Part VIII

Hypergraphs

Part VIII: Hypergraphs

Hypergraphs form a framework in which many of the min-max relations discussed before can be formulated. This is not to say that they all can be *derived* from general hypergraph theory. Rather, hypergraph theory yields relations between different min-max relations, for instance through the blocking and antiblocking relations of hypergraphs and of polyhedra. Moreover, certain hereditary min-max relations can be characterized by equivalent but weaker conditions. This can be helpful in proving min-max relations for special classes of hypergraphs.

The material in this part is grouped by the hypergraph generalizations of four notions that also played a central role in the earlier parts on graphs: matching, vertex cover, edge cover, and stable set. Among the landmarks of this part are theorems of Lehman on minimally nonideal hypergraphs and of Seymour characterizing binary Mengerian hypergraphs.

Chapters:

Chapter 77

Packing and blocking in hypergraphs: elementary notions

Packing in hypergraphs asks for a maximum number of disjoint edges. *Blocking* concerns the minimum number of vertices intersecting each edge. In this chapter we give basic concepts of hypergraphs, in particular those related to packing and blocking.

77.1. Elementary hypergraph terminology and notation

We start with some elementary definitions and notation on hypergraphs. A *hypergraph*[1] is a pair $H = (V, \mathcal{E})$, where V is a finite set and $\mathcal{E}$ is a family of subsets of V. Any element of V is called a *vertex* of H and any set in $\mathcal{E}$ an *edge* of H. We sometimes denote the vertex set and the edge set of H by VH and EH respectively. In our discussions, we can assume without loss of generality that $\mathcal{E}$ is a collection of subsets (rather than a family, with multiplicities).

Graphs are special cases of hypergraphs: they are the hypergraphs that have all its edges of size 2.

The $\mathcal{E} \times V$ *incidence matrix* of H is the $\mathcal{E} \times V$ matrix M with $M_{F,v} = 1$ if $v \in F$ and $M_{F,v} = 0$ if $v \notin F$ (for $v \in V$, $F \in \mathcal{E}$). For most of our purposes, studying a hypergraph is equivalent to studying its incidence matrix. Any result on hypergraphs is simultaneously a result on $0, 1$ matrices, and conversely. We will go back and forth between both interpretations and often choose the most appropriate one.

The *dual hypergraph* H^* of a hypergraph $H = (V, \mathcal{E})$ is the hypergraph with vertex set $\mathcal{E}$ and edges all sets $\{E \in \mathcal{E} \mid v \in E\}$ for $v \in V$. So the incidence matrix of H^* is the transpose of the incidence matrix of H.

For any hypergraph $H = (V, \mathcal{E})$ we denote

(77.1) $$r_{\min}(H) := \min\{|E| \bigm| E \in \mathcal{E}\} \text{ and } r_{\max}(H) := \max\{|E| \bigm| E \in \mathcal{E}\}.$$

[1] Berge [1996] said that the name 'hypergraph' was invented in 1969 by J.-M. Pla, after earlier attempts to call it 'graphoid' (e.g. Berge [1969]).

77.2. Deletion, restriction, and contraction

We describe two operations on a hypergraph $H = (V, \mathcal{E})$, deletion and contraction. Let $v \in V$, and define:

(77.2) $$\mathcal{E} \setminus v := \{E \in \mathcal{E} \mid v \notin E\}, \quad H \setminus v := (V \setminus \{v\}, \mathcal{E} \setminus v),$$
$$\mathcal{E}/v := \{E \setminus \{v\} \mid E \in \mathcal{E}\}, \quad H/v := (V \setminus \{v\}, \mathcal{E}/v).$$

Replacing H by $H \setminus v$ is called *deleting* v and replacing H by H/v is called *contracting* v. We say that H' is a *restriction* of H if it arises by a series of deletions, and a *contraction* of H if it arises by a series of contractions. The *restriction to* $U \subseteq V$ is $H \setminus (V \setminus U)$.

Deletions and contractions commute in the ways one may expect: for distinct $u, v \in V$ one has

(77.3) $(H/u)/v = (H/v)/u$, $(H \setminus u) \setminus v = (H \setminus v) \setminus u$, and $(H/u) \setminus v = (H \setminus v)/u$.

Deletion of an edge E means replacing $\mathcal{E}$ by $\mathcal{E} \setminus \{E\}$. A hypergraph H' is called a *minor* of H, if H' arises from H by a series of deletions and contractions of vertices, and deletions of edges that are not inclusionwise minimal edges.

77.3. Duplication and parallelization

Let $H = (V, \mathcal{E})$ be a hypergraph and let $v \in V$. *Duplicating* v means extending V by a new vertex, v' say, and replacing $\mathcal{E}$ by

(77.4) $$\mathcal{E} \cup \{(E \setminus \{v\}) \cup \{v'\} \mid v \in E \in \mathcal{E}\}.$$

A hypergraph obtained from H by a sequence of deletions and duplications of vertices, is called a *parallelization* of H. If $w : V \to \mathbb{Z}_+$, we denote by H^w the result of deleting any vertex v with $w(v) = 0$, and duplicating any vertex v $w(v) - 1$ times, if $w(v) \geq 2$. So restrictions correspond to functions $w : V \to \{0, 1\}$. In a certain sense, contractions correspond to functions $w : V \to \{1, \infty\}$.

77.4. Clutters

For any hypergraph $H = (V, \mathcal{E})$, define

(77.5) $$H^{\min} := (V, \{F \in \mathcal{E} \mid \text{there is no } E \in \mathcal{E} \text{ with } E \subset F\}) \text{ and}$$
$$H^{\uparrow} := (V, \{F \subseteq V \mid \text{there is an } E \in \mathcal{E} \text{ with } E \subseteq F\}).$$

A hypergraph $H = (V, \mathcal{E})$ is called a *clutter* if no two sets in $\mathcal{E}$ are contained in each other[2]. So for any hypergraph, $H^{\min}$ is a clutter.

[2] The term 'clutter' was introduced by Edmonds and Fulkerson [1970].

77.5. Packing and blocking

Let $H = (V, \mathcal{E})$ be a hypergraph. The following notions generalize the corresponding notions defined for graphs.

A *vertex cover* is a set of vertices intersecting each edge of H. A *matching* is a collection of pairwise disjoint edges of H. Define

(77.6) $\tau(H) :=$ the minimum size of a vertex cover in H,
$\nu(H) :=$ the maximum size of a matching in H.

Determining these numbers is NP-complete, since determining $\tau(G)$ and (the stability number) $\alpha(G)$ of a graph $G = (V, E)$ is NP-complete (cf. Theorem 64.1), and since $\alpha(G) = \nu(G^*)$.

We should note that replacing H by $H^{\min}$ or $H^{\uparrow}$ does not change the value of $\tau(H)$ or $\nu(H)$. So $\tau(H) = \tau(H^{\uparrow}) = \tau(H^{\min})$ and $\nu(H) = \nu(H^{\uparrow}) = \nu(H^{\min})$.

There is the following straightforward inequality:

(77.7) $\nu(H) \leq \tau(H)$.

In the previous parts we met several classes of hypergraphs where equality holds in (77.7), and the purpose of this and the coming chapters is to treat them in a unifying and clarifying framework.

77.6. The blocker

For any hypergraph $H = (V, \mathcal{E})$, the *blocking hypergraph*, or *blocker*, of H is the hypergraph $b(H) = (V, \mathcal{B})$ where $\mathcal{B}$ is the collection of all inclusionwise minimal vertex covers of H. So $b(H)$ is a clutter and

(77.8) $\tau(H) = r_{\min}(b(H))$.

Moreover, $b(H)^{\uparrow}$ is the collection of vertex covers.

The following important duality relation was noticed by Lawler [1966] (also by Edmonds and Fulkerson [1970]):

Theorem 77.1. *For any hypergraph $H = (V, \mathcal{E})$, $b(b(H)) = H^{\min}$. In particular, if H is a clutter, then $b(b(H)) = H$.*

Proof. It suffices to show $b(b(H))^{\uparrow} = H^{\uparrow}$. If $U \in H^{\uparrow}$, then U intersects each set in $b(H)$. Hence U is a vertex cover of $b(H)$, and so $U \in b(b(H))^{\uparrow}$.

Conversely, if $U \notin H^{\uparrow}$, then $V \setminus U$ is a vertex cover of H. So $V \setminus U \in b(H)$. Hence U is not a vertex cover of $b(H)$. So $U \notin b(b(H))^{\uparrow}$. ∎

One may check that the operations of deletion and contraction interchange when passing to the blocker. More precisely, for any vertex v of a hypergraph H one has:

(77.9) $b(H/v) = b(H) \setminus v$ and $b(H \setminus v) = (b(H)/v)^{\min}$.

77.7. Fractional matchings and vertex covers

Let $H = (V, \mathcal{E})$ be a hypergraph. A *fractional vertex cover* is a function $x : V \to \mathbb{R}_+$ satisfying

(77.10) $$\sum_{v \in F} x_v \geq 1 \text{ for each } F \in \mathcal{E}.$$

A *fractional matching* is a function $y : \mathcal{E} \to \mathbb{R}_+$ satisfying

(77.11) $$\sum_{F \ni v} y_F \leq 1 \text{ for each } v \in V.$$

(Here and below, F ranges over the edges of H.) Let $\tau^*(H)$ denote the minimum size of a fractional vertex cover and let $\nu^*(H)$ denote the maximum size of a fractional matching (where the *size* of a vector is the sum of its components).

We can describe $\tau^*(H)$ and $\nu^*(H)$ by linear programs[3]:

(77.12) $$\tau^*(H) = \min\{\mathbf{1}^{\mathsf{T}} x \mid x \in \mathbb{R}^V_+, Mx \geq \mathbf{1}\},$$

where M is the $\mathcal{E} \times V$ incidence matrix of H. Similarly,

(77.13) $$\nu^*(H) = \max\{y^{\mathsf{T}} \mathbf{1} \mid y \in \mathbb{R}^{\mathcal{E}}_+, y^{\mathsf{T}} M \leq \mathbf{1}^{\mathsf{T}}\}.$$

As these linear programs are each others dual, this gives:

(77.14) $$\nu^*(H) = \tau^*(H).$$

77.8. *k*-matchings and *k*-vertex covers

There is an alternative interpretation of the parameters $\nu^*(H)$ and $\tau^*(H)$, in terms of 'k-vertex covers' and 'k-matchings'.

A *k-vertex cover* is a function $x : V \to \mathbb{Z}_+$ such that

(77.15) $$\sum_{v \in F} x_v \geq k \text{ for each } F \in \mathcal{F}.$$

Let $\tau_k(H)$ denote the minimum size of a k-vertex cover. Since (minimal) 1-vertex covers are precisely the incidence vectors of the vertex covers, we have $\tau_1(H) = \tau(H)$.

A *k-matching* is a function $y : \mathcal{E} \to \mathbb{Z}_+$ such that

(77.16) $$\sum_{F \ni v} y_F \leq k \text{ for each } v \in V.$$

Let $\nu_k(H)$ denote the maximum size of a k-matching in H. As 1-matchings are the incidence vectors of the matchings, we have $\nu_1(H) = \nu(H)$.

Then for any $k \in \mathbb{Z}_+$:

[3] $\mathbf{1}$ stands for all-one column vectors of appropriate sizes.

(77.17) $\nu_k(H) \leq \tau_k(H)$,

since for any k-vertex cover x and any k-matching y:

(77.18) $$\sum_F y_F \leq \frac{1}{k} \sum_F y_F \sum_{v \in F} x_v = \frac{1}{k} \sum_v x_v \sum_{F \ni v} y_F \leq \sum_v x_v.$$

More extensively, one has for each $k \geq 1$:

(77.19) $$\nu(H) \leq \frac{\nu_k(H)}{k} \leq \nu^*(H) = \tau^*(H) \leq \frac{\tau_k(H)}{k} \leq \tau(H).$$

The first two inequalities follow from the facts that if y is a 1-matching, then $k \cdot y$ is a k-matching, and that if y is a k-matching, then $k^{-1} \cdot y$ is a fractional matching. The last two inequalities are shown similarly.

We will investigate classes of hypergraphs where some or all of the inequalities in (77.19) are satisfied with equality. Obviously, if $\nu(H) = \tau(H)$, then all terms in (77.19) are equal.

$\nu^*(H)$ can be described in terms of the $\nu_k(H)$ (Lovász [1974]):

(77.20) $$\nu^*(H) = \max_k \frac{\nu_k(H)}{k} = \lim_{k \to \infty} \frac{\nu_k(H)}{k}.$$

Here the left-hand side equality holds as the maximum in (77.13) is attained by a rational optimum solution y. If k is the common denominator of the components of y, then $k \cdot y$ is a k-matching, and hence $k \cdot \nu^*(H) \leq \nu_k(H)$; so equality follows by (77.19).

The right-hand side equality follows from Fekete's lemma (Theorem 2.2), using the fact that for all $k, l \geq 1$:

(77.21) $\nu_{k+l}(H) \geq \nu_k(H) + \nu_l(H)$,

since if y' and y'' are a k- and an l-matching respectively, then $y' + y''$ is a $k + l$-matching.

Similarly we have:

(77.22) $$\tau^*(H) = \min_k \frac{\tau_k(H)}{k} = \lim_{k \to \infty} \frac{\tau_k(H)}{k},$$

using (77.12) and the fact that for all $k, l \geq 1$:

(77.23) $\tau_{k+l}(H) \leq \tau_k(H) + \tau_l(H)$.

77.9. Further results and notes

77.9a. Bottleneck extrema

Edmonds and Fulkerson [1970] showed that for any clutter $H = (V, \mathcal{E})$, its blocker $(V, \mathcal{B})$ is the unique clutter with the property that for each $f : V \to \mathbb{R}$ the following equality holds:

(77.24) $$\min_{E \in \mathcal{E}} \max_{x \in E} f(x) = \max_{B \in \mathcal{B}} \min_{y \in B} f(y).$$

(These extrema are called *bottleneck extrema.*) To see that the collection $\mathcal{B}$ of minimal vertex covers of H has this property, let $E \in \mathcal{E}$ and $x \in E$ attain the first minimum and first maximum. Then each $F \in \mathcal{E}$ contains a vertex z with $f(z) \geq f(x)$. Hence $\{z \in V \mid f(z) \geq f(x)\}$ contains a set B in $\mathcal{B}$. So $f(x) \leq \min_{y \in B} f(y)$. This shows $\leq$ in (77.24). Moreover, for any $B \in \mathcal{B}$, as E intersects B, we have $f(x) \geq \min_{y \in B} f(y)$. This gives $\geq$ in (77.24).

To see that this property characterizes the blocker, let $\mathcal{B}$ be any clutter satisfying (77.24) for each $f : V \to \mathbb{R}$. Then for each $B \in \mathcal{B}$ and $E \in \mathcal{E}$ we have $B \cap E \neq \emptyset$, since otherwise we can define f such that $f(x) < 0$ for all $x \in E$ and $f(y) > 0$ for each $y \in B$, giving $<$ in (77.24), a contradiction.

Finally, each vertex cover B of $\mathcal{E}$ contains a set in $\mathcal{B}$. If not, we can define f such that $f(x) > 0$ for each $x \in B$ and $f(y) < 0$ for each $y \in V \setminus B$. Then we have $>$ in (77.24), again a contradiction.

77.9b. The ratio of τ and τ^*

The following theorem of Johnson [1974a] and Lovász [1975c] bounds $\tau(H)$ in terms of $\tau^*(H)$ and the maximum degree of H. (The *degree* of a vertex v is the number of edges containing v. The *maximum degree* of H is the maximum of the degrees of its vertices.) The method is similar to that of Theorem 64.13.

Theorem 77.2. *For any hypergraph $H = (V, \mathcal{E})$ of maximum degree d one has:*

$$\tau(H) \leq (1 + \ln d)\tau^*(H). \tag{77.25}$$

Proof. Iteratively choose vertices $v_1, v_2, \ldots$, where, for each $i = 1, 2, \ldots$, vertex v_i is chosen such that it is contained in a maximum number of edges not intersecting $\{v_1, \ldots, v_{i-1}\}$. We stop if the set $\{v_1, \ldots, v_k\}$ of chosen vertices is a vertex cover. So $\tau(H) \leq k$.

For each $i = 1, \ldots, k$, let d_i be the number of edges containing v_i but not intersecting $\{v_1, \ldots, v_{i-1}\}$. For each $F \in \mathcal{E}$, define

$$y_F := \frac{1}{d_i}, \tag{77.26}$$

where i is the smallest index with $v_i \in F$. Then

$$\sum_{F \in \mathcal{E}} y_F = \sum_{i=1}^{k} d_i \frac{1}{d_i} = k, \tag{77.27}$$

and hence

$$\tau(H) \leq \sum_{F \in \mathcal{E}} y_F. \tag{77.28}$$

We next show that $(1 + \ln d)^{-1} \cdot y$ is a fractional matching. To this end, consider any vertex v. Let $F_1, \ldots, F_t$ be the edges of H containing v, in the order by which they are intersected by $v_1, \ldots, v_k$. Then for each $j = 1, \ldots, t$, we have

$$y_{F_j} \leq \frac{1}{t - j + 1}. \tag{77.29}$$

For let i be the smallest index with $v_i \in F_j$. So $y_{F_j} = 1/d_i$. Moreover, $d_i \geq t-j+1$, since v is contained in at least $t-j+1$ edges not intersected by $\{v_1, \ldots, v_{i-1}\}$. This proves (77.29).

Hence

(77.30) $$\sum_{j=1}^{t} y_{F_j} \leq \sum_{j=1}^{t} \frac{1}{t-j+1} = \sum_{j=1}^{t} \frac{1}{j} \leq 1 + \ln t \leq 1 + \ln d.$$

As this holds for each vertex v, $(1+\ln d)^{-1} \cdot y$ is a fractional matching.

This implies

(77.31) $$\tau(H) \leq \sum_{F \in \mathcal{E}} y_F \leq (1+\ln d)\nu^*(H) = (1+\ln d)\tau^*(H),$$

as required. ∎

(Related work can be found in Balas [1984].)

The proof shows that one can find a vertex cover of size less than $(1+\ln d)\tau^*$, by iteratively selecting a vertex of maximum degree and deleting it.

The proof method of Theorem 67.17 gives that for any hypergraph $H = (V, \mathcal{E})$:

(77.32) $$\tau^*(H) = \lim_{k \to \infty} \sqrt[k]{\tau(H^k)},$$

where H^k is the hypergraph on V^k with edges all sets $E_1 \times \cdots \times E_k$ with $E_1, \ldots, E_k \in \mathcal{E}$.

77.9c. Further notes

Füredi, Kahn, and Seymour [1993] showed that each hypergraph $H = (V, \mathcal{E})$ has a matching $\mathcal{M} \subseteq \mathcal{E}$ such that

(77.33) $$\sum_{F \in \mathcal{M}} (|F| - 1 + \frac{1}{|F|}) \geq \nu^*(H).$$

In particular, for any hypergraph H:

(77.34) $$\nu(H) \geq \frac{r_{\max}}{r_{\max}^2 - r_{\max} + 1} \nu^*(H),$$

where $r_{\max} := r_{\max}(H)$ (the maximum edge size of H). (For *uniform* hypergraphs H (that is, all edges of H have the same size), this was proved by Füredi [1981] (confirming a conjecture of L. Lovász (cf. Füredi [1988])).)

Füredi, Kahn, and Seymour [1993] conjecture the following weighted extension of (77.33):

(77.35) (?) For each hypergraph $H = (V, \mathcal{E})$ and each $w : \mathcal{E} \to \mathbb{R}_+$, there exists a matching $\mathcal{M} \subseteq \mathcal{E}$ such that

$$\sum_{F \in \mathcal{M}} \big(|F| - 1 + \frac{1}{|F|}\big) w(F) \geq \nu_w^*(H),$$

where $\nu_w^*(H)$ is the maximum weight $w^{\mathsf{T}}y$ of a fractional matching $y : \mathcal{E} \to \mathbb{R}_+$. Füredi, Kahn, and Seymour [1993] proved this conjecture for uniform hypergraphs, and also for hypergraphs H with $\nu(H) = 1$.

Related work on the relations between fractional and integer packing and covering was reported by Chvátal [1979], Dobson [1982], Fisher and Wolsey [1982], Aharoni, Erdős, and Linial [1985,1988], Raghavan [1988], Feige [1996,1998], and Slavík [1996,1997].

Lovász [1975b] showed that for each choice of $\nu, \tau \in \mathbb{Z}_+$ and $r \in \mathbb{Q}_+$ satisfying $1 \leq \nu \leq r \leq \tau$ and $r > 1$, there exists a hypergraph H with $\nu(H) = \nu$, $\tau^*(H) = r$, and $\nu(H) = \nu$. Chung, Füredi, Garey, and Graham [1988] showed that for each rational number r, there exists a 3-uniform hypergraph H with $\tau^*(H) \equiv r$ (mod 1). (For each 2-uniform hypergraph (= graph) H, $\tau^*(H)$ belongs to $\frac{1}{2}\mathbb{Z}$ (cf. Section 64.6).)

Saks [1986] studied the behaviour of the parameters τ and ν under taking unions of edges and vertex covers.

The hypergraph analogue of matching augmenting paths in graphs was studied by Edmonds [1962].

Seymour [1977a] gave a forbidden minor characterization of those clutters H that come from an undirected graph $G = (V, E)$ and $s, t \in V$, by taking as edges of H all edge sets of $s - t$ paths. (Related work can be found in Novick and Sebő [1995].)

Determining the vertex cover number $\tau(H)$ of a hypergraph H is equivalent to the set covering problem. In Section 82.6b we give further references for this problem. Determining the matching number $\nu(H)$ of H is equivalent to the vertex packing (equivalently, the set packing) problem. In Section 64.9e we gave further references for this problem.

Connectivity augmentation for hypergraphs was studied by Bang-Jensen and Jackson [1999], Benczúr [1999], Benczúr and Frank [1999], Cheng [1999], and Szigeti [1999].

The problems of finding a maximum-size matching and a minimum-size vertex cover in a hypergraph are equivalent to finding a maximum-size stable set in a graph and a minimum-size edge cover in a hypergraph. For references to general methods for these problems, we refer to Sections 64.9e and 82.6b, respectively.

Extensions of Gallai's theorem (Theorem 19.1) to hypergraphs were given by Tuza [1991], and generalizations of Kőnig's and Hall's theorems to hypergraphs by Aharoni and Haxell [2000] and Aharoni, Berger, and Ziv [2002].

Surveys on packing and covering in hypergraphs were given by Berge [1973b, 1973c,1978a,1979b,1989a], Schrijver [1979b], Füredi [1988], and Cornuéjols [2001].

Chapter 78

Ideal hypergraphs

Ideal hypergraphs are those hypergraphs for which the convex hull of the vertex covers is given by the edge inequalities. They therefore form a class of hypergraphs where polyhedral methods apply. Since the relations of blocking hypergraphs and of blocking polyhedra coincide in this case, the class of ideal hypergraphs is closed under taking blockers.
The class of ideal hypergraphs is also closed under taking minors. A characterization of ideal hypergraphs in terms of forbidden minors is not known, but a theorem of Lehman gives powerful properties of minimally nonideal hypergraphs.

78.1. Ideal hypergraphs

For any hypergraph $H = (V, \mathcal{E})$, let P_H be the set of all fractional vertex covers; that is, P_H is the solution set of

(78.1) (i) $x_v \geq 0$ for $v \in V$,
(ii) $x(F) \geq 1$ for $F \in \mathcal{E}$.

A hypergraph $H = (V, \mathcal{E})$ is called *ideal* if P_H is integer[4]. Obviously, H is ideal $\iff$ $H^{\min}$ is ideal $\iff$ $H^{\uparrow}$ is ideal.

Note that each integer vertex of P_H is a 0,1 vector, and hence the incidence vector of some vertex cover of H. So H is ideal if and only if (78.1) determines the up hull of the incidence vectors of the vertex covers of H. By Theorem 5.19, H is ideal if and only if the convex hull of the incidence vectors of the vertex covers of H is determined by

(78.2) (i) $0 \leq x_v \leq 1$ for $v \in V$,
(ii) $x(F) \geq 1$ for $F \in \mathcal{E}$.

By the theory of blocking polyhedra (cf. Theorem 5.8), H is ideal if and only if each vertex of the polyhedron determined by

[4] Alternatively, such hypergraphs are called *Fulkersonian*, or said to satisfy the *length-width inequality* or the *width-length inequality*, or to have the *max-flow min-cut property*, the $\mathbb{Q}_+$*-max-flow min-cut property*, shortly the *MFMC property* or the $\mathbb{Q}_+$*-MFMC property*. Sakarovitch [1975,1976] used the term *quasi-balanced*.

(78.3) (i) $x_v \geq 0$ for $v \in V$,
(ii) $x(B) \geq 1$ for $B \in b(H)$

is integer — that is, is the incidence vector of an edge of H. This gives the following important theorem of Fulkerson [1970b,1971a]:

Theorem 78.1. *A hypergraph H is ideal $\Longleftrightarrow$ its blocker $b(H)$ is ideal.*

Proof. See above. ∎

The class of ideal hypergraphs is also closed under taking minors (Lehman [1965,1979]):

Theorem 78.2. *Any minor of an ideal hypergraph is ideal again.*

Proof. Let $H = (V, \mathcal{E})$ be ideal and let $v \in V$. Choose $x \in P_{H/v}$. Let $\tilde{x} \in \mathbb{R}^V$ be defined by $\tilde{x}_u := x_u$ for $u \in V \setminus \{v\}$ and $\tilde{x}_v := 0$. Then $\tilde{x} \in P_H$. Hence $\tilde{x}$ is a convex combination of integer vectors z in P_H. Each of these vectors z satisfies $z_v = 0$. Hence we obtain x as a convex combination of integer vectors in $P_{H/v}$.

Next choose $x \in P_{H\setminus v}$. Now let $\tilde{x} \in \mathbb{R}^V$ be defined by $\tilde{x}_u := x_u$ for $u \in V \setminus \{v\}$ and $\tilde{x}_v := 1$. Then $\tilde{x} \in P_H$. Hence $\tilde{x}$ is a convex combination of integer vectors z in P_H. Now deleting the vth component from any such z, we obtain an integer vector in $P_{H\setminus v}$. Hence we obtain x as a convex combination of integer vectors in $P_{H\setminus v}$. ∎

78.2. Characterizations of ideal hypergraphs

We will give several characterizations of ideal hypergraphs — albeit not by forbidden minors, since such a characterization is not known.

In the present section we discuss some equivalent properties each characterizing ideality. In Section 78.4, we show Lehman's theorem, which gives properties of minimally nonideal hypergraphs. From this, some further characterizations of ideality will be derived.

The definition of ideal hypergraph can be stated equivalently as:

(78.4) H is ideal if and only if for each $w : V \to \mathbb{R}_+$, the minimum of $w^{\mathsf{T}}x$ over (78.1) is attained by an integer vector x.

As we can restrict ourselves to rational-valued w, and hence to integer-valued w, we have equivalently:

(78.5) H is ideal if and only if for each $w : V \to \mathbb{Z}_+$, the minimum of $w^{\mathsf{T}}x$ over (78.1) is attained by an integer vector x.

We can formulate this in terms of the 'parallelization' H^w (defined in Section 77.3). To this end, it is good to observe that, for any 'weight' function $w : V \to \mathbb{Z}_+$:

(78.6) $\tau(H^w) =$ the minimum weight of a vertex cover of H

and

(78.7) $\nu(H^w) =$ the maximum number t of edges $E_1, \ldots, E_t$ of H such that each $v \in V$ is in at most $w(v)$ of the E_i.

The values of $\tau^*(H^w)$ and $\nu^*(H^w)$ can be described by dual linear programs:

(78.8) $$\tau^*(H^w) = \min\{w^{\mathsf{T}}x \mid x \in \mathbb{R}_+^V, Mx \geq \mathbf{1}\}$$ $$= \min\{y^{\mathsf{T}}\mathbf{1} \mid y \in \mathbb{R}_+^{\mathcal{E}}, y^{\mathsf{T}}M \leq w^{\mathsf{T}}\} = \nu^*(H^w),$$

where M is the $\mathcal{E} \times V$ incidence matrix of H. So we have:

(78.9) H is ideal if and only if $\tau^*(H^w) = \tau(H^w)$ for each $w : V \to \mathbb{Z}_+$.

The following further characterizations were found[5]:

Theorem 78.3. *For any hypergraph $H = (V, \mathcal{E})$ the following are equivalent:*

(78.10)
(i) *H is ideal, that is $\tau^*(H') = \tau(H')$ for each parallelization H' of H;*
(ii) *$\tau^*(H')$ is an integer for each parallelization H' of H;*
(iii) *$b(H)$ is ideal;*
(iv) *$\tau^*(b(H)')$ is an integer for each parallelization $b(H)'$ of $b(H)$;*
(v) *P_H and $P_{b(H)}$ form a pair of blocking polyhedra;*
(vi) *$\tau(H^w)\tau(b(H)^l) \leq w^{\mathsf{T}}l$ for all $w, l : V \to \mathbb{Z}_+$.*

Proof. The implications (i)⇒(ii) and (iii)⇒(iv) are trivial. The equivalences (i) $\iff$ (iii) $\iff$ (v) were shown above.

The implication (ii)⇒(i) is shown as follows[6]. We must show that (ii) implies that each vertex x^* of the polyhedron P_H defined by (78.1) is integer. Suppose not. Choose $v \in V$ with x^*_v not integer. As x^* is a vertex, there is a weight function $w : V \to \mathbb{R}_+$ such that the minimum of $w^{\mathsf{T}}x$ over P_H is attained uniquely by x^*. By scaling, we can assume that w is integer and that for $\tilde{w} := w + \chi^v$, also the minimum of $\tilde{w}^{\mathsf{T}}x$ over P_H is attained at x^*. So $w^{\mathsf{T}}x^*$ and $\tilde{w}^{\mathsf{T}}x^*$ are integers (by (ii)), and hence $x^*_v = \tilde{w}^{\mathsf{T}}x^* - w^{\mathsf{T}}x^*$ is an integer, contradicting our assumption.

This proves (ii)⇒(i) and similarly (iv)⇒(iii). So conditions (i), (ii), (iii), (iv), and (v) are equivalent. We finally consider condition (vi).

Necessity of (vi) can be seen as follows. Choose $w, l : V \to \mathbb{Z}_+$. Let $\alpha := \tau(H^w) = \tau^*(H^w)$ and $\beta := \tau(b(H)^l) = \tau^*(b(H)^l)$. So $w(B) \geq \alpha$ for each edge B of $b(H)$ (= minimal vertex cover of H), and hence $\alpha^{-1} \cdot w(B) \geq 1$

[5] (i)⇔(vi)⇔(iii) was shown by Lehman [1965,1979], (i)⇔(v) by Fulkerson [1970b,1971a], and (i)⇔(ii) and (iii)⇔(iv) by Lovász [1977b].

Lehman called condition (i) the *max-flow min-cut property*, and condition (vi) the *width-length inequality* (motivated by work of Moore and Shannon [1956] who proved this inequality for the width (minimum cut-capacity) and length (shortest path) of a network).

[6] It also follows directly from general polyhedral theory (Theorem 5.18).

for each edge of $b(H)$. So $\alpha^{-1} \cdot w \in P_{b(H)}$. Similarly, $\beta^{-1} \cdot l$ belongs to P_H. As $P_{b(H)}$ is the blocking polyhedron of P_H, we have $(\alpha^{-1} \cdot w)^{\mathsf{T}}(\beta^{-1} \cdot l) \geq 1$, that is $w^{\mathsf{T}} l \geq \alpha\beta$, as required.

To see sufficiency of (vi), suppose that P_H has a noninteger vertex x^*. Then there is a hyperplane separating x^* from the integer vectors in P_H. So there is a $y \in \mathbb{Q}_+^V$ with $y^{\mathsf{T}} x^* < 1$ while $y^{\mathsf{T}} x \geq 1$ for each integer vector x in P_H. Let $\alpha > 0$ and $\beta > 0$ be such that $w := \alpha \cdot y$ and $l := \beta \cdot x^*$ are integer vectors. As $y^{\mathsf{T}} x \geq 1$ for each integer vector x in P_H, we have $w^{\mathsf{T}} x \geq \alpha$ for each integer vector x in P_H, and so $w(B) \geq \alpha$ for each vertex cover B of H; that is $\tau(H^w) \geq \alpha$. Since x^* belongs to P_H, we have that $x^*(F) \geq 1$ for each $F \in \mathcal{E}$, and hence $l(F) \geq \beta$ for each $F \in \mathcal{E}$; that is $\tau(b(H)^l) \geq \beta$. This implies

(78.11) $$\tau(H^w)\tau(b(H)^l) \geq \alpha\beta > \alpha\beta \cdot y^{\mathsf{T}} x^* = w^{\mathsf{T}} l,$$

contradicting (vi). ∎

78.3. Minimally nonideal hypergraphs

A hypergraph $H = (V, \mathcal{E})$ is called *minimally nonideal* if H is nonideal and each proper minor of H is ideal. In particular, H is a clutter.

So being ideal can be characterized by not having a minimally nonideal hypergraph as a minor. Since the class of ideal hypergraphs is closed under taking the blocker, the blocker of any minimally nonideal hypergraph is minimally nonideal again.

There turn out to be infinitely many minimally nonideal hypergraphs. Known examples are[7]:

(78.12)
- (i) for each $n \geq 3$: $J_n :=$ the hypergraph with vertex set $\{1, \ldots, n\}$ and edges $\{2, \ldots, n\}$, $\{1, 2\}, \ldots, \{1, n\}$;
- (ii) the odd circuits C_{2k+1} and their blockers $b(C_{2k+1})$ $(k \geq 1)$;
- (iii) $F_7 :=$ the hypergraph with vertex set the points of the projective plane of order 2, and edges all lines (the *Fano hypergraph*)[8];
- (iv) $\mathcal{O}(K_5) :=$ the hypergraph with vertex set EK_5 and edges all odd circuits of K_5, and its blocker $b(\mathcal{O}(K_5))$ (having edges the complements of the nonempty cuts of K_5);
- (v) the hypergraph with vertex set EK_5 and edges all triangles of K_5, and its blocker;

[7] Examples (i), (ii), (iii) were given by Lehman [1965,1979], example (iv) by Seymour [1977b], examples (v) and (vi) by Cornuéjols and Novick [1994], example (vii) by P.D. Seymour (cf. Ding [1993]), $\mathcal{C}_5^3$ and $\mathcal{C}_7^4$ by Lehman [1965,1979] (they are the blockers of the circuits C_5 and C_7), $\mathcal{C}_8^3$ by Cornuéjols and Novick [1994] and Ding [1993], $\mathcal{C}_9^5$ and $\mathcal{C}_{11}^6$ by Qi [1989], and the other $\mathcal{C}_n^k$ by Cornuéjols and Novick [1994].

[8] Equivalently, $VF_7 = \{1, \ldots, 7\}$ and $EF_7 = \{\{i, i+1, i+3\} \mid i = 1, \ldots, 7\}$, taking addition mod 7.

(vi) the hypergraph with vertex set EK_5 and edges the complements of maximum-size cuts, and its blocker;
(vii) the hypergraph $\mathcal{D}_8$ with vertex set $\{1,\ldots,8\}$ and edges $\{1,2,6\}$, $\{2,3,5\}$, $\{3,4,8\}$, $\{4,5,7\}$, $\{2,5,6\}$, $\{1,6,7\}$, $\{4,7,8\}$, and $\{1,3,8\}$, and its blocker $b(\mathcal{D}_8)$;
(viii) the hypergraphs $\mathcal{C}_5^3$, $\mathcal{C}_8^3$, $\mathcal{C}_{11}^3$, $\mathcal{C}_{14}^3$, $\mathcal{C}_{17}^3$, $\mathcal{C}_7^4$, $\mathcal{C}_{11}^4$, $\mathcal{C}_9^5$, $\mathcal{C}_{11}^6$, $\mathcal{C}_{13}^7$ (where $\mathcal{C}_n^k$ has vertex set VC_n and edges all consecutive k-tuples from VC_n), and their blockers.

Note that $b(F_7) = F_7$ and $b(J_n) = J_n$ for each n. The hypergraphs given in (viii) are all the minimally nonideal hypergraphs of the form $\mathcal{C}_n^k$ with $k \geq 3$. This was proved by Cornuéjols and Novick [1994], who also gave several thousands of other minimally nonideal hypergraphs. A 'catalogue' of minimally nonideal hypergraphs was given by Lütolf and Margot [1998].

Seymour [1981a][9] conjectures that $\mathcal{O}(K_5)$, $b(\mathcal{O}(K_5))$, and F_7 are the only *binary* minimally nonideal hypergraphs (see Chapter 80).

We saw in Section 75.5 that $\mathcal{O}(K_5)$ is the unique minimally nonideal hypergraph among the hypergraphs obtained from a signed graph $G = (V, E, \Sigma)$ by taking EG as vertex set and the circuits C in G with $|C \cap \Sigma|$ odd as edges.

To see that F_7 is nonideal, the vector $x : VF_7 \to \mathbb{R}_+$ defined by $x_v := \frac{1}{3}$ for each $v \in VF_7$, is a fractional vertex cover of size $\frac{7}{3}$, but F_7 has no vertex cover of size $\leq \frac{7}{3}$. Moreover, F_7 is minimally nonideal: if we contract any vertex $v \in VF_7$, we obtain the hypergraph Q_6 isomorphic to the hypergraph $\mathcal{O}(K_4)$ (the hypergraph with vertex set EK_4 and edges all triangles). As this is a proper minor of $\mathcal{O}(K_5)$, it is ideal. Since $b(F_7) = F_7$, also deleting any vertex of F_7 results in an ideal hypergraph.

78.4. Properties of minimally nonideal hypergraphs: Lehman's theorem

A full list of minimally nonideal hypergraphs is not known, but the following theorems of Lehman [1990] show that minimally nonideal hypergraphs different from J_n ($n \geq 3$) are remarkably regular (shorter proofs were given by Padberg [1993] and Seymour [1990b] — we follow the latter):

Theorem 78.4. *Let $H = (V, \mathcal{E})$ be a minimally nonideal hypergraph with $H \neq J_n$ for $n := |V|$. Then P_H has a unique noninteger vertex, namely $r^{-1} \cdot \mathbf{1}$, where $r := r_{\min}(H)$. Moreover, H has precisely n edges of size r, and each vertex of H is contained in precisely r of them.*

Proof. Let x be a noninteger vertex of P_H. Then

(78.13) $\quad 0 < x_v < 1$ for each $v \in V$.

[9] Seymour [1981a] said that this conjecture was presented in Seymour [1977b], but the latter paper presents the three hypergraphs only as minimally nonideal hypergraphs.

For suppose first that $x_v = 0$. Then $x|V \setminus \{v\}$ is a noninteger vertex of $P_{H/v}$, contradicting the minimality of H. Similarly, if $x_v = 1$, then $x|V \setminus \{v\}$ is a noninteger vertex of $P_{H\setminus v}$, again contradicting the minimality of H. This proves (78.13).

Let $\mathcal{F}$ be the collection of sets $F \in \mathcal{E}$ having equality for x in (78.1)(ii). As x is a vertex, $\mathcal{F}$ has dimension n. (Here and below, the *dimension* of a collection of subsets of V, is the dimension of the collection of incidence vectors of these subsets.) Let $\mathcal{F}_v$ and $\mathcal{F} \setminus v$ be the collections of sets in $\mathcal{F}$ containing v and not containing v, respectively. Then

(78.14) (i) For each $F \in \mathcal{F}$ and $v \in V \setminus F$: $\dim(\mathcal{F} \setminus v) \leq n - |F|$;
(ii) for each $F \in \mathcal{F}$ and $v \in F$: $\dim(\mathcal{F}_v) \leq n - |F| + 1$.

To see (i), choose $F \in \mathcal{F}$ and $v \in V \setminus F$. Since $H \setminus v$ is ideal, $x|V \setminus \{v\}$ is a convex combination of incidence vectors of vertex covers of $H \setminus v$. For each $u \in F$, since $x_u > 0$, there is a vertex cover B_u of $H\setminus v$ having positive scalar in this convex decomposition and with $u \in B_u$. So $B_u \cap F = \{u\}$ (as $x(F) = 1$). Hence the incidence vectors χ^{B_u} for $u \in F$ are linearly independent. This implies that the vectors $\chi^{B_u} - x$ for $u \in F$ have dimension at least $|F| - 1$. As each of these vectors is orthogonal to $\chi^{F'}$ for each $F' \in \mathcal{F} \setminus v$, we have $\dim(\mathcal{F} \setminus v) \leq (n-1) - (|F| - 1) = n - |F|$, proving (78.14)(i).

We prove (ii) similarly. Choose $F \in \mathcal{F}$ and $v \in F$. Define $z := (1 - x_v)^{-1} \cdot x|V \setminus \{v\}$. Then $z \in P_{H/v}$, since $x(F' \setminus \{v\}) \geq 1 - x_v$ for each $F' \in \mathcal{F}$. Hence, since H/v is ideal, z is a convex combination of incidence vectors of vertex covers of H/v. For each $u \in F \setminus \{v\}$, since $z_u > 0$, there is a vertex cover B_u of H/v having positive scalar in this convex decomposition and with $u \in B_u$. So $B_u \cap F = \{u\}$ (since $z(F \setminus \{v\}) = 1$). Hence the incidence vectors χ^{B_u} for $u \in F \setminus \{v\}$ are linearly independent. This implies that the vectors $\chi^{B_u} - x$ for $u \in F$ have affine dimension at least $|F| - 1$. As each of these vectors is orthogonal to $\chi^{F'}$ for each $F' \in \mathcal{F}_v$, we have $\dim(\mathcal{F}_v) \leq (n-1) - (|F \setminus \{v\}| - 1) = n - |F| + 1$, proving (78.14)(ii).

Now (78.14)(i) implies:

(78.15) $|\mathcal{F}| = n$ and $|\mathcal{F} \setminus v| = n - |F|$ for each $v \in V$ and $F \in \mathcal{F} \setminus v$.

Indeed, let $\mathcal{F}'$ be a subcollection of $\mathcal{F}$ of dimension and size n. By (78.14)(i), $|\mathcal{F}' \setminus v| \leq n - |F|$ for each $v \in V$ and each $F \in \mathcal{F} \setminus v$. Let U be the set of $v \in V$ not covered by all sets in $\mathcal{F}'$. Then:

(78.16)
$$n = \sum_{F \in \mathcal{F}'} 1 = \sum_{F \in \mathcal{F}'} \sum_{v \in V \setminus F} \frac{1}{n - |F|} = \sum_{v \in U} \sum_{F \in \mathcal{F}' \setminus v} \frac{1}{n - |F|}$$
$$\leq \sum_{v \in U} \sum_{F \in \mathcal{F}' \setminus v} \frac{1}{|\mathcal{F}' \setminus v|} = \sum_{v \in U} 1 = |U| \leq n.$$

So we have equality throughout; that is, $U = V$ and $|\mathcal{F}' \setminus v| = n - |F|$ for each $v \in V$ and each $F \in \mathcal{F}' \setminus v$.

We deduce that $\mathcal{F}' = \mathcal{F}$. For suppose that there exists an $F \in \mathcal{F} \setminus \mathcal{F}'$. Then there is an $F' \in \mathcal{F}'$ such that $\mathcal{F}'' := (\mathcal{F}' \setminus \{F'\}) \cup \{F\}$ has dimension n. Choose $v \in F \setminus F'$ and $F'' \in \mathcal{F}'' \setminus v$. So $F'' \neq F$ and hence $F'' \in \mathcal{F}' \setminus v$. Hence $|\mathcal{F}'' \setminus v| = n - |F''| = |\mathcal{F}' \setminus v|$, contradicting the fact that $v \in F \setminus F'$. Concluding, $\mathcal{F}' = \mathcal{F}$ and we have (78.15).

(78.15) and (78.14)(ii) imply:

(78.17) $|F| + |F'| \leq n + 1$ for any two distinct $F, F' \in \mathcal{F}$.

For choose $v \in F' \setminus F$. Then

(78.18) $$n = |\mathcal{F}| = |\mathcal{F}\setminus v| + |\mathcal{F}_v| \leq n - |F| + n - |F'| + 1 = 2n - |F| - |F'| + 1,$$

implying (78.17).

Let G be the graph on V where distinct $u, v \in V$ are adjacent if there is an $F \in \mathcal{F}$ with $u, v \notin F$. So by (78.15), $|\mathcal{F} \setminus u| = |\mathcal{F} \setminus v|$ for adjacent u, v. Hence, if G is connected, then $|\mathcal{F} \setminus v|$ is independent of v, and hence by (78.15), all sets in $\mathcal{F}$ have the same size, p say. Hence $x = p^{-1} \cdot \mathbf{1}$ and $p \geq r$ (as r is the minimum size of the sets in $\mathcal{E}$). On the other hand, the inequality $x(E) \geq 1$ for any minimum-size $E \in \mathcal{E}$, gives that $r \geq p$. So $p = r$, and the theorem follows.

So we can assume that G is not connected. Then there exists a partition of V into nonempty sets V_1, V_2 with $V_1 \subseteq F$ or $V_2 \subseteq F$ for each $F \in \mathcal{F}$. Let $\mathcal{F}_i$ be the collection of sets $F \in \mathcal{F}$ with $V_i \subseteq F$ (for $i = 1, 2$). So $\mathcal{F}_1, \mathcal{F}_2$ partition $\mathcal{F}$. By (78.17) we can assume that $\mathcal{F}_1 \subseteq \{V_1\}$ (since $|V_1| + |V_2| = n$). Then (78.17) gives moreover that $\mathcal{F}_2 \subseteq \{V_2 \cup \{v\} \mid v \in V_1\}$ (as $|\mathcal{F}| = n \geq 3$, so $\mathcal{F} \neq \{V_1, V_2\}$). Since $|\mathcal{F}_1| + |\mathcal{F}_2| = n$, it follows that $|V_1| = n - 1$, and $(V, \mathcal{F}) = J_n$. Since any subset of V is contained in or contains one of the sets in $\mathcal{F}$, we know that $H = J_n$, a contradiction. ∎

This theorem implies:

Corollary 78.4a. *Let H be a minimally nonideal hypergraph. Define $n := |VH|$, $r := r_{\max}(H)$, and $s := \tau(H)$. Then $\tau(H) - 1 < \tau^*(H) < \tau(H)$. If moreover $H \neq J_n$, then $rs > n$ and $\tau^*(H) = n/r$.*

Proof. First assume that $H = J_n$. Then $\tau(H) = 2$ and $\tau^*(H) = (2n-3)/(n-1) = 2 - \frac{1}{n-1}$ as one easily checks. So we can assume that $H \neq J_n$.

Consider a pair $x \in P_H$ and $y \in P_{b(H)}$ minimizing $x^{\mathsf{T}}y$. So $x^{\mathsf{T}}y < 1$ (since P_H and $P_{b(H)}$ form no blocking pair of polyhedra). We can assume that x and y are vertices of P_H and $P_{b(H)}$ respectively. Moreover, x and y are noninteger, for if, say, x is integer, it is the incidence vector of a vertex cover of H, and hence $x^{\mathsf{T}}y \geq 1$, since $y \in P_{b(H)}$.

As H and $b(H)$ are minimally nonideal, we know by Theorem 78.4 that $x = r^{-1} \cdot \mathbf{1}$ and $y = s^{-1} \cdot \mathbf{1}$. Then $x^{\mathsf{T}}y < 1$ implies $rs > n$.

Let z minimize $\mathbf{1}^T z$ over $z \in P_H$, where z is a vertex of P_H. So z is a minimum-size vertex cover, and hence $\mathbf{1}^{\mathsf{T}}z = \tau^*(H)$. If z is integer, then

$\mathbf{1}^{\mathsf{T}}z \geq s > n/r$. If z is noninteger, then $z = r^{-1} \cdot \mathbf{1}$ by Theorem 78.4, and hence $\mathbf{1}^{\mathsf{T}}z = n/r$. So $\tau^*(H) = n/r$.

As $rs > n$, we have $n/r < s$ and so $\tau^*(H) < \tau(H)$. Moreover, for any $v \in VH$ we have $\tau^*(H \setminus v) \leq (n-1)/r$, since $r^{-1} \cdot \mathbf{1}_{V\setminus\{v\}}$ is a fractional vertex cover of H. Hence

(78.19) $$\tau^*(H) = \frac{n}{r} > \frac{n-1}{r} \geq \tau^*(H \setminus v) = \tau(H \setminus v) \geq \tau(H) - 1,$$

as required. ∎

Corollary 78.4a implies a number of further characterizations of ideal hypergraphs, partly sharpening Theorem 78.3 (Lehman [1990], Padberg [1993], Seymour [1990b]; the equivalence (i)⇔(iv) answers a question of P.D. Seymour (personal communication 1976)):

Corollary 78.4b. *For any hypergraph $H = (V, \mathcal{E})$, the following are equivalent:*

(78.20)
(i) *H is ideal, that is, $\tau(H^w) = \tau^*(H^w)$ for each $w : V \to \mathbb{Z}_+$;*
(ii) *$H' \neq J_n$ (for all $n \geq 3$) and $\tau(H')r_{\min}(H') \leq |VH'|$, for each minor H' of H;*
(iii) *$\tau^*(H') \in \mathbb{Z}$ for each minor H' of H;*
(iv) *$\tau(H') = \tau^*(H')$ for each minor H' of H;*
(v) *$\tau(H^w) = \tau^*(H^w)$ for each $w : V \to \{0, 1, |V|\}$.*

Proof. Condition (i) implies each of (ii)-(v), since ideality is closed under taking minors and parallelization. The implication (iv)⇒(iii) is direct. The implications (ii)⇒(i) and (iii)⇒(i) follow from Corollary 78.4a: if H is not ideal, it has a minor H' that is minimally nonideal; then Corollary 78.4a contradicts (ii) and (iii). So it suffices to show (v)⇒(iv).

Let (v) hold. Let H' be a minor obtained from H by contracting the vertices in a set U and deleting the vertices in a set W. Define $w(v) := 0$ if $v \in W$, $w(v) := |V|$ if $v \in U$, and $w(v) := 1$ otherwise. We assume that $\tau(H')$ is finite (so $\emptyset \in EH'$). We show

(78.21) $$\tau(H') \leq \tau(H^w) = \tau^*(H^w) \leq \tau^*(H'),$$

which implies (iv).

We first show the first inequality in (78.21). If $\tau(H') \geq |V|$, then $\tau(H') \geq |VH'|$, and hence each singleton is an edge of H'. So $\tau(H') = |VH'|$, and hence $\tau(H^w) \geq |VH'| = \tau(H')$. So we can assume that $\tau(H') < |V|$. Then $\tau(H') \leq \tau(H^w)$, since otherwise $\tau(H^w) < |V|$, and hence H has a vertex cover B contained in $V \setminus U$ with $|B \setminus W| = \tau(H^w)$. So $\tau(H') \leq |B \setminus W| = \tau(H^w)$. This proves the first inequality in (78.21).

To see the second inequality, let $x \in \mathbb{R}^{V\setminus(U\cup W)}$ be a minimum-size fractional vertex cover of H'. We can extend x to a fractional vertex cover $\tilde{x} \in \mathbb{R}^V$ of H by defining $\tilde{x}_v := 0$ if $v \in U$ and $\tilde{x}_v := 1$ if $v \in W$. Then

(78.22) $$\tau^*(H^w) \leq w^{\mathsf{T}}\tilde{x} = \mathbf{1}^{\mathsf{T}}x = \tau^*(H').$$

Hence $\tau^*(H^w) \leq \tau^*(H')$, proving (78.21). ∎

With Theorem 78.4 some more properties of minimally nonideal hypergraphs can be derived (Lehman [1990]), where J denotes an all-one matrix:

Theorem 78.5. *Let $H = (V, \mathcal{E})$ be a minimally nonideal hypergraph with $H \neq J_n$ where $n := |V|$. Let $r := r_{\min}(H)$ and $s = \tau(H)$. Let $\mathcal{F}$ and $\mathcal{C}$ be the collections of minimum-size edges of H and $b(H)$ respectively. Let M and N be the $\mathcal{F} \times V$ and $\mathcal{C} \times V$ incidence matrices of $\mathcal{F}$ and $\mathcal{C}$ respectively. Then the rows of M can be ordered such that*

(78.23) $$MN^{\mathsf{T}} = J + (rs - n)I = N^{\mathsf{T}}M.$$

Proof. For each $B \in \mathcal{C}$ we have:

(78.24) $$\sum_{F \in \mathcal{F}} |F \cap B| = rs,$$

since $|B| = s$ and since each $v \in V$ is in exactly r sets in $\mathcal{F}$ (by Theorem 78.4).

As $|F \cap B| \geq 1$ for each $F \in \mathcal{F}$, (78.24) gives:

(78.25) $|F \cap B| \leq rs - n + 1$ for each $F \in \mathcal{F}$, and $|F \cap B| \geq 2$ for at least one $F \in \mathcal{F}$.

Choose for each $B \in \mathcal{C}$ a set $F_B \in \mathcal{F}$ with $|B \cap F_B| \geq 2$. Then

(78.26) for each $v \in V$ there are at least $rs - n + 1$ sets $B \in \mathcal{C}$ with $v \in B \cap F_B$.

To see this, consider $H \setminus v$ and the vector $x := r^{-1} \cdot \mathbf{1}$ in $\mathbb{R}^{V \setminus \{v\}}$. Then x satisfies (78.1) for $H \setminus v$. As $H \setminus v$ is ideal, there exist distinct $B_1, \ldots, B_m \in b(H)$ and $\lambda_1, \ldots, \lambda_m > 0$ with

(78.27) $$x \geq \sum_{i=1}^{m} \lambda_i \chi^{B_i \setminus \{v\}} \text{ and } \sum_{i=1}^{m} \lambda_i = 1.$$

We can assume that $v \in B_i \in \mathcal{C}$ holds for $i = 1, \ldots, k$, and $v \notin B_i$ or $B_i \notin \mathcal{C}$ for $i > k$. So $|B_i \setminus \{v\}| \geq s$ for $i > k$. Then (78.27) implies

(78.28) $$\begin{aligned}\frac{n-1}{r} &= x^{\mathsf{T}}\mathbf{1} \geq \sum_{i=1}^{m} \lambda_i |B_i \setminus \{v\}| \geq \sum_{i=1}^{k} \lambda_i (s-1) + \sum_{i=k+1}^{m} \lambda_i s \\ &= s - \sum_{i=1}^{k} \lambda_i \geq s - \frac{k}{r},\end{aligned}$$

since $\lambda_i \leq 1/r$ for each i, by (78.27). (78.28) implies $k \geq rs - n + 1$. Now for each $i \leq k$, we have $v \in F_{B_i}$, since otherwise $x(F_{B_i}) = 1$, implying $|B_i \cap F_{B_i}| = 1$ (by (78.27)), a contradiction. So we have (78.26).

This implies

(78.29) $$n(rs-n+1) \geq \sum_{B\in\mathcal{C}} |B \cap F_B| = \sum_{v\in V}(\text{number of } B \in \mathcal{C} \text{ with } v \in B \cap F_B) \geq n(rs-n+1),$$

and hence we have equality throughout. So for each $B \in \mathcal{C}$ we have $|B \cap F_B| = rs-n+1$ and $|B \cap F| = 1$ for each $F \in \mathcal{F}$ with $F \neq F_B$. By symmetry we have, for each $F \in \mathcal{F}$, that $|B \cap F| = 1$ for all but one $B \in \mathcal{C}$, which has $|B \cap F| = rs-n+1$. So the set of pairs (B, F) with $|B \cap F| = rs-n+1$ forms a perfect matching covering $\mathcal{C}$ and $\mathcal{F}$. Hence we can reorder the rows of M such that $MN^{\mathsf{T}} = J + (rs-n)I$. In particular, M and N are nonsingular.

This implies

(78.30) $$\begin{aligned} MN^{\mathsf{T}}MN^{\mathsf{T}} &= (J+(rs-n)I)(J+(rs-n)I) \\ &= (n+2(rs-n))J+(rs-n)^2I = rsJ+(rs-n)(J+(rs-n)I) \\ &= MJN^{\mathsf{T}}+(rs-n)MN^{\mathsf{T}} = M(J+(rs-n)I)N^{\mathsf{T}}. \end{aligned}$$

So $N^{\mathsf{T}}M = J + (rs-n)I$ (as M and N are nonsingular). ∎

Notes. Seymour [1990b] asked the following related questions. Suppose that $H = (V, \mathcal{E})$ is a hypergraph without J_n minor ($n \geq 3$). Let $l, w : V \to \mathbb{Z}_+$ be such that

(78.31) $$\tau(H^w) \cdot \tau(b(H)^l) > l^{\mathsf{T}}w.$$

Is there a minor H' of H and $l', w' : VH' \to \{0, 1\}$ such that

(78.32) $$\tau((H')^{w'}) \cdot \tau(b(H')^{l'}) > l'^{\mathsf{T}}w'$$

and such that $\tau((H')^{w'}) \leq \tau(H^w)$ and $\tau(b(H')^{l'}) \leq \tau(b(H)^l)$?

A second question of Seymour is: Let $H = (V, \mathcal{E})$ be a nonideal hypergraph. Is the minimum of $\tau(H')$ over all parallelizations and minors H' of H with $\tau^*(H') < \tau(H')$ attained by a minor of H?

78.4a. Application of Lehman's theorem: Guenin's theorem

Lehman's theorem can be used as a tool in proving the characterization of Guenin [1998a,2001a] of weakly bipartite graphs (Corollary 75.4a). We follow the derivation as given in Schrijver [2002a].

Recall that a signed graph $G = (V, E, \Sigma)$ is called *weakly bipartite* if each vertex of the polyhedron (in $\mathbb{R}^E$) determined by:

(78.33)
(i) $x_e \geq 0$ for each edge e,
(ii) $x(C) \geq 1$ for each odd circuit C,

is integer, that is, the incidence vector of an odd circuit cover. Equivalently, if the hypergraph with vertex set E and edge set all odd circuits of G, is ideal.

Again, let odd-K_5 be the signed graph (VK_5, EK_5, EK_5). Then:

Theorem 78.6 (Guenin's theorem)**.** *A signed graph is weakly bipartite if and only if it has no odd-K_5 minor.*

Proof. Necessity follows from the fact that weak bipartition is closed under taking minors and that odd-K_5 is not weakly bipartite.

To see sufficiency, let $G = (V, E, \Sigma)$ be a minimally non-weakly bipartite signed graph (minimal under taking minors). We show that $G = (V, E, \Sigma)$ contains an odd-K_5 minor. Note that the operations of deletion and contraction in the signed graph G correspond to deletion and contraction in the hypergraph defined above.

Let $n := |E|$, let r be the minimum size of an odd circuit, and let s be the minimum size of an odd circuit cover. Let M (N, respectively) be the matrix whose rows are the incidence vectors of the minimum-size odd circuits (minimum-size odd circuit covers, respectively). By Lehman's theorem (Theorem 78.5), we know that both M and N have precisely n rows, that $rs > n$, and that the rows of M can be ordered such that

(78.34) $\qquad MN^{\mathsf{T}} = J + (rs - n)I = N^{\mathsf{T}}M.$

This implies that we can index the minimum-size odd circuits as $C_1, \ldots, C_n$ and the minimum-size odd circuit covers as $B_1, \ldots, B_n$ in such a way that for all $i, j = 1, \ldots, n$:

(78.35) $\qquad |C_i \cap B_j| = 1$ if $i \neq j$ and $|C_i \cap B_j| = q$ if $i = j$,

where $q := rs - n + 1$. Since $q = |C_1 \cap B_1|$ is odd and ≥ 2 (as $rs > n$), we have $q \geq 3$.

The fact that $N^{\mathsf{T}}M = J + (rs - n)I$ is equivalent to:

(78.36) (i) for each $e \in E$ there are precisely q indices i with $e \in C_i \cap B_i$,

(ii) for all distinct $e, f \in E$ there is precisely one index i with $e \in B_i$ and $f \in C_i$.

Then for all distinct $i, j = 1, \ldots, n$:

(78.37) the only odd circuits contained in $C_i \cup C_j$ are C_i and C_j; the only odd circuit covers contained in $B_i \cup B_j$ are B_i and B_j.

For let C be an odd circuit contained in $C_i \cup C_j$. Then $C_i \triangle C_j \triangle C$ contains an odd circuit, C' say. This implies that $C \cup C' \subseteq C_i \cup C_j$ and $C \cap C' \subseteq C_i \cap C_j$ (for if $e \in C \cap C'$, then $e \notin C_i \triangle C_j$). Hence $|C| + |C'| \leq |C_i| + |C_j|$. So also C and C' are minimum-size odd circuits and $C \cup C' = C_i \cup C_j$. As $|C_i \cap B_i| \geq 3$ we have $|C \cap B_i| \geq 2$ or $|C' \cap B_i| \geq 2$. Therefore, C or C' is equal to C_i, and the other is equal to C_j. The proof for odd circuit covers is analogous. This shows (78.37).

We now construct an odd-K_5 minor. Fix an edge $e \in E$, with ends v_1 and v_2, say. By (78.36)(i) we can assume that e is contained in $C_i \cap B_i$ for $i = 1, \ldots, q$. Then, by (78.36):

(78.38) any two sets among $C_1 \setminus \{e\}, \ldots, C_q \setminus \{e\}, B_1 \setminus \{e\}, \ldots, B_q \setminus \{e\}$ are disjoint, except that $|(C_i \setminus \{e\}) \cap (B_i \setminus \{e\})| = q - 1$ for $i = 1, \ldots, q$.

To see this, choose distinct $i, j = 1, \ldots, q$. Then $C_i \cap B_j = \{e\}$, as $|C_i \cap B_j| = 1$. Moreover, $C_i \cap C_j = \{e\}$, for suppose $f \in C_i \cap C_j$ with $f \neq e$. Then $f \in C_i \cap C_j$ and $e \in B_i \cap B_j$, contradicting (78.36)(ii). One similarly shows that $B_i \cap B_j = \{e\}$. This proves (78.38).

(78.37) implies:

(78.39) $VC_i \cap VC_j = \{v_1, v_2\}$ for distinct $i, j = 1, \ldots, q$.

Otherwise $(C_i \cup C_j) \setminus \{e\}$ contains a path P from v_1 to v_2 different from $C_i \setminus \{e\}$ and $C_j \setminus \{e\}$. By (78.37), $(C_i \cup C_j) \setminus \{e\}$ contains no odd circuit. Hence P and $C_i \setminus \{e\}$ have the same parity (with respect to Σ), and so $P \cup \{e\}$ is an odd circuit in $C_i \cup C_j$, contradicting (78.37). This proves (78.39).

Since $B_i \triangle \Sigma$ is a cut for each $i = 1, 2, 3$, there exist $U_1, U_2, U_3 \subseteq V$ such that

(78.40) $\delta(U_i) = B_j \triangle B_k = (B_j \cup B_k) \setminus \{e\}$

for all distinct $i, j, k \in \{1, 2, 3\}$. As $e \notin B_j \triangle B_k$, we can assume $v_1, v_2 \notin U_i$. Also

(78.41) U_i induces a connected subgraph of G.

If not, there is a $K \subseteq U_i$ such that $\delta(K)$ is a nonempty proper subset of $\delta(U_i)$. Then $B_j \triangle \delta(K)$ is an odd circuit cover contained in $B_j \cup B_k$, distinct from B_j and B_k, contradicting (78.37).

By (78.40), $\delta(U_1 \triangle U_2 \triangle U_3) = \delta(U_1) \triangle \delta(U_2) \triangle \delta(U_3) = \emptyset$, and hence $U_1 \triangle U_2 \triangle U_3 = \emptyset$ (as G is connected). So there exist pairwise disjoint sets V_1, V_2, V_3 of vertices with $U_i = V_j \cup V_k$ for all distinct $i, j, k \in \{1, 2, 3\}$. Define $V_0 := V \setminus (V_1 \cup V_2 \cup V_3)$.

(78.38) and (78.40) imply that $\delta(U_j) \cap \delta(U_k) = B_i \setminus \{e\}$ for distinct i, j, k. Hence $B_i \setminus \{e\}$ is the set of edges connecting either V_i and V_0, or V_j and V_k. So any edge not in $(B_1 \cup B_2 \cup B_3) \setminus \{e\}$ is spanned by one of the sets V_0, V_1, V_2, V_3.

Let $\{i, j, k\} = \{1, 2, 3\}$. Since C_i contains no edge in $(B_j \cup B_k) \setminus \{e\} = \delta(U_i)$, the set VC_i is disjoint from $U_i = V_j \cup V_k$. As $|C_i \cap B_i| \geq 3$ we know that VC_i intersects V_i.

We can reset Σ to an equivalent signing

(78.42) $\Sigma := B_1 \triangle B_2 \triangle B_3 \triangle \delta(V_0)$.

So Σ consists of e and all edges connecting distinct sets among V_1, V_2, V_3. For each $i = 1, 2, 3$ and $k = 1, 2$, let $e_{i,k}$ be the first edge along the path $C_i \setminus \{e\}$ that belongs to B_i, when starting from vertex v_k. So both $e_{i,1}$ and $e_{i,2}$ connect V_0 and V_i.

Let H be the minor of G obtained by deleting all edges except those in $C_1 \cup C_2 \cup C_3$ and those spanned by $V_1 \cup V_2 \cup V_3$, and contracting all remaining edges that are not in $\Sigma \cup \{e_{i,k} \mid i = 1, 2, 3; k = 1, 2\}$.

H can be described as follows. H contains the edge e, connecting the vertices $\overline{v}_1$ and $\overline{v}_2$ to which v_1 and v_2 are contracted (we have $\overline{v}_1 \neq \overline{v}_2$ by (78.39)). For each $i = 1, 2, 3$, the part of the path $C_i \setminus \{e\}$ that is between $e_{i,1}$ and $e_{i,2}$ belongs to one contracted vertex of H, call it z_i. This vertex z_i is adjacent to $\overline{v}_1$ and $\overline{v}_2$ by the edges $e_{i,1}$ and $e_{i,2}$. For each $i = 1, 2, 3$, V_i has been contracted to z_i and a number of other vertices, together forming the stable set S_i (say) in H. Any further edge of H connects S_i and S_j for some distinct $i, j \in \{1, 2, 3\}$.

By (78.41), the subgraph of H induced by $S_i \cup S_j$ is connected (for all distinct $i, j = 1, 2, 3$). So by Lemma 75.4α, the graph $H - \overline{v}_2$ has an odd K_4-subdivision as subgraph, containing the edges $\overline{v}_1 z_1$, $\overline{v}_1 z_2$, and $\overline{v}_1 z_3$. As $\overline{v}_2$ is adjacent to $\overline{v}_1$, z_1, z_2, and z_3, it follows that H has an odd-K_5 minor. ∎

78.4b. Ideality is in co-NP

Seymour [1990b] showed (upon a suggestion of J. Edmonds) that Lehman's theorem (Theorem 78.5) implies that the question 'Given a hypergraph, is it ideal?' belongs to co-NP.

In this, we should be careful in the way the hypergraph $(V, \mathcal{E})$ is given. In most classes of examples, the number of edges is exponential in the number of vertices, and we have no full list of all edges at hand. We can however assume that we have an oracle telling us, for any subset U of V, if U contains an edge of H; that is, if $U \in H^{\uparrow}$. This gives us a polynomial-time test if a subset belongs to $H^{\min}$, and also a polynomial-time test if a subset B is a vertex cover (since B is a vertex cover if and only if $V \setminus B \notin H^{\uparrow}$). So if we have such an oracle for H, we can derive one for its blocker $b(H)$, and conversely.

Moreover, for any $v \in V$, an oracle for H gives oracles for H/v and $H \setminus v$. Indeed, for any $U \subseteq V \setminus \{v\}$: $U \in (H/v)^{\uparrow} \iff U \cup \{v\} \in H^{\uparrow}$ and $U \in (H \setminus v)^{\uparrow} \iff U \in H^{\uparrow}$.

Now to certify that a hypergraph is nonideal, it is sufficient and possible to specify *either* a minor H with $H = J_n$ for $n := |VH|$, *or* a minor H together with numbers r, s, edges $F_1, \ldots, F_n$, and vertex covers $B_1, \ldots, B_n$ (where $n := |VH|$) of H such that

(78.43) (i) $rs > n$,
(ii) $|F_i| = r$, $|B_i| = s$, and $|B_i \cap F_i| = rs - n + 1$ for each $i = 1, \ldots, n$;
(iii) each $v \in VH$ is in precisely r of the F_i and in precisely s of the B_i.

This is possible by Theorem 65.2. If $H = J_n$, this can be tested easily with the oracle. If $H \neq J_n$, then the sets F_i (B_i respectively) can be taken to be minimal edges of H ($b(H)$ respectively); the oracle can tell us that they belong to H ($b(H)$ respectively).

It is also sufficient to certify nonideality: (78.43) implies that $\tau(H) \geq s$: a vertex cover B of H intersects at most $r|B|$ of the F_i, and hence $r|B| \geq n$, implying $|B| \geq s$ (since otherwise $(s-1)r \geq n$ and hence $rs - n + 1 > r$, contradicting (78.43)(ii)). Similarly, (78.43) implies that $r_{\min}(H) \geq r$. As $rs > n$, this implies that H is nonideal.

78.5. Further results and notes

78.5a. Composition of clutters

Billera [1971] described the following composition of hypergraphs. Let $H' = (V', \mathcal{E}')$ and $H'' = (V'', \mathcal{E}'')$ be hypergraphs with V' and V'' disjoint, and choose $v \in V'$. Let $V := (V' \setminus \{v\}) \cup V''$, and define $\mathcal{E}$ by:

(78.44) $$\mathcal{E} := \{E' \in \mathcal{E}' \mid v \notin E'\} \cup \{(E' \setminus \{v\}) \cup E'' \mid E' \in \mathcal{E}', v \in E', E'' \in \mathcal{E}''\}.$$

Let $H = (V, \mathcal{E})$. Then H is ideal if and only if H' and H'' are ideal. (The 'only if' part was shown by Billera [1971] and the 'if' part by Bixby [1971].)

Related results were reported by Chopra [1995]. An extension of these results to *clutter amalgam* was given by Nobili and Sassano [1993a] (cf. Nobili and Sassano [1993b]).

78.5b. Further notes

Cornuéjols and Novick [1994] conjecture that there are only finitely many minimally nonideal hypergraphs H with $r_{\min}(H) > 2$ and $\tau(H) > 2$. This would confirm the question of Ding [1993] whether there exists a number t such that each minimally nonideal hypergraph H satisfies $r_{\min}(H) \leq t$ or $\tau(H) \leq t$.

Since by Lehman's theorem, each minimally nonideal hypergraph $H \neq J_n$ satisfies $\tau^*(H) = r^{-1}\tau_r(H) < \tau(H)$, where $r := r_{\min}(H)$, the existence of such a t would imply that the following property characterizes ideality of a hypergraph H:

(78.45) H contains no J_n minor ($n \geq 3$) and satisfies $\tau_k(H') = k \cdot \tau(H')$ and $\tau_k(b(H')) = k \cdot \tau(b(H'))$ for each minor H' of H and each $k \leq t$.

Ding wondered if $t = 3$ would do.

Ding [1993] conjectures that for each fixed $k \geq 2$, each minor-minimal hypergraph H with $\tau_k(H) < k \cdot \tau(H)$, contains some J_n minor ($n \geq 3$) or satisfies the regularity conditions of Lehman's theorems (Theorem 78.4 and 78.5). Ding [1993] proved this for $k = 2$: if H is minor-minimal with the property $\tau_2 < 2\tau$ and if H has no J_n minor ($n \geq 3$), then the minimum-size vertex covers form an odd circuit on VH.

A $\{0, \pm 1\}$ matrix M is called *ideal* if the polytope

(78.46) $$\{x \mid \mathbf{0} \leq x \leq \mathbf{1}, Mx \geq \mathbf{1} - b\}$$

is integer, where b is the vector with b_i equal to the number of -1's in the ith row of M. These matrices generalize the incidence matrices of ideal hypergraphs. Guenin [1998b] and Nobili and Sassano [1995,1998] showed that they can be characterized in terms of ideal hypergraphs.

Related work on ideal hypergraphs was reported by Novick and Sebő [1996]. A survey on ideal hypergraphs was given by Cornuéjols and Guenin [2002b].

Chapter 79

Mengerian hypergraphs

Mengerian hypergraphs form a subclass of the ideal hypergraphs. They are characterized by the total dual integrality of the edge inequalities (where ideal hypergraphs require only totally *primal* integrality). So Mengerian hypergraphs satisfy min-max relations that are combinatorial at both optima.

This chapter gives a few characterizations of Mengerity. No characterization in terms of forbidden minors is known. In Chapter 80 we will give Seymour's forbidden minor characterization of *binary* Mengerian hypergraphs.

79.1. Mengerian hypergraphs

A hypergraph $H = (V, \mathcal{E})$ is called *Mengerian* if $\nu(H') = \tau(H')$ for each parallelization H' of H.[10] Equivalently:

(79.1) $\quad H$ is Mengerian $\iff$ system (78.1) is totally dual integral.

By (77.19) (or by the theory of total dual integrality), each Mengerian hypergraph is ideal. Like ideal hypergraphs, the class of Mengerian hypergraphs is closed under taking minors:

Theorem 79.1. *Any minor of a Mengerian hypergraph is Mengerian again.*

Proof. As restriction is a special case of parallelization, any restriction of a Mengerian hypergraph is again Mengerian. As for contraction, let $H = (V, \mathcal{E})$ be a Mengerian hypergraph and let $v \in V$ and $w : V \setminus \{v\} \to \mathbb{Z}_+$. Define $w' : V \to \mathbb{Z}_+$ by $w'(u) := w(u)$ if $u \in V \setminus \{v\}$ and $w'(v) := \tau((H/v)^w)$. Then

(79.2) $\quad \tau((H/v)^w) \leq \tau(H^{w'}) = \nu(H^{w'}) \leq \nu((H/v)^w).$

So $\tau((H/v)^w) = \nu((H/v)^w)$. Concluding, H/v is Mengerian. ∎

Unlike ideal hypergraphs, the class of Mengerian hypergraphs is not closed under taking blockers, as we shall see in Section 79.2.

[10] Alternatively, such hypergraphs are said to have the $\mathbb{Z}_+$*-max-flow min-cut property*, shortly the $\mathbb{Z}_+$*-MFMC property*.

Theorem 78.3 implies some characterizations of Mengerian hypergraphs (Lovász [1975a] showed (i)⇔(ii), and Lovász [1976c] (i)⇔(iii); the equivalence (i)⇔(ii) also follows from a more general theorem of Hoffman [1974]):

Theorem 79.2. *For any hypergraph $H = (V, \mathcal{E})$, the following are equivalent:*

(79.3) (i) *H is Mengerian, that is, $\nu(H') = \tau(H')$ for each parallelization H' of H;*
(ii) *$\nu^*(H') = \nu(H')$ for each parallelization H' of H;*
(iii) *$\nu_2(H') = 2\nu(H')$ for each parallelization H' of H.*

Proof. The equivalence (i) $\iff$ (ii) follows from Theorem 78.3, since (79.3)(ii) implies that $\nu^*(H')$ is an integer and since $\nu^*(H') = \tau^*(H')$. The implications (ii)⇒(iii) follows from (77.19), since $\nu^*(H') \geq \frac{1}{2}\nu_2(H') \geq \nu(H')$. So it suffices to prove (iii)⇒(ii).

First observe that for each $w : V \to \mathbb{Z}_+$ and all $j, k \in \mathbb{Z}_+$ we have $\nu_{jk}(H^w) = \nu_k(H^{jw})$. Hence, if (79.3)(iii) holds, then for each $w : V \to \mathbb{Z}_+$ and each i:

(79.4) $$\nu_{2^{i+1}}(H^w) = \nu_2(H^{2^i w}) = 2\nu(H^{2^i w}) = 2\nu_{2^i}(H^w).$$

So by induction on i we find that for all i:

(79.5) $$\nu_{2^i}(H^w) = 2^i \nu(H^w), \text{ that is, } \frac{\nu_{2^i}(H^w)}{2^i} = \nu(H^w).$$

As

(79.6) $$\nu^*(H^w) = \lim_{k \to \infty} \frac{\nu_k(H^w)}{k}$$

(by (77.20)), this gives (79.3)(ii). ∎

Another characterization, in terms of the blocker, is:

Theorem 79.3. *Let $H = (V, \mathcal{E})$ be a hypergraph. Then the blocker $b(H)$ of H is Mengerian if and only if for each natural number k, each k-vertex cover is the sum of k 1-vertex covers.*

Proof. By definition, $b(H)$ is Mengerian if and only if $\nu(b(H)^l) = \tau(b(H)^l)$ for each $l : V \to \mathbb{Z}_+$. Now $\tau(b(H)^l)$ is equal to the minimum value of $l(E)$ for $E \in \mathcal{E}$ (by Theorem 77.1). In other words, $\tau(b(H)^l)$ is equal to the maximum number k for which l is a k-vertex cover.

Moreover, $\nu(b(H)^l)$ is equal to the maximum number k of vertex covers $B_1, \ldots, B_k$ with

(79.7) $$\chi^{B_1} + \cdots + \chi^{B_k} \leq l.$$

So $\nu(b(H)^l) = \tau(b(H)^l)$ for each $l : V \to \mathbb{Z}_+$ if and only if for each k, each k-vertex cover l is the sum of k 1-vertex covers. ∎

Note that the right-hand side of the equivalence in this theorem directly implies, by definition of τ_k, that $\tau_k(H) = k \cdot \tau(H)$ for each k; that is, $\tau^*(H) = \tau(H)$.

79.1a. Examples of Mengerian hypergraphs

Bipartite graphs. Let $G = (V, E)$ be a bipartite graph. It is very easy to show that $\nu_2(G) = 2\nu(G)$. (It amounts to the fact that each bipartite graph G of maximum degree at most 2 has a matching of size at least $\frac{1}{2}|EG|$.)

Since the class of bipartite graphs is closed under parallelization, Theorem 79.2 gives $\nu(G) = \tau(G)$; that is, the matching number of G is equal to the vertex cover number of G. This is Kőnig's matching theorem (Theorem 16.2).

Network flows. Let $D = (V, A)$ be a directed graph and let $s, t \in V$. Let $\mathcal{P}$ be the collection of arc sets of $s-t$ paths. Consider the hypergraph $H = (A, \mathcal{P})$. Then $b(H)$ is the hypergraph with edge set all inclusionwise minimal $s-t$ cuts.

Now $\nu(b(H)) = \tau(b(H))$, since the minimum size k of an $s-t$ path is equal to the maximum number of pairwise disjoint $s-t$ cuts. This is an easy result, by considering the cuts $\delta^{\text{out}}(V_i)$ for $i = 1, \ldots, k$, where V_i is the set of vertices at distance $< i$ from s.

Since the class of hypergraphs $b(H)$ obtained in this way is closed under parallelization (it corresponds to replacing arcs by paths), $b(H)$ is Mengerian. Hence $b(H)$ is ideal, and hence H is ideal. That is, for each weight function $w : A \to \mathbb{Z}_+$ we have $\tau(H^w) = \tau^*(H^w) = \nu^*(H^w)$. This gives that the minimum weight of an $s-t$ cut is equal to the maximum of $\sum_{P \in \mathcal{P}} \lambda_P$ where $\lambda : \mathcal{P} \to \mathbb{R}_+$ with $\sum_{P \in \mathcal{P}} \lambda_P \chi^P \leq w$. That is, we have the max-flow min-cut theorem.

By Menger's theorem, we even know that $\nu(H) = \tau(H)$. As the class of these hypergraphs is closed under parallelization (it corresponds to adding parallel arcs to arcs), we know that H is Mengerian. By Theorem 79.2, to prove the existence of an integer maximum flow, it suffices to show that $\nu_2(H) = 2\nu(H)$, since this class of hypergraphs is closed under parallelization.

Arborescences. Let $D = (V, A)$ be a directed graph and let $r \in V$. Recall that a subset B of A is called an *r-arborescence* if (V, B) is a rooted tree with root r. An r-cut is a set $\delta^{\text{in}}(U)$ of arcs, where U is a nonempty subset of $V \setminus \{r\}$. Let H be the hypergraph with vertex set A and edges all r-arborescences. So the blocker $b(H)$ of H has edges all inclusionwise minimal r-cuts.

Since this class of hypergraphs is closed under parallelization, Edmonds' disjoint arborescences theorem (Theorem 53.1b) implies that H is Mengerian. By the optimum arborescence theorem (Theorem 52.3) also $b(H)$ is Mengerian.

Directed cuts. Let $D = (V, A)$ be a directed graph. Recall that a *directed cut* is a set of arcs of the form $\delta^{\text{in}}(U)$ where U is a nonempty proper subset of V with $\delta^{\text{out}}(U) = \emptyset$. A *directed cut cover* is a set of arcs intersecting all directed cuts. Let H be the hypergraph with vertex set A and edges all directed cuts. So the blocker $b(H)$ of H has edges all inclusionwise minimal directed cut covers.

One may show that $\nu_2(H) = 2\nu(H)$, as was done in the proof of the Lucchesi-Younger theorem (Theorem 55.2). Since again this class of hypergraphs is closed under parallelization, Theorem 79.2 implies that H is Mengerian. This is the Lucchesi-Younger theorem.

So H is ideal, and hence also $b(H)$ is ideal. The example of Figure 56.1 in Section 56.1 shows that $b(H)$ in general is not Mengerian.

79.2. Minimally non-Mengerian hypergraphs

By Theorem 79.1, the class of Mengerian hypergraphs is closed under taking minors. It is not closed under taking blockers, since the hypergraph

(79.8) $$Q_6 := \mathcal{O}(K_4)$$

(the hypergraph with vertex set EK_4 and edges all triangles of K_4) is non-Mengerian, while its blocker is Mengerian: Q_6 is non-Mengerian, since $\nu(Q_6) = 1$ (K_4 has no two edge-disjoint triangles), while $\tau(Q_6) = 2$ (no edge is contained in all triangles). Its blocker $H := b(Q_6)$ has edges all complements of nonempty cuts of K_4. To see that it is Mengerian, we show that $\nu(H^l) = \tau(H^l)$ for each 'length' function $l : EK_4 \to \mathbb{Z}_+$. Then $\tau(H^l)$ is the minimum length of a triangle in K_4. To calculate $\nu(H^l)$, observe that the edges of H are the triangles and the perfect matchings of K_4. Consider any perfect matching M of K_4 with $l(e) > 0$ for both edges $e \in M$. Then replacing l by $l - \chi^M$ reduces $\tau(H^l)$ by 1 and $\nu(H^l)$ by at least 1. So inductively we can assume that each perfect matching of K_4 contains an edge e with $l(e) = 0$. So l is 0 on all edges of a triangle, in which case $\tau(H^l) = 0 \leq \nu(H^l)$, or on all edges of a star, in which case both $\tau(H^l)$ and $\nu(H^l)$ are equal to the minimum length of the edges of the complementary triangle.

Call a hypergraph $H = (V, \mathcal{E})$ *minimally non-Mengerian* if H is a non-Mengerian hypergraph and each proper minor of H is Mengerian.

The hypergraph Q_6 is minimally non-Mengerian. Indeed, choose a vertex e of Q_6. The restriction $Q_6 \setminus e$ has only two edges, and hence is trivially Mengerian. The contraction Q_6/e is isomorphic to $b(Q_6) \setminus e$, and hence is Mengerian, as we saw above.

In Section 80.5 we will see that Q_6 is the only *binary* minimally nonideal hypergraph (binary is defined in Chapter 80). We list this and other examples of minimally non-Mengerian hypergraphs ((i) was given by Lovász [1974], and (ii)-(vi) by Seymour [1977b]):

(79.9)
- (i) $Q_6 = \mathcal{O}(K_4)$, the hypergraph with vertex set EK_4, and edges all triangles;
- (ii) any odd circuit;
- (iii) the blocker of any odd circuit;
- (iv) J_n for $n \geq 3$ (cf. (78.12));
- (v) the circuit on $1, 2, 3, 4, 5, 6, 7, 9$ (in order) added with the edge $\{3, 6, 9\}$;
- (vi) the blocker of the hypergraph in (v);

(vii) the hypergraph with vertex set $\{0, 1, 2, 3\}$ and edges $\{0, 1, 2\}$, $\{0, 1, 3\}$, $\{0, 2, 3\}$, and its blocker.

Example (vii) shows that a minimally non-Mengerian hypergraph H can satisfy $\nu(H) = \tau(H)$.

Notes. Seymour [1977b] conjectured that Q_6 is the only minimally non-Mengerian hypergraph with Mengerian blocker. However, the example of Figure 56.1 gives a minimally non-Mengerian hypergraph on 9 vertices. Its blocker is Mengerian (by the Lucchesi-Younger theorem). Two other examples of hypergraphs consisting of directed cut covers in a directed graph were given by Cornuéjols and Guenin [2002c] and yield two more minimally non-Mengerian hypergraphs with Mengerian blockers.

Seymour [1977b] indicated by a construction that it might be hard to characterize all minimally non-Mengerian hypergraphs.

79.3. Further results and notes

79.3a. Packing hypergraphs

A hypergraph $H = (V, \mathcal{E})$ is called *packing* if $\nu(H') = \tau(H')$ for each minor H' of H. So we have for any hypergraph H (using Theorem 79.1 and Corollary 78.4b):

(79.10) H Mengerian $\Rightarrow$ H packing $\Rightarrow$ H ideal.

Q_6 is an example which is ideal but not packing, but no example is known of a non-Mengerian packing hypergraph. In fact, Conforti and Cornuéjols [1993] conjecture that both concepts coincide. Cornuéjols, Guenin, and Margot [1998,2000] proved this for *dyadic hypergraphs*, that is, hypergraphs H with $|E \cap B| \leq 2$ for each edge E of $H^{\min}$ and each edge B of $b(H)$.

The definition of packing implies that it is closed under taking minors. Call a hypergraph *minimally nonpacking* if it is nonpacking, but each proper minor is packing. So it is a minor-minimal hypergraph satisfying $\nu < \tau$.

Cornuéjols, Guenin, and Margot [1998,2000] showed that if a hypergraph is both minimally nonideal and minimally nonpacking, then $H = J_n$ for some $n \geq 3$ or $r_{\min}(H)\tau(H) = |VH| + 1$. They conjecture that, conversely, each minimally nonideal hypergraph H with $r_{\min}(H)\tau(H) = |VH| + 1$ is minimally nonpacking. By a computer program, this conjecture was verified for all hypergraphs with ≤ 14 vertices.

Another conjecture of Cornuéjols, Guenin, and Margot [1998,2000] is that $\tau(H) = 2$ for each ideal minimally nonpacking hypergraph H. This implies the above conjecture of Conforti and Cornuéjols that each packing hypergraph is Mengerian. For suppose that $H = (V, \mathcal{E})$ is packing and minimally non-Mengerian. Since H is non-Mengerian, there is a $w : V \to \mathbb{Z}_+$ with H^w nonpacking. Choose w with $w(V)$ minimal. Then H^w is minimally nonpacking. So by the second conjecture, $\tau(H^w) = 2$. As H is packing, $w(v) \geq 2$ for some $v \in V$. Now $\tau(H^w/v) \geq \tau(H^w) = 2$. Hence $\nu(H^w/v) \geq 2$. So there exist edges E_1, E_2 of H with $\chi^{E_1 \setminus \{v\}} + \chi^{E_2 \setminus \{v\}} \leq w$. Since $w(v) \geq 2$, this implies $\chi^{E_1} + \chi^{E_2} \leq w$, and hence $\nu(H^w) \geq 2$. This contradicts the fact that H^w is minimally nonpacking.

For a survey, see Cornuéjols and Guenin [2002b].

79.3b. Restrictions instead of parallelizations

It is tempting to conjecture that a hypergraph H is Mengerian if and only if $\nu(H') = \tau(H')$ for each restriction H' of H (instead of for each parallelization H' of H). Similarly, one may speculate that H is ideal if and only if $\tau^*(H') = \tau(H)$ for each restriction H' of H. But these characterizations are not valid, as is shown by the hypergraph of (79.9)(vii): then $\nu(H') = \tau(H')$ for each restriction H' of H (as soon as we delete any vertex we obtain a hypergraph with at most one edge); but H is nonideal, since if we duplicate vertex 0, we obtain a hypergraph with $\tau = 2$ and $\tau^* = \frac{3}{2}$.

So in Theorem 78.3, we cannot restrict H' to restrictions of H instead of parallelizations. But the equivalence of (78.10)(i) and (ii) is maintained if we restrict H' to restrictions of H:

Theorem 79.4. *For any hypergraph, $H = (V, \mathcal{E})$, the following are equivalent:*

(79.11) (i) $\tau(H') = \tau^*(H')$ *for each restriction H' of H;*
(ii) $\tau^*(H')$ *is an integer for each restriction H' of H.*

Proof. Since obviously (i)⇒(ii), we prove (ii)⇒(i). Choose a counterexample with $|V|$ smallest. Let x be a fractional vertex cover of size $\tau^*(H)$. Choose a vertex v with $x_v > 0$. As $x|V \setminus \{v\}$ is a fractional vertex cover of $H \setminus v$, we know:

(79.12) $$\tau^*(H \setminus v) \leq x(V \setminus \{v\}) < x(V) = \tau^*(H).$$

As $\tau^*(H)$ and $\tau^*(H \setminus v)$ are integer, this implies that $\tau^*(H \setminus v) \leq \tau^*(H) - 1$. By the minimality of V we know $\tau^*(H \setminus v) = \tau(H \setminus v)$. Therefore,

(79.13) $$\tau(H) \leq 1 + \tau(H \setminus v) = 1 + \tau^*(H \setminus v) \leq \tau^*(H),$$

and so $\tau(H) = \tau^*(H)$. ∎

As a direct consequence one has (Lovász [1974]):

Corollary 79.4a. *For any hypergraph, $H = (V, \mathcal{E})$, the following are equivalent:*

(79.14) (i) $\tau(H') = \nu(H')$ *for each restriction H' of H;*
(ii) $\nu^*(H') = \nu(H')$ *for each restriction H' of H.*

Proof. Directly from Theorem 79.4. ∎

(Lovász [1974] called hypergraphs with the properties (79.14) *seminormal*.)

Symmetry suggests the question if we can replace in Theorem 79.4 or Corollary 79.4a 'restriction' by 'contraction'.

79.3c. Equivalences for *k*-matchings and *k*-vertex covers

Some of the equivalences in Theorems 78.3 and 79.2 can be generalized as follows (Lovász [1977b] ($k \leq 2$), Schrijver and Seymour [1979] (general k)).

Theorem 79.5. *For any hypergraph $H = (V, \mathcal{E})$ and any $k \in \mathbb{Z}_+$, the following are equivalent:*

(79.15) (i) *$k \cdot \tau^*(H') = \tau_k(H')$ for each parallelization H' of H;*
(ii) *$k \cdot \tau^*(H')$ is an integer for each parallelization H' of H.*

Proof. Similar to the proof of Theorem 78.3. ∎

This implies a result proved by Schrijver and Seymour [1979] (just by adapting the proof methods of Lovász [1975a] for the equivalence (i)⇔(ii) for $k \leq 2$ and of Lovász [1977b] for the equivalence (i)⇔(iii) for $k = 1$):

Corollary 79.5a. *For any hypergraph $H = (V, \mathcal{E})$ and any $k \in \mathbb{Z}_+$, the following are equivalent:*

(79.16) (i) *$\nu_k(H') = \tau_k(H')$ for each parallelization H' of H;*
(ii) *$k \cdot \nu^*(H') = \nu_k(H')$ for each parallelization H' of H;*
(iii) *$\nu_{2k}(H') = 2\nu_k(H')$ for each parallelization H' of H.*

Proof. Similar to the proof of Theorem 79.2. ∎

As an application, let $G = (V, E)$ be an undirected graph. Then $\nu_4(G) = 2\nu_2(G)$ is not difficult to show. Since the class of graphs is closed under parallelization, Corollary 79.5a implies that $\nu_2(G) = \tau_2(G)$, which is Theorem 30.1.

79.3d. A general technique

The following general result (derived with a method given by Lovász [1976c]) gives some more equivalences:

Theorem 79.6. *Let $H = (V, \mathcal{E})$ be a hypergraph and $w : V \to \mathbb{Z}_+$. Let $f : \mathbb{Z}_+^V \longrightarrow \mathbb{R}_+$ satisfy*

(79.17) (i) *$f(x + y) \geq f(x) + f(y)$ for all $x, y \in \mathbb{Z}_+^V$;*
(ii) *if $u \leq w$, then $f(u) \in \mathbb{Z}_+$;*
(iii) *if $x \leq w + \mathbf{1}$, then $f(2x) = 2f(x)$;*
(iv) *$f(\chi^U) > 0$ for each $U \in \mathcal{E}$.*

Then $\tau(H^w) \leq f(w)$.

Proof. By induction on $\mathbf{1}^\mathsf{T} w$, the case where $\tau(H^w) = 0$ being trivial. Assume $\tau(H^w) > 0$. That is, the support U of w contains an edge of H. Choose $x \in \mathbb{Z}_+^V$ with $w \leq x \leq w + \chi^U$ such that $f(x) = f(w)$ and such that $\mathbf{1}^\mathsf{T} x$ is as large as possible. Then $x \neq w + \chi^U$, since $f(w + \chi^U) \geq f(w) + f(\chi^U) > f(w)$, by (i) and (iv) (note that f is monotone by (i)). So $x_v < w_v + 1$ for some $v \in U$. By the maximality of x we know that $f(x + \chi^v) > f(x)$. Moreover, by induction, $\tau(H^{w-\chi^v}) \leq f(w - \chi^v)$, and hence

$$\begin{aligned}(79.18)\qquad \tau(H^w) &\leq 1 + \tau(H^{w-\chi^v}) \leq 1 + f(w - \chi^v) \leq 1 + f(x - \chi^v)\\ &\leq 1 + f(2x) - f(x + \chi^v) = 1 + 2f(x) - f(x + \chi^v) < 1 + f(x) = 1 + f(w),\end{aligned}$$

and hence, since $f(w) \in \mathbb{Z}$ we have $\tau(H^w) \leq f(w)$. ∎

This gives the following equivalent form of Theorem 79.4:

Corollary 79.6a. *Let $H = (V, \mathcal{E})$ be a hypergraph and let $w \in \mathbb{Z}_+^V$ be such that $\tau^*(H^x) \in \mathbb{Z}$ for each $x \leq w$. Then $\tau(H^w) = \tau^*(H^w)$.*

Proof. Define $f(x) := \tau^*(H^x)$ for $x \in \mathbb{Z}_+^V$ and apply Theorem 79.6. ∎

We also obtain a generalization of Theorem 79.2:

Corollary 79.6b. *Let $H = (V, \mathcal{E})$ be a hypergraph and let $w \in \mathbb{Z}_+^V$ be such that $\nu(H^x) = \frac{1}{2}\nu_2(H^x)$ for each $x \leq w + \mathbf{1}$. Then $\tau(H^w) = \nu(H^w)$.*

Proof. Define $f(x) := \nu(H^x)$ for $x \in \mathbb{Z}_+^V$ and apply Theorem 79.6. ∎

A special case of this is:

Corollary 79.6c. *Let $H = (V, \mathcal{E})$ be a hypergraph with $\nu_2(H^w) = 2\nu(H^w)$ for each $w : V \to \{0, 1, 2\}$. Then $\nu(H) = \tau(H)$.*

Proof. This follows by taking $w = \mathbf{1}$ in Corollary 79.6b. ∎

This gives a generalization to arbitrary k (instead of $k = 2$), since if $\nu(H) = \frac{1}{k}\nu_k(H)$ for some $k \geq 2$, then $\nu(H) = \frac{1}{k-1}\nu_{k-1}(H)$. This follows from

$$\nu_{k-1}(H) \leq \nu_k(H) - \nu(H) = k \cdot \nu(H) - \nu(H) = (k-1)\nu(H). \tag{79.19}$$

Hence $\nu(H) = \frac{1}{2}\nu_2(H)$.

Another consequence of Theorem 79.6 is:

Corollary 79.6d. *For any hypergraph $H = (V, \mathcal{E})$ and any $k \in \mathbb{Z}_+$, the following are equivalent:*

(79.20) (i) *$\tau_k(H') = k \cdot \tau(H')$ for each restriction H' of H;*
(ii) *$\frac{1}{k}\tau_k(H') \in \mathbb{Z}$ for each restriction H' of H.*

Proof. Define $f(x) := \frac{1}{k}\tau_k(H^x)$ for $x \in \mathbb{Z}_+^V$ and apply Theorem 79.6 to $w = \mathbf{1}$. ∎

79.3e. Further notes

Seymour [1979a] gave the following example of an ideal hypergraph H with $\tau(H) \neq \frac{1}{2}\nu_2(H)$. Replace each edge of the Petersen graph by a path of length 2, making the graph G. Let $T := VG \setminus \{v\}$, where v is an arbitrary vertex of v of degree 3. Let $\mathcal{E}$ be the collection of T-joins. Let $T_{30} := (EG, \mathcal{E})$. Then $\tau(H') = \nu^*(H')$ for each parallelization H' of T_{30}, by Theorem 29.5. On the other hand, $\tau(T_{30}) = 2$ while G has no T-joins J_1, J_2, J_3, J_4 containing each edge of G at most twice. Otherwise, the sets $J_1 \triangle J_2$, $J_1 \triangle J_3$, and $J_1 \triangle J_4$ are cycles, together containing every edge of G precisely twice. Hence their complements give a 3-edge-colouring of the Petersen graph. This is not possible.

In this example, T_{30} is not only ideal, but also satisfies $\tau(b(T_{30})') = \frac{1}{2}\nu_2(b(T_{30})')$ for each parallelization $b(T_{30})'$ of $b(T_{30})$ (by Corollary 29.2a). Seymour [1981a] conjectures that T_{30} is the unique minor-minimal binary ideal hypergraph with the property $\nu_2 < 2\tau$.

P.D. Seymour (personal communication 1975) conjectures that for each ideal hypergraph H one has $\nu_k(H) = k \cdot \tau(H)$ where k is some power of 2. He also asks if $k = 4$ would do in all cases. Moreover, Seymour [1979a] conjectures that for each ideal hypergraph H, the g.c.d. of those k with $\nu_k(H) = k \cdot \tau(H)$ is equal to 1 or 2.

In Schrijver and Seymour [1979] it is shown that, for each hypergraph H there is an integer k such that $\nu_k(H') = \tau_k(H')$ for each parallelization H' of H.

Chapter 80

Binary hypergraphs

Several hypergraphs coming from graphs are binary. Binary hypergraphs are hypergraphs such that the symmetric difference of any odd number of edges contains an edge as subset.
Binary hypergraphs have a convenient algebraic structure, that enables to handle packing and blocking problems better than for general hypergraphs. Key result of this chapter is Seymour's characterization of binary Mengerian hypergraphs.

80.1. Binary hypergraphs

A hypergraph $H = (V, \mathcal{E})$ is called *binary* if

(80.1) for all odd s and $E_1, \ldots, E_s \in \mathcal{E}$ there is an $E \in \mathcal{E}$ with $E \subseteq E_1 \triangle \cdots \triangle E_s$.

Trivially, for each binary hypergraph $H = (V, \mathcal{E})$, the hypergraph $H^{\min}$ is again binary.

In previous chapters we have seen several examples of binary hypergraphs: given an undirected graph $G = (V, E)$, binary hypergraphs on E are formed by the odd circuits, by the complements of cuts, by the $s-t$ paths, by the $s-t$ cuts (given $s, t \in V$), by the T-joins, by the T-cuts (given $T \subseteq V$), and by the paths that connect either s_1 and t_1, or s_2 and t_2 (given $s_1, t_1, s_2, t_2 \in V$).

It is not difficult to show that the class of binary hypergraphs is closed under taking minors, parallelizations, and blockers (see also Section 80.3 below).

80.2. Binary hypergraphs and binary matroids

Binary hypergraphs have a strong linear algebraic structure over the field GF(2), and are strongly related to binary matroids. It will be good to understand these relations.

For a binary hypergraph $H = (V, \mathcal{E})$, a *cycle* is the symmetric difference of any number of edges of H. Call the cycle *odd* (*even*, respectively), if it is the symmetric difference of an odd (even, respectively) number of edges of H.

The odd cycles of H form again a binary hypergraph, say H'. By definition of binarity, the inclusionwise minimal edges of H' coincide with the inclusionwise minimal edges of H. So from a packing and blocking point of view there is no difference in considering any of the binary hypergraphs H, H', or $H^{\min}$.

If $\emptyset \notin \mathcal{E}$, there is no cycle that is both odd and even. The even cycles form a subspace of the boolean space $\mathcal{P}(V)$, and (if $\emptyset \notin \mathcal{E}$) the odd cycles form a cospace of it.

A clutter $H = (V, \mathcal{E})$ is binary if and only if there is a binary matroid $M = (V, \mathcal{I})$ such that, for some $B \subseteq V$:

(80.2) $\mathcal{E}$ is equal to the collection of circuits C of M with $|C \cap B|$ odd.

The matroid M is unique and is equal to the binary matroid whose circuits are the minimal nonempty cycles of H. The set B (generally) is not unique: any set B qualifies for it if and only if $|B \cap E|$ is odd for each $e \in E$.

Another way of obtaining a binary hypergraph H from a binary matroid $M = (V, \mathcal{I})$ is by choosing a vertex $v \in V$, and taking as edges of H the sets $C \setminus \{v\}$ where C is a circuit of M containing v. This hypergraph will be denoted by $H_{M,v}$ and is called a *matroid port*. Each binary clutter can be obtained in this way.

80.3. The blocker of a binary hypergraph

The following is an important observation:

(80.3) The blocker $b(H)$ of a binary hypergraph $H = (V, \mathcal{E})$ is equal to the collection of all inclusionwise minimal sets B satisfying $|B \cap E|$ odd for each $E \in \mathcal{E}$.

To see this, if $|B \cap E|$ is odd for each $E \in \mathcal{E}$, then B is a vertex cover, and hence it contains a set in $b(H)$. Conversely, if $B \in b(H)$, then $|B \cap E|$ is odd for each $E \in \mathcal{E}$. For suppose that $|B \cap E|$ is even. As B is a minimal vertex cover, for each $v \in B \cap E$ there is an $E_v \in \mathcal{E}$ with $E_v \cap B = \{v\}$. Then by (80.1) the symmetric difference of E and the sets E_v for $v \in B \cap E$ contains a set $F \in \mathcal{E}$. Then $F \cap B = \emptyset$, a contradiction.

This proves (80.3), which implies that:

(80.4) if H is binary, then $b(H)$ is binary; if H is a clutter, then: H is binary $\iff$ $b(H)$ is binary.

The second statement follows from the fact that $b(b(H)) = H$ if H is a clutter.

If $H = H_{M,v}$ for some binary matroid M and $v \in VM$, then the blocker satisfies $b(H_{M,v}) = H_{M^*,v}$.

80.3a. Further characterizations of binary clutters

Lehman [1964] and Seymour [1976b] gave further characterizations of binary clutters. They showed that the following are equivalent for any clutter $H = (V, \mathcal{E})$:

(80.5)
(i) H is binary, that is, satisfies (80.1);
(ii) for all $E_1, E_2, E_3 \in \mathcal{E}$ there is an $E \in \mathcal{E}$ with $E \subseteq E_1 \triangle E_2 \triangle E_3$;
(iii) $|B \cap E|$ is odd for all $E \in H$ and $B \in b(H)$;
(iv) $|B \cap E| \neq 2$ for all $E \in H$ and $B \in b(H)$;
(v) H has no minor equal to $P_4 := (\{1,2,3,4\}, \{\{1,2\},\{2,3\},\{3,4\}\})$ or to J_n for any $n \geq 3$ (defined in (78.12)).

(The equivalence of (i) and (iii) was shown by Lehman [1964], the equivalence of (i) and (ii) by A. Lehman (unpublished) and Seymour [1976b], and the other equivalences by Seymour [1976b].)

80.4. On characterizing binary ideal hypergraphs

Since the class of binary ideal hypergraphs is closed under taking minors, it can be characterized by specifying the collection of binary minimally nonideal hypergraphs. Seymour [1981a] offered the conjecture that this collection consists precisely of $\mathcal{O}(K_5)$, $b(\mathcal{O}(K_5))$, and F_7 (see (78.12)).

This conjecture is still open. As we saw in Sections 75.5 and 78.4a, the conjecture has been proved for the class of hypergraphs formed by the odd circuits of signed graphs, by Guenin [1998a,2001a]. For this class, only $\mathcal{O}(K_5)$ is a forbidden minor, since $b(\mathcal{O}(K_5))$ and F_7 do not arise in this way.

By Corollary 29.2b, the conjecture is also true for the class of hypergraphs of T-joins since neither of the three proposed forbidden minors comes from T-joins.

This was extended by Cornuéjols and Guenin [2002a] to binary hypergraphs without Q_6^+ or Q_7^+ minor. Here for any hypergraph $H = (V, \mathcal{E})$, the hypergraph H^+ arises by adding a new vertex u to V and by taking as edges all sets $E \cup \{u\}$ with $E \in \mathcal{E}$. The hypergraph Q_7 arises from Q_6^+ by adding as edges the perfect matchings of K_4.

This implies that for any regular matroid $M = (V, \mathcal{I})$ and any $\Sigma \subseteq V$, the collection of circuits C of M with $|C \cap \Sigma|$ odd, form a hypergraph for which Seymour's conjecture holds. Other cases where Seymour's conjecture holds were given by Guenin [2001c,2002c].

Adding an Eulerian condition. In Section 79.3e we saw that the following ideal hypergraph H does not satisfy $\nu_2(H) = 2\tau(H)$. Let G be obtained from the Petersen graph $\mathbf{P}_{10}$ by replacing each edge by a path of length 2. Let $T := VG \setminus \{v\}$ for some degree-3 vertex v of G. Let T_{30} be the hypergraph of T-joins on EG.

Seymour [1981a] conjectured that any binary ideal hypergraph $H = (V, \mathcal{E})$ without T_{30} minor satisfies $\frac{1}{2}\nu_2(H) = \tau(H)$. If moreover all edges of $b(H)$ have the same parity, then $\nu(H) = \tau(H)$. However, as A.M.H. Gerards and B. Guenin observed, the Petersen graph gives a simpler counterexample to the second conjecture: the hypergraph T_{15} of $V\mathbf{P}_{10}$-joins in $\mathbf{P}_{10}$ is a binary ideal hypergraph without T_{30} minor and having all edges of $b(T_{15})$ odd, while $\nu(T_{15}) = 2 < 3 = \tau(T_{15})$. This suggests the question if for each binary hypergraph $H = (V, \mathcal{E})$:

(80.6) (?) $\nu(H^w) = \tau(H^w)$ for each $w : V \to \mathbb{Z}_+$ with $w(B)$ even for all $B \in b(H) \iff \nu_2(H^w) = 2\tau(H^w)$ for each $w : V \to \mathbb{Z}_+ \iff H$ has no $\mathcal{O}(K_5)$, $b(\mathcal{O}(K_5))$, F_7, or T_{15} minor. (?)

80.5. Seymour's characterization of binary Mengerian hypergraphs

For binary Mengerian hypergraphs, the forbidden minors are known: Seymour [1977b] showed that the only binary minimally non-Mengerian hypergraph is $Q_6 = \mathcal{O}(K_4)$. In this section we give a proof based on Seymour [1977b] and on the short proof by Guenin [2002a].

Theorem 80.1. *A binary hypergraph is Mengerian if and only if it has no Q_6 minor.*

Proof. We have seen necessity in Section 79.2. We prove sufficiency.

Call a hypergraph $H = (V, \mathcal{E})$ *critical* if each vertex is contained in a vertex cover of size $\tau(H)$. Call a subset of V a *cycle* if it is a symmetric difference of edges, a *k-cycle* if it is a symmetric difference of k edges, an *even cycle* if it is a k-cycle for some even k, and a *circuit* if it is a minimal nonempty cycle. Call two vertices x, y *parallel* if $\{x, y\}$ is a 2-cycle. By the definition of binary hypergraph, for each cycle C and each edge E, the set $C \triangle E$ contains an edge. Hence, if x and y are parallel, then for any inclusionwise minimal edge F of H with $x \in F$, one has $y \notin F$ and $(F \setminus \{x\}) \cup \{y\}$ is again an edge. So any minimal vertex cover containing x also contains y.

To see sufficiency, it suffices to show that $\nu(H) = \tau(H)$ for each binary hypergraph without Q_6 minor (since the class of binary hypergraphs without Q_6 minor is closed under parallelization). Choose a counterexample $H = (V, \mathcal{E})$ to this with $|V|$ minimal. So H has no Q_6 minor while $\nu(H) < \tau(H)$. Choose H moreover such that the number of pairs of parallel elements is as large as possible.

Note that the minimality of V implies that H is critical (as any vertex that belongs to no minimum-size vertex cover can be deleted to obtain a smaller counterexample). Define $\tau := \tau(H)$ and, for each $v \in V$ define:

(80.7) $\beta_v := \{B \mid B \text{ vertex cover}, |B| = \tau, v \in B\}$.

Define U to be the set of vertices u with β_u inclusionwise minimal:

(80.8) $U := \{u \mid \text{there is no } v \in V \text{ with } \beta_v \subset \beta_u\}$.

So U is nonempty. Note that if u and v are parallel, then $\beta_u = \beta_v$. Let M be the set of pairs of nonparallel elements u, v in U with $\beta_u = \beta_v$. Then:

(80.9) each element $u \in U$ is contained in a pair in M.

Suppose not. By the minimality of V, $\nu(H \setminus u) = \tau(H \setminus u) = \tau - 1$ (since $\tau - 1 \leq \tau(H \setminus u) = \nu(H \setminus u) \leq \nu(H) < \tau$). So $V \setminus \{u\}$ contains a subset Y that is the union of $\tau - 1$ disjoint edges of H. So Y is a $(\tau - 1)$-cycle.

Let K be the parallel class of u. By the minimality of V, $\nu(H/K) = \tau(H/K) \geq \tau$. So $V \setminus K$ contains a collection $\mathcal{F}$ of disjoint edges of H/K with $|\mathcal{F}| = \tau$. Then $\mathcal{F}$ partitions $V \setminus K$, since for each $v \in V \setminus K$ there exists a $B \in \beta_v \setminus \beta_u$ (since $\beta_v \not\subseteq \beta_u$, as u is contained in no pair in M, by assumption). So $u \notin B$, hence $B \cap K = \emptyset$. As $|B| = \tau$, B intersects each edge in $\mathcal{F}$ precisely once. Hence $\mathcal{F}$ covers v. Concluding, $\mathcal{F}$ partitions $V \setminus K$.

This implies that $V \setminus K$ is contained in some τ-cycle L. So $L \triangle Y$ is a $(2\tau - 1)$-cycle, and hence contains a minimal edge E of H. As K is a parallel class, $|E \cap K| \leq 1$. If $E \cap K = \emptyset$ let $E' := E$; if $E \cap K \neq \emptyset$, let $E' := (E \setminus K) \cup \{u\}$. Then E' is disjoint from Y, so $\nu(H) \geq \tau$, contradicting our assumption. This proves (80.9).

Next:

(80.10) each pair $e \in M$ contains a vertex u such that H has edges $E_1, \ldots, E_\tau$ with $E_1 \cap E_2 = \{u\}$ and with $E_1 \setminus \{u\}, E_2 \setminus \{u\}, E_3, \ldots, E_\tau$ partitioning $V \setminus e$.

Indeed, let $e = \{u, v\}$ be such that u has at least as many parallel elements as v has. Let $\widetilde{H}$ be obtained from H by deleting v and adding an extra parallel element to u (that is, we duplicate u). This increases the number of pairs of parallel elements. So, by the choice of H, $\nu(\widetilde{H}) = \tau(\widetilde{H})$. Moreover, since $\beta_u = \beta_v$, we have that $\tau(\widetilde{H}) = \tau$ and $\widetilde{H}$ is critical. So $V\widetilde{H}$ can be partitioned into τ edges of $\widetilde{H}$. This gives (80.10).

A consequence of (80.10) is that $V \setminus e$ is a τ-cycle of H. Hence

(80.11) $e \triangle f$ is an even cycle of H, for all $e, f \in M$,

since $e \triangle f = (V \setminus e) \triangle (V \setminus f)$.

Now fix a pair $e \in M$, and let $u, E_1, \ldots, E_\tau$ be as in (80.10). Then

(80.12) $E_1 \triangle E_2$ contains no edge E of H,

since otherwise replacing E_1 and E_2 by E and $E_1 \triangle E_2 \triangle E$ would show $\nu(H) \geq \tau$.

Let H' be a smallest minor of H such that $H' = H \setminus Y/X$ for some disjoint subsets X, Y of $E_1 \triangle E_2$ and such that, defining $C := (E_1 \triangle E_2) \setminus (X \cup Y)$:

(80.13) (i) $X \cup Y$ is a union of circuits of H;
(ii) C is a cycle of H';
(iii) $C \cup \{u\}$ contains an edge of H';
(iv) $\tau(H') \geq \tau$;

(v) each edge E of H' contained in $C \cup \{u\}$ satisfies $\tau(H' \setminus E) \leq \tau - 2$.

Such an H' exists, since H has these properties. Then:

Claim 1. C is a circuit of H'.

Proof of Claim 1. Suppose that C is not a circuit of H'. Then, as C is a cycle, C can be partitioned into two nonempty cycles C_1 and C_2 of H'. By (80.12), C_1 and C_2 are even cycles.

Define

(80.14) $\quad H_1 := H'/C_2.$

We show that H_1 satisfies (80.13)(i)-(iv).

To see (80.13)(i) for H_1, $X \cup Y \cup C_2$ is a union of cycles of H, since $C_2 = C' \setminus (X \cup Y)$ for some cycle C' of H. To see (80.13)(ii) for H_1, C_1 is a cycle of H', hence also of H_1. To see (80.13)(iii) for H_1, $C \cup \{u\}$ contains an edge of H', hence $C_1 \cup \{u\} = (C \cup \{u\}) \setminus C_2$ contains an edge of H_1. To see (80.13)(iv) for H_1, we have $\tau(H'/C_2) \geq \tau(H') \geq \tau$.

So H_1 satisfies (80.13)(i)-(iv). Hence, by the minimality of H', H_1 has an edge $E \subseteq C_1 \cup \{u\}$ with $\tau(H_1 \setminus E) \geq \tau - 1$. Define $P := E \setminus \{u\}$, $Q = C_1 \setminus E$, and

(80.15) $\quad H_2 := H' \setminus P/Q.$

We show that H_2 satisfies (80.13), which contradicts the minimality of H'. To see (80.13)(i) for H_2, $X \cup Y \cup C_1$ is a union of circuits of H, since $C_1 = C' \setminus (X \cup Y)$ for some cycle C' of H. To see (80.13)(ii) for H_2, C_2 is a cycle of H', hence also of H_2. To see (80.13)(iii) for H_2, $E = E' \setminus C_2$ for some edge E' of H'. Then $E' \triangle C_1$ contains an edge E'' of H'. Then $E'' \cap P = \emptyset$, since $E' \triangle C_1$ is disjoint from P (as $P \subseteq E' \cap C_1$). So $E'' \setminus Q = E'' \setminus C_1 \subseteq (E' \triangle C_1) \setminus C_1 \subseteq C_2 \cup \{u\}$. This proves (80.13)(iii) for H_2.

To see (80.13)(iv) for H_2, suppose to the contrary that B is a minimum-size vertex cover of H_2 of size $\leq \tau - 1$. Then B intersects each of $E_3, \ldots, E_\tau$ at least once, and hence does not intersect C_2 (as $|B \cap C_2|$ is even). So B is a vertex cover of H_2/C_2. As $C_2 \cup \{u\}$ contains an edge of H_2, we also know that $u \in B$. So $B \setminus \{u\}$ is a vertex cover of $H_2/C_2 \setminus \{u\} = H' \setminus E/(C_2 \cup Q) = H_1 \setminus E/Q$, and hence of $H_1 \setminus E$. This contradicts the fact that $\tau(H_1 \setminus E) \geq \tau - 1$. So H_2 satisfies (80.13)(iv).

To see (80.13)(v) for H_2, let F be an edge of H_2 contained in $C_2 \cup \{u\}$. As $H_2 = H' \setminus P/Q$, there exists a $Q' \subseteq Q$ such that $F \cup Q'$ is an edge of H'.

Suppose that $Q' \neq Q$. Choose $r \in Q \setminus Q'$. Let B be a vertex cover of H of size τ containing r. Then B intersects each of $E_3, \ldots, E_\tau$ at least once. As $r \in B$, B intersects $E_1 \triangle E_2$ at least once, hence at least twice (as $E_1 \triangle E_2$ is an even cycle). Hence, as $|B| = \tau$, $u \notin B$ and B intersects each E_i in precisely one element. Moreover, B is disjoint from $C_2 \cup X \cup Y$, as this last set is a union of circuits of H, implying that if B intersects $C_2 \cup X \cup Y$ at least once, then at least twice; since $r \notin C_2 \cup X \cup Y$, this is not possible. So B is a vertex cover of H_1. Hence $B \cap E \neq \emptyset$. So B contains a second element in C_1, say s. As $F \cup Q' \cup X$ contains an edge of H, it intersects B. So $s \in Q'$. This would mean that B is disjoint from E, a contradiction.

So $Q' = Q$. Then $R := F \cup P = (F \cup Q) \triangle C_1$ contains an edge of H'. By (80.13)(v), $H' \setminus R$ has a vertex cover B of size $\tau - 2$. As B intersects each of $E_3, \ldots, E_\tau$, B is disjoint from Q. So B is a vertex cover of $H' \setminus R/Q = H_2 \setminus F$, proving $\tau(H_2 \setminus F) \leq \tau - 2$. So H_2 satisfies (80.13)(v), contradicting the minimality of H'. *End of Proof of Claim 1*

By (80.13)(ii)(iii), H' has edges F_1, F_2 with $F_1 \cap F_2 = \{u\}$ and $F_1 \cup F_2 = C \cup \{u\}$. By (80.13)(v), there exist vertex covers B_1 and B_2 of H' with $|B_i \setminus F_i| \leq \tau - 2$. So $B_1 \cap F_2 = B_2 \cap F_1 = \{u\}$. Then

(80.16) H' has an edge F_3 disjoint from $(B_1 \setminus F_1) \cup (B_2 \setminus F_2) \cup \{u\}$.

Otherwise, the latter set contains a minimal vertex cover B of H'. Now each B_i intersects each of the edges $E_3, \ldots, E_\tau$ precisely once. So B intersects each of these E_i at most twice, hence precisely once. Therefore $|B| \leq \tau - 1$, contradicting (80.13)(iv). This proves (80.16).

Choose F_3 in (80.16) with $F_3 \setminus C$ minimal. Let H'' arise from H' by deleting all vertices not in $F_1 \cup F_2 \cup F_3$. Then $\tau(H'') \geq 2$, since $F_1 \cap F_2 \cap F_3 = \{u\} \cap F_3 = \emptyset$. Moreover, $\nu(H'') = 1$, for suppose that H'' has disjoint edges F and F', with $u \notin F$. By the minimality of $F_3 \setminus C$ we know that $F \setminus C = F_3 \setminus C$. So $F' \subseteq C \cup \{u\}$. But then, since C is a circuit, $F' = F_1$ or $F' = F_2$ (otherwise $F' \triangle F_1$ is a nonempty cycle properly contained in C). However, F intersects B_1 and B_2, hence F intersects $B_1 \cap F_1$ and $B_2 \cap F_2$, and hence it intersects F_1 and F_2, a contradiction. So $\nu(H'') = 1 < \tau(H'')$.

The minimality of H implies $H'' = H' = H$ and $\tau = 2$. The equality $\tau = 2$ implies that $U = V$ and M forms a perfect matching on V: if $v \in V \setminus U$, then $\beta_u \subset \beta_v$ for some $u \in U$, and hence (using (80.9)) any minimum-size vertex cover containing u has at least three elements — a contradiction, since $\tau = 2$; similarly, if $u \in V$ would be in two pairs in M, there is a minimum-size vertex cover of size ≥ 3 — again a contradiction.

Also, $V \setminus e = E_1 \triangle E_2 = F_1 \triangle F_2$, whence it is a circuit (by Claim 1). As any two pairs from M form an even cycle (by (80.11)), we know $|V \setminus e| = 4$. So $|V| = 6$, $|M| = 3$, giving $H = Q_6$. ∎

Notes. Tseng and Truemper [1986] gave a decomposition theorem for binary Mengerian hypergraphs. It implies that the property of being Mengerian belongs to NP for binary hypergraphs. Shorter proofs of the decomposition result were given by Bixby and Rajan [1989] and Truemper [1987]. The latter paper also gives polynomial-time algorithms for testing Mengerity of a binary hypergraph and for finding a minimum-weight vertex cover and a maximum packing of edges subject to a weight function in binary Mengerian hypergraphs. A description of this algorithm was given in Bixby and Cunningham [1995]. Also Hartvigsen and Wagner [1988] gave a polynomial-time algorithm to test Mengerity of a binary hypergraph.

More background can be found in the book of Truemper [1992].

80.5a. Applications of Seymour's theorem

We describe a number of applications of Theorem 80.1, some of which we have seen in previous parts of this book. Except for those in the first two applications below, the theorems are due to Seymour [1977b].

$s-t$ cuts. Let $G = (V, E)$ be a graph and let $s, t \in V$. The collection of $s-t$ cuts forms a binary hypergraph on E, without Q_6 minor. Hence Theorem 80.1 implies the edge-disjoint version of (the easy) Theorem 6.1: the maximum number of edge-disjoint $s-t$ cuts is equal to the minimum length of an $s-t$ path (the max-potential min-work theorem).

$s-t$ paths. Let $G = (V, E)$ be a graph and let $s, t \in V$. The collection of $s-t$ paths forms a binary hypergraph on E, without Q_6 minor. Hence Theorem 80.1 implies the edge-disjoint undirected version of Menger's theorem (Corollary 9.1b): the maximum number of edge-disjoint $s-t$ paths is equal to the minimum size of an $s-t$ cut.

***T*-cuts.** Let $G = (V, E)$ be a graph and let $T \subseteq V$. The collection of T-cuts forms a binary hypergraph on E. If it is Q_6, then $G = K_4$ and $T = VK_4$. Hence Theorem 80.1 implies Corollary 29.9a: If K_4, VK_4 is not a minor of G, T (in the sense of Section 29.11b), then the minimum size of a T-join is equal to the maximum number of disjoint T-cuts.

***T*-joins.** Let $G = (V, E)$ be a graph and let $T \subseteq V$. The collection of T-joins forms a binary hypergraph on E. If it is Q_6, then $G = K_{2,3}$ and $T = VK_{2,3} \setminus \{u\}$, where u is a vertex of degree 3. Hence Theorem 80.1 implies Theorem 29.10: If $K_{2,3}, VK_{2,3} \setminus \{u\}$ is not a minor of G, T (in the sense of Section 29.11b), then the minimum size of a T-cut is equal to the maximum number of disjoint T-joins.

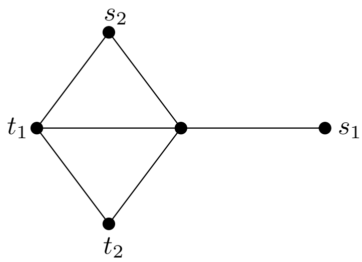

Figure 80.1
s_1 and t_1, and s_2 and t_2, have distance 2, but there exist no two disjoint cuts each separating both s_1 and t_1, and s_2 and t_2.

$s_1 - t_1$ and $s_2 - t_2$ cuts. Let $G = (V, E)$ be a graph and let $s_1, t_1, s_2, t_2 \in V$. The collection of cuts that separate *both* s_1 and t_1, *and* s_2 and t_2 forms a binary hypergraph on E. If it is Q_6, then G is the graph in Figure 80.1 up to permuting indices and exchanging s_1 and t_1. Hence Theorem 80.1 implies Theorem 71.4: if G has no subgraph contractible to the graph in Figure 80.1 up to permuting indices and exchanging s_1 and t_1, then the minimum length of a path connecting either s_1 and t_1, or s_2 and t_2 is equal to the maximum number of pairwise disjoint cuts each separating both s_1 and t_1, and s_2 and t_2. (Here we assume that the subgraph contains the s_i, t_i, and that these vertices are contracted to the vertices indicated by s_i and t_i in the figure.)

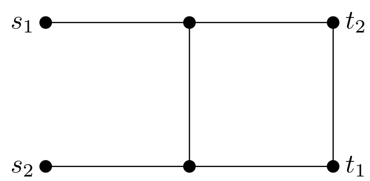

Figure 80.2
The maximum total value of a 2-commodity flow (subject to capacity 1) is equal to 2, but the maximum total value of an integer 2-commodity flow is equal to 1.

$s_1 - t_1$ and $s_2 - t_2$ paths. Let $G = (V, E)$ be a graph and let $s_1, t_1, s_2, t_2 \in V$. The collection of paths that connect *either* s_1 and t_1, *or* s_2 and t_2 forms a binary hypergraph on E. If it is Q_6, then it is the graph of Figure 80.2 up to exchanging s_1 and t_1, and s_2 and t_2. Hence Theorem 80.1 implies Theorem 71.2: If G has no subgraph contractible to the graph of Figure 80.2 up to exchanging s_1 and t_1, and s_2 and t_2, then the maximum number of edge-disjoint paths, each connecting either s_1 and t_1, or s_2 and t_2, is equal to the minimum size of a cut separating both s_1 and t_1, and s_2 and t_2.

Odd circuits. Let $G = (V, E, \Sigma)$ be a signed graph; that is $G = (V, E)$ is an undirected graph and $\Sigma \subseteq E$. Call a circuit C *odd* if $|C \cap \Sigma|$ is odd. The collection of odd circuits forms a binary hypergraph on E. If it is Q_6, then $G = (V, E, \Sigma)$ is the odd-K_4; that is, $V = VK_4$ and $E = \Sigma = EK_4$. Hence Theorem 80.1 implies Corollary 75.3a: if $G = (V, E, \Sigma)$ has no odd-K_4 minor, then the maximum number of edge-disjoint odd circuits is equal to the minimum size of an odd circuit cover. In other words, if $G = (V, E, \Sigma)$ has no odd-K_4 minor, then G is strongly bipartite.

Odd circuit covers. Let $G = (V, E, \Sigma)$ be a signed graph. The collection of inclusionwise minimal odd circuit covers forms a binary hypergraph on E. If it is Q_6, then $G = K_3^2$ and $\Sigma = \Delta$, where Δ is a triangle in K_3^2. Here K_3^2 is the graph with three vertices, each pair of which connected by two parallel edges. Hence Theorem 80.1 implies: if $G = (V, E, \Sigma)$ has no (VK_3^2, EK_3^2, Δ) as minor, then the maximum number of edge-disjoint odd circuit covers is equal to the minimum length of an odd circuit.

Notes. Gan and Johnson [1989] developed a framework that includes the above examples on T-joins, T-cuts, odd circuits, and odd circuit covers, and derived algorithms for the corresponding optimization problems.

80.6. Mengerian matroids

Seymour's characterization of binary Mengerian hypergraphs implies a full characterization of matroids that have the corresponding matroidal Mengerian property. In this case, we need not restrict the characterization to binary matroids, since binarity of the matroid *follows* from the Mengerian property.

Again, for any matroid $M = (V, \mathcal{I})$ and any $v \in V$, let $H_{M,v}$ be the hypergraph on $V \setminus \{v\}$ with edges all sets $C \setminus \{v\}$, where C is a circuit of M containing v. Call M *Mengerian* if $H_{M,v}$ is a Mengerian hypergraph for each $v \in V$.

Theorem 80.1 implies the following conjecture of Th. Chang of the late 1960s (cf. Seymour [1977b]):

Corollary 80.1a. *A matroid is Mengerian if and only if it is binary and has no F_7^* minor.*

Proof. As the 2-uniform matroid U_4^2 on 4 elements and F_7^* are not Mengerian (since $H_{M,v} = K_3$ or $H_{M,v} = Q_6$ for these matroids), necessity follows (using the fact that each matroid without U_4^2 minor is binary (Theorem 39.4)). To see sufficiency, observe that, if $M = (V, \mathcal{I})$ is a binary matroid, then for each $v \in V$, the hypergraph $H_{M.v}$ is binary, and that if M contains no F_7^* minor (in the matroidal sense), then $H_{M,v}$ contains no Q_6 minor (in the hypergraphical sense). ∎

Notes. As was outlined by Seymour [1980a,1981a] and Bixby [1982], there is an easier direct proof of this corollary, based on the fact that binary matroids without F_7^* minor can be decomposed into regular matroids (coming from totally unimodular matrices) and copies of F_7. (This follows from the 'splitter theorem' of Seymour [1980a].)

Mengerity of regular matroids follows from the total unimodularity of the matrix representing the matroid, as was shown by Gallai [1959b] (and also by Minty [1966] (an alternative proof was given by Fulkerson [1968])).

Bixby [1982] described that this decomposition gives a polynomial-time algorithm finding the optima in Corollary 80.1a. For background we refer to the book of Truemper [1992].

80.6a. Oriented matroids

Matroids generalize *undirected* graphs, and one may ask for an extension of matroid theory to include directed structures, in order to investigate the max-flow min-cut theorem in greater generality. Bland and Las Vergnas [1978] and Folkman and Lawrence [1978], following work of Minty [1966], Fulkerson [1968], Rockafellar [1969], and Lawrence [1975], developed a theory of oriented matroids. It may be seen as the abstraction of a linear subspace of $\mathbb{R}^n$; the abstraction of any vector is a $\{0,+1,-1\}$ vector, having 0, +1, or −1 in the positions where the original vector has a zero, positive, or negative entry, respectively. If we have a digraph $D=(V,A)$ and we take as $\{0,+1,-1\}$ vectors all $x \in \{0,\pm1\}^A$ for which there is an undirected circuit C with $x_a=1$ for forward arcs a of C, $x_a=-1$ for backward arcs a of C, and $x_a=0$ for all other arcs a, then we obtain an oriented matroid. Again, one may define the Mengerian property for oriented matroids; its characterization by excluded minors is unsolved.

More on oriented matroids can be found in Bachem and Kern [1992] and Björner, Las Vergnas, Sturmfels, White, and Ziegler [1993].

A different approach to extending Menger's theorem to matroids was given by Tutte [1965b] — see Section 41.5a.

80.7. Further results and notes

80.7a. $\tau_2(H)=2\tau(H)$ for binary hypergraphs H

Lovász [1975a] showed:

Theorem 80.2. $\tau_2(H)=2\tau(H)$ *for each binary hypergraph* H.

Proof. Let x be a minimum-size 2-vertex cover of H. Let $U:=\{v\in V \mid x_v=0\}$ and $W:=\{v\in V \mid x_v=2\}$. Let $H':=H/U\setminus W$ and $V':=V\setminus(U\cup W)$. Then H' is binary and each edge of H' has size at least 2, since for any edge F of H' there is an edge E of H with $E\cap W=\emptyset$ and $E\setminus U=F$. Then $|F|=x(E)\geq 2$.

As $r_{\max}(H')\geq 2$, for each $v\in V'$ there is a $B\in b(H')$ with $v\notin B$. Consider now the cospace

(80.17) $\qquad \mathcal{C}:=\{B\subseteq V' \bigm| |B\cap F| \text{ is odd for each edge } F \text{ of } H'\}.$

Then for each $v\in V'$ there is a $B\in\mathcal{C}$ with $v\notin B$. As $\mathcal{C}$ is a cospace, it follows that v is in at most half of the sets in $\mathcal{C}$. As this is true for each $v\in V'$, $\mathcal{C}$ contains a set B of size at most $\frac{1}{2}|V'|$. Then $W\cup B$ is a vertex cover of H of size at most $\frac{1}{2}x(V)=\frac{1}{2}\tau_2(H)$. So $\tau(H)\leq\frac{1}{2}\tau_2(H)$ as required. ∎

Lovász [1975a] showed more generally:

Theorem 80.3. *Let* $H=(V,\mathcal{E})$ *be a hypergraph such that*

(80.18) *if* $X,Y,Z\in\mathcal{E}$, $y\in(X\cap Y)\setminus Z$ *and* $z\in(X\cap Z)\setminus Y$, *then there is an* $F\in\mathcal{E}$ *satisfying* $F\subseteq(X\cup Y\cup Z)\setminus\{y,z\}$.

Then $\tau_2(H)=2\tau(H)$.

Proof. Consider a counterexample with $|V|$ minimal. Let x be a minimum-size 2-vertex cover of H. Then

(80.19) $\qquad x_v=1$ for each $v\in V$.

For if $x_v = 0$, then $x|V \setminus \{v\}$ is a 2-vertex cover of H/v, and hence, since H/v again satisfies (80.18):

(80.20) $$2\tau(H) \leq 2\tau(H/v) = \tau_2(H/v) \leq x(V \setminus \{v\}) = x(V) = \tau_2(H),$$

contradicting the fact that H is a counterexample. Similarly, if $x_v = 2$, then $x|V \setminus \{v\}$ is a 2-vertex cover of $H \setminus v$, and hence, since $H \setminus v$ again satisfies (80.18):

(80.21) $$2\tau(H) \leq 2\tau(H \setminus v) + 2 = \tau_2(H \setminus v) + 2 \leq x(V \setminus \{v\}) + 2 = x(V) = \tau_2(H),$$

again contradicting the fact that H is a counterexample.

This proves (80.19). Hence $|E| \geq 2$ for each $E \in \mathcal{E}$ and we must show that there is a vertex cover of size $\leq \frac{1}{2}|V|$. By the minimality of x, there is an edge X of size 2 (otherwise we can reset $x_v := 0$ for some $v \in V$), say $X = \{y, z\}$. Then $\tau_2(H \setminus \{y, z\}) \leq |V| - 2 = \tau_2(H) - 2$, and so, by the minimality of $|V|$:

(80.22) $$\tau(H \setminus \{y, z\}) = \tfrac{1}{2}\tau_2(H \setminus \{y, z\}) \leq \tfrac{1}{2}\tau_2(H) - 1 < \tau(H) - 1,$$

and hence $\tau(H \setminus \{y, z\}) \leq \tau(H) - 2$. Let $U \subseteq V \setminus \{y, z\}$ be a minimum-size vertex cover of $H \setminus \{y, z\}$. Since $U \cup \{y\}$ and $U \cup \{z\}$ are not vertex covers of H (since $\tau(H) \geq |U| + 2$), there are edges Y and Z in H disjoint from $U \cup \{z\}$ and $U \cup \{y\}$ respectively. As $U \cup \{y, z\}$ does intersect all edges, we know $y \in Y$ and $z \in Z$. Then X, Y, Z contradict (80.18). ∎

Condition (80.18) is closed under taking minors. The hypergraph $(\{1, 2\}, \{\{1\}, \{2\}, \{1, 2\}\})$ is the unique minor-minimal hypergraph violating (80.18).

80.7b. Application: *T*-joins and *T*-cuts

Let $G = (V, E)$ be an undirected graph and let $T \subseteq V$ with $|T|$ even. Let $\mathcal{C}$ be the collection of T-cuts. Then $H = (E, \mathcal{C})$ is a binary hypergraph, and its blocker consists of the minimal T-joins.

We will derive

(80.23) $\nu_2(H) = 2\tau(H)$, and $\nu(H) = \tau(H)$ if G is bipartite,

from general hypergraph theory and from the result that

(80.24) $\nu_2(H) = 2\nu(H)$ if G is bipartite

(Seymour [1981d]).

We first give Seymour's proof of (80.24). Let $U_1, \ldots, U_t$ be subsets of V with each $|U_i \cap T|$ odd such that each edge of G is in at most two of the $\delta(U_i)$ and such that $t = \nu_2(H)$. Such U_i exist by the definition of $\nu_2(H)$. Choose them such that

(80.25) $$\sum_{i=1}^{t} |U_i||V \setminus U_i|$$

is as small as possible. Then the U_i are *cross-free*, that is, for all $i, j = 1, \ldots, t$ one has

(80.26) $U_i \subseteq U_j$ or $U_j \subseteq U_i$ or $U_i \cap U_j = \emptyset$ or $U_i \cup U_j = V$.

If this would not hold, we can replace U_i and U_j either by $U_i \cap U_j$ and $U_i \cup U_j$ (if $|U_i \cap U_j \cap T|$ is odd) or by $U_i \setminus U_j$ and $U_j \setminus U_i$ (otherwise), therewith decreasing the sum (80.25) — a contradiction.

So (80.26) holds. By symmetry, we can assume that $|U_i| \leq |V \setminus U_i|$ for each i. If each U_i is a singleton, then the U_i form a 2-stable set in the subgraph $G[T]$ induced by T, and hence $G[T]$ has a stable set of size at least $\frac{1}{2}t$ (as $G[T]$ is bipartite). This implies $\nu(H) \geq \frac{1}{2}\nu_2(H)$.

If some U_i is a singleton and $U_i = U_j$ for some $j \neq i$, we can contract $\delta(U_i)$ and obtain a bipartite graph $G' = (V', E')$ and $T' \subseteq V'$, and a 2-packing of T'-cuts of G' of size $t - 2$. Hence, inductively, G' has a packing of T-cuts of size at least $\frac{1}{2}(t-2)$. With $\delta(U_i)$ this gives a packing of T cuts in G, of size at least $\frac{1}{2}t$.

So we can assume that each singleton occurs at most once among the U_i and that not each U_i is a singleton. Then we can assume that U_1 is a minimal nonsingleton set among the U_i. Let $U_2, \ldots, U_r$ be the sets properly contained in U_1. So $U_2, \ldots, U_r$ are singletons from $T \cap U_1$. Hence $r - 1 \leq |T \cap U_1|$. As $|T \cap U_1|$ is odd and G is bipartite, there is a stable set $S \subseteq T \cap U_i$ with $2|S| \geq |T \cap U_i| + 1 \geq r$. Replacing $U_1, \ldots, U_r$ by twice the singletons from S, gives a 2-packing of t T-cuts with smaller sum (80.25) — a contradiction. This proves (80.24).

Now (80.24) implies:

(80.27) $\nu_4(H) = 2\nu_2(H)$ for any graph G.

Indeed, replace each edge by a path of length 2, thus obtaining the bipartite graph $G' = (V', E')$, with $T \subseteq V'$. Let H' be the corresponding hypergraph of T-cuts. Then by (80.24):

(80.28) $\nu_4(H) = \nu_2(H') = 2\nu(H') = 2\nu_2(H)$,

which is (80.27).

As the class of hypergraphs H obtained in this way from graphs is closed under parallelization (since it corresponds to replacing edges by paths), Corollary 79.5a then implies $\nu_2(H) = \tau_2(H)$. Hence, with Theorem 80.2 we obtain $\nu_2(H) = 2\tau(H)$, and, using (80.24), we have (80.23).

80.7c. Box-integrality of $k \cdot P_H$

A polyhedron P is called *box-integer* if for all $c, d \in \mathbb{Z}^V$, the polytope

(80.29) $P \cap \{x \in \mathbb{R}^V \mid d \leq x \leq c\}$

is integer. Gerards and Laurent [1995] showed that the following are equivalent for any binary hypergraph $H = (V, \mathcal{E})$, where P_H is defined by (78.1):

(80.30)
(i) $k \cdot P_H$ is box-integer for each $k \geq 1$;
(ii) $k \cdot P_H$ is box-integer for some $k \geq 2$;
(iii) H has no Q_6 or $b(Q_6)^+$ minor.

As in Section 80.4, the hypergraph H^+ arises from a hypergraph $H = (V, \mathcal{E})$ by adding a new vertex, u say, and taking as edges all sets $F \cup \{u\}$ for $F \in \mathcal{E}$.

This characterization extends results of Laurent and Poljak [1995b] for the bipartite subgraph polytope.

Chapter 81

Matroids and multiflows

Corollary 80.1a gives a forbidden minor characterization of matroids for which the corresponding generalization of the integer max-flow min-cut theorem holds. Seymour [1981a] showed that several theorems on multi-flows can be generalized similarly to the level of matroids. We give a survey of these results, without proofs.

81.1. Multiflows in matroids

Let $M = (E, \mathcal{I})$ be a matroid, let $R \subseteq E$, and let $c : E \to \mathbb{R}_+$. Let $\mathcal{C}_R$ be the collection of circuits C of M with $|C \cap R| = 1$.

The *multiflow problem* in M asks for a function $y : \mathcal{C}_R \to \mathbb{R}_+$ satisfying

(81.1)
$$\sum_{\substack{C \in \mathcal{C}_R \\ e \in C}} y_C \geq c_e \quad \text{if } e \in R,$$
$$\sum_{\substack{C \in \mathcal{C}_R \\ e \in C}} y_C \leq c_e \quad \text{if } e \in E \setminus R.$$

We call any y satisfying (81.1) a *multiflow* in M (relative to R and c). So R plays the role of the 'demand edges', and $E \setminus R$ the role of the 'supply edges'.

The corresponding *cut condition* is:

(81.2) (cut condition) $c(D \cap R) \leq c(D \setminus R)$ for each cocircuit D of M.

This condition is necessary for the existence of a multiflow y, since

(81.3)
$$c(D \cap R) \leq \sum_{C \in \mathcal{C}_R} y_C |C \cap D \cap R| \leq \sum_{C \in \mathcal{C}_R} y_C |C \cap D \setminus R| \leq c(D \setminus R).$$

Here we use that $|C \cap D| \neq 1$ for any circuit C and cocircuit D, implying $|C \cap D \cap R| \leq |C \cap D \setminus R|$ if $|C \cap R| = 1$.

In this terminology, Corollary 80.1a can be formulated as:

(81.4) for each $R \subseteq E$ with $|R| = 1$ and each $c : E \to \mathbb{Z}_+$, the cut condition implies the existence of an integer multiflow $\iff$ M has no U_4^2 or F_7^* minor.

So Corollary 80.1a concerns integer 1-commodity flows in a matroid.

Let $k \in \mathbb{Z}_+$. Seymour [1981a] called a matroid $M = (V, \mathcal{I})$ *k-flowing* if

(81.5) for each $R \subseteq E$ with $|R| \leq k$ and for each $c : E \to \mathbb{R}_+$, the cut condition implies the existence of a multiflow y.

The matroid is *integer k-flowing* if for integer c we can take y integer.

As was the case for multiflows in graphs, the following *Euler condition* (for $c : E \to \mathbb{Z}_+$) will turn out to be helpful:

(81.6) (Euler condition) $c(D)$ is even for each cocircuit D.

M is called *k-cycling* if

(81.7) for each $R \subseteq E$ with $|R| \leq k$ and for each $c : E \to \mathbb{Z}_+$, the cut and Euler condition implies the existence of an integer multiflow y.

For each k, there are the following direct implications:

(81.8) integer k-flowing $\Longrightarrow$ k-cycling $\Longrightarrow$ k-flowing.

As Seymour [1981a] showed, for each fixed $k \geq 2$ the concepts of k-cycling and k-flowing are equivalent.

M is called *∞-flowing*, *integer ∞-flowing*, *∞-cycling*, respectively, if M is k-flowing, integer k-flowing, k-cycling, respectively, for each k. Seymour [1981a] showed that the concepts of 4-flowing, 4-cycling, ∞-flowing, and ∞-flowing are equivalent.

We will now discuss Seymour's results in some greater detail.

81.2. Integer k-flowing

By definition, a matroid is integer 1-flowing if and only if M is Mengerian (Section 80.6). Corollary 80.1a therefore characterizes integer 1-flowing matroids, by forbidding U_4^2 and F_7^* as minors.

Also for other values of k, a forbidden minor characterization of binary integer k-flowing matroids is known. In fact, Seymour [1981a] proved that for binary matroids, the concepts of integer ∞-flowing and integer 2-flowing coincide:

Theorem 81.1. *For any binary matroid $M = (E, \mathcal{I})$ the following are equivalent:*

(81.9) (i) *M is integer ∞-flowing, that is, for each $R \subseteq E$ and each $c : E \to \mathbb{Z}_+$, the cut condition implies the existence of an integer multiflow;*

(ii) *M is integer 2-flowing, that is, for each $R \subseteq E$ with $|R| \leq 2$ and each $c : E \to \mathbb{Z}_+$, the cut condition implies the existence of an integer multiflow;*

(iii) *M has no $M(K_4)$ minor.*

Restricted to graphic and cographic matroids, this bears upon series-parallel graphs.

In Theorem 81.1, the implications (i)$\Rightarrow$(ii)$\Rightarrow$(iii) are easy. The proof of (iii)$\Rightarrow$(i) is based on showing that each binary matroid without $M(K_4)$ minor can be decomposed into matroids with at most 3 elements.

81.3. 1-flowing and 1-cycling

As before, for any matroid $M = (V, \mathcal{I})$ and any $v \in V$, let $H_{M,v}$ be the hypergraph on $V \setminus \{v\}$ with edges all sets $C \setminus \{v\}$, where C is a circuit of M containing v (like in Section 80.6). Then M is 1-flowing if and only if $H_{M,v}$ is ideal for each $v \in V$ (cf. Chapter 78). Since no forbidden minor characterization of ideal hypergraphs is known, we cannot infer a characterization of 1-flowing matroids. While the latter characterization yet might be easier to prove, no such characterization is known. Similarly, no characterization of 1-cycling matroids is known. Seymour [1981a] conjectures that for binary matroids both concepts are equivalent; in fact, that for any binary matroid M:

(81.10) (?) M is 1-cycling $\iff$ M is 1-flowing $\iff$ M has no AG(3,2), T_{11}, or T_{11}^* minor. (?)

Here T_{11} is the binary matroid represented by the 11 vectors in $\{0,1\}^5$ with precisely 3 or 5 ones. Moreover, AG(3,2) is the matroid with 8 elements, obtained from the 3-dimensional affine geometry over GF(2); equivalently, AG(3,2) is the binary matroid represented by the columns of the matrix[11]:

(81.11) $$\begin{pmatrix} 1&0&0&0&0&1&1&1 \\ 0&1&0&0&1&0&1&1 \\ 0&0&1&0&1&1&0&1 \\ 0&0&0&1&1&1&1&0 \end{pmatrix}.$$

The second equivalence in conjecture (81.10) is a consequence of Seymour's conjecture that $\mathcal{O}(K_5)$, $b(\mathcal{O}(K_5))$, and F_7 are the only binary minimally nonideal hypergraphs.

81.4. 2-flowing and 2-cycling

The next theorem of Seymour [1981a] lifts Hu's 2-commodity flow theorem to matroids. It shows that for binary matroids, the concepts of 2-flowing and 2-cycling coincide.

[11] Seymour [1981a] used the notation AG(2,3) instead of the (more standard) AG(3,2) (for the 3-dimensional affine geometry over GF(2)).

Theorem 81.2. *For any binary matroid $M = (E, \mathcal{I})$ the following are equivalent:*

(81.12)
(i) *M is 2-cycling, that is, for each $R \subseteq E$ with $|R| \leq 2$ and each $c : E \to \mathbb{Z}_+$, the Euler and cut conditions imply the existence of an integer multiflow;*
(ii) *M is 2-flowing, that is, for each $R \subseteq E$ with $|R| \leq 2$ and each $c : E \to \mathbb{R}_+$, the cut condition implies the existence of a multiflow;*
(iii) *M has no* AG(3,2) *or S_8 minor.*

Here S_8 is the binary matroid represented by the columns of the matrix

$$(81.13)\qquad \begin{pmatrix} 1&0&0&0&0&1&1&1 \\ 0&1&0&0&1&0&1&1 \\ 0&0&1&0&1&1&0&1 \\ 0&0&0&1&1&1&1&1 \end{pmatrix}.$$

Since AG(3,2) and S_8 are self-dual, this describes a self-dual property. These matroids are nongraphic and (hence) noncographic. For graphic matroids, Theorem 81.2 amounts to the results on 2-commodity flows described in Chapter 71. For cographic matroids, it amounts to the theorem mentioned in the Notes at the end of Section 71.3.

In Theorem 81.2, the implications (i)$\Rightarrow$(ii)$\Rightarrow$(iii) are easy. The proof of (iii)$\Rightarrow$(i) is based on showing that each binary matroid without AG(3,2) or S_8 minor can be decomposed into regular matroids and copies of F_7 and F_7^*.

81.5. 3-flowing and 3-cycling

Also the concepts of 3-flowing and 3-cycling are equivalent, as follows from the following characterization, again of Seymour [1981a]:

Theorem 81.3. *For any binary matroid $M = (E, \mathcal{I})$ the following are equivalent:*

(81.14)
(i) *M is 3-cycling, that is, for each $R \subseteq E$ with $|R| \leq 3$ and each $c : E \to \mathbb{Z}_+$, the Euler and cut conditions imply the existence of an integer multiflow;*
(ii) *M is 3-flowing, that is, for each $R \subseteq E$ with $|R| \leq 3$ and each $c : E \to \mathbb{R}_+$, the cut condition implies the existence of a multiflow;*
(iii) *M has no F_7, R_{10}, or $M(H_6)$ minor.*

Here H_6 is the graph obtained from $K_{3,3}$ by adding in each colour class one additional edge.

For graphic matroids, Theorem 81.3 gives a theorem on 3-commodity flows. For cographic matroids, this gives nothing new compared with Theorem 81.4 below.

In Theorem 81.3, the implications (i)$\Rightarrow$(ii)$\Rightarrow$(iii) are easy. The proof of (iii)$\Rightarrow$(i) is based on showing that each binary matroid without F_7, R_{10}, or $M(H_6)$ minor, can be decomposed into cographic matroids and copies of $M(K_5)$.

81.6. 4-flowing, 4-cycling, ∞-flowing, and ∞-cycling

Trivially, one has the implications:

(81.15) ∞-cycling $\Longrightarrow$ ∞-flowing $\Longrightarrow$ 4-flowing.

Seymour [1981a] showed that these implications can be reversed for binary matroids, and gave the following characterization:

Theorem 81.4. *For any binary matroid $M = (E, \mathcal{I})$ the following are equivalent:*

(81.16) (i) *M is ∞-cycling, that is, for each $R \subseteq E$ and each $c : E \to \mathbb{Z}_+$, the Euler and cut conditions imply the existence of an integer multiflow;*

(ii) *M is ∞-flowing, that is, is for each $R \subseteq E$ and each $c : E \to \mathbb{R}_+$, the cut condition implies the existence of a multiflow;*

(iii) *M is 4-flowing, that is, for each $R \subseteq E$ with $|R| \leq 4$ and each $c : E \to \mathbb{R}_+$, the cut condition implies the existence of a multiflow;*

(iv) *M has no F_7, R_{10}, or $M(K_5)$ minor.*

The matroid R_{10} is the matroid on EK_5 with all minimally nonempty even cycles of K_5 as circuits. (Equivalently, the circuits of R_{10} are the even circuits of K_5 and their complements.) An alternative characterization is that R_{10} is the binary matroid represented by all vectors in $\{0,1\}^5$ with precisely three 1's. So R_{10} arises from T_{11} by deleting one element.

For graphic matroids, Theorem 81.4 implies Corollary 75.4d on multiflows if the underlying graph added with the demand edges has no K_5 minor. For cographic matroids, this gives Theorem 29.2 that in bipartite graphs the minimum-size of a T-join is equal to the maximum number of disjoint T-cuts.

In Theorem 81.4, the implications (i)$\Rightarrow$(ii)$\Rightarrow$(iii)$\Rightarrow$(iv) are easy. The proof of (iv)$\Rightarrow$(i) is based on showing that each binary matroid without F_7, R_{10}, or $M(K_5)$ minor can be decomposed into cographic matroids and copies of F_7^* and of $M(V_8)$ (Figure 3.2).

Notes. Schwärzler and Sebő [1993] extended Theorem 81.4 so as to include Karzanov's Theorem 72.5 that characterizes when the $K_{2,3}$-metric condition suffices for the existence of a multiflow. Related work can also be found in Marcus and Sebő [2001].

81.7. The circuit cone and cycle polytope of a matroid

Circuits in matroids generalize both circuits and cuts in graphs. Hence studying the cone generated by the circuits in a matroid, bears on the circuit cone of a graph considered in Section 29.7 (sums of circuits) and on the cut cone of a graph considered in Section 75.7.

Studying the circuit cone of a matroid relates to multiflows, as it concerns the question under which conditions equality holds in the inequalities (81.1).

Let $M = (E, \mathcal{I})$ be a matroid. The *circuit cone* is the convex cone generated by the incidence vectors of the circuits. Each vector x in the circuit cone satisfies:

(81.17) $$\begin{aligned} & x_e \geq 0 && \text{for } e \in E, \\ & x_f \leq x(D \setminus \{f\}) && \text{for each cocircuit } D \text{ and each } f \in D. \end{aligned}$$

Indeed, if $x = \chi^C$ for some circuit C, then

(81.18) $$x(D \setminus \{f\}) = \sum_{C \in \mathcal{C}} |(C \cap D) \setminus \{f\}| \geq \sum_{C \in \mathcal{C}} |C \cap \{f\}| = x_f,$$

since $C \cap D \neq \{f\}$.

Seymour [1981a] says that M has the *sums of circuits property* if the circuit cone is determined by (81.17). He derived from Theorem 81.4 the following characterization of this property:

Corollary 81.4a. *For any matroid M the following are equivalent:*

(81.19)
(i) *M has the sums of circuits property;*
(ii) *M is binary and ∞-flowing;*
(iii) *M is binary and has no F_7^*, R_{10}, or $M^*(K_5)$ minor.*

Since none of these forbidden minors are graphic, this generalizes Corollary 29.2f (due to Seymour [1979b]). For cographic matroids, this generalizes Corollary 75.4e.

The derivation of Corollary 81.4a from Theorem 81.4 is similar to the derivation of Corollary 75.4e from Corollary 75.4d.

The *cycle polytope* of a binary matroid $M = (E, \mathcal{I})$ is the convex hull of the incidence vectors of cycles. (A *cycle* is the disjoint union of circuits.)

For each x in the cycle polytope the following is necessary:

(81.20) $$\begin{aligned} & 0 \leq x_e \leq 1 && \text{for each } e \in E, \\ & x(F) - x(D \setminus F) \leq |F| - 1 && \text{for each cocircuit } D \text{ and each} \\ & && F \subseteq D \text{ with } |F| \text{ odd.} \end{aligned}$$

Barahona and Grötschel [1986] showed that Corollary 81.4a gives (in fact, is equivalent to) the following cycle polytope result:

Corollary 81.4b. *For any binary matroid M the following are equivalent:*

(81.21) (i) *the cycle polytope of M is determined by* (81.20)*;*
(ii) *M is ∞-flowing;*
(iii) *M has no F_7^*, R_{10}, or $M^*(K_5)$ minor.*

The derivation of this from Corollary 81.4a is similar to the derivation of Corollary 75.4f from Corollary 75.4e.

Barahona and Grötschel [1986] characterized adjacency and facets of the cycle polytope of matroids with the sums of circuits property. Grötschel and Truemper [1989] gave an alternative proof of Corollary 81.4b.

81.8. The circuit space and circuit lattice of a matroid

The *circuit space* and the *circuit lattice* of $M = (E, \mathcal{I})$ are the linear space and the lattice, respectively, generated by the incidence vectors of the circuits of M.

Barahona and Grötschel [1986] showed that for any matroid $M = (E, \mathcal{I})$, a vector $x \in \mathbb{R}^E$ belongs to the circuit space of M if and only if

(81.22) $x_e = 0$ if e is a bridge; $x_e = x_f$ if e and f are in series.

The proof is based on an idea of Seymour [1981a]. Necessity being direct, we prove sufficiency, by induction on $|E|$. We may assume that M has no bridges. For each series class P of M, the vector $\mathbf{1}_{E \setminus P}$ belongs to the circuit space of $M \setminus P$. (Here $\mathbf{1}_X$ denotes the all-one vector in $\mathbb{R}^X$.) This follows by induction, as $M \setminus P$ has no bridges. Hence $\mathbf{1}_E - \chi^P$ belongs to the circuit space of M. As this is true for each series class P, we have the theorem.

Now let M be binary. Then each vector x in the circuit lattice satisfies (81.22) and the *Euler condition*:

(81.23) $x(D)$ is even for each cocircuit D.

Lovász and Seress [1993] showed that for any binary matroid M this is enough to characterize the circuit lattice if and only if M^* has no restriction that is a binary sum of copies of the Fano matroid F_7. In particular, if M has no F_7^* minor, then the circuit lattice is characterized by (81.22) and (81.23). (Further work on this in Goddyn [1993], Lovász and Seress [1995], and Fleiner, Hochstättler, Laurent, and Loebl [1999].)

81.9. Nonnegative integer sums of circuits

A necessary condition that a vector x is a nonnegative *integer* combination of incidence vectors of circuits is that x is integer and satisfies the Euler condition (81.23). This is not sufficient, as is shown by the cycle matroid $M(\mathbf{P}_{10})$ of the Petersen graph $\mathbf{P}_{10}$ (which is graphic, and hence has the sums of circuits property): choose a perfect matching N in $\mathbf{P}_{10}$ and let x be 2 on the edges of N, and 1 on the other edges.

Fu and Goddyn [1999] characterized when this necessary condition is sufficient, thus proving a conjecture of Seymour [1981a]:

Theorem 81.5. *For any matroid $M = (E, \mathcal{I})$ the following are equivalent:*

(81.24) (i) *each vector $x \in \mathbb{Z}_+^E$ satisfying* (81.17) *and* (81.23) *is a nonnegative integer combination of incidence vectors of circuits;*
(ii) *M is binary and has no F_7^*, R_{10}, $M^*(K_5)$, or $M(\mathbf{P}_{10})$ minor.*

For graphic matroids, this reduces to Theorem 29.4 of Alspach, Goddyn, and Zhang [1994], and for cographic matroids, results on the cut cone mentioned in Section 75.7.

The proof of Theorem 81.5 is by decomposing any matroid satisfying (81.24)(ii) into graphic matroids without $M(\mathbf{P}_{10})$ minor (to which Theorem 29.4 applies), and copies of F_7 and $M^*(V_8)$ (cf. Figure 3.2).

Goddyn [1993] conjectured (more strongly than Theorem 81.5) that for each matroid without $\mathbf{P}_{10}$ minor, the circuits form a Hilbert base. However, Laurent [1996b] showed that this is not true for $M^*(K_6)$.

A survey on this type of problems was given by Goddyn [1993].

81.10. Nowhere-zero flows and circuit double covers in matroids

Let $M = (E, \mathcal{I})$ be a binary matroid. A *flow over* GF(4) is a function $f : E \to$ GF(4) with $f(D) = 0$ for each cocircuit D of M. The flow is *nowhere-zero* if $f(e) \neq 0$ for each $e \in E$. By linear algebra, each flow over GF(4) can be decomposed as a sum of vectors $\alpha \cdot \chi^C$, where $\alpha \in$ GF(4) and C is a circuit.

Seymour [1981c] proved that the 4-flow conjecture of Tutte [1966] ('each bridgeless graph without a Petersen graph minor has a nowhere-zero 4-flow' — see Section 28.4) is equivalent to the following stronger conjecture, also given by Tutte [1966]:

(81.25) (?) each bridgeless matroid without F_7^*, $M^*(K_5)$, or $M(\mathbf{P}_{10})$ minor has a nowhere-zero flow over GF(4). (?)

For graphic matroids, this clearly includes the 4-flow conjecture. For cographic matroids, the existence of a nowhere-zero flow over GF(4) is equivalent to the 4-vertex-colourability of the underlying graph G. By the four-colour theorem and Wagner's theorem (cf. Section 64.3b), any graph without K_5 minor is 4-vertex-colourable — so conjecture (81.25) includes this.

The existence of a nowhere-zero 4-flow is equivalent to the existence of three cycles (= disjoint unions of circuits) that cover each $e \in E$ precisely twice. Indeed, for each nonzero $z \in \mathrm{GF}(4)$, let $C_z := \{e \in E \mid f(e) \neq z\}$. Then the C_z are cycles as required, and the construction can be reversed.

Weaker is the concept of a *circuit double cover* in a binary matroid, which is a family of circuits covering each element precisely twice. Trivially, each bridgeless cographic matroid has a circuit double cover (just take all stars in the corresponding (loopless) graph). The *circuit double cover conjecture* (cf. Sections 29.8 and 38.8) asserts that also each bridgeless graphic matroid has a circuit double cover. Jamshy and Tarsi [1989] proved that this conjecture is equivalent to a generalization to matroids:

(81.26) (?) each bridgeless binary matroid without F_7^* minor has a circuit double cover. (?)

The property of having a circuit double cover need not be closed under taking deletions. So (81.26) gives no necessary and sufficient conditions. One may not relax the condition in (81.26) to requiring that M is binary and $\mathbf{2}$ belongs both to the circuit lattice and to the circuit cone, as is shown by the matroid whose circuits are the even-size cuts of K_{12} (M. Laurent (cf. Goddyn [1993])). This matroid M has an F_7^* minor, and hence does not contradict (81.26).

What has been proved by Jamshy and Tarsi [1989] is:

(81.27) each bridgeless binary matroid without F_7^* minor has a family of circuits covering each element precisely four times.

This extends the corresponding result for graphic matroids of Bermond, Jackson, and Jaeger [1983].

More on nowhere-zero flows and circuit covers in matroids can be found in Tarsi [1985,1986], Jamshy, Raspaud, and Tarsi [1987], and Jamshy and Tarsi [1989].

Chapter 82

Covering and antiblocking in hypergraphs

In this chapter we study the notions of stable set and edge cover in hypergraphs. These concepts are dual to those of matching and vertex cover, by taking the dual hypergraph. Yet, the way we study them is not dual: the classes of hypergraphs considered are closed under operations performed on the *vertex set* (like contraction), while when dualizing the results obtained above, would lead to operations on the *edge set*.
So, although several of the concepts considered in this chapter are just the duals of concepts considered before, we do not dualize the way we studied them above.
As it will turn out, the antiblocking analogues corresponding to the blocking concepts of ideal and Mengerian hypergraphs, all boil down to perfect graph theory.

82.1. Elementary concepts

Let $H = (V, \mathcal{E})$ be a hypergraph. A subset S of V is called *stable* if $|F \cap S| \leq 1$ for each $F \in \mathcal{E}$. An *edge cover* is a collection of edges covering V. So a stable set of H can be considered as a matching of the dual hypergraph H^*, and an edge cover of H as a vertex cover of H^*.

For any hypergraph $H = (V, \mathcal{E})$, define

(82.1) $\alpha(H) :=$ the maximum size of a stable set in H,
$\rho(H) :=$ the minimum size of an edge cover in H.

Determining these numbers is NP-complete, since finding a maximum-size stable set or a minimum-size vertex cover in a graph can be easily reduced to it.

There is the following straightforward inequality:

(82.2) $\alpha(H) \leq \rho(H)$.

For any hypergraph $H = (V, \mathcal{E})$, define

(82.3) $H^{\max} := (V, \{F \in \mathcal{E} \mid \text{there is no } E \in \mathcal{E} \text{ with } E \supset F\})$ and
$H^{\downarrow} := (V, \{F \mid \text{there is an } E \in \mathcal{E} \text{ with } E \supseteq F\})$

So for any hypergraph, $H^{\max}$ is a clutter. Moreover, we have $\alpha(H) = \alpha(H^{\max}) = \alpha(H^{\downarrow})$ and $\rho(H) = \rho(H^{\max}) = \rho(H^{\downarrow})$.

82.2. Fractional edge covers and stable sets

Let $H = (V, \mathcal{E})$ be a hypergraph. A *fractional stable set* is a function $x : V \to \mathbb{R}_+$ satisfying

(82.4) $$\sum_{v \in F} x_v \leq 1 \text{ for each } F \in \mathcal{F}.$$

A *fractional edge cover* is a function $y : \mathcal{E} \to \mathbb{R}_+$ satisfying

(82.5) $$\sum_{F \ni v} y_F \geq 1 \text{ for each } v \in V.$$

(Here and below, F ranges over the edges of H.) Let $\alpha^*(H)$ denote the maximum size of a fractional stable set and let $\rho^*(H)$ denote the minimum size of a fractional edge cover (where the *size* of a vector is the sum of its components).

So $\rho^*(H)$ can be described as

(82.6) $$\rho^*(H) = \min\{y^{\mathsf{T}}\mathbf{1} \mid y \in \mathbb{R}_+^{\mathcal{E}}, y^{\mathsf{T}}M \geq \mathbf{1}^{\mathsf{T}}\},$$

where M is the $\mathcal{E} \times V$ incidence matrix of H. Similarly,

(82.7) $$\alpha^*(H) = \max\{\mathbf{1}^{\mathsf{T}}x \mid x \in \mathbb{R}_+^V, Mx \leq \mathbf{1}\}.$$

As these represent dual linear programs, this gives:

(82.8) $$\rho^*(H) = \alpha^*(H).$$

82.3. *k*-edge covers and *k*-stable sets

Like in the blocking case, there is an alternative interpretation of the parameters $\rho^*(H)$ and $\alpha^*(H)$. A *k-stable set* is a function $x : V \to \mathbb{Z}_+$ such that

(82.9) $$\sum_{v \in F} x_v \leq k \text{ for each } F \in \mathcal{F}.$$

Let $\alpha_k(H)$ denote the maximum size of a k-stable set. As 1-stable sets are precisely the incidence vectors of the stable sets, $\alpha_1(H) = \alpha(H)$.

A *k-edge cover* is a function $y : \mathcal{E} \to \mathbb{Z}_+$ such that

(82.10) $$\sum_{F \ni v} y_F \geq k \text{ for each } v \in V.$$

Let $\rho_k(H)$ denote the minimum size of a k-edge cover in H. The minimal 1-edge covers are precisely the incidence vectors of the edge covers, and hence $\rho_1(H) = \rho(H)$.

One easily checks that, for any $k \in \mathbb{Z}_+$:

(82.11) $$\alpha_k(H) \leq \rho_k(H).$$

In fact, for each $k \geq 1$:

(82.12) $$\alpha(H) \leq \frac{\alpha_k(H)}{k} \leq \alpha^*(H) = \rho^*(H) \leq \frac{\rho_k(H)}{k} \leq \rho(H).$$

Also one has (Lovász [1974]):

(82.13) $$\rho^*(H) = \min_k \frac{\rho_k(H)}{k} = \lim_{k\to\infty} \frac{\rho_k(H)}{k}.$$

Here the left-hand side equality holds as the minimum in (82.6) is attained by a rational optimum solution y. The right-hand side equality follows from Fekete's lemma (Theorem 2.2), using the fact that for all $k, l \geq 1$:

(82.14) $$\rho_{k+l}(H) \leq \rho_k(H) + \rho_l(H),$$

since the sum of a k-edge cover and an l-edge cover is a $k + l$-edge cover. Similarly we have:

(82.15) $$\alpha^*(H) = \max_k \frac{\alpha_k(H)}{k} = \lim_{k\to\infty} \frac{\alpha_k(H)}{k},$$

using (82.7) and the fact that for all $k, l \geq 1$:

(82.16) $$\alpha_{k+l}(H) \geq \alpha_k(H) + \alpha_l(H).$$

82.4. The antiblocker and conformality

For any hypergraph $H = (V, \mathcal{E})$, the *antiblocking hypergraph*, or the *antiblocker*, of H is the hypergraph $a(H)$ with vertex set V and edges all inclusionwise maximal stable sets of H. So $a(H)$ is a clutter, and $\alpha(H) = r_{\max}(a(H))$ (=the maximum edge-size of $a(H)$).

In Section 77.6 we saw that for any clutter H we have $b(b(H)) = H$. A similar duality phenomenon does not hold for antiblockers. For instance, for the hypergraph $H = K_3$ (with $V := \{1, 2, 3\}$ and $\mathcal{E} := \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$) one has $a(H) = (V, \{\{1\}, \{2\}, \{3\}\})$, and hence $a(a(H)) = (V, \{\{1, 2, 3\}\}) \neq H$.

However, by adding a further condition, we can restore this duality relation for the antiblocking operation. Call a hypergraph $H = (V, \mathcal{E})$ *conformal* if for each $U \subseteq V$:

(82.17) if each pair in U is contained in some edge of H, then U is contained in some edge of H.

So H is conformal $\iff$ $H^{\max}$ is conformal $\iff$ $H^{\downarrow}$ is conformal. Moreover:

(82.18) H is conformal $\iff$ there exists a graph G on V such that $H^{\max}$ consists of the inclusionwise maximal cliques of G.

One may check that for each hypergraph H, the hypergraph $a(H)$ is conformal. Also:

Theorem 82.1. *A hypergraph H is conformal if and only if $a(a(H)) = H^{\max}$. In particular, if H is a conformal clutter, then $a(a(H)) = H$.*

Proof. If H is conformal, there is a graph G on V such that $H^{\max}$ is the collection of inclusionwise maximal cliques of G. Then $a(H)$ is the collection of inclusionwise maximal stable sets of G. Hence $a(a(H))$ is the collection of inclusionwise maximal cliques of G. So $a(a(H)) = H^{\max}$. ∎

82.4a. Gilmore's characterization of conformality

Conformality of hypergraphs has been characterized by Gilmore [1962] as follows:

Theorem 82.2. *A hypergraph $H = (V, \mathcal{E})$ is conformal if and only if $V = \cup\mathcal{E}$ and for all $E_1, E_2, E_3 \in \mathcal{E}$ there is an $E \in \mathcal{E}$ with*

(82.19) $E \supseteq (E_1 \cap E_2) \cup (E_1 \cap E_3) \cup (E_2 \cap E_3)$.

Proof. Necessity follows from the definition of conformality, since any two vertices in $(E_1 \cap E_2) \cup (E_1 \cap E_3) \cup (E_2 \cap E_3)$ are contained in some E_i.

To see sufficiency, suppose that the condition is satisfied, but that H is not conformal. Let U be a minimal set such that any pair of vertices in U is contained in some edge of H, but U is contained in no edge of H. So $|U| \geq 3$. Choose distinct $u_1, u_2, u_3 \in U$ and let $F_i := U \setminus \{u_i\}$ for $i = 1, 2, 3$. By the minimality of U, each F_i is contained in some edge, E_i say, of H. Now $U = (F_1 \cap F_2) \cup (F_1 \cap F_3) \cup (F_2 \cap F_3) \subseteq (E_1 \cap E_2) \cup (E_1 \cap E_3) \cup (E_2 \cap E_3)$. By the condition, the latter set is contained in an edge of H, and hence also U is contained in an edge of H. This contradicts our assumption. ∎

As was noted by M. Conforti, Theorem 82.2 implies a polynomial-time test of conformality of a hypergraph, if all maximal edges are given.

82.5. Perfect hypergraphs

We now define the antiblocking analogue of the blocking concept of ideal hypergraph. A hypergraph $H = (V, \mathcal{E})$ is called *perfect*, if $\bigcup \mathcal{E} = V$ and each vertex of the polyhedron Q_H in $\mathbb{R}^V$ determined by:

(82.20)
(i) $x_v \geq 0$ for $v \in V$,
(ii) $x(F) \leq 1$ for $F \in \mathcal{E}$

is integer. (Lovász [1972c] called a hypergraph H *normal* if its dual H^* is perfect.)

We first observe:

Theorem 82.3. *A perfect hypergraph is conformal.*

Proof. Suppose that $H = (V, \mathcal{E})$ is perfect but not conformal. Let U be a minimal subset of V such that any two vertices are contained in an edge of H, but U is contained in no edge of H. So $|U| \geq 3$ and $U \setminus \{u\}$ is contained in an edge of H, for each $u \in U$. Define $z : V \to \mathbb{R}_+$ by:

(82.21) $$z := \frac{1}{|U|-1}\chi^U.$$

Then z belongs to Q_H, and hence z is a convex combination of integer vectors in Q_H. However, each integer vector x satisfies $x(U) \leq 1$ (since $x(U \setminus \{u\}) \leq 1$ for each $u \in U$ and since $|U| \geq 3$). As $z(U) = |U|/(|U|-1) > 1$, this is a contradiction. ∎

Note that each integer vector in Q_H is a 0,1 vector, and hence is the incidence vector of a stable set of H. So H is perfect if and only if Q_H is the convex hull of the incidence vectors of stable sets of H. By the theory of antiblocking polyhedra, this implies that if H is perfect, then each vertex of the polytope $Q_{a(H)}$, by definition determined by

(82.22)
$$\begin{array}{lll} \text{(i)} & x_v \geq 0 & \text{for } v \in V, \\ \text{(ii)} & x(S) \leq 1 & \text{for } S \in a(H), \end{array}$$

is integer — hence $a(H)$ is perfect.

We cannot simply reverse this implication: if H is the complete graph K_3, then H is not perfect (as $\frac{1}{2} \cdot \mathbf{1}$ is a noninteger vertex of Q_H), but $a(H)$ is perfect: its edges are all singleton vertices of K_3.

However, if we require H to be conformal, the duality is restored (Fulkerson [1971a,1972a]):

Corollary 82.3a. *A hypergraph H is perfect $\iff$ H is conformal and its antiblocker $a(H)$ is perfect.*

Proof. If H is perfect, then H is conformal by Theorem 82.3. Moreover, $a(H)$ is perfect, by the theory of antiblocking polyhedra.

Conversely, if $a(H)$ is perfect, then $a(a(H))$ is perfect. As H is conformal, $H = a(a(H))$, and hence H is perfect. ∎

The following theorem implies that most of hypergraph theory related to antiblocking boils down to the theory of perfect graphs (the 'only if' part is due to Fulkerson [1972a] and the 'if' part to Lovász [1972c]):

Corollary 82.3b. *A hypergraph $H = (V, \mathcal{E})$ is perfect if and only if $H^{\max}$ consists of the maximal cliques of some perfect graph $G = (V, E)$.*

Proof. To see necessity, as H is perfect, it is conformal, and hence $H^{\max}$ consists of the maximal cliques of some graph $G = (V, E)$. Then G is a perfect graph, by Corollary 65.2e.

To see sufficiency, if $H^{\max}$ consists of the maximal cliques of a perfect graph, then (82.20) has integer vertices (again by Corollary 65.2e) and hence H is perfect. ∎

Perfect hypergraphs can be characterized by a weaker, and also by stronger, conditions than the definition. In the following corollary we collect some of them (Fulkerson [1972a]: (iii)⇔(iv)⇔(v), Lovász [1972c]: (i)⇔(ii)⇔(iv)⇔(vi), Lovász [1972a]: (i)⇔(viii), Berge [1973a]: (i)⇔(vii)).

Theorem 82.4. *For any hypergraph $H = (V, \mathcal{E})$ with $\bigcup \mathcal{E} = V$ the following are equivalent, where M denotes the $\mathcal{E} \times V$ incidence matrix of H:*

(82.23)
(i) *$H^{\max}$ consists of the maximal cliques of some perfect graph;*
(ii) *$\alpha(H') = \rho(H')$ for each contraction H' of H;*
(iii) *H is perfect, that is, $\{x \geq \mathbf{0} \mid Mx \leq \mathbf{1}\}$ is an integer polytope;*
(iv) *the system $x \geq \mathbf{0}, Mx \leq \mathbf{1}$ is totally dual integral;*
(v) *$a(H)$ is perfect;*
(vi) *$\alpha^*(H')$ is an integer for each contraction H' of H;*
(vii) *$\rho_2(H') = 2\rho(H')$ for each contraction H' of H;*
(viii) *$\alpha(H')r_{\max}(H') \geq |VH'|$ for each contraction H' of H.*

Proof. The equivalence of (i) and (iii) is Corollary 82.3b. The equivalence of (i), (iii), and (v) then follows from the perfect graph theorem (Corollary 65.2a). The implication (i)⇒(iv) follows from Corollary 65.2f. Since contractions of H correspond to taking induced subgraphs of G, the implication (i)⇒(ii) is the definition of perfect graph. The implication (ii)⇒(viii) is direct, as $\rho(H')r_{\max}(H') \geq |VH'|$ for any hypergraph H'. The implications (ii)⇒(vi) and (ii)⇒(vii) follow from (82.12). The implication (iv)⇒(iii) is general polyhedral theory (Theorem 5.22).

So it suffices to show that each of (vi), (vii), and (viii) implies (i). We first show that each of (vi), (vii), and (viii) implies that H is conformal.

Suppose that H is not conformal. Then there is a minimal subset U of V such that each pair in U is covered by an edge of H, but U is not covered by any edge of H. So $|U| \geq 3$. Let H' be obtained from H by contracting $V \setminus U$.

Then H' has a 2-edge cover of size 3 (taking $U \setminus \{u\}$ for three vertices $u \in U$), while $\rho(H') \geq 2$, contradicting (vii). Moreover, $\alpha(H') = 1$ and $r_{\max}(H') = |U| - 1 < |U| = |VH'|$, contradicting (viii).

As U is contained in no edge of H, we know that $\alpha^*(H') \geq |U|/(|U| - 1)$, since $(|U|-1)^{-1} \cdot \mathbf{1}$ is a fractional stable set of H'. Also, $\alpha^*(H') \leq |U|/(|U|-1)$, since for any fractional stable set x of H' we have $x(U \setminus \{u\}) \leq 1$ for each $u \in U$ (as $U \setminus \{u\}$ is contained in an edge of H), and hence

(82.24) $$x(U) = \frac{1}{|U|-1} \sum_{u \in U} x(U \setminus \{u\}) \leq \frac{|U|}{|U|-1}.$$

So $\alpha^*(H')$ is not an integer, contradicting (vi).

So each of (vi), (vii), and (viii) implies that H is conformal. Knowing that H is conformal, let $H^{\max}$ consist of the maximal cliques of a graph $G = (V, E)$. To show that H is perfect, it suffices to show that G is perfect if (vi), (vii), or (viii) holds. This follows from Theorems 65.10, 65.11, and 65.2, respectively (using that G is perfect if $\overline{G}$ is perfect, and that $\alpha^*(H) = \chi^*(\overline{G})$). ∎

By definition, 'perfect hypergraph' is the antiblocking analogue of 'ideal hypergraph'. By Theorem 82.4, we know that the antiblocking analogue of 'Mengerian hypergraph' coincides with 'perfect hypergraph' (since (82.23)(iii) and (iv) are equivalent). So perfect hypergraph theory reduces to perfect graph theory, and minimally imperfect hypergraphs can be characterized with the strong perfect graph theorem. We will not expand further on this but refer to the chapters in Part VI on perfect graphs.

82.6. Further notes

82.6a. Some equivalences for the k-parameters

Let $H = (V, \mathcal{E})$ be a hypergraph and let $v \in V$. Adding a *serial* vertex to v means extending V by a new vertex v' and replacing $\mathcal{E}$ by

(82.25) $$\{E \mid v \notin E \in \mathcal{E}\} \cup \{E \cup \{v'\} \mid v \in E \in \mathcal{E}\}.$$

A hypergraph obtained from H by a sequence of contractions of vertices and adding serial vertices, is called a *serialization* of H. If $w : V \to \mathbb{Z}_+$ indicates the size of the final series classes of the vertices, we denote the serialization by H_w. So contractions are special cases of serializations and correspond to functions $w : V \to \{0, 1\}$. In a certain sense, also restrictions are special cases of parallelizations and correspond to functions $w : V \to \{1, \infty\}$.

Theorem 82.5. *For any hypergraph $H = (V, \mathcal{E})$ with $\cup\mathcal{E} = V$ and any $k \in \mathbb{Z}_+$, the following are equivalent:*

(82.26)
(i) *$k \cdot \alpha^*(H') = \alpha_k(H')$ for each serialization H' of H;*
(ii) *$k \cdot \alpha^*(H')$ is an integer for each serialization H' of H.*

Proof. Similar to the proof of Theorem 78.3. ∎

This is used in proving:

Theorem 82.6. *For any hypergraph $H = (V, \mathcal{E})$ with $\cup\mathcal{E} = V$ and any $k \in \mathbb{Z}_+$, the following are equivalent:*

(82.27) (i) $\rho_k(H') = \alpha_k(H')$ *for each serialization H' of H;*
(ii) $k \cdot \rho^*(H') = \rho_k(H')$ *for each serialization H' of H;*
(iii) $\rho_{2k}(H') = 2\rho_k(H')$ *for each serialization H' of H.*

Proof. Similar to the proof of Theorem 79.2. ∎

Are Theorems 82.5 and 82.6 maintained if serializations are replaced by just contractions? As we will see, this is the case for $k = 2$ and $k = 3$ but not for general k.

As for $k = 2$, Lovász [1977b] showed:

Theorem 82.7. *For any hypergraph $H = (V, \mathcal{E})$ with $\cup\mathcal{E} = V$ the following are equivalent:*

(82.28) (i) $\alpha^*(H') = \frac{1}{2}\alpha_2(H')$ *for each contraction H' of H;*
(ii) $\alpha^*(H') \in \frac{1}{2}\mathbb{Z}$ *for each contraction H' of H.*

Proof. The implication (i)⇒(ii) is trivial. To see the reverse implication, we can assume that (ii) holds and that $\alpha^*(H') = \frac{1}{2}\alpha_2(H')$ for each contraction $H' \neq H$ of H, while $\alpha^*(H) > \frac{1}{2}\alpha_2(H)$.

Since $\alpha^*(H) > \frac{1}{2}\alpha_2(H)$ and $\alpha^*(H) \in \frac{1}{2}\mathbb{Z}$, we know $\alpha^*(H) \geq \frac{1}{2}\alpha_2(H) + \frac{1}{2}$. Let x be a fractional stable set of H of size $\alpha^*(H)$. Then for each $v \in V$, $x|V \setminus \{v\}$ is a fractional stable set of H/v, and so

(82.29) $$x(V \setminus \{v\}) \leq \alpha^*(H/v) = \tfrac{1}{2}\alpha_2(H/v) \leq \tfrac{1}{2}\alpha_2(H) \leq \alpha^*(H) - \tfrac{1}{2} = x(V) - \tfrac{1}{2}.$$

So $x_v \geq \frac{1}{2}$ for each $v \in V$. Hence $|F| \leq 2$ for each $F \in \mathcal{E}$. So H is (essentially) a graph, and hence $\alpha_2(H) = \rho_2(H)$ (by Corollary 30.9a). This implies $\alpha_2(H) = \frac{1}{2}\alpha^*(H)$. ∎

As a consequence one has (Lovász [1975a]: (i)⇔(ii)):

Corollary 82.7a. *For any hypergraph $H = (V, \mathcal{E})$ with $\cup\mathcal{E} = V$ the following are equivalent:*

(82.30) (i) $\alpha_2(H') = \rho_2(H')$ *for each contraction H' of H;*
(ii) $2\alpha^*(H') = \rho_2(H')$ *for each contraction H' of H;*
(iii) $\rho_6(H') = 3\rho_2(H')$ *for each contraction H' of H.*

Proof. The equivalence of (i) and (ii) follows directly from Theorem 82.7. Also the implication (i)⇒(iii) is direct, since $\alpha_2(H) \leq \frac{1}{3}\rho_6(H) \leq \rho_2(H)$ for any hypergraph H.

To see (iii)⇒(i), let $H = (V, \mathcal{E})$ be a counterexample with $|V|$ minimal. So $\rho_2(H') = \alpha_2(H')$ for each contraction $H' \neq H$ of H, and $\rho_6(H) = 3\rho_2(H)$. If each edge of H has size at most 2, then $\rho_2(H) = \alpha_2(H)$, by Corollary 30.9a. So H has an edge F of size at least 3. Choose distinct $v_1, v_2, v_3 \in F$. Then for each $i = 1, 2, 3$ we have;

(82.31) $$\rho_2(H/v_i) = \alpha_2(H/v_i) \leq \alpha_2(H) < \rho_2(H).$$

Hence $\rho_2(H/v_i) \leq \rho_2(H) - 1$.

For $i = 1, 2, 3$, let y_i be a 2-edge cover of H/v_i of size $\rho_2(H/v_i)$. Then $y_1 + y_2 + y_3 + 2\chi^{\{F\}}$ is a 6-edge cover of H of size

(82.32) $$\rho_2(H/v_1) + \rho_2(H/v_2) + \rho(H/v_3) + 2 \leq 3(\rho_2(H) - 1) + 2 < 3\rho_2(H).$$

This contradicts the fact that $\rho_6(H) = 3\rho_2(H)$. ∎

Lovász [1977b] showed that Theorem 82.7 also holds if we replace 2 by 3:

Theorem 82.8. *For any hypergraph $H = (V, \mathcal{E})$ with $\cup\mathcal{E} = V$ the following are equivalent:*

(82.33) (i) $\alpha^*(H') = \frac{1}{3}\alpha_3(H')$ *for each contraction H' of H;*
(ii) $\alpha^*(H') \in \frac{1}{3}\mathbb{Z}$ *for each contraction H' of H.*

Proof. The implication (i)⇒(ii) being direct, we prove (ii)⇒(i). Let $H = (V, \mathcal{E})$ be a counterexample with $|V|$ minimal. So $\alpha^*(H) \in \frac{1}{3}\mathbb{Z}$, $\alpha^*(H) > \frac{1}{3}\alpha_3(H)$, and $\alpha^*(H') = \frac{1}{3}\alpha_3(H')$ for each contraction $H' \neq H$ of H. So $\alpha^*(H) \geq \frac{1}{3}\alpha_3(H) + \frac{1}{3}$.

Let x be a fractional stable set of H with $x(V) = \alpha^*(H)$. Then for each $v \in V$, $x|V \setminus \{v\}$ is a fractional stable set of H/v, and hence:

(82.34) $$\begin{aligned} x(V \setminus \{v\}) &\leq \alpha^*(H/v) = \tfrac{1}{3}\alpha_3(H/v) \leq \tfrac{1}{3}\alpha_3(H) \leq \alpha^*(H) - \tfrac{1}{3} \\ &= x(V) - \tfrac{1}{3}. \end{aligned}$$

So $x_v \geq \frac{1}{3}$ for each $v \in V$. Therefore, $|F| \leq 3$ for each $F \in \mathcal{E}$. Let U be the union of the edges of H of size 3. Then $x_v = \frac{1}{3}$ for each $v \in U$.

Let $W := V \setminus U$. Then the edges of H contained in W form a bipartite graph. Otherwise, it contains an odd circuit C, and then $H' := H/(V \setminus VC)$ satisfies $\alpha^*(H') = \frac{1}{2}|VC|$. So $\alpha^*(H')$ does not belong to $\frac{1}{3}\mathbb{Z}$, a contradiction.

Let N be the set of vertices w in W for which there is a $u \in U$ with $\{u, w\} \in \mathcal{E}$. Since x is a maximum-size fractional stable set of H and since $x_v = \frac{1}{3}$ for each $v \in U$, we know that $x|W$ attains the maximum in the linear program of maximizing $z(W)$ over $z \in \mathbb{R}^W$ satisfying

(82.35) $$\begin{array}{ll} 0 \leq z(v) \leq 1 & \text{for each } v \in V, \\ z(v) \leq \frac{2}{3} & \text{for each } v \in N, \\ z(u) + z(v) \leq 1 & \text{for each edge } \{u, v\} \subseteq W \text{ of } H. \end{array}$$

Since the constraint matrix of this LP-problem is totally unimodular and since the right-hand side is $\frac{1}{3}$-integer, there is a $\frac{1}{3}$-integer optimum solution z. We can assume that $x|W = z$. So $x \in \frac{1}{3}\mathbb{Z}^V$, implying that $3x$ is a 3-stable set. Hence $\alpha_3(H) \geq 3\alpha^*(H)$, contradicting our assumption. ∎

This implies (Lovász [1977b]):

Corollary 82.8a. *For any hypergraph $H = (V, \mathcal{E})$ with $\cup\mathcal{E} = V$ the following are equivalent:*

(82.36) (i) $\alpha_3(H') = \rho_3(H')$ *for each contraction H' of H;*
(ii) $3\alpha^*(H') = \rho_3(H')$ *for each contraction H' of H.*

Proof. Directly from Theorem 82.8. ∎

Lovász [1977b] raised the question if in these results 3 can be replaced by any arbitrary integer k. However, Schrijver and Seymour [1979] gave the following example of a hypergraph $H = (V, \mathcal{E})$ satisfying $\alpha_{60}(H) < \rho_{60}(H)$ while $60\alpha^*(H') = \rho_{60}(H')$ for each contraction H' of H:

(82.37) $V := \{1,2,3,4,5,6,7\}$, $\mathcal{E} := \{V \setminus \{1,2\}, V \setminus \{1,3\}, V \setminus \{1,4\}, V \setminus \{2,3\}, V \setminus \{2,4\}, V \setminus \{3,4\}, V \setminus \{5\}, V \setminus \{6\}, V \setminus \{7\}\}$.

To see that $\rho_{60}(H') = 60\alpha^*(H')$ for each contraction H' of H, observe that if we contract two of the vertices $1,2,3,4$ or one of the vertices $5,6,7$, there is an edge covering all vertices, and $\alpha = \rho$ follows. So by symmetry it suffices to show that $\rho_{60}(H') = 60\alpha^*(H')$ for $H' := H$ and for $H' := H/1$.

The fractional stable set x of $H/1$ defined by $x := \frac{1}{5} \cdot \mathbf{1}$ shows that $\alpha^*(H/1) \geq \frac{6}{5}$. Then the 5-edge cover y of $H/1$ defined by: $y(V \setminus \{1,i\}) := 1$ for $i = 2, \ldots, 7$, and $y(E) := 0$ for any other edge E of $H/1$, shows that $\rho_5(H/1) \leq 6$. Hence $\rho_{60}(H/1) \leq 12\rho_5(H/1) \leq 72 \leq 60\alpha^*(H/1)$.

Finally we consider H. Let x be the fractional stable set defined by:

(82.38) $x(1) := x(2) := x(3) := x(4) := \frac{1}{8}$, $x(5) := x(6) := x(7) := \frac{1}{4}$,

and let y be the fractional edge cover defined by:

(82.39) $y(V \setminus \{i,j\}) := \frac{1}{12}$ for all $1 \leq i < j \leq 4$ and $y(V \setminus \{i\}) := \frac{1}{4}$ for $i = 5,6,7$.

So $x(V) = \frac{5}{4} = y(\mathcal{E})$. Hence $\alpha^*(H) = \frac{5}{4}$. However, x is the only fractional stable set of size $\frac{5}{4}$. Indeed, for any fractional stable set x of size $\frac{5}{4}$ one has $x(\{i,j\}) \geq \frac{1}{4}$ for all $1 \leq i < j \leq 4$ and $x(\{i\}) \geq \frac{1}{4}$ for all $5 \leq i \leq 7$. So $x(\{5,6,7\}) \geq \frac{3}{4}$, hence $x(\{1,2,3,4\}) \leq \frac{1}{2}$. Therefore, $x(\{i\}) = \frac{1}{4}$ for all $5 \leq i \leq 7$ and $x(\{i,j\}) = \frac{1}{4}$ for all $1 \leq i < j \leq 4$. This gives $x(\{i\}) = \frac{1}{8}$ for each $1 \leq i \leq 4$.

As $60x \notin \mathbb{Z}$, this shows that $\alpha_{60}(H) < 60\alpha^*(H)$.

82.6b. Further notes

The complete graphs show that $\rho(H)$ cannot be bounded in terms of $\alpha(H)$. Ding, Seymour, and Winkler [1994] showed that for each fixed k, $\rho(H)$ is bounded by a polynomial in $\alpha(H)$ if we restrict H to hypergraphs not having the complete graph on k vertices as partial subhypergraph. Here, a*partial subhypergraph* arises by deleting edges and contracting vertices.

A $\{0, \pm 1\}$ matrix M is *perfect* if the polytope

(82.40) $\{x \mid \mathbf{0} \leq x \leq \mathbf{1}, Mx \leq \mathbf{1} - b\}$

is integer, where b is the vector with b_i equal to the number of -1's in the ith row of M. These matrices generalize the incidence matrices of perfect hypergraphs and were studied by Conforti, Cornuéjols, and de Francesco [1997] (who gave a characterization in terms of perfect graphs), Boros and Čepek [1997], Guenin [1998b], and Tamura [2000].

An extension of the equivalence of (iii) and (iv) in Theorem 82.4 was proved by Korach [1982]: Let M_1 and M_2 be integer matrices such that each row of M_2 is a nonnegative linear combination of rows of M_1. Consider the system

(82.41) $M_1x \geq \mathbf{0}, M_2x \leq \mathbf{1}.$

Then (82.41) is TDI if and only if $M_1x \geq \mathbf{0}$ is TDI and (82.41) determines an integer polyhedron.

The intersection of the polyhedra made by perfect and ideal hypergraphs was investigated by Sebő [1998]. Related results were given by Shepherd [1994a] and Gasparyan [1998]. Monma and Trotter [1979] gave an alternative proof of the relation between perfect graphs and perfect hypergraphs.

Determining the stable set number $\alpha(H)$ of a hypergraph H is equivalent to the vertex packing problem (equivalently, the set packing problem). In Section 64.9e we gave further references for this problem. Determining the edge cover number $\rho(H)$ of H amounts to the *set covering problem*. This NP-complete problem is studied by Lawler [1966], Roth [1969], Lemke, Salkin, and Spielberg [1971], Thiriez [1971], Balas and Padberg [1972,1975a], Garfinkel and Nemhauser [1972b] (survey), Even [1973], Guha [1973], Salkin and Koncal [1973], Christofides and Korman [1974], Fulkerson, Nemhauser, and Trotter [1974], Johnson [1974a], Gondran and Laurière [1975], Lovász [1975c], Etcheberry [1977], Chvátal [1979], Padberg [1979], Avis [1980a], Balas [1980], Balas and Ho [1980], Baker [1981], Bar-Yehuda and Even [1981], Ho [1982], Hochbaum [1982,1983b], Lifschitz and Pittel [1983], Vasko and Wilson [1984a,1984b], Beasley [1987,1990], Bertolazzi and Sassano [1987, 1988], Balas and Ng [1989a,1989b], Cornuéjols and Sassano [1989], Feo and Resende [1989], Nobili and Sassano [1989,1992], Sassano [1989], Fisher and Kedia [1990], Karmarkar, Resende, and Ramakrishnan [1991], El-Darzi and Mitra [1992], Goldschmidt, Hochbaum, and Yu [1993], Khuller, Vishkin, and Young [1993,1994], Lorena and Lopes [1994], Mannino and Sassano [1995], Halldórsson [1995,1996], Caprara, Fischetti, and Toth [1996,1999], Feige [1996,1998], Duh and Fürer [1997], Bar-Yehuda [2000], Halperin [2000,2002], and Holmerin [2002].

The related *set partitioning problem* was investigated by Garfinkel and Nemhauser [1969], Michaud [1972], Marsten [1973], Nemhauser, Trotter, and Nauss [1973], Gondran and Laurière [1974], Balas and Padberg [1975a,1975b,1976], Balas [1977], Nemhauser and Weber [1979], Johnson [1980], Hwang, Sun, and Yao [1985], John [1988], Fisher and Kedia [1990], El-Darzi and Mitra [1992], and Sherali and Lee [1996].

Chapter 83

Balanced and unimodular hypergraphs

In the preceding chapters we investigated conditions under which $\tau(H) = \nu(H)$ or $\alpha(H) = \rho(H)$ holds for all hypergraphs H obtained by deleting or multiplying vertices of some hypergraph. Although these parameters transfer to each other by taking the dual hypergraph, the study was unsymmetric as we considered only deleting or multiplying of vertices, not of edges. In the applications, generally the number of edges is exponentially large in the number of vertices.

In the present chapter we study hypergraphs for which these equalities hold in a symmetric fashion. This leads to the classes of balanced and unimodular matrices.

83.1. Balanced hypergraphs

A 0,1 matrix M is called *balanced* if M has no submatrix which is the incidence matrix of an odd circuit. A hypergraph H is *balanced* if its incidence matrix is balanced.

Another way of characterizing balancedness of a hypergraph $H = (V, \mathcal{E})$ is by the associated bipartite graph G with colour classes V and $\mathcal{E}$, and $v \in V$ and $F \in \mathcal{E}$ adjacent if and only if $v \in F$:

(83.1) H is balanced $\iff$ the length of each chordless circuit in G is a multiple of 4.

The class of balanced hypergraphs is closed under taking 'partial subhypergraphs'. A *partial hypergraph* of a hypergraph H is a hypergraph $(V, \mathcal{E}')$ with $\mathcal{E}' \subseteq \mathcal{E}$. A *partial subhypergraph* of H is a contraction of a partial hypergraph of H. So the incidence matrices of partial subhypergraphs of H arise by deleting rows and columns of the incidence matrix of H. In this terminology,

(83.2) a hypergraph H is balanced $\iff$ H has no odd circuit as partial subhypergraph.

Trivially, the dual of a balanced hypergraph is again balanced. Also, the class of balanced hypergraphs is closed under contractions and restrictions. More generally, it is closed under parallelization and serialization. Hence also the class of blockers of balanced hypergraphs is closed under parallelization and serialization.

Note that for graphs (that is, hypergraphs with each edge of size 2), balancedness coincides with bipartiteness.

In a deep theorem, Conforti, Cornuéjols, and Rao [1999] showed that balancedness of a hypergraph can be tested in polynomial time. The method is based on decomposition of balanced matrices into totally unimodular matrices. An outline of the method was given by Conforti and Cornuéjols [1990]. Related work is reported in Conforti, Cornuéjols, and Rao [1995].

83.2. Characterizations of balanced hypergraphs

Balanced hypergraphs can be characterized in several ways in terms of polyhedra and optimization, as in the following theorem. As before, the hypergraphs $b(H)$ and $a(H)$ denote the blocker and antiblocker of H, respectively. (Berge and Las Vergnas [1970] proved (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii) and Berge [1972] proved (i)$\Leftrightarrow$(iv). Given the equivalence of (i) and (iii), the pluperfect graph theorem of Fulkerson [1971a] implies the equivalence of (i), (iii), and (v) (conjectured by Berge [1969]), since balancedness is closed under parallelization.)

Theorem 83.1. *For any hypergraph $H = (V, \mathcal{E})$, the following are equivalent:*

(83.3) (i) *H is balanced;*
(ii) *$\nu(H') = \tau(H')$ for each partial subhypergraph H' of H;*
(iii) *$\alpha(H') = \rho(H')$ for each partial subhypergraph H' of H;*
(iv) *$\nu(b(H')) = r_{\min}(H')$ for each partial subhypergraph H' of H;*
(v) *$\rho(a(H')) = r_{\max}(H')$ for each partial subhypergraph H' of H.*

Proof. Each of (ii), (iii), (iv), (v) implies (i), since if H is not balanced, it has a partial subhypergraph that is an odd circuit. It is easy to see that none of (ii)-(v) hold for any odd circuit. To show the reverse implications, it suffices to derive from (i) that each of the equalities holds for $H' = H$, since the class of balanced matrices is closed under taking partial subhypergraphs.

We first show (i)$\Rightarrow$(ii). Since the class of balanced hypergraphs is closed under parallelization, by Theorem 79.2 it suffices to show that $\nu_2(H) = 2\nu(H)$. Let $y : \mathcal{E} \to \mathbb{Z}_+$ be a 2-matching of size $\nu_2(H)$. Let $\mathcal{M} := \{E \in \mathcal{E} \mid y(E) = 2\}$ and $\mathcal{F} := \{E \in \mathcal{E} \mid y(E) = 1\}$. The dual of the hypergraph $(V, \mathcal{F})$ is a graph G, added with some edges of size ≤ 1. Since H is balanced, G is bipartite. Let $\mathcal{N}$ be the largest of the two colour classes of G. Then $|\mathcal{N}| \geq \frac{1}{2}|\mathcal{F}|$, and hence $\mathcal{M} \cup \mathcal{N}$ is a matching of size $\geq \frac{1}{2}\nu_2(H)$.

This shows (i)$\Rightarrow$(ii). By taking the dual of H, we see (i)$\Rightarrow$(iii). By Theorem 82.4, (iii) implies that the maximal edges of H are the maximal cliques of some perfect graph G on V. Then $\chi(G) = \omega(G)$ implies $\rho(a(H)) = r_{\max}(H)$.

We finally show (i)$\Rightarrow$(iv). We first show that

(83.4) the vertex set V of a balanced hypergraph $H = (V, \mathcal{E})$ can be partitioned into two sets, each intersecting each edge of size ≥ 2.

The proof is by induction on $|V|$. Let E be the collection of pairs in $\mathcal{E}$. Then the graph $G = (V, E)$ contains a vertex u such that any two neighbours of u belong to the same component of $G - u$. (This is true for any graph. To see it, we can assume that G is connected. Then choose an arbitrary vertex v and let u be a vertex at maximum distance from v.)

By induction, we can partition $V \setminus \{u\}$ into two sets V_1, V_2 each intersecting each edge F of H with $|F \setminus \{u\}| \geq 2$. Now any two neighbours of u in G are connected by a path in $G - u$ of even length, since G is bipartite (as H is balanced). Hence the neighbours belong either all to V_1 or all to V_2. By symmetry, we can assume that they all belong to V_1. Then $V_1, V_2 \cup \{u\}$ is a partition as required. This shows (83.4).

To show (i)⇒(iv), we prove $\nu(b(H)) = r_{\min}(H)$, that is, the maximum number of disjoint vertex covers of H is equal to the minimum edge size r. This is shown by induction on $|\mathcal{E}|$. Choose $F \in \mathcal{E}$ and define $\mathcal{E}' := \mathcal{E} \setminus \{F\}$. Then, by induction, the hypergraph $(V, \mathcal{E}')$ has r disjoint vertex covers $B_1, \ldots, B_r$. We can assume that they partition V. Choose $B_1, \ldots, B_r$ such that a maximum number of the B_i intersect F.

If each B_i intersects F we are done, so we may assume that $B_1 \cap F = \emptyset$. As $|F| \geq r$, we can assume that $|B_2 \cap F| \geq 2$. Now apply (83.4) to the contraction H' of H to $B_1 \cup B_2$. Then $r_{\min}(H') \geq 2$. So, by (83.4), $B_1 \cup B_2$ can be partitioned into two vertex covers of H', hence of H. Replacing B_1, B_2 by B_1', B_2' gives a partition of V into vertex covers of H' thereby increasing the number of them intersecting F. This contradicts our assumption. ∎

Since the incidence matrix of a bipartite graph is balanced, Theorem 83.1 generalizes several theorems of Kőnig, like Kőnig's matching theorem (Theorem 16.2), the Kőnig-Rado edge cover theorem (Theorem 19.4), and Kőnig's edge-colouring theorem (Theorem 20.1).

Theorem 83.1 implies some more extensive characterizations (cf. Fulkerson, Hoffman, and Oppenheim [1974], Berge [1980]):

Corollary 83.1a. *For any hypergraph $H = (V, \mathcal{E})$, the following are equivalent:*

(83.5)
(i) *H is balanced;*
(ii) *$\tau^*(H') \in \mathbb{Z}$ for each partial subhypergraph H' of H;*
(iii) *each partial hypergraph of H is ideal;*
(iv) *each partial hypergraph of H is Mengerian;*
(v) *the blocker of each partial hypergraph of H is Mengerian;*
(vi) *$\alpha^*(H') \in \mathbb{Z}$ for each partial subhypergraph H' of H;*
(vii) *each partial hypergraph of H is perfect.*

Proof. We know the implications (iv)⇒(iii) (Section 79.1), (iii)⇒(ii) (Corollary 78.4b), (vii)⇒(vi) (Theorem 82.4), and (v)⇒(iii) (Theorem 78.1). Since the class of balanced hypergraphs is closed under parallelization, (i)⇒(ii) and (i)⇒(iii) in Theorem 83.1 give (i)⇒(iv) and (i)⇒(vii) in (83.5). Also the class of blockers of balanced hypergraphs is closed under parallelization (as the class of balanced matrices is closed under duplicating columns); so (i)⇒(iv) in Theorem 83.1 gives (i)⇒(v) in (83.5).

So it suffices to show (ii)⇒(i) and (vi)⇒(i). Suppose that H is not balanced. Let $U \subseteq V$ and $\mathcal{E}' \subseteq \mathcal{E}$ induce a partial subhypergraph that is an odd circuit. We can assume that $U = V$ and $\mathcal{E}' = \mathcal{E}$. Then $\frac{1}{2} \cdot \mathbf{1}$ is a minimum-size vertex cover and a maximum-size stable set of H, and hence $\tau^*(H)$ and $\alpha^*(H)$ are noninteger. ∎

These characterizations imply that certain linear programs have integer optimum solutions (taking $\infty \cdot 0 = 0$):

Corollary 83.1b. *For any $\{0,1\}$-valued $m \times n$ matrix M, the following are equivalent:*

(83.6)
(i) *M is balanced;*
(ii) $\forall b \in \{1,\infty\}^m \;\; \forall w \in \{0,1\}^n : \min\{y^{\mathsf{T}}b \mid y \geq \mathbf{0}, y^{\mathsf{T}}M \geq w^{\mathsf{T}}\}$ *has an integer optimum solution y;*
(iii) $\forall b \in \{1,\infty\}^m \;\; \forall w \in \mathbb{Z}_+^n : \min\{y^{\mathsf{T}}b \mid y \geq \mathbf{0}, y^{\mathsf{T}}M \geq w^{\mathsf{T}}\}$ *has an integer optimum solution y;*
(iv) $\forall b \in \mathbb{Z}_+^m \;\; \forall w \in \{0,1\}^n : \min\{y^{\mathsf{T}}b \mid y \geq \mathbf{0}, y^{\mathsf{T}}M \geq w^{\mathsf{T}}\}$ *has an integer optimum solution y;*
(v) $\forall b \in \{1,\infty\}^m \;\; \forall w \in \{0,1\}^n : \max\{w^{\mathsf{T}}x \mid x \geq \mathbf{0}, Mx \leq b\}$ *has an integer optimum solution x;*
(vi) $\forall b \in \{1,\infty\}^m \;\; \forall w \in \mathbb{Z}_+^n : \max\{w^{\mathsf{T}}x \mid x \geq \mathbf{0}, Mx \leq b\}$ *has an integer optimum solution x;*
(vii) $\forall b \in \mathbb{Z}_+^m \;\; \forall w \in \{0,1\}^n : \max\{w^{\mathsf{T}}x \mid x \geq \mathbf{0}, Mx \leq b\}$ *has an integer optimum solution x;*
(viii) $\forall b \in \{0,1\}^m \;\; \forall w \in \{1,\infty\}^n : \min\{w^{\mathsf{T}}x \mid x \geq \mathbf{0}, Mx \geq b\}$ *has an integer optimum solution x;*
(ix) $\forall b \in \mathbb{Z}_+^m \;\; \forall w \in \{1,\infty\}^n : \min\{w^{\mathsf{T}}x \mid x \geq \mathbf{0}, Mx \geq b\}$ *has an integer optimum solution x.*
(x) $\forall b \in \{0,1\}^m \;\; \forall w \in \mathbb{Z}_+^n : \min\{w^{\mathsf{T}}x \mid x \geq \mathbf{0}, Mx \geq b\}$ *has an integer optimum solution x;*
(xi) $\forall b \in \{0,1\}^m \;\; \forall w \in \{1,\infty\}^n : \max\{y^{\mathsf{T}}b \mid y \geq \mathbf{0}, y^{\mathsf{T}}M \leq w^{\mathsf{T}}\}$ *has an integer optimum solution y;*
(xii) $\forall b \in \mathbb{Z}_+^m \;\; \forall w \in \{1,\infty\}^n : \max\{y^{\mathsf{T}}b \mid y \geq \mathbf{0}, y^{\mathsf{T}}M \leq w^{\mathsf{T}}\}$ *has an integer optimum solution y;*
(xiii) $\forall b \in \{0,1\}^m \;\; \forall w \in \mathbb{Z}_+^n : \max\{y^{\mathsf{T}}b \mid y \geq \mathbf{0}, y^{\mathsf{T}}M \leq w^{\mathsf{T}}\}$ *has an integer optimum solution y.*

Proof. Observe that each of (ii)-(vii) is equivalent to each of (viii)-(xiii), respectively, after replacing M by M^{T}. The implications (iii)⇒(ii)⇒(i), (iv)⇒(ii), (vi)⇒(v)⇒(i), and (vii)⇒(v) are direct. Here we use that (ii) and (v) are closed under taking submatrices, and that the incidence matrix of an odd circuit does not satisfy (ii) and (v) for $b = \mathbf{1}$ and $w = \mathbf{1}$.

Finally, (i)⇒(iii) and (i)⇒(vi) follow from (83.5)(i)⇒(vii), (i)⇒(x) (hence (i)⇒(iv)) follows from (83.5)(i)⇒(iii), and (i)⇒(xiii) (hence (i)⇒(vii)) follows from (83.5)(i)⇒(iv). ∎

Berge [1970] gave the following further characterization:

(83.7) a hypergraph is balanced $\Longleftrightarrow$ each partial subhypergraph is bicolourable,

where a hypergraph is *bicolourable* if its vertex set can be coloured with two colours such that each edge of size at least 2 gets both colours. While ⇐ in (83.7) is easy, ⇒ can be shown with the proof of (83.4).

More generally, Theorem 83.1 gives the following generalization of Theorem 20.6 for bipartite graphs (Berge [1973b]):

Corollary 83.1c. *Let $H = (V, \mathcal{E})$ be a balanced hypergraph and let $k \in \mathbb{Z}_+$. Then V can be partitioned into $V_1, \ldots, V_k$ such that each $E \in \mathcal{E}$ is intersected by $\min\{k, |E|\}$ of the V_i.*

Proof. Choose $F \in \mathcal{E}$. By induction on $|\mathcal{E}|$, there is a partition $V_1, \ldots, V_k$ of V such that each $E \in \mathcal{E}$ with $E \neq F$ is intersected by $\min\{k, |E|\}$ of the V_i. Choose the partition such that F is intersected by a maximum number of the V_i. If F is not intersected by $\min\{k, |F|\}$ of the V_i, there exist V_i, V_j with $V_i \cap F = \emptyset$ and $|V_j \cap F| \geq 2$. The hypergraph H' obtained from H by contracting $V \setminus (V_i \cup V_j)$ and after that deleting all edges of size ≤ 1, has $r_{\min}(H') \geq 2$. Hence by Theorem 83.1, $\nu(b(H')) \geq 2$, that is $V_i \cup V_j$ can be partitioned into two vertex covers V_i' and V_j' of H'. Then replacing V_i, V_j by V_i', V_j' increases the number of intersections with F, a contradiction. ∎

Another consequence was given by Conforti, Cornuéjols, Kapoor, and Vušković [1996]. Call a matching $\mathcal{M}$ in a hypergraph $H = (V, \mathcal{E})$ *perfect* if $\mathcal{M}$ covers all vertices — that is, if $\mathcal{M}$ is a partition of V.

Corollary 83.1d. *Let $H = (V, \mathcal{E})$ be a balanced hypergraph. Then H has a perfect matching if and only if there are no disjoint vertex sets B, R with $|B| > |R|$ and $|B \cap E| \leq |R \cap E|$ for each $E \in \mathcal{E}$.*

Proof. Necessity is easy, since if $\mathcal{M}$ is a perfect matching, then

(83.8) $$|B| = \sum_{E \in \mathcal{M}} |B \cap E| \leq \sum_{E \in \mathcal{M}} |R \cap E| = |R|.$$

To see sufficiency, let M be the $\mathcal{E} \times V$ incidence matrix of H. Suppose that H has no perfect matching. Since $\{y \geq \mathbf{0} \mid y^{\mathsf{T}} M \leq \mathbf{1}^{\mathsf{T}}\}$ is an integer polytope (by (83.6)(xii)), it implies that there is no vector $y \geq \mathbf{0}$ with $y^{\mathsf{T}} M = \mathbf{1}^{\mathsf{T}}$. Hence, by Farkas' lemma, there is an x with $Mx \geq \mathbf{0}$ and $\mathbf{1}^{\mathsf{T}} x < 0$. We can assume $-\mathbf{1} \leq x \leq \mathbf{1}$. Set $z := \mathbf{1} - x$. Then $\mathbf{0} \leq z \leq \mathbf{2}$, $Mz \leq M\mathbf{1}$, and $\mathbf{1}^{\mathsf{T}} z > \mathbf{1}^{\mathsf{T}} \mathbf{1}$. By (83.6)(vii), applied to the balanced matrix

(83.9) $$\begin{pmatrix} I \\ M \end{pmatrix},$$

we can assume that z is integer. Hence we can assume that x is integer and $-\mathbf{1} \leq x \leq \mathbf{1}$. Then $B := \{v \in V \mid x_v = -1\}$ and $R := \{v \in V \mid x_v = +1\}$ contradict the condition of the corollary. ∎

A combinatorial proof of this theorem was given by Huck and Triesch [2002].

83.2a. Totally balanced matrices

A 0,1 matrix is called *totally balanced* if it has no submatrix that is the incidence matrix of a circuit of length at least 3. Obviously, each totally balanced matrix is balanced.

Totally balanced matrices have several nice properties so as to apply 'perfect elimination' and 'greedy' methods when solving optimization problems. They might be considered as the bipartite analogue of chordal graphs.

Call a bipartite graph *totally balanced* (or *chordal bipartite*) if it has no chordless circuit of length at least 6. So a 0,1 matrix is totally balanced if and only if the associated bipartite graph is totally balanced. (The bipartite graph *associated* to an $m \times n$ matrix M is the bipartite graph with colour classes $\{u_1, \ldots, u_m\}$ and $\{v_1, \ldots, v_n\}$, where u_i and v_j are adjacent if and only if $M_{i,j} \neq 0$.)

The first important property of totally balanced matrices was found by Golumbic and Goss [1978]. Call an entry M_{i_0,j_0} of a $\{0,1\}$-valued $m \times n$ matrix M *simplicial* if

(83.10) (i) $M_{i_0,j_0} = 1$,
(ii) for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$: if $M_{i_0,j} = M_{i,j_0} = 1$, then $M_{i,j} = 1$.

Theorem 83.2. *Each nonzero totally balanced matrix M has a simplicial entry.*

Proof. Let $G = (V, E)$ be the bipartite graph associated to M, with colour classes U and W. To prove that M has a simplicial entry, we must show that G has an edge uw such that each vertex in $N(u)$ is adjacent to each vertex in $N(w)$.

We can assume that G is not a complete bipartite graph, since otherwise M trivially has a simplicial entry. Choose an inclusionwise maximal nonempty set $X \subseteq V$ such that

(83.11) the subgraph $G[X]$ of G induced by X is connected and G has an edge disjoint from $X \cup N(X)$.

Such a set X exists, since $X := \{u\}$ satisfies (83.11) for any vertex u that is isolated or (if no isolated vertices exist) any vertex $u \in U$ nonadjacent to at least one vertex in W.

Define $Z := V \setminus (X \cup N(X))$. The maximality of X gives:

(83.12) each vertex y in $N(X)$ is adjacent to one of the ends of any edge contained in Z,

since otherwise we can add y to X without violating (83.11), contradicting the maximality of X.

Also we have:

(83.13) each vertex in $N(X) \cap U$ is adjacent to each vertex in $N(X) \cap W$.

For choose $y \in N(X) \cap U$ and $z \in N(X) \cap W$. Let uw be an edge in Z, with $u \in U$ and $w \in W$. As $G[X]$ is connected, there is a path P in $G[X]$ connecting $N(y)$ and $N(z)$. Choose P shortest. Then y, P, z, u, w, y is a circuit of length at least 6 in G. Hence it has a chord. It cannot connect $\{u, w\}$ and P, since $u, w \notin N(X)$. So it is a chord of the path y, P, z. Since P is shortest, it follows that y and z are adjacent. This proves (83.13).

Now by induction we know that Z contains an edge uw such that $N(\{u, w\}) \cap Z$ induces a complete bipartite graph. Then (83.12) and (83.13) imply that $N(\{u, w\})$ induces a complete bipartite graph. ∎

Most of the properties of totally balanced matrices (including that described in the theorem above, which however is used in the proof) follow from the next theorem, saying that the rows and columns of a totally balanced matrix can be permuted such that it has no submatrix

(83.14) $$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

(in this order). Following Lubiw [1982], we call such a matrix *Γ-free*. In other words, M is Γ-free if for all row indices $i < i'$ and column indices $j < j'$ one has

(83.15) if $M_{i,j} = M_{i',j} = M_{i,j'} = 1$, then $M_{i',j'} = 1$.

The following was shown by Hoffman, Kolen, and Sakarovitch [1985] and Lubiw [1982]:

Theorem 83.3. *The rows and columns of a totally balanced matrix M can be permuted such that the matrix becomes Γ-free.*

Proof. We apply induction on the number of nonzero entries of M. If M is all-zero, the theorem is trivial. So we can assume that M has at least one nonzero entry. By Theorem 83.2, M has a simplicial entry M_{i_0,j_0}.

Reset M_{i_0,j_0} to 0, to obtain matrix $\widetilde{M}$. Then $\widetilde{M}$ is again totally balanced. For suppose that $\widetilde{M}$ has a submatrix C that is the incidence matrix of a circuit of length ≥ 3. Since M is totally balanced, C contains the entry $\widetilde{M}_{i_0,j_0}$. Row i_0 has two 1's in C and column j_0 has two 1's in C. Hence, by (83.10), C has a row with three 1's, a contradiction.

So $\widetilde{M}$ is totally balanced again. By induction, we can permute the rows and columns of $\widetilde{M}$ such that it becomes Γ-free. We can assume that entry $\widetilde{M}_{i_0,j_0}$ of $\widetilde{M}$

has moved to position i_0, j_0. We can also assume that among all valid permutations, we have chosen one which minimizes $i_0 + j_0$. Then

(83.16) $\qquad M_{i,j_0} = 0$ for each $i < i_0$ and $M_{i_0,j} = 0$ for each $j < j_0$.

For suppose that $M_{i,j_0} = 1$ for some $i < i_0$. By the minimality of $i_0 + j_0$, we cannot exchange rows i_0 and $i_0 - 1$ of $\widetilde{M}$ without violating Γ-freeness. Hence there exist j, j' with $j < j'$ with $M_{i_0,j} = M_{i_0,j'} = 1$ and $M_{i_0-1,j} = 1$, $M_{i_0-1,j'} = 0$. Since $M_{i_0,j} = M_{i_0,j'} = 1$ and $M_{i,j_0} = 1$ we know by (83.10) that $M_{i,j} = M_{i,j'} = 1$.

So $i \neq i_0 - 1$ and hence $i < i_0 - 1$. But then $M_{i,j} = M_{i,j'} = M_{i_0-1,j} = 1$ while $M_{i_0-1,j'} = 0$, contradicting the Γ-freeness of $\widetilde{M}$.

This proves (83.16). Then resetting the (i_0, j_0)th entry to its original value 1, the matrix remains Γ-free (by (83.10) and (83.16)). ∎

Call a hypergraph $H = (V, \mathcal{E})$ *totally balanced* if its incidence matrix is totally balanced. Call two sets X and Y *comparable* if $X \subseteq Y$ or $Y \subseteq X$. Then (Brouwer and Kolen [1980], Anstee and Farber [1984]):

Corollary 83.3a. *Each totally balanced hypergraph $H = (V, \mathcal{E})$ with $V \neq \emptyset$, has a vertex v such that any two edges containing v are comparable.*

Proof. By Theorem 83.3, we can assume that the incidence matrix M of H is Γ-free. Then the vertex of H corresponding to the first column of M is as required. ∎

Other consequences of Theorem 83.3 are algorithmic. It gives a good characterization of total balancedness. In fact, the method gives a polynomial-time algorithm to test total balancedness: we iteratively find a simplicial entry and set it to 0. If we succeed in this until the matrix is all-zero, the original matrix is totally balanced, and otherwise not.

Lubiw [1982] gave the following simple algorithm to permute the rows and columns of a totally balanced matrix such that it becomes Γ-free. Iteratively, choose a column j such that the supports of the rows with a 1 in column j form a chain, and remove column j. The order in which we remove the columns, gives the permutation of the columns. Next order the rows lexicographically (reading from right to left). The final matrix is Γ-free. (Hoffman, Kolen, and Sakarovitch [1985] gave an $O(nm^2)$-time algorithm to transform a totally balanced $m \times n$ matrix to a Γ-free matrix, speeded up by Paige and Tarjan [1987] and Spinrad [1993].)

Also, if A is a nonsingular matrix whose support is totally balanced, then we can solve a system $Ax = b$ of linear equalities with Gaussian elimination, by repeatedly choosing a simplicial entry and pivoting on it. If we create no 0's 'by accident', then we can keep pivoting on simplicial entries throughout the process (since then we never change any zero entry to nonzero). So if the initial matrix is sparse, it remains sparse during the Gaussian elimination.

Lubiw [1982], Farber [1984], and Hoffman, Kolen, and Sakarovitch [1985] gave polynomial-time algorithms for optimization problems over a totally balanced matrix.

Call a hypergraph $H = (V, \mathcal{E})$ a *tree-hypergraph* if V is the vertex set of a tree T and each edge $E \in \mathcal{E}$ of H induces a subtree of T. Lehel [1985] showed that a hypergraph H is totally balanced if and only if each contraction of H is a tree-hypergraph.

Lubiw [1982] showed that for any totally balanced hypergraph $H = (V, \mathcal{E})$, its intersection matrix (the $\{0,1\}$-valued $\mathcal{E} \times \mathcal{E}$ matrix N with $N_{E,E'} = 1$ if and only if $E \cap E' \neq \emptyset$) is totally balanced.

Chvátal [1993] pointed out that a bipartite graph is totally balanced ($\equiv$ chordal bipartite) if and only if its complementary graph is perfectly orderable (cf. Hoàng [1996a]). More results on totally balanced matrices are reported by Golumbic [1980], Lubiw [1982,1987], Anstee and Farber [1984], and Dragan and Voloshin [1996], and applications by Tamir [1987].

83.2b. Examples of balanced hypergraphs

A graph $G = (V, E)$ is balanced if and only if it is bipartite. This follows directly from the definition of balancedness.

A second example was given by Frank [1977]. Let $D = (V, A)$ be a rooted tree. Let $P_1, \ldots, P_m$ and $Q_1, \ldots, Q_n$ be directed paths in D. Define the $m \times n$ matrix M by:

(83.17) $$M_{i,j} := \begin{cases} 1 & \text{if } VP_i \cap VQ_j \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

for $i = 1, \ldots, m$ and $j = 1, \ldots, n$. Then M is a balanced matrix, as one easily checks. M need not be totally unimodular, as example (83.22) below shows. As Lubiw [1982] observed, these matrices are even totally balanced. The fact that for the corresponding hypergraphs $\alpha(H) = \rho(H)$ and (equivalently) $\nu(H) = \tau(H)$ hold was shown by Meir and Moon [1975]. Related results can be found in Slater [1977].

A third example was given by Giles [1978a]. Let $G = (V, E)$ be an (undirected) tree. For each $a : V \to \mathbb{Z}_+$, define

(83.18) $$U_v := \{u \in V \mid \text{dist}_G(v, u) \leq a_v\}.$$

Then $(V, \{U_v \mid v \in V\})$ is a balanced hypergraph. Lubiw [1982] showed that these hypergraphs are in fact totally balanced.

83.2c. Balanced $0, \pm 1$ matrices

Truemper [1982] extended the concept of balancedness to $0, \pm 1$ matrices: A $0, \pm 1$ matrix is *balanced* if in each square submatrix with precisely two nonzeros in each row and in each column, the sum of the entries is a multiple of 4.

Most of the results described above for balanced $0, 1$ matrices, can be extended to $0, \pm 1$ matrices. Conforti and Cornuéjols [1995b] showed that for any balanced $0, \pm 1$ matrix M the following systems are TDI, and hence determine an integer polytope:

(83.19) $$\mathbf{0} \leq x \leq \mathbf{1}, Mx \leq \mathbf{1} - b,$$

and

(83.20) $$\mathbf{0} \leq x \leq \mathbf{1}, Mx \geq \mathbf{1} - b,$$

where b is the vector with b_i equal to the number of negative entries in the ith row of M. So balanced matrices are both perfect and ideal. By requiring this for each submatrix, each of this characterizes balancedness.

Conforti and Cornuéjols [1995b] also proved a bicolouring theorem extending Corollary 83.1c:

(83.21) the columns of a balanced $0, \pm 1$ matrix M can be split into two sets such that each row of M with at least two nonzeros, has nonzero entries of the same sign in both sets, or of opposite signs in one of the two sets.

Again, by requiring this for each submatrix, this characterizes balancedness.

Finally, the decomposition results and algorithms for balanced $0, 1$ matrices were extended to $0, \pm 1$ matrices by Conforti, Cornuéjols, Kapoor, and Vušković [1994,2001a,2001b]. For surveys, see Conforti, Cornuéjols, Kapoor, Vušković, and Rao [1994], Conforti and Cornuéjols [2001], and Cornuéjols [2001].

83.3. Unimodular hypergraphs

A hypergraph $H = (V, \mathcal{E})$ is called *unimodular* if its incidence matrix M is totally unimodular; that is, each square submatrix of M has determinant 0, +1, or −1.

Since the incidence matrix of an odd circuit has determinant ± 2, each unimodular hypergraph is balanced. Not every balanced hypergraph is unimodular, as is shown by the hypergraph with incidence matrix

(83.22)
$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

Trivially, the dual of a unimodular hypergraph is again unimodular. Also, contracting vertices and deleting edges maintain unimodularity of a hypergraph.

For graphs (that is, hypergraphs with each edge of size 2), the concept of unimodular coincides with bipartite.

Characterizations of totally unimodular matrices imply corresponding characterizations of unimodular hypergraphs. We describe some of them in the following theorem. (The equivalence of (i)-(vii) is due to Hoffman and Kruskal [1956], characterization (viii) to Ghouila-Houri [1962b], characterization (ix) to Camion [1963,1965], and characterization (x) to R.E. Gomory (cf. Camion [1965]).)

For the proof we refer to Chapter 19 of Schrijver [1986b].

Theorem 83.4. *Let $H = (V, \mathcal{E})$ be a hypergraph, with incidence matrix M. Then the following are equivalent:*

(83.23)
(i) *H is unimodular, that is, each square submatrix of M has determinant in $\{0, \pm 1\}$;*
(ii) *for each $b \in \mathbb{Z}_+^{\mathcal{E}}$, the polyhedron $\{x \geq \mathbf{0} \mid Mx \leq b\}$ is integer;*
(iii) *for each $b \in \mathbb{R}_+^{\mathcal{E}}$, the system $x \geq \mathbf{0}$, $Mx \leq b$ is totally dual integral;*
(iv) *for each $b \in \mathbb{Z}_+^{\mathcal{E}}$, the polyhedron $\{x \geq \mathbf{0} \mid Mx \geq b\}$ is integer;*
(v) *for each $b \in \mathbb{R}_+^{\mathcal{E}}$, the system $x \geq \mathbf{0}$, $Mx \geq b$ is totally dual integral;*
(vi) *for all $a, b \in \mathbb{Z}^{\mathcal{E}}$ and $c, d \in \mathbb{Z}^V$, the polyhedron $\{x \mid c \leq x \leq d, a \leq Mx \leq b\}$ is integer;*
(vii) *for all $a, b \in \mathbb{R}^{\mathcal{E}}$ and $c, d \in \mathbb{R}^V$, the system $c \leq x \leq d, a \leq Mx \leq b$ is totally dual integral;*
(viii) *each $U \subseteq V$ can be partitioned into sets U_1 and U_2 such that each $E \in \mathcal{E}$ satisfies $\big||E \cap U_1| - |E \cap U_2|\big| \leq 1$;*
(ix) *the sum of the entries in any square submatrix of M with even row and column sums, is divisible by 4;*
(x) *no square submatrix of M has determinant ± 2.*

Proof. See Chapter 19 of Schrijver [1986b]. ∎

This implies a characterization similar to Corollary 83.1b:

Corollary 83.4a. *For any $\{0, 1\}$-valued $m \times n$ matrix M, the following are equivalent:*

(83.24)
(i) *M is totally unimodular;*
(ii) $\forall b \in \mathbb{Z}_+^m \;\; \forall w \in \mathbb{Z}_+^n : \min\{y^\mathsf{T} b \mid y \geq \mathbf{0}, y^\mathsf{T} M \geq w^\mathsf{T}\}$ *has an integer optimum solution y;*
(iii) $\forall b \in \mathbb{Z}_+^m \;\; \forall w \in \mathbb{Z}_+^n : \max\{w^\mathsf{T} x \mid x \geq \mathbf{0}, Mx \leq b\}$ *has an integer optimum solution x;*
(iv) $\forall b \in \mathbb{Z}_+^m \;\; \forall w \in \mathbb{Z}_+^n : \max\{y^\mathsf{T} b \mid y \geq \mathbf{0}, y^\mathsf{T} M \leq w^\mathsf{T}\}$ *has an integer optimum solution y;*
(v) $\forall b \in \mathbb{Z}_+^m \;\; \forall w \in \mathbb{Z}_+^n : \min\{w^\mathsf{T} x \mid x \geq \mathbf{0}, Mx \geq b\}$ *has an integer optimum solution x.*

Proof. From Theorem 83.4. ∎

Unimodular hypergraphs have the following property stronger than was shown for balanced hypergraphs in Corollary 83.1c:

Theorem 83.5. *Let $H = (V, \mathcal{E})$ be a unimodular matrix and let $k \in \mathbb{Z}_+$ with $k \geq 1$. Then V can be partitioned into sets $V_1, \ldots, V_k$ such that*

(83.25)
$$\left\lfloor \frac{|E|}{k} \right\rfloor \leq |E \cap V_i| \leq \left\lceil \frac{|E|}{k} \right\rceil$$

for each $E \in \mathcal{E}$ and each $i = 1, \ldots, k$.

Proof. Choose $F \in \mathcal{E}$. By induction, there is a partition $V_1, \ldots, V_k$ as required for the hypergraph $H' := (V, \mathcal{E} \setminus \{F\})$. Choose the partition with

$$\sum_{i=1}^{k} |F \cap V_i|^2 \tag{83.26}$$

as small as possible. Suppose that (83.25) does not hold for $E := F$. Then there exist i and j such that

$$|F \cap V_i| \geq |F \cap V_j| + 2. \tag{83.27}$$

Consider the contraction of H to $V_i \cup V_j$. By (83.23)(viii), we can split $V_i \cup V_j$ into V_i' and V_j' such that $\big||E \cap V_i'| - |E \cap V_j'|\big| \leq 1$ for each $E \in \mathcal{E}$. So replacing V_i, V_j by V_i', V_j' gives again a valid partition, but decreases the sum (83.26), a contradiction. ∎

A basic theorem of Seymour [1980a] states that each totally unimodular matrix can be decomposed into network matrices, their transposes, and two special 5×5 matrices. As J. Edmonds noted, it yields a polynomial-time test of total unimodularity of matrices, and hence of unimodularity of hypergraphs — Bixby [1982], Schrijver [1986b], and Truemper [1990,1992] described implementations.

83.3a. Further notes

Truemper and Chandrasekaran [1978] proved the following characterization, that includes the polyhedral characterizations of both the balanced and the totally unimodular matrices. For any pair of an $\{0,1\}$-valued $m \times n$ matrix A and a vector $b \in \mathbb{Z}_+^n$, the following are equivalent:

(83.28)
- (i) the polyhedron $\{x \geq \mathbf{0} \mid A'x \leq d'\}$ is integer for each row submatrix A' of A and each integer vector d' with $\mathbf{0} \leq d' \leq b'$, where b' is the part of b corresponding to A';
- (ii) A has no square submatrix M with the following properties: $\det M = \pm 2$, each entry of M^{-1} is $\pm\frac{1}{2}$, and $M\mathbf{1} \leq 2b'$, where b' is the part of b corresponding to M.

For $b = \mathbf{1}$ this characterizes balanced matrices. For b sufficiently large, it characterizes total unimodularity. Related results can be found in Conforti, Cornuéjols, and Truemper [1994] and Conforti, Cornuéjols, and Zambelli [2002a].

Conforti and Rao [1992c] reduced testing if a hypergraph is balanced, to testing if some derived hypergraphs are perfect. Conforti and Rao [1993] gave a polynomial-time algorithm to test if a given hypergraph H is balanced, provided that any two edges of H intersect in at most one vertex. Related results can be found in Lubiw [1988] and Conforti and Rao [1989,1992d].

Berge and Hoffman [1978] gave a formula for the minimum number of stable vertex covers needed to cover the vertex set of a unimodular hypergraph. Dahlhaus, Kratochvil, Manuel, and Miller [1997] described a polynomial-time algorithm to find a maximum number of disjoint vertex covers of a balanced hypergraph.

Conforti and Cornuéjols [1995a] applied balanced matrices to logic problems. Conforti, Cornuéjols, and Vušković [1999] gave a linear-time algorithm to find a chordless circuit in a bipartite graph of length $\equiv 0 \pmod 4$.

Survey of Problems, Questions, and Conjectures

We here collect unsolved problems, questions, and conjectures mentioned in this book. For terminology and background, we refer to the pages indicated.

1 (page 41). Is NP≠P?

2 (page 42). Is P=NP∩co-NP?

3 (page 65). The *Hirsch conjecture*: A d-dimensional polytope with m facets has diameter at most $m-d$.

4 (page 161). Is there an $O(nm)$-time algorithm for finding a maximum flow?

5 (page 232). Berge [1982b] posed the following conjecture generalizing the Gallai-Milgram theorem. Let $D=(V,A)$ be a digraph and let $k \in \mathbb{Z}_+$. Then for each path collection $\mathcal{P}$ partitioning V and minimizing

$$(1) \qquad \sum_{P \in \mathcal{P}} \min\{|VP|, k\},$$

there exist disjoint stable sets $C_1, \ldots, C_k$ in D such that each $P \in \mathcal{P}$ intersects $\min\{|VP|, k\}$ of them. This was proved by Saks [1986] for acyclic graphs.

6 (page 403). The following open problem was mentioned by Fulkerson [1971b]: Let $\mathcal{A}$ and $\mathcal{B}$ be families of subsets of a set S and let $w \in \mathbb{Z}_+^S$. What is the maximum number k of common transversals $T_1, \ldots, T_k$ of $\mathcal{A}$ and $\mathcal{B}$ such that

$$(2) \qquad \chi^{T_1} + \cdots + \chi^{T_k} \leq w?$$

7 (page 459). Can the weighted matching problem be formulated as a linear programming problem of size bounded by a polynomial in the size of the graph, by extending the set of variables? That is, is the matching polytope of a graph $G=(V,E)$ equal to the projection of some polytope $\{x \mid Ax \leq b\}$ with A and b having size bounded by a polynomial in $|V|+|E|$?

8 (pages 472,646). The 5-*flow conjecture* of Tutte [1954a]:

(3) (?) each bridgeless graph has a nowhere-zero 5-flow. (?)

(A *nowhere-zero k-flow* is a flow over $\mathbb{Z}_k$ in some orientation of the graph, taking value 0 nowhere.)

9 (pages 472,498,645,1426). The 4-*flow conjecture* of Tutte [1966]:

(4) (?) each bridgeless graph without Petersen graph minor has a nowhere-zero 4-flow. (?)

This implies the four-colour theorem. For cubic graphs, (4) was proved by Robertson, Seymour, and Thomas [1997], Sanders, Seymour, and Thomas [2000], and Sanders and Thomas [2000].

Seymour [1981c] showed that the 4-flow conjecture is equivalent to the following more general conjecture, also due to Tutte [1966]:

(5) (?) each bridgeless matroid without F_7^*, $M^*(K_5)$, or $M(\mathbf{P}_{10})$ minor has a nowhere-zero flow over GF(4). (?)

Here $\mathbf{P}_{10}$ denotes the Petersen graph.

10 (page 472). The 3-*flow conjecture* (W.T. Tutte, 1972 (cf. Bondy and Murty [1976], Unsolved problem 48)):

(6) (?) each 4-edge-connected graph has a nowhere-zero 3-flow. (?)

11 (page 473). The *weak* 3-*flow conjecture* of Jaeger [1988]:

(7) (?) there exists a number k such that each k-edge-connected graph has a nowhere-zero 3-flow. (?)

12 (page 473). The following *circular flow conjecture* of Jaeger [1984] generalizes both the 3-flow and the 5-flow conjecture:

(8) (?) for each $k \geq 1$, any $4k$-connected graph has an orientation such that in each vertex, the indegree and the outdegree differ by an integer multiple of $2k+1$. (?)

13 (pages 475,645). The *generalized Fulkerson conjecture* of Seymour [1979a]:

(9) (?) $\lceil \chi'^*(G) \rceil = \lceil \frac{1}{2}\chi'(G_2) \rceil$ (?)

for each graph G. (Here $\chi'^*(G)$ denotes the fractional edge-colouring number of G, and G_2 the graph obtained from G by replacing each edge by two parallel edges.) This is equivalent to the conjecture that

(10) (?) for each k-graph G there exists a family of $2k$ perfect matchings, covering each edge precisely twice. (?)

(A *k-graph* is a k-regular graph $G = (V, E)$ with $|\delta(U)| \geq k$ for each odd-size subset U of V.)

14 (pages 476,645). Fulkerson [1971a] asked if in each bridgeless cubic graph there exist 6 perfect matchings, covering each edge precisely twice (the *Fulkerson conjecture*). It is a special case of Seymour's generalized Fulkerson conjecture.

15 (page 476). Berge [1979a] conjectures that the edges of any bridgeless cubic graph can be covered by 5 perfect matchings. (This would follow from the Fulkerson conjecture.)

16 (page 476). Gol'dberg [1973] and Seymour [1979a] conjecture that for each (not necessarily simple) graph G one has

(11) $$\text{(?)}\ \chi'(G) \leq \max\{\Delta(G)+1, \lceil \chi'^{*}(G) \rceil\}.\ \text{(?)}$$

An equivalent conjecture was stated by Andersen [1977].

17 (page 476). Seymour [1981c] conjectures the following generalization of the four-colour theorem:

(12) (?) each planar k-graph is k-edge-colourable. (?)

For $k = 3$, this is equivalent to the four-colour theorem. For $k = 4$ and $k = 5$, it was derived from the case $k = 3$ by Guenin [2002b].

18 (pages 476,644). Lovász [1987] conjectures more generally:

(13) (?) each k-graph without Petersen graph minor is k-edge-colourable. (?)

This is equivalent to stating that the incidence vectors of perfect matchings in a graph without Petersen graph minor, form a Hilbert base.

19 (page 481). The following question was asked by Vizing [1968]: Is there a simple planar graph of maximum degree 6 and with edge-colouring number 7?

20 (page 481). Vizing [1965a] asked if a minimum edge-colouring of a graph can be obtained from an arbitrary edge-colouring by iteratively swapping colours on a colour-alternating path or circuit and deleting empty colours.

21 (page 482). Vizing [1976] conjectures that the list-edge-colouring number of any graph is equal to its edge-colouring number.

(The *list-edge-colouring number* $\chi^l(G)$ of a graph $G = (V, E)$ is the minimum number k such that for each choice of sets L_e for $e \in E$ with $|L_e| = k$, one can select $l_e \in L_e$ for $e \in E$ such that for any two incident edges e, f one has $l_e \neq l_f$.)

22 (page 482). Behzad [1965] and Vizing [1968] conjecture that the total colouring number of a simple graph G is at most $\Delta(G)+2$. (The *total colouring* *number* of a graph $G = (V, E)$ is a colouring of $V \cup E$ such that each colour consists of a stable set and a matching, vertex-disjoint.)

23 (page 482). More generally, Vizing [1968] conjectures that the total colouring number of a graph G is at most $\Delta(G) + \mu(G) + 1$, where $\mu(G)$ is the maximum edge multiplicity of G.

24 (pages 497,645). Seymour [1979b] conjectures that each even integer vector in the circuit cone of a graph is a nonnegative integer combination of incidence vectors of circuits.

25 (pages 497,645,1427). A special case of this is the *circuit double cover conjecture* (asked by Szekeres [1973] and conjectured by Seymour [1979b]): each bridgeless graph has circuits such that each edge is covered by precisely two of them.

Jamshy and Tarsi [1989] proved that the circuit double cover conjecture is equivalent to a generalization to matroids:

(14) (?) each bridgeless binary matroid without F_7^* minor has a circuit double cover. (?)

26 (page 509). Is the system of T-join constraints totally dual quarter-integral?

27 (page 517). L. Lovász asked for the complexity of the following problem: given a graph $G = (V, E)$, vertices $s, t \in V$, and a length function $l : E \to \mathbb{Q}$ such that each circuit has nonnegative length, find a shortest odd $s-t$ path.

28 (page 545). What is the complexity of deciding if a given graph has a 2-factor without circuits of length at most 4?

29 (page 545). What is the complexity of finding a maximum-weight 2-factor without circuits of length at most 3?

30 (page 646). Tarsi [1986] mentioned the following strengthening of the circuit double cover conjecture:

(15) (?) in each bridgeless graph there exists a family of at most 5 cycles covering each edge precisely twice. (?)

31 (page 657). Is the dual of any algebraic matroid again algebraic?

32 (page 892). A special case of a question asked by A. Frank (cf. Schrijver [1979b], Frank [1995]) amounts to the following:

(16) (?) Let $G = (V, E)$ be an undirected graph and let $s \in V$. Suppose that for each vertex $t \neq s$, there exist k internally vertex-disjoint $s-t$ paths. Then G has k spanning trees such that for each vertex

$t \neq s$, the $s-t$ paths in these trees are internally vertex-disjoint. (?)

(The spanning trees need not be edge-disjoint — otherwise $G = K_3$ would form a counterexample.) For $k = 2$, (16) was proved by Itai and Rodeh [1984, 1988], and for $k = 3$ by Cheriyan and Maheshwari [1988] and Zehavi and Itai [1989].

33 (page 962). Can a maximum number of disjoint directed cut covers in a directed graph be found in polynomial time?

34 (page 962). Woodall [1978a,1978b] conjectures (*Woodall's conjecture*):

(17) (?) In a digraph, the minimum size of a directed cut is equal to the maximum number of disjoint directed cut covers. (?)

35 (page 985). Let $G = (V, E)$ be a complete undirected graph, and consider the system

(18)
$$\begin{aligned} &0 \leq x_e \leq 1 \quad \text{for each edge } e, \\ &x(\delta(v)) = 2 \quad \text{for each vertex } v, \\ &x(\delta(U)) \geq 2 \text{ for each } U \subseteq V \text{ with } \emptyset \neq U \neq V. \end{aligned}$$

Let $l : E \to \mathbb{R}_+$ be a length function. Is the minimum length of a Hamiltonian circuit at most $\frac{4}{3}$ times the minimum value of $l^{\mathsf{T}}x$ over (18)?

36 (page 990). Padberg and Grötschel [1985] conjecture that the diameter of the symmetric traveling salesman polytope of a complete graph is at most 2.

37 (page 1076). Frank [1994a] conjectures:

(19) (?) Let $D = (V, A)$ be a simple acyclic directed graph. Then the minimum size of a k-vertex-connector for D is equal to the maximum of $\sum_{v \in V} \max\{0, k - \deg^{\text{in}}(v)\}$ and $\sum_{v \in V} \max\{0, k - \deg^{\text{out}}(v)\}$. (?)

(A *k-vertex-connector* for D is a set of (new) arcs whose addition to D makes it k-vertex-connected.)

38 (page 1087). *Hadwiger's conjecture* (Hadwiger [1943]): If $\chi(G) \geq k$, then G contains K_k as a minor.

Hadwiger's conjecture is trivial for $k = 1, 2, 3$, was shown by Hadwiger [1943] for $k = 4$ (also by Dirac [1952]), is equivalent to the four-colour theorem for $k = 5$ (by a theorem of Wagner [1937a]), and was derived from the four-colour theorem for $k = 6$ by Robertson, Seymour, and Thomas [1993]. For $k \geq 7$, the conjecture is unsettled.

39 (page 1099). Chvátal [1973a] asked if for each fixed t, the stable set problem for graphs for which the stable set polytope arises from $P(G)$ by at most

t rounds of cutting planes, is polynomial-time solvable. Here $P(G)$ is the polytope determined by the nonnegativity and clique inequalities.

40 (page 1099). Chvátal [1975b] conjectures that there is no polynomial $p(n)$ such that for each graph G with n vertices we can obtain the inequality $x(V) \leq \alpha(G)$ from the system defining $Q(G)$ by adding at most $p(n)$ cutting planes. Here $Q(G)$ is the polytope determined by the nonnegativity and edge inequalities. (This conjecture would be implied by NP≠co-NP.)

41 (page 1105). Gyárfás [1987] conjectures that there exists a function $g : \mathbb{Z}_+ \to \mathbb{Z}_+$ such that $\chi(G) \leq g(\omega(G))$ for each graph G without odd holes.

42 (page 1107). Can perfection of a graph be tested in polynomial time?

43 (page 1131). Berge [1982a] conjectures the following. A directed graph $D = (V, A)$ is called *α-diperfect* if for every induced subgraph $D' = (V', A')$ and each maximum-size stable set S in D' there is a partition of V' into directed paths each intersecting S in exactly one vertex. Then for each directed graph D:

(20) (?) D is α-diperfect if and only if D has no induced subgraph C whose underlying undirected graph is a chordless odd circuit of length ≥ 5, say with vertices $v_1, \ldots, v_{2k+1}$ (in order) such that each of $v_1, v_2, v_3, v_4, v_6, v_8, \ldots, v_{2k}$ is a source or a sink. (?)

44 (page 1170). Is $\vartheta(C_n) = \Theta(C_n)$ for each odd n?

45 (page 1170). Can Haemers' bound $\eta(G)$ on the Shannon capacity of a graph G be computed in polynomial time?

46 (page 1187). Is every t-perfect graph strongly t-perfect?

Here a graph is *t-perfect* if its stable set polytope is determined by the nonnegativity, edge, and odd circuit constraints. It is *strongly t-perfect* if this system is totally dual integral.

47 (page 1195). T-perfection is closed under taking induced subgraphs and under contracting all edges in $\delta(v)$ where v is a vertex not contained in a triangle. What are the minimally non-t-perfect graphs under this operation?

48 (page 1242). For any k, let $f(k)$ be the smallest number such that in any $f(k)$-connected undirected graph, for any choice of distinct vertices $s_1, t_1, \ldots, s_k, t_k$ there exist vertex-disjoint $s_1 - t_1, \ldots, s_k - t_k$ paths. Thomassen [1980] conjectures that $f(k) = 2k + 2$ for $k \geq 2$.

49 (page 1242). For any k, let $g(k)$ be the smallest number such that in any $g(k)$-edge-connected undirected graph, for any choice of vertices $s_1, t_1, \ldots, s_k, t_k$ there exist edge-disjoint $s_1 - t_1, \ldots, s_k - t_k$ paths. Thomassen [1980] conjectures that $g(k) = k$ if k is odd and $g(k) = k + 1$ if k is even.

50 (page 1243). What is the complexity of the k arc-disjoint paths problem in directed planar graphs, for any fixed $k \geq 2$? This is even unknown for $k = 2$, also if we restrict ourselves to two opposite nets.

51 (page 1274). Karzanov [1991] conjectures that if the nets in a multiflow problem form two disjoint triangles and if the capacities and demands are integer and satisfy the Euler condition, then the existence of a fractional multiflow implies the existence of a half-integer multiflow.

52 (page 1274). The previous conjecture implies that for each graph $H = (T, R)$ without three disjoint edges, there is an integer k such that for each graph $G = (V, E)$ with $V \supseteq T$ and any $c : E \to \mathbb{Z}_+$ and $d : R \to \mathbb{Z}_+$, if there is a feasible multiflow, then there exists a $\frac{1}{k}$-integer multiflow.

53 (page 1276). Okamura [1998] conjectures the following. Let $G = (V, E)$ be an l-edge-connected graph (for some l). Let $H = (T, R)$ be a 'demand' graph, with $T \subseteq V$, such that $d_R(U) \leq l$ for each $U \subseteq V$. Then the edge-disjoint paths problem has a half-integer solution.

54 (page 1293). Is each Mader matroid a gammoid?

55 (page 1294). Is each Mader matroid linear?

56 (page 1299). Is the undirected edge-disjoint paths problem for planar graphs polynomial-time solvable if all terminals are on the outer boundary? Is it NP-complete?

57 (page 1310). Is the integer multiflow problem polynomial-time solvable if the graph and the nets form a planar graph such that the nets are spanned by a fixed number of faces?

58 (page 1310). Pfeiffer [1990] raised the question if the edge-disjoint paths problem has a half-integer solution if the graph $G + H$ (the union of the supply graph and the demand graph) is embeddable in the torus and there exists a quarter-integer solution.

59 (page 1320). Let $G = (V, E)$ be a planar bipartite graph and let q be a vertex on the outer boundary. Do there exist disjoint cuts $C_1, \ldots, C_p$ such that any pair s, t of vertices with s and t on the outer boundary, or with $s = q$, is separated by $\mathrm{dist}_G(s, t)$ cuts?

60 (page 1345). Fu and Goddyn [1999] asked: Is the class of graphs for which the incidence vectors of cuts form a Hilbert base, closed under taking minors?

61 (page 1382). Füredi, Kahn, and Seymour [1993] conjecture that for each hypergraph $H = (V, \mathcal{E})$ and each $w : \mathcal{E} \to \mathbb{R}_+$, there exists a matching $\mathcal{M} \subseteq \mathcal{E}$ such that

(21) $$\sum_{F\in\mathcal{M}} \big(|F|-1+\frac{1}{|F|}\big)w(F) \geq \nu_w^*(H),$$

where $\nu_w^*(H)$ is the maximum weight $w^{\mathsf{T}}y$ of a fractional matching $y:\mathcal{E}\to\mathbb{R}_+$.

62 (pages 1387,1408). Seymour [1981a] conjectures:

(22) (?) a binary hypergraph is ideal if and only if it has no $\mathcal{O}(K_5)$, $b(\mathcal{O}(K_5))$, or F_7 minor. (?)

63 (page 1392). Seymour [1990b] asked the following. Suppose that $H = (V,\mathcal{E})$ is a hypergraph without J_n minor ($n \geq 3$). Let $l, w : V \to \mathbb{Z}_+$ be such that

(23) $$\tau(H^w)\cdot\tau(b(H)^l) > l^{\mathsf{T}}w.$$

Is there a minor H' of H and $l', w' : VH' \to \{0,1\}$ such that

(24) $$\tau((H')^{w'})\cdot\tau(b(H')^{l'}) > l'^{\mathsf{T}}w'$$

and such that $\tau((H')^{w'}) \leq \tau(H^w)$ and $\tau(b(H')^{l'}) \leq \tau(b(H)^l)$?

Here, for each $n \geq 3$: $J_n :=$ the hypergraph with vertex set $\{1,\ldots,n\}$ and edges $\{2,\ldots,n\}$, $\{1,2\},\ldots,\{1,n\}$.

64 (page 1392). Seymour [1990b] also asked the following. Let $H = (V,\mathcal{E})$ be a nonideal hypergraph. Is the minimum of $\tau(H')$ over all parallelizations and minors H' of H with $\tau^*(H') < \tau(H')$ attained by a minor of H?

65 (page 1395). Cornuéjols and Novick [1994] conjecture that there are only finitely many minimally nonideal hypergraphs H with $r_{\min}(H) > 2$ and $\tau(H) > 2$.

66 (page 1396). Ding [1993] asked whether there exists a number t such that each minimally nonideal hypergraph H satisfies $r_{\min}(H) \leq t$ or $\tau(H) \leq t$.

(The above conjecture of Cornuéjols and Novick [1994] implies a positive answer to this question.)

67 (page 1396). Ding [1993] conjectures that for each fixed $k \geq 2$, each minor-minimal hypergraph H with $\tau_k(H) < k\cdot\tau(H)$, contains some J_n minor ($n \geq 3$) or satisfies the regularity conditions of Lehman's theorems (Theorem 78.4 and 78.5).

68 (page 1401). Conforti and Cornuéjols [1993] conjecture:

(25) (?) a hypergraph is Mengerian if and only if it is packing. (?)

69 (page 1401). Cornuéjols, Guenin, and Margot [1998,2000] conjecture:

(26) (?) each minimally nonideal hypergraph H with $r_{\min}(H)\tau(H) = |VH|+1$ is minimally nonpacking. (?)

70 (page 1401). Cornuéjols, Guenin, and Margot [1998,2000] conjecture that $\tau(H) = 2$ for each ideal minimally nonpacking hypergraph H.

71 (page 1404). Seymour [1981a] conjectures that T_{30} is the unique minor-minimal binary ideal hypergraph H with the property $\nu_2(H) < 2\tau(H)$.

Here the hypergraph T_{30} arises as follows. Replace each edge of the Petersen graph by a path of length 2, making the graph G. Let $T := VG \setminus \{v\}$, where v is an arbitrary vertex of v of degree 3. Let $\mathcal{E}$ be the collection of T-joins. Then $T_{30} := (EG, \mathcal{E})$.

72 (page 1405). P.D. Seymour (personal communication 1975) conjectures that for each ideal hypergraph H there exists an integer k such that $\nu_k(H) = k \cdot \tau(H)$ and such that $k = 2^i$ for some i. He also asks if $k = 4$ would do in all cases.

73 (page 1405). Seymour [1979a] conjectures that for each ideal hypergraph H, the g.c.d. of those k with $\nu_k(H) = k \cdot \tau(H)$ is equal to 1 or 2.

74 (page 1409). Is the following true for binary hypergraphs H:

(27) (?) $\nu(H^w) = \tau(H^w)$ for each $w : V \to \mathbb{Z}_+$ with $w(B)$ even for all $B \in b(H) \iff \frac{1}{2}\nu_2(H^w) = \tau(H^w)$ for each $w : V \to \mathbb{Z}_+ \iff$ H has no $\mathcal{O}(K_5)$, $b(\mathcal{O}(K_5))$, F_7, or T_{15} minor. (?)

Here T_{15} is the hypergraph of $V\mathbf{P}_{10}$-joins in the Petersen graph $\mathbf{P}_{10}$.

75 (page 1421). Seymour [1981a] conjectures that for any binary matroid M:

(28) (?) M is 1-cycling $\iff$ M is 1-flowing $\iff$ M has no AG(3,2), T_{11}, or T_{11}^* minor. (?)

Here T_{11} is the binary matroid represented by the 11 vectors in $\{0,1\}^5$ with precisely 3 or 5 ones. Moreover, AG(3,2) is the matroid with 8 elements obtained from the 3-dimensional affine geometry over GF(2).