Exact Algorithms for NP-hard problems

Homework 1

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An *odd cycle transversal*, OCT in short, of a graph $G = (V, E)$ is a vertex set X such that $G - X$ is bipartite.

> **Input:** a graph $G = (V, E)$, an odd cycle transversal X of size at most k, and a non-negative integer ℓ.
>
> **Parameter:** k.
>
> **Task:** find a vertex cover of size at most ℓ.

1. Design a $2^k n^{O(1)}$-time algorithm for the problem VERTEX COVER/OCT.

A tournament is a directed graph containing exactly one arc (=edge with orientation) between every pair of vertices. The problem FEEDBACK ARC SET IN TOURNAMENTS is defined as below.

> **Input:** a tournament $T = (V, A)$ on n vertices.
>
> **Parameter:** k.
>
> **Task:** find a set of at most k arcs whose reversal (=reversing the orientation of an arc) makes T acyclic.

2. Consider the problem FEEDBACK ARC SET IN TOURNAMENTS for (a)-(c).

1(a). Establish that $F \subseteq A(T)$ is a minimal feedback arc set of T if and only if it is a minimal arc set intersecting every directed cycle of T.

1(b). Devise reduction rules with safeness proof. Use them to establish that FEEDBACK ARC SET IN TOURNAMENTS admits a kernel on $O(k^2)$ vertices. (Hint: similar to $O(k^2)$ kernelization for VERTEX COVER.)

3. We consider a polynomial-time algorithm for constructing a half-integral optimal solution to LP for VERTEX COVER which does not solve LP directly. Let G be an input instance to VERTEX COVER and G^* be an auxiliary bipartite graph so that:

- $\{v_1, v_2 : v \in V(G)\}$ is the vertex set, and
- for every edge $(u, v) \in E(G)$, (u_1, v_2) and (u_2, v_1) are edges of G^*.

Let S^* be a minimum vertex cover of G^*. We define a solution x^* to LP from S^* as follows:

$$x_u^* = \begin{cases} 0.5 & \text{if exactly one of } u_1 \text{ and } u_2 \text{ belong to } S^*. \\ 1 & \text{if both of } u_1 \text{ and } u_2 \text{ belong to } S^*. \\ 0 & \text{if none of } u_1 \text{ and } u_2 \text{ belongs to } S^*. \end{cases}$$

2(a). Show that for any matching M of a graph and any feasible solution z to LP, the objective value of z is at least $|M|$.

2(b). Show that for an arbitrary feasible solution z to LP(G), the solution z' defined as $z'_{u_1} = z'_{u_2} = z_u$, for all $u \in V(G)$ is also feasible to LP(G^*), where LP formulation for VERTEX COVER of G is denoted as LP(G).

2(c). Show that x^* is an optimal solution to LP(G). (Hint: Use König theorem which says that in a bipartite graph, the size of a maximum matching equals the size of a vertex cover.)

2(d). Neatly present a kernelization implied by this exercise and the size bound of the obtained kernel, and estimate the running time of the kernelization (search for existing literature if necessary).

4. In the problem CLUSTER EDITING, we are given a graph G and a nonnegative integer k and want to find a set of at most k pairs of vertices $F \subseteq \binom{V(G)}{2}$ so that the graph $(V(G), E(G) \triangle F)$ obtained by editing G with F is a cluster graph. Here, $X \triangle Y$ denotes the symmetric difference of the sets X and Y. Hence, editing G with F is equivalent to adding a pair $(u, v) \in F$ to G if (u, v) is a non-edge in G, and removing $(u, v) \in F$ if it is an edge in G, thereby obtaining $G = (V(G), (E(G) \setminus F) \cup (F \setminus E(G))$. A *cluster graph* is a graph in which each connected component is a clique[1].
Show that CLUSTER EDITING admits a kernel on $O(k^2)$ vertices.

♣ Submit your solution via email (eunjungkim78@gmail.com) by 25 January 2021, midnight.

[1] A *clique* is a complete graph, i.e. between every pair of vertices there is an edge.