

Table of Contents

Volume A

1	Introduction	1
1.1	Introduction	1
1.2	Matchings	2
1.3	But what about nonbipartite graphs?	4
1.4	Hamiltonian circuits and the traveling salesman problem	5
1.5	Historical and further notes	6
1.5a	Historical sketch on polyhedral combinatorics	6
1.5b	Further notes	8
2	General preliminaries	9
2.1	Sets	9
2.2	Orders	11
2.3	Numbers	11
2.4	Vectors, matrices, and functions	11
2.5	Maxima, minima, and infinity	14
2.6	Fekete's lemma	14
3	Preliminaries on graphs	16
3.1	Undirected graphs	16
3.2	Directed graphs	28
3.3	Hypergraphs	36
3.3a	Background references on graph theory	37
4	Preliminaries on algorithms and complexity	38
4.1	Introduction	38
4.2	The random access machine	39
4.3	Polynomial-time solvability	39
4.4	P	40
4.5	NP	40
4.6	co-NP and good characterizations	42
4.7	Optimization problems	42
4.8	NP-complete problems	43
4.9	The satisfiability problem	44

X Table of Contents

4.10	NP-completeness of the satisfiability problem	44
4.11	NP-completeness of some other problems	46
4.12	Strongly polynomial-time	47
4.13	Lists and pointers	48
4.14	Further notes	49
4.14a	Background literature on algorithms and complexity .	49
4.14b	Efficiency and complexity historically	49
5	Preliminaries on polyhedra and linear and integer programming	59
5.1	Convexity and halfspaces	59
5.2	Cones	60
5.3	Polyhedra and polytopes	60
5.4	Farkas' lemma	61
5.5	Linear programming	61
5.6	Faces, facets, and vertices	63
5.7	Polarity	65
5.8	Blocking polyhedra	65
5.9	Antiblocking polyhedra	67
5.10	Methods for linear programming	67
5.11	The ellipsoid method	68
5.12	Polyhedra and NP and co-NP	71
5.13	Primal-dual methods	72
5.14	Integer linear programming	73
5.15	Integer polyhedra	74
5.16	Totally unimodular matrices	75
5.17	Total dual integrality	76
5.18	Hilbert bases and minimal TDI systems	81
5.19	The integer rounding and decomposition properties . .	82
5.20	Box-total dual integrality	83
5.21	The integer hull and cutting planes	83
5.21a	Background literature	84
Part I: Paths and Flows		85
6	Shortest paths: unit lengths	87
6.1	Shortest paths with unit lengths	87
6.2	Shortest paths with unit lengths algorithmically: breadth-first search	88
6.3	Depth-first search	89
6.4	Finding an Eulerian orientation	91
6.5	Further results and notes	91
6.5a	All-pairs shortest paths in undirected graphs	91
6.5b	Complexity survey	93

6.5c	Ear-decomposition of strongly connected digraphs	93
6.5d	Transitive closure	94
6.5e	Further notes	94
7	Shortest paths: nonnegative lengths	96
7.1	Shortest paths with nonnegative lengths	96
7.2	Dijkstra's method	97
7.3	Speeding up Dijkstra's algorithm with k -heaps	98
7.4	Speeding up Dijkstra's algorithm with Fibonacci heaps	99
7.5	Further results and notes	101
7.5a	Weakly polynomial-time algorithms	101
7.5b	Complexity survey for shortest paths with nonnegative lengths	103
7.5c	Further notes	105
8	Shortest paths: arbitrary lengths	107
8.1	Shortest paths with arbitrary lengths but no negative circuits	107
8.2	Potentials	107
8.3	The Bellman-Ford method	109
8.4	All-pairs shortest paths	110
8.5	Finding a minimum-mean length directed circuit	111
8.6	Further results and notes	112
8.6a	Complexity survey for shortest path without negative-length circuits	112
8.6b	NP-completeness of the shortest path problem	114
8.6c	Nonpolynomiality of Ford's method	115
8.6d	Shortest and longest paths in acyclic graphs	116
8.6e	Bottleneck shortest path	117
8.6f	Further notes	118
8.6g	Historical notes on shortest paths	119
9	Disjoint paths	131
9.1	Menger's theorem	131
9.1a	Other proofs of Menger's theorem	133
9.2	Path packing algorithmically	134
9.3	Speeding up by blocking path packings	135
9.4	A sometimes better bound	136
9.5	Complexity of the vertex-disjoint case	137
9.6	Further results and notes	138
9.6a	Complexity survey for the disjoint $s - t$ paths problem	138
9.6b	Partially disjoint paths	140
9.6c	Exchange properties of disjoint paths	140
9.6d	Further notes	141

9.6e	Historical notes on Menger's theorem	142
10	Maximum flow	148
10.1	Flows: concepts	148
10.2	The max-flow min-cut theorem	150
10.3	Paths and flows	151
10.4	Finding a maximum flow	151
10.4a	Nontermination for irrational capacities	152
10.5	A strongly polynomial bound on the number of iterations	153
10.6	Dinitz' $O(n^2m)$ algorithm	154
10.6a	Karzanov's $O(n^3)$ algorithm	155
10.7	Goldberg's push-relabel method	156
10.8	Further results and notes	159
10.8a	A weakly polynomial bound	159
10.8b	Complexity survey for the maximum flow problem	160
10.8c	An exchange property	162
10.8d	Further notes	162
10.8e	Historical notes on maximum flow	164
11	Circulations and transshipments	170
11.1	A useful fact on arc functions	170
11.2	Circulations	171
11.3	Flows with upper and lower bounds	172
11.4	b -transhipments	173
11.5	Upper and lower bounds on excess $_f$	174
11.6	Finding circulations and transhipments algorithmically	175
11.6a	Further notes	176
12	Minimum-cost flows and circulations	177
12.1	Minimum-cost flows and circulations	177
12.2	Minimum-cost circulations and the residual graph D_f	178
12.3	Strongly polynomial-time algorithm	179
12.4	Related problems	182
12.4a	A dual approach	183
12.4b	A strongly polynomial-time algorithm using capacity-scaling	186
12.5	Further results and notes	190
12.5a	Complexity survey for minimum-cost circulation	190
12.5b	Min-max relations for minimum-cost flows and circulations	191
12.5c	Dynamic flows	192
12.5d	Further notes	195

13 Path and flow polyhedra and total unimodularity	198
13.1 Path polyhedra	198
13.1a Vertices, adjacency, and facets	202
13.1b The $s - t$ connector polytope	203
13.2 Total unimodularity	204
13.2a Consequences for flows	205
13.2b Consequences for circulations	207
13.2c Consequences for transshipments	207
13.2d Unions of disjoint paths and cuts	210
13.3 Network matrices	213
13.4 Cross-free and laminar families	214
14 Partially ordered sets and path coverings	217
14.1 Partially ordered sets	217
14.2 Dilworth's decomposition theorem	218
14.3 Path coverings	219
14.4 The weighted case	220
14.5 The chain and antichain polytopes	221
14.5a Path coverings algorithmically	222
14.6 Unions of directed cuts and antichains	224
14.6a Common saturating collections of chains	226
14.7 Unions of directed paths and chains	227
14.7a Common saturating collections of antichains	229
14.7b Conjugacy of partitions	230
14.8 Further results and notes	232
14.8a The Gallai-Milgram theorem	232
14.8b Partially ordered sets and distributive lattices	233
14.8c Maximal chains	235
14.8d Further notes	236
15 Connectivity and Gomory-Hu trees	237
15.1 Vertex-, edge-, and arc-connectivity	237
15.2 Vertex-connectivity algorithmically	239
15.2a Complexity survey for vertex-connectivity	241
15.2b Finding the 2-connected components	242
15.3 Arc- and edge-connectivity algorithmically	243
15.3a Complexity survey for arc- and edge-connectivity	246
15.3b Finding the 2-edge-connected components	247
15.4 Gomory-Hu trees	248
15.4a Minimum-requirement spanning tree	251
15.5 Further results and notes	252
15.5a Ear-decomposition of undirected graphs	252
15.5b Further notes	253

Part II: Bipartite Matching and Covering	257
16 Cardinality bipartite matching and vertex cover	259
16.1 M -augmenting paths	259
16.2 Frobenius' and König's theorems	260
16.2a Frobenius' proof of his theorem	262
16.2b Linear-algebraic proof of Frobenius' theorem	262
16.2c Rizzi's proof of König's matching theorem	263
16.3 Maximum-size bipartite matching algorithm	263
16.4 An $O(n^{1/2}m)$ algorithm	264
16.5 Finding a minimum-size vertex cover	265
16.6 Matchings covering given vertices	265
16.7 Further results and notes	267
16.7a Complexity survey for cardinality bipartite matching	267
16.7b Finding perfect matchings in regular bipartite graphs	267
16.7c The equivalence of Menger's theorem and König's theorem	275
16.7d Equivalent formulations in terms of matrices	276
16.7e Equivalent formulations in terms of partitions	276
16.7f On the complexity of bipartite matching and vertex cover	277
16.7g Further notes	277
16.7h Historical notes on bipartite matching	278
17 Weighted bipartite matching and the assignment problem	285
17.1 Weighted bipartite matching	285
17.2 The Hungarian method	286
17.3 Perfect matching and assignment problems	288
17.4 Finding a minimum-size w -vertex cover	289
17.5 Further results and notes	290
17.5a Complexity survey for maximum-weight bipartite matching	290
17.5b Further notes	290
17.5c Historical notes on weighted bipartite matching and optimum assignment	292
18 Linear programming methods and the bipartite matching polytope	301
18.1 The matching and the perfect matching polytope	301
18.2 Totally unimodular matrices from bipartite graphs	303
18.3 Consequences of total unimodularity	304

18.4	The vertex cover polytope	305
18.5	Further results and notes	305
18.5a	Derivation of Konig's matching theorem from the matching polytope	305
18.5b	Dual, primal-dual, primal?	305
18.5c	Adjacency and diameter of the matching polytope . .	307
18.5d	The perfect matching space of a bipartite graph . . .	308
18.5e	Up and down hull of the perfect matching polytope . .	309
18.5f	Matchings of given size	310
18.5g	Stable matchings	311
18.5h	Further notes	314
19	Bipartite edge cover and stable set	315
19.1	Matchings, edge covers, and Gallai's theorem	315
19.2	The Konig-Rado edge cover theorem	317
19.3	Finding a minimum-weight edge cover	317
19.4	Bipartite edge covers and total unimodularity	318
19.5	The edge cover and stable set polytope	318
19.5a	Some historical notes on bipartite edge covers	319
20	Bipartite edge-colouring	321
20.1	Edge-colourings of bipartite graphs	321
20.1a	Edge-colouring regular bipartite graphs	322
20.2	The capacitated case	322
20.3	Edge-colouring polyhedrally	323
20.4	Packing edge covers	324
20.5	Balanced colours	325
20.6	Packing perfect matchings	326
20.6a	Polyhedral interpretation	327
20.6b	Extensions	328
20.7	Covering by perfect matchings	329
20.7a	Polyhedral interpretation	330
20.8	The perfect matching lattice of a bipartite graph	331
20.9	Further results and notes	333
20.9a	Some further edge-colouring algorithms	333
20.9b	Complexity survey for bipartite edge-colouring . . .	334
20.9c	List-edge-colouring	335
20.9d	Further notes	336
21	Bipartite b-matchings and transportation	337
21.1	b -matchings and w -vertex covers	337
21.2	The b -matching polytope and the w -vertex cover polyhedron	338
21.3	Simple b -matchings and b -factors	339
21.4	Capacitated b -matchings	341

XVI Table of Contents

21.5	Bipartite b -matching and w -vertex cover algorithmically	342
21.6	Transportation	343
21.6a	Reduction of transshipment to transportation	345
21.6b	The transportation polytope	346
21.7	b -edge covers and w -stable sets	347
21.8	The b -edge cover and the w -stable set polyhedron	348
21.9	Simple b -edge covers	349
21.10	Capacitated b -edge covers	350
21.11	Relations between b -matchings and b -edge covers	351
21.12	Upper and lower bounds	353
21.13	Further results and notes	355
21.13a	Complexity survey on weighted bipartite b -matching and transportation	355
21.13b	The matchable set polytope	359
21.13c	Existence of matrices	359
21.13d	Further notes	361
21.13e	Historical notes on the transportation and transshipment problems	362
22	Transversals	378
22.1	Transversals	378
22.1a	Alternative proofs of Hall's marriage theorem	379
22.2	Partial transversals	380
22.3	Weighted transversals	382
22.4	Min-max relations for weighted transversals	382
22.5	The transversal polytope	383
22.6	Packing and covering of transversals	385
22.7	Further results and notes	387
22.7a	The capacitated case	387
22.7b	A theorem of Rado	389
22.7c	Further notes	389
22.7d	Historical notes on transversals	390
23	Common transversals	393
23.1	Common transversals	393
23.2	Weighted common transversals	395
23.3	Weighted common partial transversals	397
23.4	The common partial transversal polytope	399
23.5	The common transversal polytope	401
23.6	Packing and covering of common transversals	402
23.7	Further results and notes	407
23.7a	Capacitated common transversals	407
23.7b	Exchange properties	407
23.7c	Common transversals of three families	408
23.7d	Further notes	409

Part III: Nonbipartite Matching and Covering	411
24 Cardinality nonbipartite matching	413
24.1 Tutte's 1-factor theorem and the Tutte-Berge formula	413
24.1a Tutte's proof of his 1-factor theorem	415
24.1b Petersen's theorem	415
24.2 Cardinality matching algorithm	415
24.2a An $O(n^3)$ algorithm	418
24.3 Matchings covering given vertices	421
24.4 Further results and notes	422
24.4a Complexity survey for cardinality nonbipartite matching	422
24.4b The Edmonds-Gallai decomposition of a graph	423
24.4c Strengthening of Tutte's 1-factor theorem	425
24.4d Ear-decomposition of factor-critical graphs	425
24.4e Ear-decomposition of matching-covered graphs	426
24.4f Barriers in matching-covered graphs	427
24.4g Two-processor scheduling	428
24.4h The Tutte matrix and an algebraic matching algorithm	429
24.4i Further notes	430
24.4j Historical notes on nonbipartite matching	431
25 The matching polytope	438
25.1 The perfect matching polytope	438
25.2 The matching polytope	439
25.3 Total dual integrality: the Cunningham-Marsh formula	440
25.3a Direct proof of the Cunningham-Marsh formula	442
25.4 On the total dual integrality of the perfect matching constraints	443
25.5 Further results and notes	444
25.5a Adjacency and diameter of the matching polytope	444
25.5b Facets of the matching polytope	446
25.5c Polynomial-time solvability with the ellipsoid method	448
25.5d The matchable set polytope	450
25.5e Further notes	452
26 Weighted nonbipartite matching algorithmically	453
26.1 Introduction and preliminaries	453
26.2 Weighted matching algorithm	454
26.2a An $O(n^3)$ algorithm	456
26.3 Further results and notes	458

XVIII Table of Contents

26.3a	Complexity survey for weighted nonbipartite matching	458
26.3b	Derivation of the matching polytope characterization from the algorithm	459
26.3c	Further notes	459
27	Nonbipartite edge cover	461
27.1	Minimum-size edge cover	461
27.2	The edge cover polytope and total dual integrality	462
27.3	Further notes on edge covers	464
27.3a	Further notes	464
27.3b	Historical notes on edge covers	464
28	Edge-colouring	465
28.1	Vizing's theorem for simple graphs	465
28.2	Vizing's theorem for general graphs	467
28.3	NP-completeness of edge-colouring	468
28.4	Nowhere-zero flows and edge-colouring	470
28.5	Fractional edge-colouring	474
28.6	Conjectures	475
28.7	Edge-colouring polyhedrally	477
28.8	Packing edge covers	478
28.9	Further results and notes	480
28.9a	Shannon's theorem	480
28.9b	Further notes	480
28.9c	Historical notes on edge-colouring	482
29	<i>T</i>-joins, undirected shortest paths, and the Chinese postman	485
29.1	<i>T</i> -joins	485
29.2	The shortest path problem for undirected graphs	487
29.3	The Chinese postman problem	487
29.4	<i>T</i> -joins and <i>T</i> -cuts	488
29.5	The up hull of the <i>T</i> -join polytope	490
29.6	The <i>T</i> -join polytope	491
29.7	Sums of circuits	493
29.8	Integer sums of circuits	494
29.9	The <i>T</i> -cut polytope	498
29.10	Finding a minimum-capacity <i>T</i> -cut	499
29.11	Further results and notes	500
29.11a	Minimum-mean length circuit	500
29.11b	Packing <i>T</i> -cuts	501
29.11c	Packing <i>T</i> -joins	507
29.11d	Maximum joins	510
29.11e	Odd paths	515

29.11f Further notes	517
29.11g On the history of the Chinese postman problem	519
30 2-matchings, 2-covers, and 2-factors	520
30.1 2-matchings and 2-vertex covers	520
30.2 Fractional matchings and vertex covers	521
30.3 The fractional matching polytope	522
30.4 The 2-matching polytope	522
30.5 The weighted 2-matching problem	523
30.5a Maximum-size 2-matchings and maximum-size matchings	524
30.6 Simple 2-matchings and 2-factors	526
30.7 The simple 2-matching polytope and the 2-factor polytope ..	528
30.8 Total dual integrality.....	531
30.9 2-edge covers and 2-stable sets	531
30.10 Fractional edge covers and stable sets	532
30.11 The fractional edge cover polyhedron.....	533
30.12 The 2-edge cover polyhedron	533
30.13 Total dual integrality of the 2-edge cover constraints	534
30.14 Simple 2-edge covers	535
30.15 Graphs with $\nu(G) = \tau(G)$ and $\alpha(G) = \rho(G)$	536
30.16 Excluding triangles	539
30.16a Excluding higher polygons	544
30.16b Packing edges and factor-critical subgraphs	544
30.16c 2-factors without short circuits	545
31 b-matchings.....	546
31.1 b -matchings	546
31.2 The b -matching polytope	547
31.3 Total dual integrality.....	550
31.4 The weighted b -matching problem	554
31.5 If b is even	556
31.6 If b is constant	558
31.7 Further results and notes	559
31.7a Complexity survey for the b -matching problem	559
31.7b Facets and minimal systems for the b -matching polytope	559
31.7c Regularizable graphs	560
31.7d Further notes	561
32 Capacitated b-matchings	562
32.1 Capacitated b -matchings	562
32.2 The capacitated b -matching polytope.....	564
32.3 Total dual integrality.....	566
32.4 The weighted capacitated b -matching problem	567

32.4a	Further notes	567
33	Simple b-matchings and b-factors	569
33.1	Simple b -matchings and b -factors	569
33.2	The simple b -matching polytope and the b -factor polytope	570
33.3	Total dual integrality	570
33.4	The weighted simple b -matching and b -factor problem	571
33.5	If b is constant	572
33.6	Further results and notes	573
33.6a	Complexity results	573
33.6b	Degree-sequences	573
33.6c	Further notes	574
34	b-edge covers	575
34.1	b -edge covers	575
34.2	The b -edge cover polyhedron	576
34.3	Total dual integrality	576
34.4	The weighted b -edge cover problem	577
34.5	If b is even	578
34.6	If b is constant	578
34.7	Capacitated b -edge covers	579
34.8	Simple b -edge covers	581
34.8a	Simple b -edge covers and b -matchings	582
34.8b	Capacitated b -edge covers and b -matchings	583
35	Upper and lower bounds	584
35.1	Upper and lower bounds	584
35.2	Convex hull	586
35.3	Total dual integrality	589
35.4	Further results and notes	591
35.4a	Further results on subgraphs with prescribed degrees	591
35.4b	Odd walks	593
36	Bidirected graphs	594
36.1	Bidirected graphs	594
36.2	Convex hull	597
36.3	Total dual integrality	598
36.4	Including parity conditions	600
36.5	Convex hull	604
36.5a	Convex hull of vertex-disjoint circuits	605
36.6	Total dual integrality	605
36.7	Further results and notes	607
36.7a	The Chvátal rank	607
36.7b	Further notes	608

37	The dimension of the perfect matching polytope	609
37.1	The dimension of the perfect matching polytope	609
37.2	The perfect matching space	611
37.3	The brick decomposition	612
37.4	The brick decomposition of a bipartite graph	613
37.5	Braces	614
37.6	Bricks	614
37.7	Matching-covered graphs without nontrivial tight cuts	617
38	The perfect matching lattice	619
38.1	The perfect matching lattice	619
38.2	The perfect matching lattice of the Petersen graph	620
38.3	A further fact on the Petersen graph	621
38.4	Various useful observations	622
38.5	Simple barriers	624
38.6	The perfect matching lattice of a brick	630
38.7	Synthesis and further consequences of the previous results	643
38.8	What further might (not) be true	644
38.9	Further results and notes	646
38.9a	The perfect 2-matching space and lattice	646
38.9b	Further notes	647

Volume B

Part IV: Matroids and Submodular Functions	649
39 Matroids	651
39.1 Matroids	651
39.2 The dual matroid	652
39.3 Deletion, contraction, and truncation	653
39.4 Examples of matroids	654
39.4a Relations between transversal matroids and gammoids	659
39.5 Characterizing matroids by bases	662
39.6 Characterizing matroids by circuits	662
39.6a A characterization of Lehman	663
39.7 Characterizing matroids by rank functions	664
39.8 The span function and flats	666
39.8a Characterizing matroids by span functions	666
39.8b Characterizing matroids by flats	667
39.8c Characterizing matroids in terms of lattices	668
39.9 Further exchange properties	669
39.9a Further properties of bases	671
39.10 Further results and notes	671
39.10a Further notes	671
39.10b Historical notes on matroids	672
40 The greedy algorithm and the independent set polytope ..	688
40.1 The greedy algorithm	688
40.2 The independent set polytope	690
40.3 The most violated inequality	693
40.3a Facets and adjacency on the independent set polytope	698
40.3b Further notes	699
41 Matroid intersection	700
41.1 Matroid intersection theorem	700
41.1a Applications of the matroid intersection theorem ..	702
41.1b Woodall's proof of the matroid intersection theorem..	704
41.2 Cardinality matroid intersection algorithm	705
41.3 Weighted matroid intersection algorithm	707
41.3a Speeding up the weighted matroid intersection algorithm	710
41.4 Intersection of the independent set polytopes	712
41.4a Facets of the common independent set polytope ..	717
41.4b Up and down hull of the common base polytope ..	719

41.5	Further results and notes	720
41.5a	Menger's theorem for matroids	720
41.5b	Exchange properties	721
41.5c	Jump systems	722
41.5d	Further notes	723
42	Matroid union	725
42.1	Matroid union theorem	725
42.1a	Applications of the matroid union theorem	727
42.1b	Horn's proof	729
42.2	Polyhedral applications	730
42.3	Matroid union algorithm	731
42.4	The capacitated case: fractional packing and covering of bases	732
42.5	The capacitated case: integer packing and covering of bases	734
42.6	Further results and notes	736
42.6a	Induction of matroids	736
42.6b	List-colouring	737
42.6c	Strongly base orderable matroids	738
42.6d	Blocking and antblocking polyhedra	741
42.6e	Further notes	743
42.6f	Historical notes on matroid union	743
43	Matroid matching	745
43.1	Infinite matroids	745
43.2	Matroid matchings	746
43.3	Circuits	747
43.4	A special class of matroids	747
43.5	A min-max formula for maximum-size matroid matching	751
43.6	Applications of the matroid matching theorem	753
43.7	A Gallai theorem for matroid matching and covering	756
43.8	Linear matroid matching algorithm	757
43.9	Matroid matching is not polynomial-time solvable in general	762
43.10	Further results and notes	763
43.10a	Optimal path-matching	763
43.10b	Further notes	764
44	Submodular functions and polymatroids	766
44.1	Submodular functions and polymatroids	766
44.1a	Examples	768
44.2	Optimization over polymatroids by the greedy method	771
44.3	Total dual integrality	773
44.4	f is determined by EP_f	773
44.5	Supermodular functions and contrapolymatroids	774

XXIV Table of Contents

44.6	Further results and notes	775
44.6a	Submodular functions and matroids	775
44.6b	Reducing integer polymatroids to matroids	776
44.6c	The structure of polymatroids	776
44.6d	Characterization of polymatroids	779
44.6e	Operations on submodular functions and polymatroids	781
44.6f	Duals of polymatroids	782
44.6g	Induction of polymatroids	782
44.6h	Lovász's generalization of König's matching theorem . .	783
44.6i	Further notes	784
45	Submodular function minimization	786
45.1	Submodular function minimization	786
45.2	Orders and base vectors	787
45.3	A subroutine	787
45.4	Minimizing a submodular function	789
45.5	Running time of the algorithm	790
45.6	Minimizing a symmetric submodular function	792
45.7	Minimizing a submodular function over the odd sets	793
46	Polymatroid intersection	795
46.1	Box-total dual integrality of polymatroid intersection . .	795
46.2	Consequences	796
46.3	Contrapolyomatroid intersection	797
46.4	Intersecting a polymatroid and a contrapolyomatroid . .	798
46.5	Frank's discrete sandwich theorem	799
46.6	Integer decomposition	800
46.7	Further results and notes	801
46.7a	Up and down hull of the common base vectors	801
46.7b	Further notes	804
47	Polymatroid intersection algorithmically	805
47.1	A maximum-size common vector in two polymatroids . .	805
47.2	Maximizing a coordinate of a common base vector . . .	807
47.3	Weighted polymatroid intersection in polynomial time .	809
47.4	Weighted polymatroid intersection in strongly polynomial time	811
47.5	Contrapolyomatroids	818
47.6	Intersecting a polymatroid and a contrapolyomatroid . .	818
47.6a	Further notes	819

48 Dilworth truncation	820
48.1 If $f(\emptyset) < 0$	820
48.2 Dilworth truncation	821
48.2a Applications and interpretations	823
48.3 Intersection	825
49 Submodularity more generally	826
49.1 Submodular functions on a lattice family	826
49.2 Intersection	828
49.3 Complexity	829
49.4 Submodular functions on an intersecting family	832
49.5 Intersection	833
49.6 From an intersecting family to a lattice family	834
49.7 Complexity	835
49.8 Intersecting a polymatroid and a contrapolyomatroid	837
49.9 Submodular functions on a crossing family	838
49.10 Complexity	840
49.10a Nonemptiness of the base polyhedron	841
49.11 Further results and notes	842
49.11a Minimizing a submodular function over a subcollection of a lattice family	842
49.11b Generalized polymatroids	845
49.11c Supermodular colourings	849
49.11d Further notes	851

Part V: Trees, Branchings, and Connectors	853
--	------------

50 Shortest spanning trees	855
50.1 Shortest spanning trees	855
50.2 Implementing Prim's method	857
50.3 Implementing Kruskal's method	858
50.3a Parallel forest-merging	859
50.3b A dual greedy algorithm	859
50.4 The longest forest and the forest polytope	860
50.5 The shortest connector and the connector polytope	862
50.6 Further results and notes	864
50.6a Complexity survey for shortest spanning tree	864
50.6b Characterization of shortest spanning trees	865
50.6c The maximum reliability problem	866
50.6d Exchange properties of forests	867
50.6e Uniqueness of shortest spanning tree	868
50.6f Forest covers	869
50.6g Further notes	870
50.6h Historical notes on shortest spanning trees	871

XXVI Table of Contents

51	Packing and covering of trees	877
51.1	Unions of forests	877
51.2	Disjoint spanning trees	877
51.3	Covering by forests	878
51.4	Complexity	879
51.5	Further results and notes	889
51.5a	Complexity survey for tree packing and covering	889
51.5b	Further notes	892
52	Longest branchings and shortest arborescences	893
52.1	Finding a shortest r -arborescence	893
52.1a	r -arborescences as common bases of two matroids	895
52.2	Related problems	895
52.3	A min-max relation for shortest r -arborescences	896
52.4	The r -arborescence polytope	897
52.4a	Uncrossing cuts	899
52.5	A min-max relation for longest branchings	900
52.6	The branching polytope	901
52.7	The arborescence polytope	901
52.8	Further results and notes	902
52.8a	Complexity survey for shortest r -arborescence	902
52.8b	Concise LP-formulation for shortest r -arborescence	902
52.8c	Further notes	903
53	Packing and covering of branchings and arborescences	904
53.1	Disjoint branchings	904
53.2	Disjoint r -arborescences	905
53.3	The capacitated case	907
53.4	Disjoint arborescences	908
53.5	Covering by branchings	908
53.6	An exchange property of branchings	909
53.7	Covering by r -arborescences	911
53.8	Minimum-length unions of k r -arborescences	913
53.9	The complexity of finding disjoint arborescences	918
53.10	Further results and notes	921
53.10a	Complexity survey for disjoint arborescences	921
53.10b	Arborescences with roots in given subsets	923
53.10c	Disclaimers	925
53.10d	Further notes	926
54	Biconnectors and bibranchings	928
54.1	Shortest $R - S$ biconnectors	928
54.2	Longest $R - S$ biforests	930
54.3	Disjoint $R - S$ biconnectors	931
54.4	Covering by $R - S$ biforests	934

54.5	Minimum-size bibranchings	934
54.6	Shortest bibranchings	935
54.6a	Longest bifurcations	937
54.7	Disjoint bibranchings.....	940
54.7a	Proof using supermodular colourings	943
54.7b	Covering by bifurcations.....	943
54.7c	Disjoint $R - S$ biconnectors and $R - S$ bibranchings..	944
54.7d	Covering by $R - S$ biforests and by $R - S$ bifurcations	944
55	Minimum directed cut covers and packing directed cuts ..	946
55.1	Minimum directed cut covers and packing directed cuts.....	946
55.2	The Lucchesi-Younger theorem	947
55.3	Directed cut k -covers.....	949
55.4	Feedback arc sets	951
55.5	Complexity	953
55.5a	Finding a dual solution.....	954
55.6	Further results and notes	956
55.6a	Complexity survey for minimum-size directed cut cover	956
55.6b	Feedback arc sets in linklessly embeddable digraphs ..	956
55.6c	Feedback vertex sets	958
55.6d	The bipartite case	959
55.6e	Further notes	960
56	Minimum directed cuts and packing directed cut covers ..	962
56.1	Minimum directed cuts and packing directed cut covers.....	962
56.2	Source-sink connected digraphs	964
56.3	Other cases where Woodall's conjecture is true	967
56.3a	Further notes	968
57	Strong connectors ..	969
57.1	Making a directed graph strongly connected.....	969
57.2	Shortest strong connectors	970
57.3	Polyhedrally	973
57.4	Disjoint strong connectors	973
57.5	Complexity	975
57.5a	Crossing families	976
58	The traveling salesman problem ..	981
58.1	The traveling salesman problem	981
58.2	NP-completeness of the TSP	982
58.3	Branch-and-bound techniques	982
58.4	The symmetric traveling salesman polytope	983
58.5	The subtour elimination constraints	984

XXVIII Table of Contents

58.6	1-trees and Lagrangean relaxation	985
58.7	The 2-factor constraints	986
58.8	The clique tree inequalities.....	987
58.8a	Christofides' heuristic for the TSP	989
58.8b	Further notes on the symmetric traveling salesman problem	990
58.9	The asymmetric traveling salesman problem	992
58.10	Directed 1-trees	993
58.10a	An integer programming formulation	993
58.10b	Further notes on the asymmetric traveling salesman problem	994
58.11	Further notes on the traveling salesman problem	995
58.11a	Further notes	995
58.11b	Historical notes on the traveling salesman problem	996
59	Matching forests	1005
59.1	Introduction	1005
59.2	The maximum size of a matching forest	1006
59.3	Perfect matching forests	1007
59.4	An exchange property of matching forests.....	1008
59.5	The matching forest polytope	1011
59.6	Further results and notes	1015
59.6a	Matching forests in partitionable mixed graphs	1015
59.6b	Further notes	1017
60	Submodular functions on directed graphs	1018
60.1	The Edmonds-Giles theorem	1018
60.1a	Applications	1020
60.1b	Generalized polymatroids and the Edmonds-Giles theorem	1020
60.2	A variant	1021
60.2a	Applications	1023
60.3	Further results and notes	1025
60.3a	Lattice polyhedra.....	1025
60.3b	Polymatroidal network flows	1028
60.3c	A general model	1029
60.3d	Packing cuts and Győri's theorem	1030
60.3e	Further notes	1034
61	Graph orientation	1035
61.1	Orientations with bounds on in- and outdegrees	1035
61.2	2-edge-connectivity and strongly connected orientations	1037
61.2a	Strongly connected orientations with bounds on degrees	1038
61.3	Nash-Williams' orientation theorem	1040

61.4	<i>k</i> -arc-connected orientations of $2k$ -edge-connected graphs	1044
61.4a	Complexity	1045
61.4b	<i>k</i> -arc-connected orientations with bounds on degrees	1045
61.4c	Orientations of graphs with lower bounds on indegrees of sets	1046
61.4d	Further notes	1047
62	Network synthesis	1049
62.1	Minimal <i>k</i> -(edge-)connected graphs	1049
62.2	The network synthesis problem	1051
62.3	Minimum-capacity network design	1052
62.4	Integer realizations and <i>r</i> -edge-connected graphs	1055
63	Connectivity augmentation	1058
63.1	Making a directed graph <i>k</i> -arc-connected	1058
63.1a	<i>k</i> -arc-connectors with bounds on degrees	1061
63.2	Making an undirected graph 2-edge-connected	1062
63.3	Making an undirected graph <i>k</i> -edge-connected	1063
63.3a	<i>k</i> -edge-connectors with bounds on degrees	1066
63.4	<i>r</i> -edge-connectivity and <i>r</i> -edge-connectors	1067
63.5	Making a directed graph <i>k</i> -vertex-connected	1074
63.6	Making an undirected graph <i>k</i> -vertex-connected	1077
63.6a	Further notes	1078

Part VI: Cliques, Stable Sets, and Colouring	1081
---	-------------

64	Cliques, stable sets, and colouring	1083
64.1	Terminology and notation	1083
64.2	NP-completeness	1084
64.3	Bounds on the colouring number	1085
64.3a	Brooks' upper bound on the colouring number	1086
64.3b	Hadwiger's conjecture	1086
64.4	The stable set, clique, and vertex cover polytope	1088
64.4a	Facets and adjacency on the stable set polytope	1088
64.5	Fractional stable sets	1090
64.5a	Further on the fractional stable set polytope	1091
64.6	Fractional vertex covers	1093
64.6a	A bound of Lorentzen	1095
64.7	The clique inequalities	1095
64.8	Fractional and weighted colouring numbers	1096
64.8a	The ratio of $\chi(G)$ and $\chi^*(G)$	1098
64.8b	The Chvátal rank	1098
64.9	Further results and notes	1099

64.9a	Graphs with polynomial-time stable set algorithm	1099
64.9b	Colourings and orientations	1101
64.9c	Algebraic methods	1102
64.9d	Approximation algorithms	1103
64.9e	Further notes	1104
65	Perfect graphs: general theory	1106
65.1	Introduction to perfect graphs	1106
65.2	The perfect graph theorem	1108
65.3	Replication	1109
65.4	Perfect graphs and polyhedra	1110
65.4a	Lovász's proof of the replication lemma	1111
65.5	Decomposition of Berge graphs	1112
65.5a	0- and 1-joins	1112
65.5b	The 2-join	1113
65.6	Pre-proof work on the strong perfect graph conjecture	1115
65.6a	Partitionable graphs	1116
65.6b	More characterizations of perfect graphs	1118
65.6c	The stable set polytope of minimally imperfect graphs	1118
65.6d	Graph classes	1120
65.6e	The P_4 -structure of a graph and a semi-strong perfect graph theorem	1122
65.6f	Further notes on the strong perfect graph conjecture	1123
65.7	Further results and notes	1125
65.7a	Perz and Rolewicz's proof of the perfect graph theorem	1125
65.7b	Kernel solvability	1126
65.7c	The amalgam	1130
65.7d	Diperfect graphs	1131
65.7e	Further notes	1133
66	Classes of perfect graphs	1135
66.1	Bipartite graphs and their line graphs	1135
66.2	Comparability graphs	1137
66.3	Chordal graphs	1138
66.3a	Chordal graphs as intersection graphs of subtrees of a tree	1142
66.4	Meyniel graphs	1143
66.5	Further results and notes	1145
66.5a	Strongly perfect graphs	1145
66.5b	Perfectly orderable graphs	1146
66.5c	Unimodular graphs	1147
66.5d	Further classes of perfect graphs	1148

66.5e	Further notes	1149
67	Perfect graphs: polynomial-time solvability	1152
67.1	Optimum clique and colouring in perfect graphs algorithmically	1152
67.2	Weighted clique and colouring algorithmically	1155
67.3	Strong polynomial-time solvability	1159
67.4	Further results and notes	1159
67.4a	Further on $\vartheta(G)$	1159
67.4b	The Shannon capacity $\Theta(G)$	1167
67.4c	Clique cover numbers of products of graphs	1172
67.4d	A sharper upper bound $\vartheta'(G)$ on $\alpha(G)$	1173
67.4e	An operator strengthening convex bodies	1173
67.4f	Further notes	1175
67.4g	Historical notes on perfect graphs	1176
68	T-perfect graphs	1186
68.1	T-perfect graphs	1186
68.2	Strongly t-perfect graphs	1187
68.3	Strong t-perfection of odd- K_4 -free graphs	1188
68.4	On characterizing t-perfection	1194
68.5	A combinatorial min-max relation	1196
68.6	Further results and notes	1200
68.6a	The w -stable set polyhedron	1200
68.6b	Bidirected graphs	1201
68.6c	Characterizing odd- K_4 -free graphs by mixing stable sets and vertex covers	1203
68.6d	Orientations of discrepancy 1	1204
68.6e	Colourings and odd K_4 -subdivisions	1206
68.6f	Homomorphisms	1207
68.6g	Further notes	1207
69	Claw-free graphs	1208
69.1	Introduction	1208
69.2	Maximum-size stable set in a claw-free graph	1208
69.3	Maximum-weight stable set in a claw-free graph	1213
69.4	Further results and notes	1216
69.4a	On the stable set polytope of a claw-free graph	1216
69.4b	Further notes	1217

Volume C

Part VII: Multiflows and Disjoint Paths	1219
<hr/>	
70 Multiflows and disjoint paths	1221
70.1 Directed multiflow problems	1221
70.2 Undirected multiflow problems	1222
70.3 Disjoint paths problems	1223
70.4 Reductions	1223
70.5 Complexity of the disjoint paths problem	1224
70.6 Complexity of the fractional multiflow problem	1225
70.7 The cut condition for directed graphs	1227
70.8 The cut condition for undirected graphs	1228
70.9 Relations between fractional, half-integer, and integer solutions.....	1230
70.10 The Euler condition	1233
70.11 Survey of cases where a good characterization has been found	1234
70.12 Relation between the cut condition and fractional cut packing.....	1236
70.12a Sufficiency of the cut condition sometimes implies an integer multiflow.....	1238
70.12b The cut condition and integer multiflows in directed graphs	1241
70.13 Further results and notes	1242
70.13a Fixing the number of commodities in undirected graphs	1242
70.13b Fixing the number of commodities in directed graphs	1243
70.13c Disjoint paths in acyclic digraphs	1244
70.13d A column generation technique for multiflows	1245
70.13e Approximate max-flow min-cut theorems for multiflows	1247
70.13f Further notes	1248
70.13g Historical notes on multicommodity flows	1249
71 Two commodities	1251
71.1 The Rothschild-Whinston theorem and Hu's 2-commodity flow theorem	1251
71.1a Nash-Williams' proof of the Rothschild-Whinston theorem	1254
71.2 Consequences	1255
71.3 2-commodity cut packing	1257
71.4 Further results and notes	1261

71.4a	Two disjoint paths in undirected graphs	1261
71.4b	A directed 2-commodity flow theorem	1262
71.4c	Kleitman, Martin-Löf, Rothschild, and Whinston's theorem	1263
71.4d	Further notes	1265
72	Three or more commodities	1266
72.1	Demand graphs for which the cut condition is sufficient	1266
72.2	Three commodities	1271
72.2a	The $K_{2,3}$ -metric condition	1273
72.2b	Six terminals	1275
72.3	Cut packing	1276
73	T-paths	1279
73.1	Disjoint T -paths	1279
73.1a	Disjoint T -paths with the matroid matching algorithm	1283
73.1b	Polynomial-time findability of edge-disjoint T -paths	1285
73.1c	A feasibility characterization for integer K_3 -flows	1286
73.2	Fractional packing of T -paths	1287
73.2a	Direct proof of Corollary 73.2d	1288
73.3	Further results and notes	1289
73.3a	Further notes on Mader's theorem	1289
73.3b	A generalization of fractionally packing T -paths	1290
73.3c	Lockable collections	1291
73.3d	Mader matroids	1292
73.3e	Minimum-cost maximum-value multiflows	1294
73.3f	Further notes	1295
74	Planar graphs	1296
74.1	All nets spanned by one face: the Okamura-Seymour theorem	1296
74.1a	Complexity survey	1299
74.1b	Graphs on the projective plane	1299
74.1c	If only inner vertices satisfy the Euler condition	1302
74.1d	Distances and cut packing	1304
74.1e	Linear algebra and distance realizability	1305
74.1f	Directed planar graphs with all terminals on the outer boundary	1307
74.2	$G + H$ planar	1307
74.2a	Distances and cut packing	1308
74.2b	Deleting the Euler condition if $G + H$ is planar	1309
74.3	Okamura's theorem	1311
74.3a	Distances and cut packing	1313

XXXIV Table of Contents

74.3b	The Klein bottle	1314
74.3c	Commodities spanned by three or more faces	1316
74.4	Further results and notes	1318
74.4a	Another theorem of Okamura	1318
74.4b	Some other planar cases where the cut condition is sufficient	1320
74.4c	Vertex-disjoint paths in planar graphs	1320
74.4d	Grid graphs	1323
74.4e	Further notes	1325
75	Cuts, odd circuits, and multiflows	1326
75.1	Weakly and strongly bipartite graphs	1326
75.1a	NP-completeness of maximum cut	1328
75.1b	Planar graphs	1328
75.2	Signed graphs	1329
75.3	Weakly, evenly, and strongly bipartite signed graphs	1330
75.4	Characterizing strongly bipartite signed graphs	1331
75.5	Characterizing weakly and evenly bipartite signed graphs	1334
75.6	Applications to multiflows	1341
75.7	The cut cone and the cut polytope	1342
75.8	The maximum cut problem and semidefinite programming	1345
75.9	Further results and notes	1348
75.9a	Cuts and stable sets	1348
75.9b	Further notes	1350
76	Homotopy and graphs on surfaces	1352
76.1	Graphs, curves, and their intersections: terminology and notation	1352
76.2	Making curves minimally crossing by Reidemeister moves	1353
76.3	Decomposing the edges of an Eulerian graph on a surface	1354
76.4	A corollary on lengths of closed curves	1356
76.5	A homotopic circulation theorem	1357
76.6	Homotopic paths in planar graphs with holes	1361
76.7	Vertex-disjoint paths and circuits of prescribed homotopies	1367
76.7a	Vertex-disjoint circuits of prescribed homotopies	1367
76.7b	Vertex-disjoint homotopic paths in planar graphs with holes	1368
76.7c	Disjoint trees	1371

Part VIII: Hypergraphs	1373
77 Packing and blocking in hypergraphs: elementary notions	1375
77.1 Elementary hypergraph terminology and notation	1375
77.2 Deletion, restriction, and contraction	1376
77.3 Duplication and parallelization	1376
77.4 Clutters	1376
77.5 Packing and blocking	1377
77.6 The blocker	1377
77.7 Fractional matchings and vertex covers	1378
77.8 k -matchings and k -vertex covers	1378
77.9 Further results and notes	1379
77.9a Bottleneck extrema	1379
77.9b The ratio of τ and τ^*	1380
77.9c Further notes	1381
78 Ideal hypergraphs	1383
78.1 Ideal hypergraphs	1383
78.2 Characterizations of ideal hypergraphs	1384
78.3 Minimally nonideal hypergraphs	1386
78.4 Properties of minimally nonideal hypergraphs: Lehman's theorem	1387
78.4a Application of Lehman's theorem: Guenin's theorem	1392
78.4b Ideality is in co-NP	1394
78.5 Further results and notes	1395
78.5a Composition of clutters	1395
78.5b Further notes	1395
79 Mengerian hypergraphs	1397
79.1 Mengerian hypergraphs	1397
79.1a Examples of Mengerian hypergraphs	1399
79.2 Minimally non-Mengerian hypergraphs	1400
79.3 Further results and notes	1401
79.3a Packing hypergraphs	1401
79.3b Restrictions instead of parallelizations	1402
79.3c Equivalences for k -matchings and k -vertex covers	1402
79.3d A general technique	1403
79.3e Further notes	1404

XXXVI Table of Contents

80 Binary hypergraphs	1406
80.1 Binary hypergraphs	1406
80.2 Binary hypergraphs and binary matroids	1406
80.3 The blocker of a binary hypergraph	1407
80.3a Further characterizations of binary clutters	1408
80.4 On characterizing binary ideal hypergraphs	1408
80.5 Seymour's characterization of binary Mengerian hypergraphs	1409
80.5a Applications of Seymour's theorem	1413
80.6 Mengerian matroids	1415
80.6a Oriented matroids	1415
80.7 Further results and notes	1416
80.7a $\tau_2(H) = 2\tau(H)$ for binary hypergraphs H	1416
80.7b Application: T -joins and T -cuts	1417
80.7c Box-integrality of $k \cdot P_H$	1418
81 Matroids and multiflows	1419
81.1 Multiflows in matroids	1419
81.2 Integer k -flowing	1420
81.3 1-flowing and 1-cycling	1421
81.4 2-flowing and 2-cycling	1421
81.5 3-flowing and 3-cycling	1422
81.6 4-flowing, 4-cycling, ∞ -flowing, and ∞ -cycling	1423
81.7 The circuit cone and cycle polytope of a matroid	1424
81.8 The circuit space and circuit lattice of a matroid	1425
81.9 Nonnegative integer sums of circuits	1425
81.10 Nowhere-zero flows and circuit double covers in matroids	1426
82 Covering and antiblocking in hypergraphs	1428
82.1 Elementary concepts	1428
82.2 Fractional edge covers and stable sets	1429
82.3 k -edge covers and k -stable sets	1429
82.4 The antiblocker and conformality	1430
82.4a Gilmore's characterization of conformality	1431
82.5 Perfect hypergraphs	1431
82.6 Further notes	1434
82.6a Some equivalences for the k -parameters	1434
82.6b Further notes	1437
83 Balanced and unimodular hypergraphs	1439
83.1 Balanced hypergraphs	1439
83.2 Characterizations of balanced hypergraphs	1440
83.2a Totally balanced matrices	1444
83.2b Examples of balanced hypergraphs	1447
83.2c Balanced $0, \pm 1$ matrices	1447

Table of Contents XXXVII

83.3 Unimodular hypergraphs	1448
83.3a Further notes	1450
Survey of Problems, Questions, and Conjectures	1453
References	1463
Name Index	1767
Subject Index	1807
Greek graph and hypergraph functions	1880