

A **matching** in a graph $G = (V, E)$ is a set $M \subseteq E$ such that no two elements of M share a vertex.

Definition

A matching M is **perfect** if all vertices are incident to an edge of M.

Definition

A matching M is **maximum** if all sets of edges of size $|M| + 1$ or more contain two edges incident on the same vertex.

Note: These definitions are given for *general* graphs, but we mostly care about bipartite graphs.

Augmenting Paths

Definition

Given $G = (V, E)$ and a matching M, an **alternating path** is a path made up of edges $e_1, e_2, \ldots, e_k$ such that for all $i \in [k-1]$ we have $e_i \in M \Leftrightarrow e_{i+1} \notin M$.

Definition

An **augmenting** path is an alternating path where the first and last vertices are not incident to edges of M.

Augmenting Paths

Definition

Given $G = (V, E)$ and a matching M, an **alternating path** is a path made up of edges $e_1, e_2, \ldots, e_k$ such that for all $i \in [k-1]$ we have $e_i \in M \Leftrightarrow e_{i+1} \notin M$.

Definition

An **augmenting** path is an alternating path where the first and last vertices are not incident to edges of M.

Theorem (Berge 1957)

A matching M is maximum if and only if no augmenting path exists.

Augmenting Paths – Proof

Theorem (Berge 1957)

A matching M is maximum if and only if no augmenting path exists.

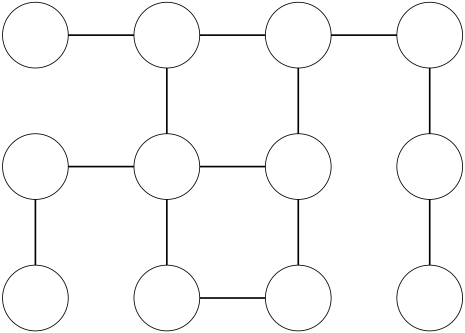
Augmenting Paths – Proof

Theorem (Berge 1957)

A matching M is maximum if and only if no augmenting path exists.

Proof.

Augmenting Path $\Rightarrow$ M is not maximum



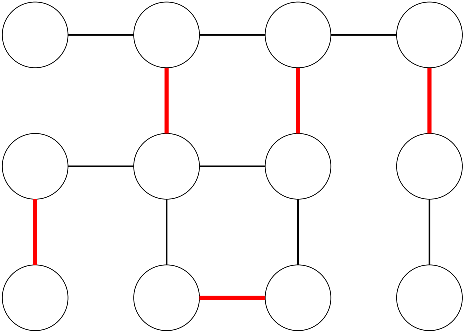
Augmenting Paths – Proof

Theorem (Berge 1957)

A matching M is maximum if and only if no augmenting path exists.

Proof.

Augmenting Path $\Rightarrow$ M is not maximum



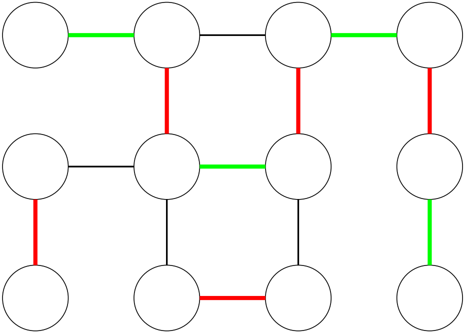
Augmenting Paths – Proof

Theorem (Berge 1957)

A matching M is maximum if and only if no augmenting path exists.

Proof.

Augmenting Path $\Rightarrow$ M is not maximum



Augmenting Paths – Proof

Theorem (Berge 1957)

A matching M is maximum if and only if no augmenting path exists.

Proof.

Augmenting Path $\Leftarrow$ M is not maximum

- Let M' be a matching larger than M.
- $M \cup M'$ induces a graph of maximum degree 2
- $\Rightarrow$ union of paths and cycles
- $\Rightarrow$ one of the paths must be augmenting to give $|M'| > |M|$

□

Perfect Matchings

Problem

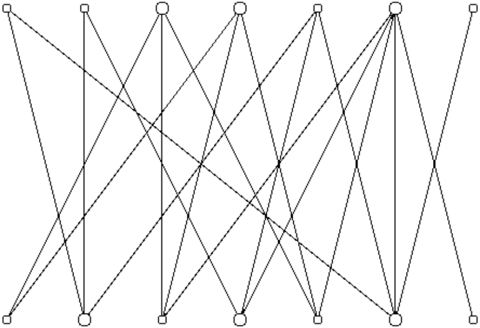
Given a bipartite graph $G = (A, B, E)$, decide if G has a perfect matching.

Perfect Matchings

Problem

Given a bipartite graph $G = (A, B, E)$, decide if G has a perfect matching.

NB: Take a moment to convince yourself that this is not trivial...

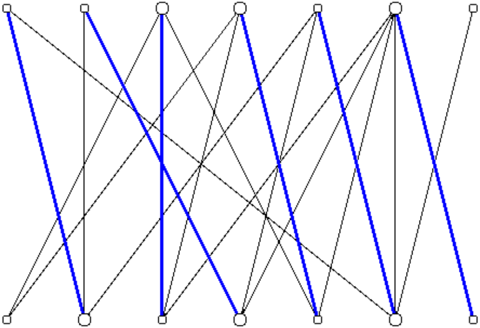


Perfect Matchings

Problem

Given a bipartite graph $G = (A, B, E)$, decide if G has a perfect matching.

NB: Take a moment to convince yourself that this is not trivial...

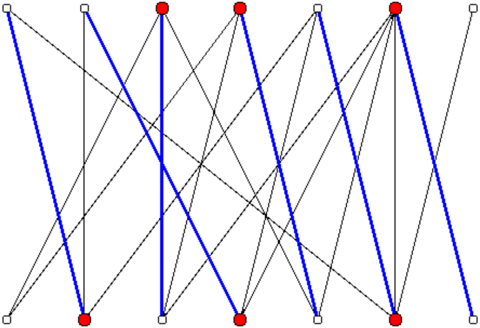


Perfect Matchings

Problem

Given a bipartite graph $G = (A, B, E)$, decide if G has a perfect matching.

NB: Take a moment to convince yourself that this is not trivial...



Perfect Matchings

Problem

Given a bipartite graph $G = (A, B, E)$, decide if G has a perfect matching.

Theorem (Hall 1935)

A bipartite graph $G = (A, B, E)$ contains a perfect matching if and only if for all $S \subseteq A$ we have $|N(S)| \geq |S|$.

Perfect Matchings

Problem

Given a bipartite graph $G = (A, B, E)$, decide if G has a perfect matching.

Theorem (Hall 1935)

A bipartite graph $G = (A, B, E)$ contains a perfect matching if and only if for all $S \subseteq A$ we have $|N(S)| \geq |S|$.

- Establishes that BIPARTITE PERFECT MATCHING $\in$ NP$\cap$coNP (why?)
- We will in fact show that it is in P...

Hall's Theorem

Theorem (Hall 1935)

A bipartite graph $G = (A, B, E)$ contains a perfect matching if and only if for all $S \subseteq A$ we have $|N(S)| \geq |S|$.

Hall's Theorem

Theorem (Hall 1935)

A bipartite graph $G = (A, B, E)$ contains a perfect matching if and only if for all $S \subseteq A$ we have $|N(S)| \geq |S|$.

Proof.

Perfect matching $\Rightarrow \forall S$ we have $|N(S)| \geq |S|$

- Easy: all elements of S have a distinct neighbor in the matching.

□

Hall's Theorem

Theorem (Hall 1935)

A bipartite graph $G = (A, B, E)$ *contains a perfect matching if and only if for all* $S \subseteq A$ *we have* $|N(S)| \geq |S|$.

Proof.

Perfect matching $\Leftarrow \forall S$ we have $|N(S)| \geq |S|$

- Suppose that max matching M is not perfect.
- Take an unmatched vertex u
- Find all vertices reachable from u via alternating paths
- M maximum $\Rightarrow$ cannot reach another unmatched vertex
- u plus reachable vertices give S with $|N(S)| < |S|$

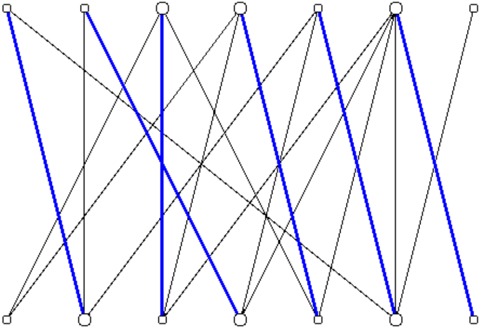
□

Hall's Theorem

Theorem (Hall 1935)

A bipartite graph $G = (A, B, E)$ contains a perfect matching if and only if for all $S \subseteq A$ we have $|N(S)| \geq |S|$.

Proof.

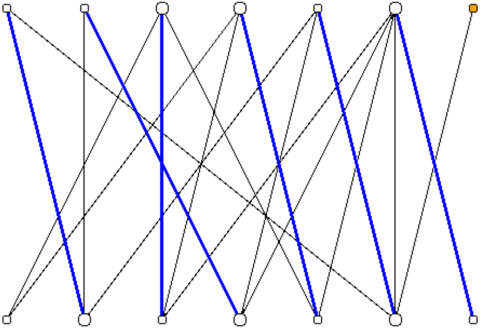


Hall's Theorem

Theorem (Hall 1935)

A bipartite graph $G = (A, B, E)$ contains a perfect matching if and only if for all $S \subseteq A$ we have $|N(S)| \geq |S|$.

Proof.

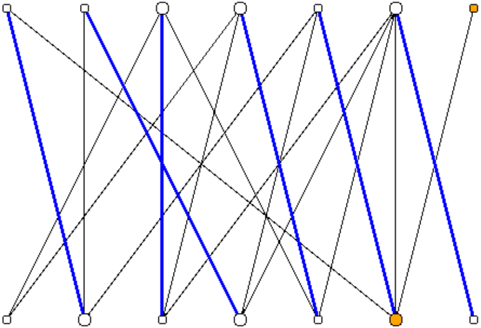


Hall's Theorem

Theorem (Hall 1935)

A bipartite graph $G = (A, B, E)$ contains a perfect matching if and only if for all $S \subseteq A$ we have $|N(S)| \geq |S|$.

Proof.

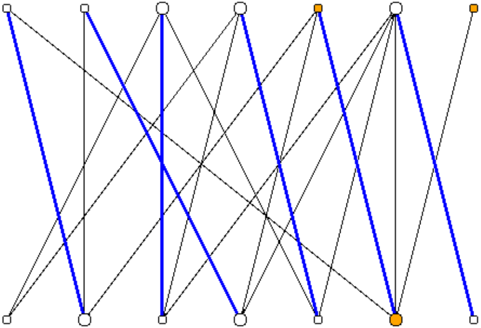


Hall's Theorem

Theorem (Hall 1935)

A bipartite graph $G = (A, B, E)$ contains a perfect matching if and only if for all $S \subseteq A$ we have $|N(S)| \geq |S|$.

Proof.

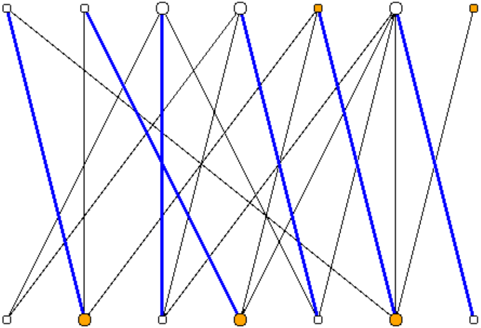


Hall's Theorem

Theorem (Hall 1935)

A bipartite graph $G = (A, B, E)$ contains a perfect matching if and only if for all $S \subseteq A$ we have $|N(S)| \geq |S|$.

Proof.

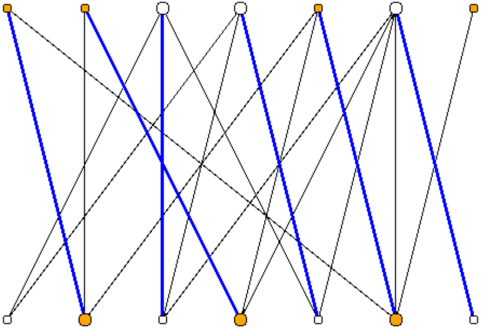


Hall's Theorem

Theorem (Hall 1935)

A bipartite graph $G = (A, B, E)$ contains a perfect matching if and only if for all $S \subseteq A$ we have $|N(S)| \geq |S|$.

Proof.



The Hungarian Method

Theorem

There is a polynomial-time algorithm for computing the maximum matching of a bipartite graph.

The Hungarian Method

Theorem

There is a polynomial-time algorithm for computing the maximum matching of a bipartite graph.

1. $G = (A, B, E)$ and start with an empty matching M
2. For each unmatched $u \in A$ attempt to find an augmenting path starting at u.
   - If successful, augment M, goto 2.
   - If unsuccessful for all u, declare M maximum.

The Hungarian Method

Theorem

There is a polynomial-time algorithm for computing the maximum matching of a bipartite graph.

1. $G = (A, B, E)$ and start with an empty matching M
2. For each unmatched $u \in A$ attempt to find an augmenting path starting at u.
   - If successful, augment M, goto 2.
   - If unsuccessful for all u, declare M maximum.

Correctness:

- If step 2 can be performed correctly, algorithm runs in polynomial-time.
- Correctness follows from Berge's theorem.

Finding Augmenting Paths

Lemma

Given $G = (A, B, E)$, matching M, unmatched $u \in A$, we can in polynomial time decide if there is an augmenting path starting at u.

Finding Augmenting Paths

Lemma

Given $G = (A, B, E)$, matching M, unmatched $u \in A$, we can in polynomial time decide if there is an augmenting path starting at u.

Algorithm:

1. $X \subseteq A$, $Y \subseteq B$ vertices reachable by alternating path from u. Initially, $X = \{u\}$ and $Y = \emptyset$.
2. Repeat n times, for all edges e
   1. If $e = ab$, $e \notin M$, $a \in X$ and $b \notin Y$, set $Y := Y \cup \{b\}$.
   2. If $e = ab$, $e \in M$, $b \in Y$ and $a \notin X$, set $X := X \cup \{a\}$.
3. If Y contains an unmatched vertex (of B), say Yes, otherwise No.

Finding Augmenting Paths

Lemma

Given $G = (A, B, E)$, matching M, unmatched $u \in A$, we can in polynomial time decide if there is an augmenting path starting at u.

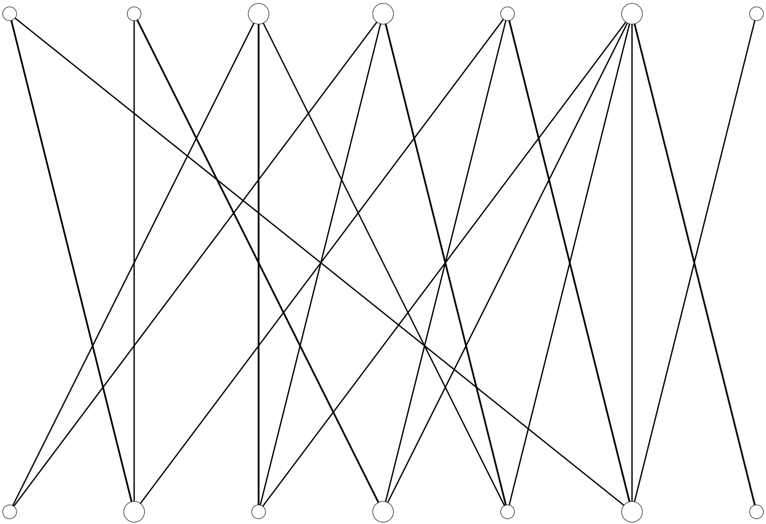
Algorithm:

1. $X \subseteq A$, $Y \subseteq B$ vertices reachable by alternating path from u. Initially, $X = \{u\}$ and $Y = \emptyset$.
2. Repeat n times, for all edges e
   1. If $e = ab$, $e \notin M$, $a \in X$ and $b \notin Y$, set $Y := Y \cup \{b\}$.
   2. If $e = ab$, $e \in M$, $b \in Y$ and $a \notin X$, set $X := X \cup \{a\}$.
3. If Y contains an unmatched vertex (of B), say Yes, otherwise No.

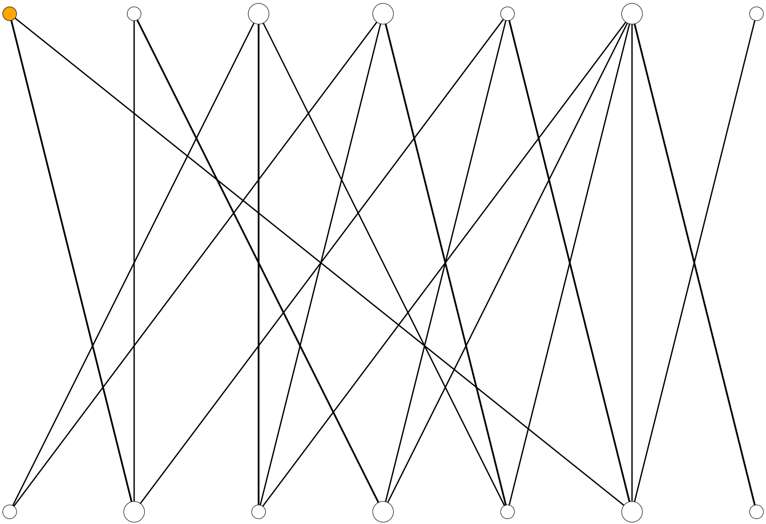
Correctness:

- G is bipartite, so X may contain only matched vertices. Paths $u \to X$ have even length, paths $u \to Y$ have odd length.

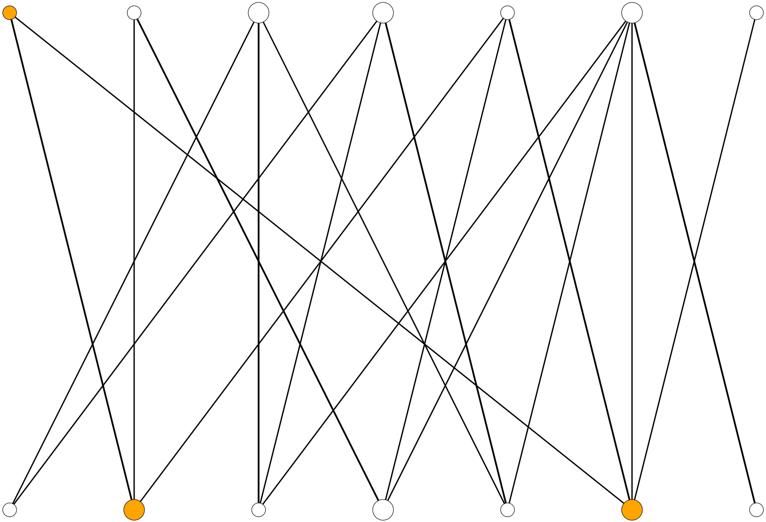
Hungarian Method: Example



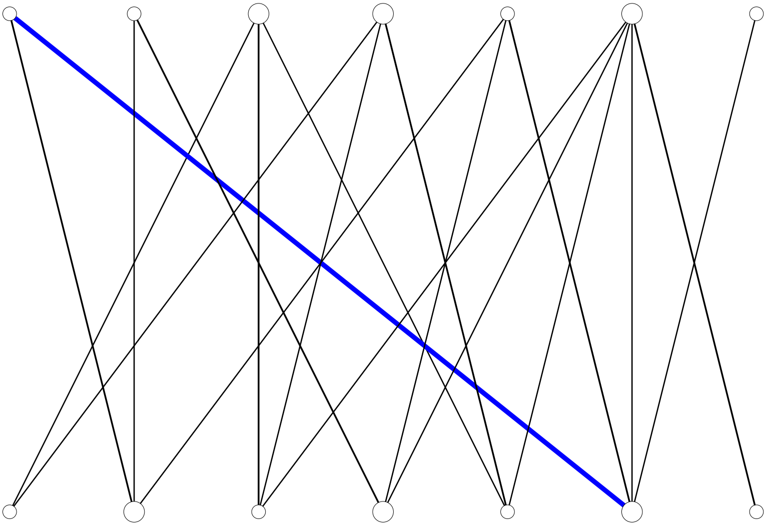
Hungarian Method: Example



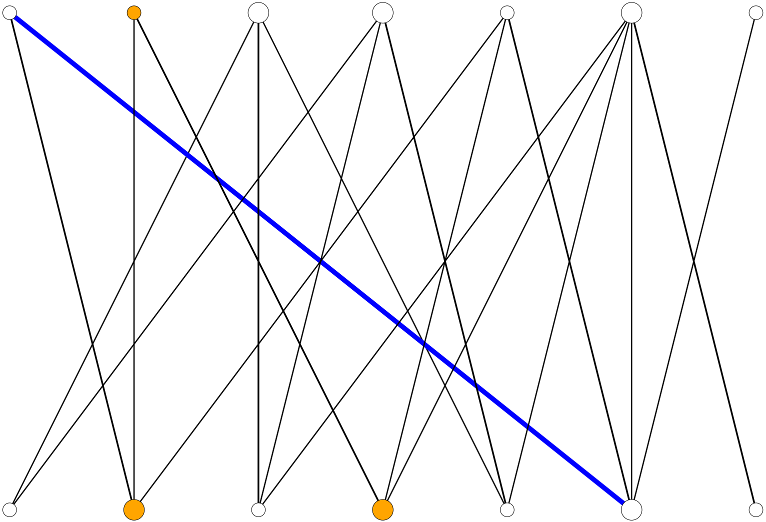
Hungarian Method: Example



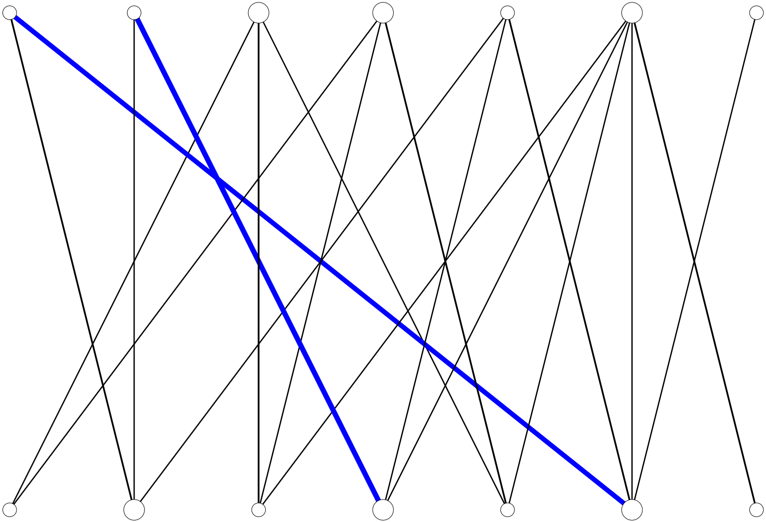
Hungarian Method: Example



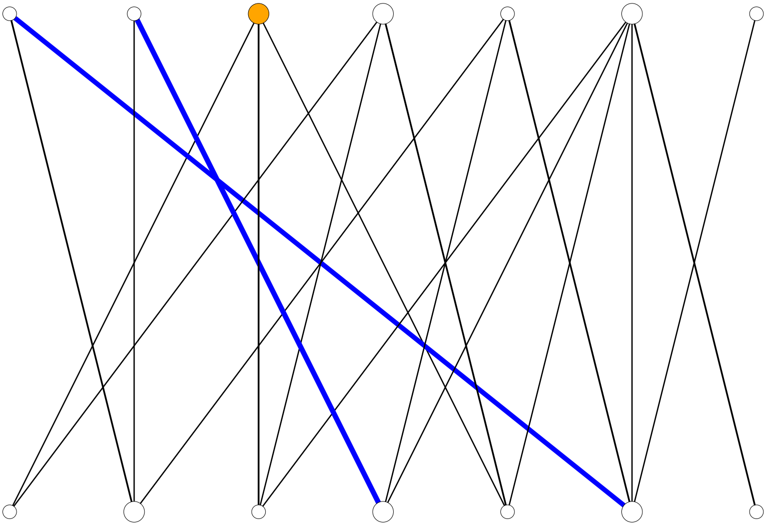
Hungarian Method: Example



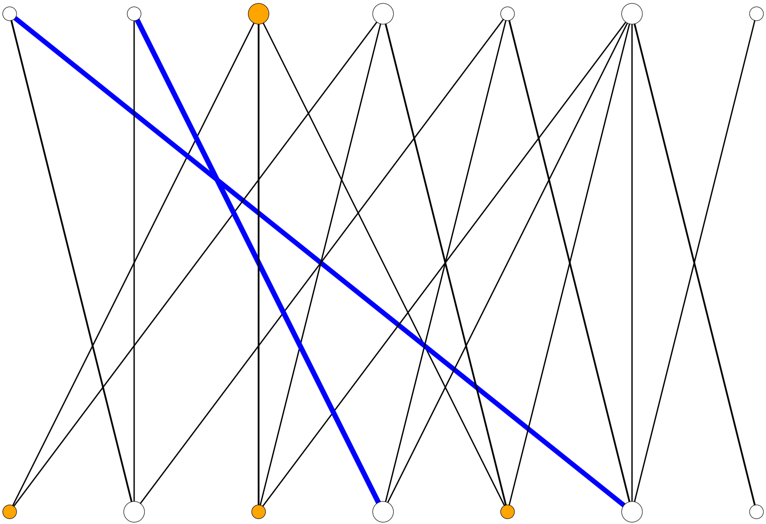
Hungarian Method: Example



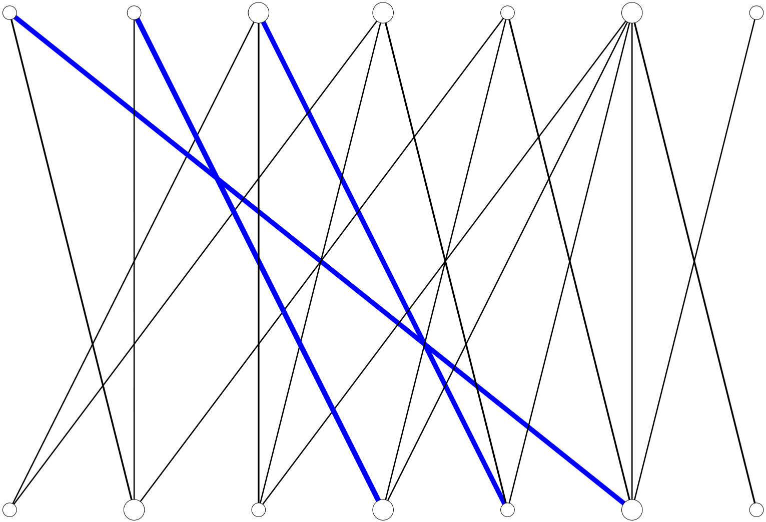
Hungarian Method: Example



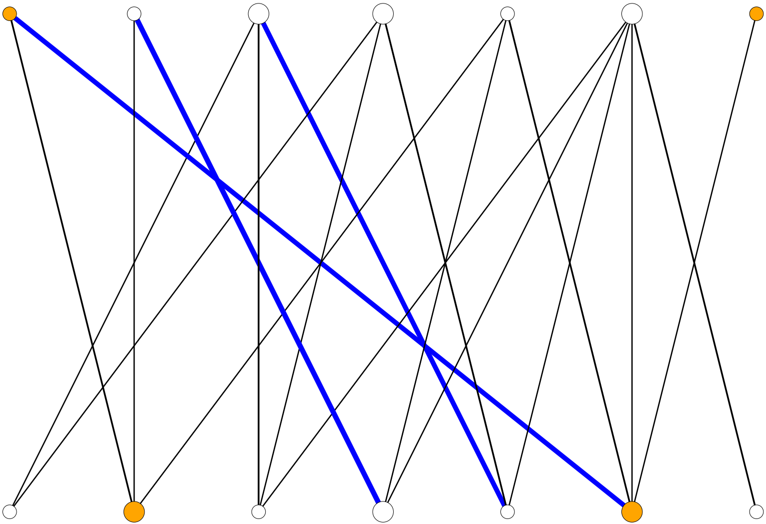
Hungarian Method: Example



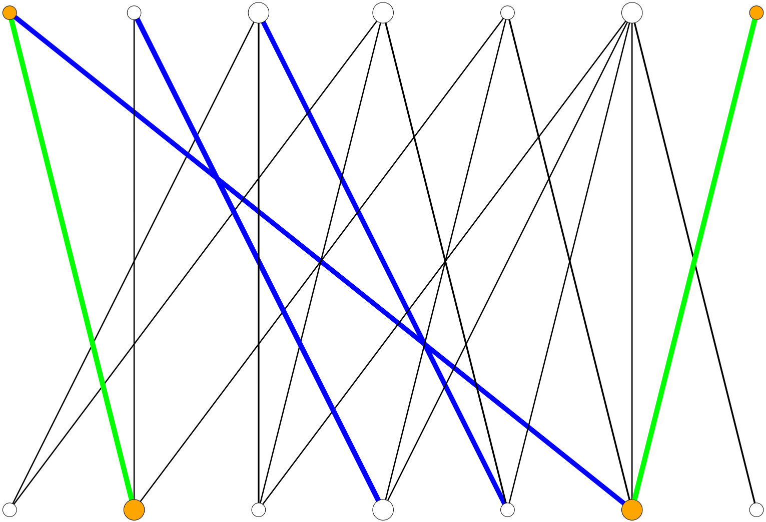
Hungarian Method: Example



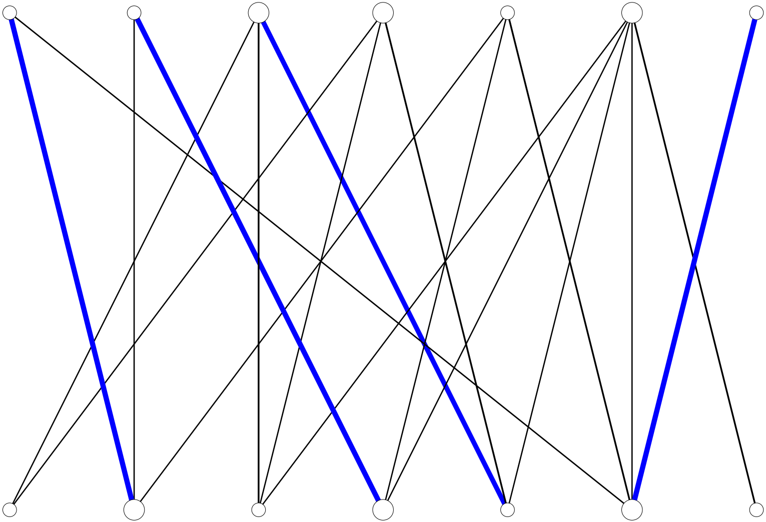
Hungarian Method: Example



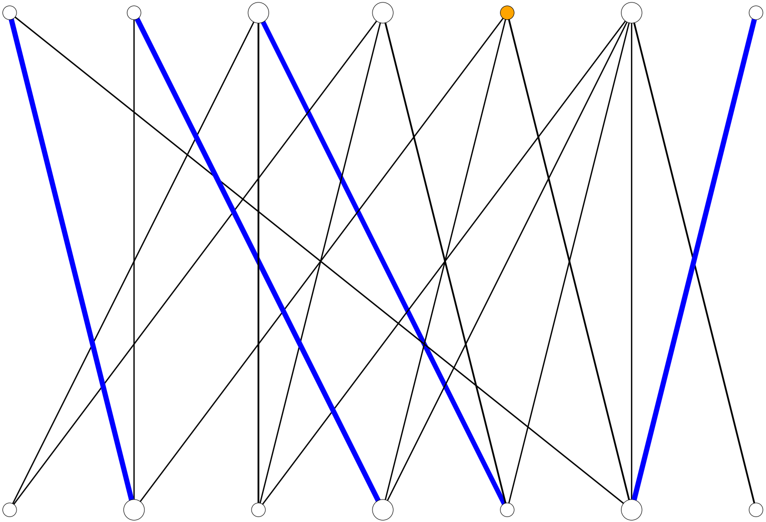
Hungarian Method: Example



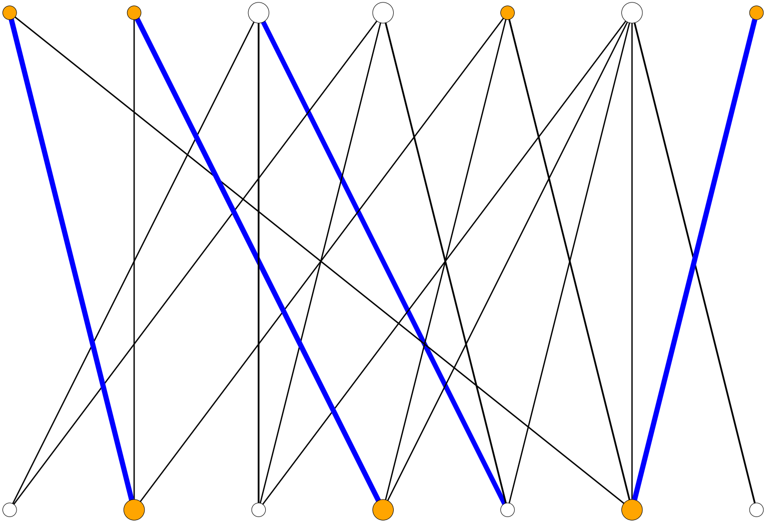
Hungarian Method: Example



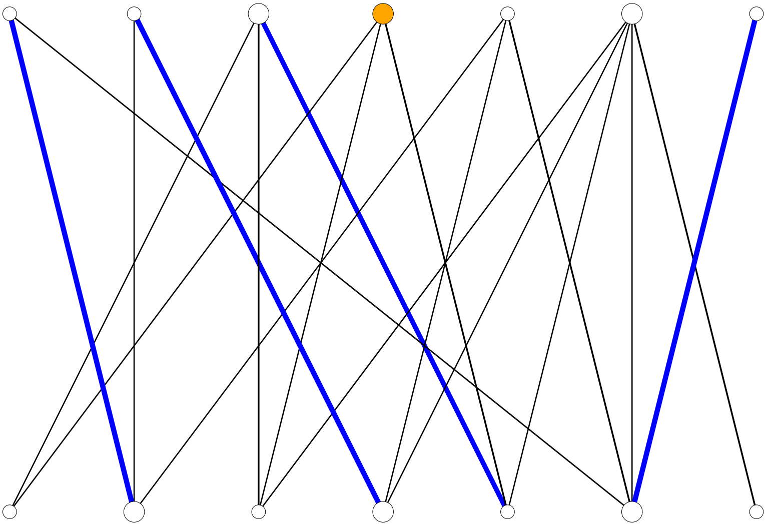
Hungarian Method: Example



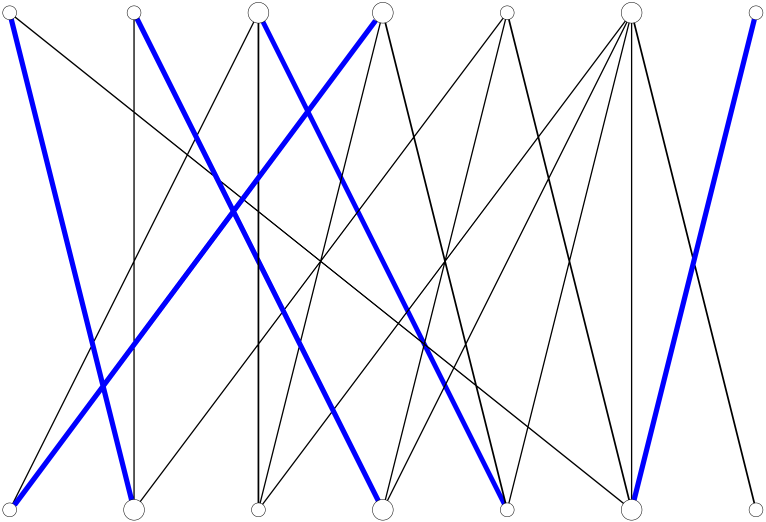
Hungarian Method: Example



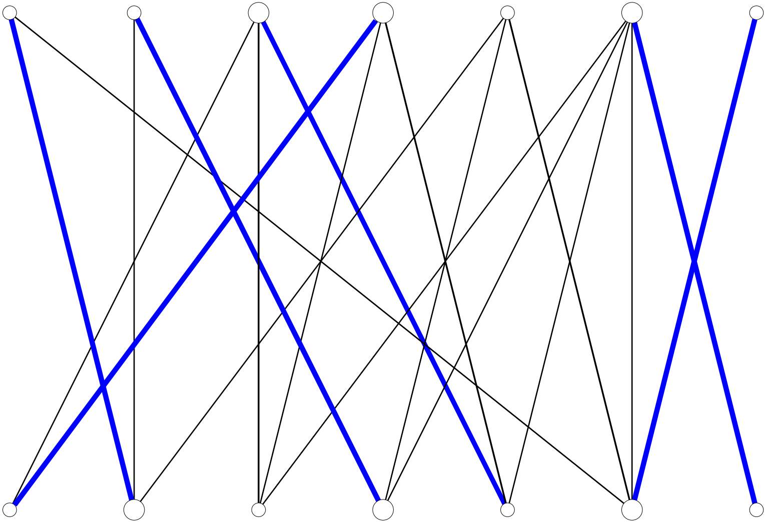
Hungarian Method: Example



Hungarian Method: Example



Hungarian Method: Example



Matchings and Vertex Covers

Vertex Covers

Definition

In a graph $G = (V, E)$ a **vertex cover** is a set $S \subseteq V$ such that all edges of E have at least an endpoint in S.

Vertex Covers

Definition

In a graph $G = (V, E)$ a **vertex cover** is a set $S \subseteq V$ such that all edges of E have at least an endpoint in S.

Problem

In the Minimum Vertex Cover *problem we take as input G, k and want to decide if G has a vertex cover of size $\leq k$.*

Theorem

In all graphs G, $\alpha(G) + \mathrm{vc}(G) = n$.

Minimum vertex cover of

- Paths P_n? Cycles C_n? Cliques K_n? Complete bipartite graphs $K_{n,m}$?

Vertex Cover and Matchings

Theorem

In all graphs G we have $\mathrm{vc}(G) \geq \mathrm{mm}(G)$.

Note: $\mathrm{vc}(G)$: min vertex cover, $\mathrm{mm}(G)$: max matching

Vertex Cover and Matchings

Theorem

In all graphs G we have $\mathrm{vc}(G) \geq \mathrm{mm}(G)$.

Note: $\mathrm{vc}(G)$: min vertex cover, $\mathrm{mm}(G)$: max matching

Proof.

Any cover must hit all edges of a maximum matching, no vertex covers two such edges. □

Vertex Cover and Matchings

Theorem

In all graphs G we have $\mathrm{vc}(G) \geq \mathrm{mm}(G)$.

Note: $\mathrm{vc}(G)$: min vertex cover, $\mathrm{mm}(G)$: max matching

Proof.

Any cover must hit all edges of a maximum matching, no vertex covers two such edges. □

Is VERTEX COVER in...

- NP?

Vertex Cover and Matchings

Theorem

In all graphs G we have $\mathrm{vc}(G) \geq \mathrm{mm}(G)$.

Note: $\mathrm{vc}(G)$: min vertex cover, $\mathrm{mm}(G)$: max matching

Proof.

Any cover must hit all edges of a maximum matching, no vertex covers two such edges. □

Is VERTEX COVER in. . .

- NP?
  - Yes. Certificate is the cover S.
- coNP?

Vertex Cover and Matchings

Theorem

In all graphs G we have $\mathrm{vc}(G) \geq \mathrm{mm}(G)$.

Note: $\mathrm{vc}(G)$: min vertex cover, $\mathrm{mm}(G)$: max matching

Proof.

Any cover must hit all edges of a maximum matching, no vertex covers two such edges. □

Is VERTEX COVER in...

- NP?
  - Yes. Certificate is the cover S.
- coNP?
  - No!! (Unless NP=coNP !!)
  - Why doesn't maximum matching work as a certificate?

Kőnig's theorem

Theorem

If G is bipartite, then $\mathrm{mm}(G) = \mathrm{vc}(G)$.

Proof.

- $G = (A, B, E)$, M a max matching, U set of unmatched vertices of A.
- Define Z to be set of vertices reachable from U via alternating paths.
- Claim: $(A \setminus Z) \cup (B \cap Z)$ is a vertex cover that contains one endpoint of each edge of M.

□

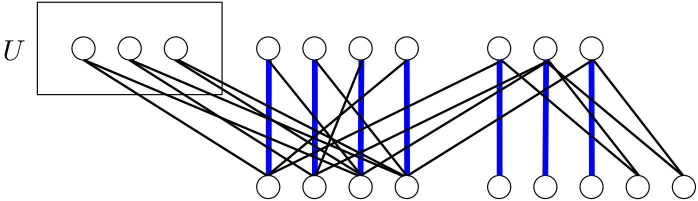
Kőnig's theorem

Theorem

If G is bipartite, then $\mathrm{mm}(G) = \mathrm{vc}(G)$.

Proof.

- $G = (A, B, E)$, M a max matching, U set of unmatched vertices of A.
- Define Z to be set of vertices reachable from U via alternating paths.
- Claim: $(A \setminus Z) \cup (B \cap Z)$ is a vertex cover that contains one endpoint of each edge of M.



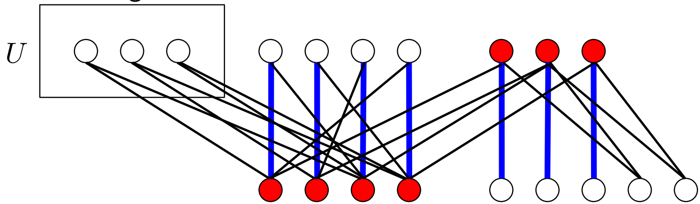
Kőnig's theorem

Theorem

If G is bipartite, then $\mathrm{mm}(G) = \mathrm{vc}(G)$.

Proof.

- $G = (A, B, E)$, M a max matching, U set of unmatched vertices of A.
- Define Z to be set of vertices reachable from U via alternating paths.
- Claim: $(A \setminus Z) \cup (B \cap Z)$ is a vertex cover that contains one endpoint of each edge of M.



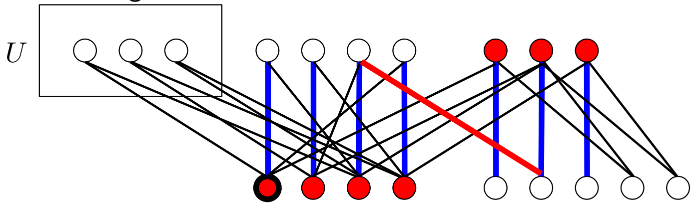
Kőnig's theorem

Theorem

If G is bipartite, then $\mathrm{mm}(G) = \mathrm{vc}(G)$.

Proof.

- $G = (A, B, E)$, M a max matching, U set of unmatched vertices of A.
- Define Z to be set of vertices reachable from U via alternating paths.
- Claim: $(A \setminus Z) \cup (B \cap Z)$ is a vertex cover that contains one endpoint of each edge of M.



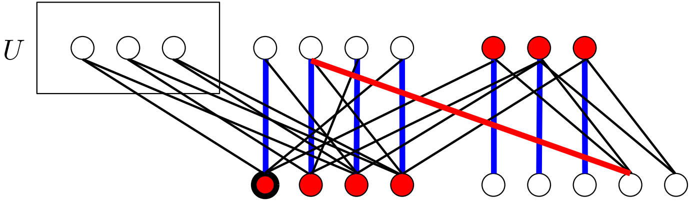
Kőnig's theorem

Theorem

If G is bipartite, then $\mathrm{mm}(G) = \mathrm{vc}(G)$.

Proof.

- $G = (A, B, E)$, M a max matching, U set of unmatched vertices of A.
- Define Z to be set of vertices reachable from U via alternating paths.
- Claim: $(A \setminus Z) \cup (B \cap Z)$ is a vertex cover that contains one endpoint of each edge of M.



Kőnig's theorem – Implications

Theorem

If G is bipartite, then $\mathrm{mm}(G) = \mathrm{vc}(G)$.

Corollary

MINIMUM VERTEX COVER *is in coNP for bipartite graphs.*

Kőnig's theorem – Implications

Theorem

If G is bipartite, then $\mathrm{mm}(G) = \mathrm{vc}(G)$.

Corollary

MINIMUM VERTEX COVER *is in coNP for bipartite graphs.*

Corollary

MINIMUM VERTEX COVER *is in P for bipartite graphs. (Using Hungarian Method).*

- On general graphs, MINIMUM VERTEX COVER is NP-complete, so not in P, nor in coNP ...unless P=NP or NP=coNP...