Algorithms M2–IF Homework 1

October 4, 2021

1 Rolling the dice

We roll a standard 6-sided die 100 times. Let X be the sum of the numbers that appear.

1. What is $E[X]$?
2. Use Markov's inequality to upper bound $Pr[X \geq 400]$.
3. Use Chebyshev's inequality to upper bound $Pr[X \geq 400]$.

2 Rolling more dice

[MU 2.6] We roll two standard 6-sided dice, let X_1, X_2 be the two numbers we obtain and $X = X_1 + X_2$. Calculate the following:

- $E[X \mid X_1 \text{ is even}]$
- $E[X \mid X_1 = X_2]$
- $E[X_1 \mid X = 9]$
- $E[X_1 - X_2 \mid X = k]$, for $k \in [2, 12]$.

3 Coupon collector revisited

[MU 2.12] We have a deck of n cards. We perform the following experiment $2n$ times: we shuffle the deck, pull out a card, look at it, then put it back in.

1. What is the expected number of cards that we have seen at the end of the experiment?
2. What is the expected number of cards that we have seen exactly once?

Hint: you may use the approximation $(1 - \frac{1}{n})^n \approx \frac{1}{e}$.

4 Elections

Alice and Bob are running for class president. There are 100 voters. 80 voters prefer Alice and 20 voters prefer Bob. During the election each voter gets confused (independently of other voters) with probability $\frac{1}{100}$ and votes for the wrong candidate (i.e. the candidate he likes less).

If A is the number of votes Alice receives and B is the number of votes Bob receives:

1. Calculate $E[A]$ and $E[B]$.
2. Use Markov's inequality to upper bound the probability that Bob wins the election.
3. Use Chebyshev's inequality to upper bound the probability that Bob wins the election.

5 Variance Properties

Prove the following:

- For any random variable X, real number c, we have $Var[cX] = c^2 Var[X]$.
- For any two independent random variables X, Y, we have $Var[X - Y] = Var[X] + Var[Y]$. (Recall that $E[X - Y] = E[X] - E[Y]$).

6 Markov's inequality

Recall that Markov's inequality states the following: if X is a random variable that only takes non-negative values, then for all $\alpha > 0$ we have

$$Pr[X \geq \alpha E[X]] \leq \frac{1}{\alpha}$$

Give an example of a random variable X and a value α such that the above inequality is tight (that is, it becomes an equality). Your random variable X should be non-trivial (that is, it should have positive probability for at least two values).

Furthermore, for the same variable, give another value of α for which the inequality is **not** tight.

7 Independence vs Pair-wise independence

Let $A = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix}$. Each variable $A_{i,j}$ is set independently at random to value 0 or 1 with probability 1/2. We define R_i to be the **exclusive or** of row

i, that is $R_i = A_{i,1} \oplus A_{i,2}$. We define C_j to be the exclusive or of column j, that is, $C_j = A_{1,j} \oplus A_{2,j}$.

Give answers with justifications for the following questions:

1. Are R_1 and R_2 independent?
2. Are R_1 and C_1 independent?
3. Are R_1, R_2, C_1, C_2 pair-wise independent?
4. Are R_1, R_2, C_1, C_2 independent?
5. Are R_1, R_2, C_1 independent?

8 Bubblesort

Bubblesort is the prototypical **bad** sorting algorithm (if you don't remember this algorithm see `https://en.wikipedia.org/wiki/Bubble_sort`). For this exercise we are interested in analyzing the performance of Bubblesort on average, that is, for an array of numbers that is given in a random permutation. Recall that the algorithm will at each step compare two numbers in consecutive positions and exchange them if they are in the wrong order.

Calculate the asymptotic expected complexity of Bubble sort when given a random array of n distinct integers.