Algorithms M2–IF TD 2

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1 A crowded theater

A theater has n seats, numbered $1, \ldots, n$. Tonight's show is sold out, so there are n spectators who want to enter. Every spectator has a ticket with the number of her seat. The spectators enter the theater in a random order. We assume that the following happens:

- The first person to enter ignores the number on his ticket and picks uniformly at random a seat to sit in. (What an idiot!)
- Every other spectator is nice, but a little shy. They all do the following: first, they go to their assigned seat and if it is free, they sit there. If not, they pick a seat uniformly at random and sit there.

Calculate the following:

1. The probability that the first spectator (the idiot) sits in someone else's seat.
2. The probability that the second spectator (the one who enters after the idiot) sits in his own seat.
3. The probability that the second spectator (the one who enters after the idiot) sits in the idiot's seat.
4. The probability that the last spectator sits in his seat. (If it helps, start with small values of n first).
5. Consider now a variation where the spectators go in in a completely random order (that is, the idiot is not necessarily the first to enter). Calculate the probability that the last spectator sits in his seat in this case.

2 Another crowded theater

This is similar to the previous exercise, except now all spectators behave like the first spectator: when they enter, they pick uniformly at random a random seat, and sit there (ignoring their assigned number).

1. What is the probability that the i-th spectator sits in her assigned place?
2. What is the expected number of spectators sitting in their assigned places?

3 Dice and Expectations

[MU Ex 2.1] Suppose we roll a fair k-sided die with the numbers $1, \ldots, k$ on its faces. Let X be the number that appears. What is $E[X]$?
[MU Ex 2.9] Suppose we roll a fair k-sided die **twice**. Let X_1, X_2 be the two values we obtained. What is $E[\min\{X_1, X_2\}]$? What is $E[\max\{X_1, X_2\}]$. Calculate also $E[\min\{X_1, X_2\}] + E[\max\{X_1, X_2\}]$.

4 Fair coins again

We flip a fair coin $2n$ times. Let X_1 be the number of times the result was heads, and X_2 the number of times the result was tails. Prove that for any $\epsilon > 0$ there exists a c such that we have $Pr[X_1 - X_2 > c\sqrt{n}] < \epsilon$.

5 Secretary Problem

We are interviewing candidates for a job. Suppose there are n candidates overall. If we had perfect information, we could assign each candidate a score from $\{1, \ldots, n\}$ such that all candidates have distinct scores and the best candidate has score n. (In other words, if we could interview everyone before deciding, we could produce a ranking of the candidates).

The problem is that we see candidates one by one, in a random order, and after seeing a candidate we have to immediately decide if this candidate is hired. If not, this candidate leaves (and gets a job somewhere else).

We want to adopt a strategy that maximizes the probability of hiring the best candidate. Suppose we adopt the following strategy: we interview m candidates just to get a feeling of their level, but we don't hire anyone among them; then we hire the first candidate who is better than *all* the m candidates in our initial sample.

- Let E be the event that we hire the best candidate. Show that $Pr[E] = \frac{m}{n} \sum_{j=m+1}^{n} \frac{1}{j-1}$.
- Using the approximation $\sum_{i=1}^{n} \frac{1}{n} \approx \ln n$ show that

$$\frac{m}{n}(\ln n - \ln m) \leq Pr[E] \leq \frac{m}{n}(\ln(n-1) - \ln(m-1))$$

- Show that $Pr[E]$ is maximized when $m = n/e$. How much is $Pr[E]$ for this value?