Algorithms M2–IF Homework 1

November 3, 2021

1 Rolling the dice

We roll a standard 6-sided die 100 times. Let X be the sum of the numbers that appear.

1. What is $E[X]$?
2. Use Markov's inequality to upper bound $Pr[X \geq 400]$.
3. Use Chebyshev's inequality to upper bound $Pr[X \geq 400]$.

Solution:

1. Let X_i be the number appearing in the i-th roll. Then $E[X_i] = 3.5$ and $E[X] = E[\sum X_i] = \sum E[X_i] = 350$.
2. $Pr[X \geq 400] = Pr[X \geq \frac{400}{350} \cdot 350] = Pr[X \geq \frac{8}{7}E[X]] \leq \frac{7}{8}$.
3. We have $Var[X] = 100Var[X_i]$ because the X_i are independent. We therefore need to calculate $Var[X_i] = E[X_i^2] - (E[X_i])^2$. We have

$$E[X_i^2] = \frac{1}{6}\left(1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2\right) = \frac{91}{6}$$

Therefore $Var[X_i] = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{91}{6} - \frac{49}{4} = \frac{182-147}{12} = \frac{35}{12}$ which gives $Var[X] = \frac{35}{6} \cdot 50$. We now have

$$Pr[X \geq 400] = Pr[X - 350 \geq 50] \leq Pr[|X - E[X]| \geq 50] \leq \frac{Var[X]}{50^2} = \frac{7}{60}$$

2 Rolling more dice

[MU 2.6] We roll two standard 6-sided dice, let X_1, X_2 be the two numbers we obtain and $X = X_1 + X_2$. Calculate the following:

- $E[X \mid X_1 \text{ is even}]$

- $E[X \mid X_1 = X_2]$
- $E[X_1 \mid X = 9]$
- $E[X_1 - X_2 \mid X = k]$, for $k \in [2, 12]$.

Solution:

- $E[X \mid X_1 \text{ is even}] = E[X_1 + X_2 \mid X_1 \text{ is even}] = E[X_1 \mid X_1 \text{ is even}] + E[X_2 \mid X_1 \text{ is even}]$. The second term is simply $E[X_2] = 3.5$ since X_1, X_2 are independent. The first term is $\frac{1}{3}(2 + 4 + 6) = 4$. So the result is 7.5.
- $E[X_1 + X_2 \mid X_1 = X_2] = E[X_1 \mid X_1 = X_2] + E[X_2 \mid X_1 = X_2] = E[X_1] + E[X_2] = 7$.
- $E[X_1 \mid X_1 + X_2 = 9] = \sum_{i=1}^{6} iPr[X_1 = i \mid X_1 + X_2 = 9] = \frac{1}{4}(3+4+5+6) = 4.5$.
- $E[X_1 - X_2 \mid X = k] = E[X_1 \mid X_1 + X_2 = k] - E[X_2 \mid X_1 + X_2 = k]$. We now observe that X_1, X_2 are identically distributed variables, therefore $E[X_1 \mid A] = E[X_2 \mid A]$, for any event A. We conclude that the answer for this question is 0.

3 Coupon collector revisited

[MU 2.12] We have a deck of n cards. We perform the following experiment $2n$ times: we shuffle the deck, pull out a card, look at it, then put it back in.

1. What is the expected number of cards that we have seen at the end of the experiment?
2. What is the expected number of cards that we have seen exactly once?

Hint: you may use the approximation $(1 - \frac{1}{n})^n \approx \frac{1}{e}$.

Solution:

1. Let X_i be a random variable that is 1 if we have seen card i at the end of the experiment, 0 otherwise. We have $Pr[X_i = 0] = (1 - \frac{1}{n})^{2n}$. To see this, note that the probability of not seeing card i in one try of the experiment is $(1 - \frac{1}{n})$, and the repetitions are independent. To simplify things, we use the approximation $Pr[X_i = 0] = ((1 - \frac{1}{n})^n)^2 = \frac{1}{e^2}$. Therefore, $E[X_i] = Pr[X_i = 1] = 1 - \frac{1}{e^2}$. Hence, $E[X] = n - \frac{n}{e^2}$.
2. Let X_i be a random variable that is 1 if we saw card i *exactly* once. Then $Pr[X_i = 1] = 2n \cdot \frac{1}{n}(1 - \frac{1}{n})^{2n-1}$. To see this, note that the probability we saw card i only in round j is $\frac{1}{n}(1 - \frac{1}{n})^{2n-1}$ and that there are $2n$ possible such events (depending on j), all of which are mutually disjoint. The remaining calculations are similar to the previous case.

4 Elections

Alice and Bob are running for class president. There are 100 voters. 80 voters prefer Alice and 20 voters prefer Bob. During the election each voter gets confused (independently of other voters) with probability $\frac{1}{100}$ and votes for the wrong candidate (i.e. the candidate he likes less).

If A is the number of votes Alice receives and B is the number of votes Bob receives:

1. Calculate $E[A]$ and $E[B]$.
2. Use Markov's inequality to upper bound the probability that Bob wins the election.
3. Use Chebyshev's inequality to upper bound the probability that Bob wins the election.

Solution:

1. We have $E[A] = 80\frac{99}{100} + 20\frac{1}{100} = 79.4$. Furthermore, $E[B] = 100 - E[A] = 20.6$, because $A + B = 100$.
2. $Pr[\text{Bob wins}] = Pr[B > 50] \leq \frac{20.6}{50} \leq 40\%$.
3. We have to calculate the variance of B. Let X_i be a random variable that is 1 if the i-th Bob voter voted for Bob. Let Y_i be a random variable that is 1 if the i-th Alice voter voted for Bob. We have $B = \sum_{i=1}^{20} X_i + \sum_{i=1}^{80} Y_i$. Because all variables are independent we have $Var[B] = 20Var[X_i] + 80Var[Y_i]$.

   We have $Var[X_i] = E[X_i^2] - E[X_i]^2 = 0.99 - 0.99^2 \approx 0.01$ and $Var[Y_i] = E[Y_i^2] - E[Y_i]^2 = 0.01 - 0.01^2 \approx 0.01$. Therefore, $Var[B] \approx 1$.

   We now have $Pr[\text{Bob wins}] = Pr[B > 50] = Pr[B - 20.6 > 29.4] \leq Pr[|B - 20.6| > 29.4] \leq \frac{Var[B]}{29.4^2} \leq 0.2\%$.

5 Variance Properties

Prove the following:

- For any random variable X, real number c, we have $Var[cX] = c^2 Var[X]$.
- For any two independent random variables X, Y, we have $Var[X - Y] = Var[X] + Var[Y]$. (Recall that $E[X - Y] = E[X] - E[Y]$).

Solution:

- $Var[cX] = E[(cX)^2] - (E[cX])^2 = E[c^2X^2] - (cE[X])^2 = c^2E[X^2] - c^2(E[X])^2 = c^2Var[X]$

- $Var[X-Y] = Var[X+(-Y)] = Var[X] + Var[-Y] = Var[X] + Var[Y]$, where we have used the fact that $X, -Y$ are independent and the previous question.

6 Markov's inequality

Recall that Markov's inequality states the following: if X is a random variable that only takes non-negative values, then for all $\alpha > 0$ we have

$$Pr[X \geq \alpha E[X]] \leq \frac{1}{\alpha}$$

Give an example of a random variable X and a value α such that the above inequality is tight (that is, it becomes an equality). Your random variable X should be non-trivial (that is, it should have positive probability for at least two values).

Furthermore, for the same variable, give another value of α for which the inequality is **not** tight.

Solution:

Consider the variable X which is 1 with probability 1/2 and 0 with probability 1/2 (i.e. a random bit). Then $E[X] = \frac{1}{2}$. Let $\alpha = 2$.

We get $\alpha E[X] = 1$. According to Markov's inequality $Pr[X \geq \alpha E[X]] = Pr[X \geq 1] \leq \frac{1}{2}$. But $Pr[X \geq 1] = Pr[X = 1] = \frac{1}{2}$, so the inequality is tight in this case.

Now, let $\alpha = 3$. Markov's inequality gives $Pr[X \geq \frac{3}{2}] \leq \frac{1}{3}$. However, for this variable X we have $Pr[X \geq \frac{3}{2}] = 0$, since this variable only takes values 0 and 1. So the inequality is not tight in this case.

7 Independence vs Pair-wise independence

Let $A = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix}$. Each variable $A_{i,j}$ is set independently at random to value 0 or 1 with probability 1/2. We define R_i to be the **exclusive or** of row i, that is $R_i = A_{i,1} \oplus A_{i,2}$. We define C_j to be the exclusive or of column j, that is, $C_j = A_{1,j} \oplus A_{2,j}$.

Give answers with justifications for the following questions:

1. Are R_1 and R_2 independent?
2. Are R_1 and C_1 independent?
3. Are R_1, R_2, C_1, C_2 pair-wise independent?
4. Are R_1, R_2, C_1, C_2 independent?
5. Are R_1, R_2, C_1 independent?

Solution:

1. Yes, since R_1, R_2 depend on distinct independent random variables.

2. Yes. We have $Pr[R_1 = 1] = Pr[R_1 = 0] = 1/2$ and $Pr[C_1 = 1] = Pr[C_1 = 0] = 1/2$. We now observe that $Pr[R_1 = 1 \wedge C_1 = 1] = 1/4$. To see this, fix a value for $A_{1,1}$. Then we have $R_1 = C_1 = 1$ if and only if $A_{1,2} = A_{2,1} = 1 - A_{1,1}$, and $A_{1,2}, A_{2,1}$ are independent.

3. Yes. By similar reasoning as above R_i, C_j are independent, for any i, j.

4. No. Take the probability $p = Pr[R_1 = 0 \wedge R_2 = 0 \wedge C_1 = 0 \wedge C_2 = 0]$. If these four variables were independent we would have $p = \frac{1}{16}$. However, there are two matrices which satisfy these conditions (the all-0 matrix and the all-1 matrix), so $p = \frac{1}{8}$.

5. We know that any subset of size 2 of $\{R_1, R_2, C_1\}$ is independent. So we need to prove that $Pr[R_1 = 0 \wedge R_2 = 0 \wedge C_1 = 0] = Pr[R_1 = 0]Pr[R_2 = 0]Pr[C_1 = 0] = \frac{1}{8}$. We have

$$
\begin{aligned}
Pr[R_1 = 0 \wedge R_2 = 0 \wedge C_1 = 0] &= Pr[R_1 = 0 \wedge R_2 = 0 \wedge C_1 = 0 \mid A_{1,1} = 0]Pr[A_{1,1} = 0] \\
&+ Pr[R_1 = 0 \wedge R_2 = 0 \wedge C_1 = 0 \mid A_{1,1} = 1]Pr[A_{1,1} = 1] \\
&= \frac{1}{2}Pr[A_{1,2} = 0 \wedge A_{2,1} = 0 \wedge A_{2,2} = 0] \\
&+ \frac{1}{2}Pr[A_{1,2} = 1 \wedge A_{2,1} = 1 \wedge A_{2,2} = 1] \\
&= \frac{1}{16} + \frac{1}{16} = \frac{1}{8}
\end{aligned}
$$

8 Bubblesort

Bubblesort is the prototypical **bad** sorting algorithm (if you don't remember this algorithm see `https://en.wikipedia.org/wiki/Bubble_sort`). For this exercise we are interested in analyzing the performance of Bubblesort on average, that is, for an array of numbers that is given in a random permutation. Recall that the algorithm will at each step compare two numbers in consecutive positions and exchange them if they are in the wrong order.

Calculate the asymptotic expected complexity of Bubble sort when given a random array of n distinct integers.

Solution:

Suppose we are given an array A of n integers. Let $W = \{(i,j) \mid i < j \wedge A[i] > A[j]\}$ be the set of pairs of indices which are out of order. Every step of Bubble sort will decrease $|W|$ by exactly 1 and in the end we have $|W| = 0$. What we need to calculate is therefore $E[|W|]$.

Let $X_{i,j}$ be an indicator random variable for the event $(i,j) \in W$. Then $W = \sum_{i<j} X_{i,j}$, so we need to calculate $E[X_{i,j}]$. However, since we have a

random permutation, for all $i < j$, $E[X_{i,j}] = \frac{1}{2}$. Therefore, $E[|W|] = \binom{n}{2}\frac{1}{2} = \frac{n(n-1)}{4} = \Theta(n^2)$.