Graph Theory: Lecture 2

Trees and Forests

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Trees

Acyclic Graphs

Definition

A graph G that does not contain any cycles is called a **forest**. If G is a connected forest, then we say that G is a **tree**.

Acyclic Graphs

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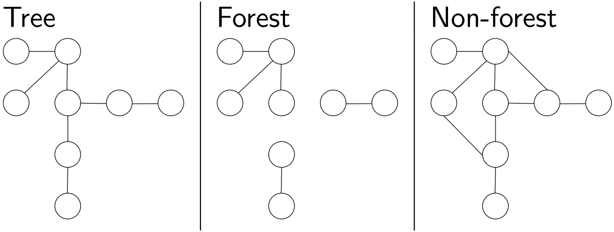
Examples:



Acyclic Graphs

Definition

A graph G that does not contain any cycles is called a **forest**. If G is a connected forest, then we say that G is a **tree**.

Questions:

- Is P_n as tree? Is $\overline{P}_n$ a tree?
- Is the complement of a tree a tree?
- Is every (induced) subgraph of a tree a tree?
- Is every (induced) subgraph of a forest a forest?

Characterizations of Trees

Theorem

The following are equivalent for any graph $G = (V, E)$:

1. *G is a tree.*
2. *Any two vertices of G are connected by a unique path.*
3. *G is minimally connected.*
4. *G is maximally acyclic.*
5. *G is connected and $|E(G)| = |V(G)| - 1$.*
6. *G is acyclic and $|E(G)| = |V(G)| - 1$.*

1⇒2

Lemma

If G is a tree then any two vertices are connected by a unique path.

1⇒2

Lemma

If G is a tree then any two vertices are connected by a unique path.

Proof.

- G is a tree $\Rightarrow$ G is connected $\Rightarrow$ any two vertices are connected by at least one path.
- If u, v were connected by two distinct paths, we would have a cycle, contradiction.

1⇒2

Lemma

If G is a tree then any two vertices are connected by a unique path.

Proof.

- G is a tree ⇒ G is connected ⇒ any two vertices are connected by at least one path.
- If u, v were connected by two distinct paths, we would have a cycle, contradiction.
  - Let u, v be the two vertices connected by two distinct paths such that $\mathrm{dist}(u, v)$ is minimum.
  - Let $P_1 = (u, x_1, x_2, \ldots, x_k, v)$, $P_2 = (u, y_1, y_2, \ldots, y_\ell, v)$ be two such paths and P_1 be a shortest $u - v$ path.
  - If $x_i = y_j$ for some i, j, then x_i, v is another pair, with shorter distance, contradiction!
  - If not, $(u, x_1, \ldots, v, y_\ell, \ldots, u)$ is a cycle.

2⇒1

Lemma

If any two vertices of G are connected by a unique path, then G is a tree.

2⇒1

Lemma

If any two vertices of G are connected by a unique path, then G is a tree.

Proof.

- G is connected, so must prove it is acyclic.
- For the sake of contradiction, suppose G has a cycle subgraph $(x_1, x_2, \ldots, x_k, x_1)$.
- Then, there exist two distinct paths $x_1 - x_k$: (x_1, x_k) and $(x_1, x_2, \ldots, x_k)$, contradiction!

□

2⇔3

Lemma

Any two vertices of G are connected by a unique path if and only if G is minimally connected.

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Minimally connected: connected but removing any edge disconnects the graph.

2⇔3

Lemma

Any two vertices of G are connected by a unique path if and only if G is minimally connected.

Proof.

- $2 \Rightarrow 3$
  - G is connected by assumption.
  - For $e = xy$, $G - e$ cannot be connected, because we would have two $x - y$ paths in G.
- $3 \Rightarrow 2$
  - Any two vertices are connected by at least one path.
  - If x, y have two paths, we have a cycle, any edge e of this cycle can be removed without disconnecting the graph.

□

$1 \Leftrightarrow 4$

Lemma

G is a tree if and only if G is maximally acyclic.

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G is a tree if and only if G is maximally acyclic.

Maximally acyclic: acyclic but adding any edge creates a cycle.

$1 \Leftrightarrow 4$

Lemma

G is a tree if and only if G is maximally acyclic.

Proof.

- $1 \Rightarrow 4$
  - G is a tree, so acyclic.
  - Adding the edge uv adds a cycle, as G is connected, so there is already a $u - v$ path.
- $4 \Rightarrow 1$
  - G is acyclic, so need to prove it is connected.
  - Suppose not, and there is no path $u \to v$.
  - Then, the edge uv does not create a cycle, contradicting maximality.

□

$(1, 3) \Rightarrow 5$

Lemma

If $G = (V, E)$ is minimally connected, then $|E| = |V| - 1$.

$(1,3) \Rightarrow 5$

Lemma

If $G = (V, E)$ is minimally connected, then $|E| = |V| - 1$.

Proof.

By induction:

- $n \leq 2$: trivial.
- Larger n: let $ab \in E$, consider the **two** (?) connected components G_1, G_2 for $G - ab$.
- By induction $|E(G_1)| = |V(G_1)| - 1$ and $|E(G_2)| = |V(G_2)| - 1$.
- $|E(G)| = |E(G_1)| + |E(G_2)| + 1 = |V(G)| - 1$.

□

$5 \Rightarrow 1$

Lemma

If for $G = (V, E)$, G is connected and $|E| \leq |V| - 1$, then G is a tree.

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Lemma

If for $G = (V, E)$, G is connected and $|E| \leq |V| - 1$, then G is a tree.

Proof.

By minimal counter-example:

- Among all counter-examples, take G to have minimum $|E|$.
- Since G is a counter-example, it must have a cycle, let e be an edge of the cycle.
- $G' = G - e$ is connected and has fewer edges, so it is **not** a counter-example.
- $\Rightarrow$ G' is a tree, and by previous slide $|E(G')| = |V(G')| - 1$.
- We have $|E(G)| = |E(G')| + 1 = |V(G')| = |V(G)|$, contradiction!

□

$(1,5) \Rightarrow 6$

Lemma

If $G = (V, E)$ is a tree and $|E| = |V| - 1$, then G is acyclic and $|E| = |V| - 1$.

$(1, 5) \Rightarrow 6$

Lemma

If $G = (V, E)$ is a tree and $|E| = |V| - 1$, then G is acyclic and $|E| = |V| - 1$.

Proof.

Obvious! □

$6 \Rightarrow 1$

Lemma

If $G = (V, E)$ *is acyclic and* $|E| = |V| - 1$*, then* G *is a tree.*

$6 \Rightarrow 1$

Lemma

If $G = (V, E)$ is acyclic and $|E| = |V| - 1$, then G is a tree.

Proof.

Need to show that G is connected.

- Let $G_1, \ldots, G_k$ be the connected components.
- Each G_i is a tree, so $|E(G_i)| = |V(G_i)| - 1$.
- $|E| = \sum_{i \in [k]} |E(G_i)| = \sum_{i \in [k]} (|V(G_i)| - 1) = |V| - k$
- Therefore, $k = 1$.

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$6 \Rightarrow 1$

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If $G = (V, E)$ is acyclic and $|E| = |V| - 1$, then G is a tree.

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$\square$

Phew!

Characterizations of Trees (Recap)

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Trees have leaves!

Definition

A vertex of degree 1 is called a **leaf**.

Theorem

If $G = (V, E)$ is a tree with $|V| \geq 2$, then G contains at least two distinct leaves.

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Proof.

- $|E| = |V| - 1$
- $2|E| = \sum_{v \in V} \deg(v)$
- If for some $v \in V$, $\deg(v) = 0$, G is disconnected, contradiction.
- If for at most one $v \in V$, $\deg(v) = 1$, then $2|E| \geq 2|V| - 1 \Rightarrow |E| \geq |V|$, contradiction!
- So, for at least two vertices $v \in V$, $\deg(v) = 1$.

□

Algorithmic Part: Recognition

Problem

Given graph $G = (V, E)$, decide if G is a tree/forest.

- Naïve algorithm:

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  - For each edge $e \in E$ verify that $G - e$ is diconnected.
  - (Tree) Verify that G is connected.

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- Alternative: check if graph is 1-degenerate

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  - (Tree) Verify that G is connected.
- Alternative: check if graph is 1-degenerate

Definition

G is k-degenerate iff every (induced) subgraph of G has a vertex of degree at most k.

Degenerate Graphs

Theorem

We can decide in polynomial time if given G is k-degenerate.

Theorem

G is a forest if and only if G is 1-degenerate.

Degenerate Graphs

Theorem

We can decide in polynomial time if given G is k-degenerate.

Proof.

Algorithm:

- If G is empty $\rightarrow$ Yes.
- If G has no vertex of degree $\leq k \rightarrow$ No.
- If v has degree $\leq k$ it suffices to check $G - v$ is k-degenerate, recurse.
  - ...because all subgraphs that contain v are OK.

□

Theorem

G is a forest if and only if G is 1-degenerate.

Degenerate Graphs

Theorem

We can decide in polynomial time if given G is k-degenerate.

Theorem

G is a forest if and only if G is 1-degenerate.

Proof.

- Forest $\Rightarrow$ 1-degenerate
  - Forests contain leaves, are closed under subgraphs
- 1-degenerate $\Rightarrow$ forest
  - If not forest $\rightarrow$ contains cycle $\rightarrow$ not 1-degenerate, contradiction!

□

Separations

- Trees are **algorithmically** important.
- One key property (among many): **balanced separators**

Separations

- Trees are **algorithmically** important.
- One key property (among many): **balanced separators**

Definition

For graph G a vertex v is called a $\frac{1}{2}$-separator if all connected components of $G - v$ contain at most $\frac{|V|}{2}$ vertices.

Theorem

If G is a tree, then G has a $\frac{1}{2}$-separator.

- Algorithmic application: **Divide&Conquer**

Trees have Balanced Separators

Theorem

If G is a tree, then G has a $\frac{1}{2}$-separator.

(**NB**): Every non-leaf vertex is a separator, but not necessarily balanced.

Trees have Balanced Separators

Theorem

If G is a tree, then G has a $\frac{1}{2}$-separator.

Proof.

- $i = 1$
- Take a vertex v_i of degree ≥ 2
  - If v_1 is a $\frac{1}{2}$-separator, done!
  - Otherwise, $G - v_1$ has **exactly one** large component.
  - Let v_{i+1} be the neighbor of v_i in that component, repeat.
- $\Rightarrow$ forms a path $v_1, v_2, \ldots, v_k$.
- Vertices cannot be repeated and graph contains no cycle, so we must end with a $\frac{1}{2}$-separator.

□