Algorithms M2–IF TD 1

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1 Big-O notation

Consider the following functions:

$$2^n, 3^n, n^3, n\log n, n!, n^n, 2^{\sqrt{\log n}}, 3^{\log n}, 2^{\log\log n}, \log^2 n, (\log n)^{\log n}, \log(n!), 2^{\sqrt{n}}, 2^{n^2}$$

Sort them in increasing order of Big-O notation.

2 Basic Probabilities

[MU Ex. 1.1] We flip a fair coin ten times. Calculate the following probabilities:

1. The number of heads is equal to the number of tails.
2. There were more heads than tails. (What if we flip 11 times?)
3. The 2nd and 9th flip produced the same outcome.
4. We flipped at least eight consecutive heads.

3 Basic Probabilities 2: Modulo arithmetic

[MU Ex. 1.15] We roll ten standard dice (six-sided). What is the probability that the sum of our dice is divisible by 6?

4 Basic Probabilities 3: Independence

Give an example of three events A, B, C which are pair-wise independent but not independent as a set.

5 Generating Randomness

Suppose that we are given a fair coin. Explain how to use repeated flips of this coin to:

1. Generate a number in $\{2, \ldots, 12\}$ which has the same distribution as the sum of two normal dice.

2. Generate a random number in $\{0, \ldots, 10\}$.

How many flips will you need (in expectation)?

Suppose that we are given a non-fair coin that has probability $p \neq \frac{1}{2}$ of giving heads. Show how you can use this coin to generate a random bit with probability $\frac{1}{2}$ of being 1. How many coin flips do you need (in expectation)?

6 False-Positive Paradox

Algorithmitis is a rare disease that affects roughly $\frac{1}{100}$ of the general population. Prof. Chaos has come up with an innovative new test to detect this disease in the population. This test has a success rate of $\frac{98}{100}$, but possibly two-sided error: with probability $\frac{2}{100}$ it may misidentify a sick person as healthy, or a healthy person as suffering from algorithmitis.

We randomly select a volunteer from the general population and perform the test on her. Suppose that the test came back positive. What is the probability that the volunteer has algorithmitis?

7 Monty Hall Paradox

In the game show "Let's make a deal" the contestant is given three closed boxes A,B,C. One of the boxes contains a prize, while the others are empty. The game is played as follows:

1. The contestant picks a box, say A.

2. The host considers the remaining boxes B and C. He publicly opens one of the two, always selecting to open a box that is empty. **NB:** This is always possible (why?)

3. The contestant must either hold his box or exchange with the remaining un-opened one. No other information is given.

What is the probability that the contestant wins if she decides to exchange her box?

8 Min Cut

For the Min Cut algorithm that we saw in class, suppose that we make the following modification: at each step, instead of picking a random edge (u, v) to contract, the algorithm picks two random vertices $u, v \in V$ and collapses them into a single vertex. Prove (by giving a counter-example) that this algorithm has, in some instances, exponentially small probability of finding a minimum cut.