# Algorithms M2–IF TD 3

October 4, 2021

## 1 Blood tests

[MU 2.25] We want to perform a blood test to detect a virus. We have a test with perfect accuracy: if we perform the test on a blood sample that contains the virus (even in small amounts), the test always detects the virus; on the other hand the test never gives false positives (i.e. if the virus is not there, it says that it's not there).

The only problem now is that the test is expensive. We have $n$ people that we wish to test, but running the test on each one costs too much. So we consider the following scheme: we take a group of $k$ people and construct a blood sample by mixing the samples of these $k$ people; then we test this mixed sample. If the test does not detect the disease, all of the $k$ people are healthy. If it does, we must check each of them individually.

Assume that each person has probability $p$ of having the virus, independently of the others.

1. For a specific group of $k$ people, what is the probability that the test on their mixed sample will be positive?
2. What is the expected number of tests we will perform?
3. For which values of $p$ is it better to forget about this scheme and simply test everyone from the beginning?

## 2 More random coins

[MU 3.22] We flip $n$ random coins. Observe that there are $m = \binom{n}{2}$ pairs of coin flips in this experiment. For each pair of coin flips we define a random variable $Y_i$ which is equal to 1 if the two flips of this pair gave different outcomes (so, $Y_i$ is the exclusive or of the two random bits of the pair). Let $Y = \sum_{i=1}^{m} Y_i$.

1. Show that $Pr[Y_i = 0] = Pr[Y_i = 1] = 1/2$.
2. Show that the $Y_i$ are not mutually independent.

3. Show that the $Y_i$ are pair-wise independent and satisfy the property $E[Y_iY_j] = E[Y_i]E[Y_j]$.
4. Find $Var[Y]$.
5. Use Chebyshev's inequality to upper bound $Pr[|Y - E[Y]| \geq n]$.