Complexity analysis in optimization

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Since the 1980s, complexity analysis has been a key tool to analyze the performance of algorithms for continuous optimization. Such results include the development of accelerated methods in convex optimization, as well as the rise of interior-point methods for constrained optimization. The past 15 years saw an upsurge in complexity results for *nonconvex* optimization, partially fulled by the prevalence of these problems in machine learning and the interest for non-asymptotic results. Moreover, recent effort has been put into reconciling the notions of complexity in the discrete and continuous settings.

This course will begin with a crash course on continuous optimization, with a focus on algorithmic formulations. We will then review the main complexity results in convex and nonconvex unconstrained optimization, by analyzing standard algorithms and mentioning open questions associated with algorithmic design and attaining lower bounds. Finally, we will illustrate how those results carry over to more complex settings arising in machine learning, game theory and discrete optimization.

Course organization The lectures will focus on the theoretical aspects of complexity analysis, yet notebooks will be provided to illustrate the connections between theoretical guarantees and practical performance.

Pre-requisites The course will not require strong familiarity with continuous optimization. Basic knowledge in vector and linear algebra is however desirable.

Detailed (tentative) syllabus:

- **Part 1** Basics of continuous optimization: Derivatives, optimality conditions, classes of functions and algorithms. Introduction to complexity in continuous optimization: convergence rates, worst-case complexity, connection to NP-hardness.
- **Part 2** Complexity in convex optimization: basic complexity results for convex and strongly convex functions, acceleration, upper/lower bounds. Complexity in nonconvex optimization: results for first-order methods and limitations, second-order methods and beyond, upper/lower bounds.
- **Part 3** Stochastic optimization: complexity results for stochastic gradient methods, lower/upper bounds. Games and saddle point problems: Complexity of gradient descent-ascent, open problems. Complexity in continuous VS discrete optimization: submodular optimization, recent advances in integer programming.

References

[1] A. Basu. Complexity of optimizing over the integers. *Math. Program.*, 200:739–780, 2023.

[2] C. Cartis, N. I. M. Gould, and Ph. L. Toint. *Evaluation Complexity of Algorithms for Nonconvex Optimization: Theory, Computation and Perspectives*, volume MO30 of *MOS-SIAM Series on Optimization*. SIAM, 2022.

[3] A. d'Aspremont, D. Scieur, and A. Taylor. Acceleration methods. *Foundations and Trends in Optimization*, 5:1–245, 2021.