Part VII

Multiflows and Disjoint Paths

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The problem of finding a maximum flow from one source s to one sink t in a directed graph is highly tractable. There is a very efficient algorithm, which outputs an integer maximum flow if all capacities are integer. Moreover, the maximum flow value is equal to the minimum capacity of a cut separating s and t. If all capacities are equal to 1, the problem reduces to finding arc-disjoint paths. Some direct transformations give similar results for vertex capacities, for vertex-disjoint paths, and for undirected graphs.

Often in practice however, one is not interested in connecting only one pair of source and sink by a flow or by paths, but several pairs of sources and sinks simultaneously. One may think of a large communication or transportation network, where several messages or goods must be transmitted all at the same time over the same network, between different pairs of terminals. Also railway circulation with different types of rolling stock gives a multicommodity flow problem. A recent application is the design of *very large-scale integrated* (VLSI) circuits, where several pairs of pins must be interconnected by wires on a chip, in such a way that the wires follow a given grid and that the wires connecting different pairs of pins do not intersect each other.

This leads to the area of multicommodity flows (briefly: multiflows) and disjoint paths. Most polyhedral and polynomial-time methods for 1-commodity flows and paths do not extend to multicommodity flows and paths. Yet a number of cases can be solved efficiently, in particular when the terminals have a special structure or when the graph is planar or, more generally, can be embedded in a specific surface.

Chapters:

Chapter 70

Multiflows and disjoint paths

We discuss basic, general facts and terminology on multiflows and disjoint paths. In particular, we study general interrelations between fractional multiflows, integer multiflows, disjoint paths, the 'cut condition', and the 'Euler condition'.

70.1. Directed multiflow problems

Given two directed graphs, a *supply digraph* $D = (V, A)$ and a *demand digraph* $H = (T, R)$ with $T \subseteq V$, a *multiflow* is a function f on R where f_r is an $s-t$ flow in D for each $r = (s, t) \in R$.[1] In this context, each pair in R is called a *net*, and each vertex covered by R is called a *terminal*.

For $k := |R|$, we also speak of a *k-commodity flow*. Occasionally, we will list the nets as $(s_1, t_1), \ldots, (s_k, t_k)$. Then for $r = (s_i, t_i)$ we denote f_r also by f_i. The indices $1, \ldots, k$ are called the *commodities*.

The *value* of f is the function $\phi : R \to \mathbb{R}_+$ where ϕ_r is the value of f_r. The *total value*, or (if no confusion may arise) just the *value*, is $\sum_{r\in R} \phi_r$.

Given a 'capacity' function $c : A \to \mathbb{R}_+$, we say that a multiflow f is *subject to* c if

(70.1) $$\sum_{r\in R} f_r(a) \leq c(a)$$

for each arc a.

The *multiflow problem* or *k-commodity flow problem* (for $k := |R|$) is:

(70.2) given: a supply digraph $D = (V, A)$, a demand digraph $H = (T, R)$ with $T \subseteq V$, a capacity function $c : A \to \mathbb{R}_+$, and a demand function $d : R \to \mathbb{R}_+$,
find: a multiflow subject to c of value d.

Given c and d, a multiflow subject to c of value d is called a *feasible multiflow*, or just a *multiflow* if no confusion is expected to arise. We call the problem *feasible* if there exists a feasible multiflow.

[1] Throughout, we use the terms 'multicommodity flow' and 'multiflow' as synonyms.

If we require each f_r to be an integer flow, the problem is called the *integer multiflow problem* or the *integer k-commodity flow problem*. Similarly for *half-integer*, *quarter-integer*, etc. For clarity, we sometimes add the adjective *fractional* if no integrality is required.

Related is the *maximum-value multiflow problem* or *maximum-value k-commodity flow problem*:

(70.3) given: a supply digraph $D = (V, A)$, a demand digraph $H = (T, R)$ with $T \subseteq V$, and a capacity function $c : A \to \mathbb{R}_+$,
find: a multiflow subject to c, of maximum total value.

Again we add *integer* (*half-integer*, etc) if we require the f_r to be integer (half-integer, etc.).

We can reduce a multiflow problem with demands $d_1, \ldots, d_k$ to a maximum-value multiflow problem, by extending the graph by an arc from a new vertex s'_i to s_i of capacity d_i (for $i = 1, \ldots, k$). Then the multiflow problem in the original graph is feasible if and only if the maximization problem in the new graph, with nets (s'_i, t_i), has maximum total value equal to $d_1 + \cdots + d_k$.

70.2. Undirected multiflow problems

The problems described above have a natural analogue for undirected graphs. Let be given two undirected graphs, a *supply graph* $G = (V, E)$ and a *demand graph* $H = (T, R)$ with $T \subseteq V$. Again, each pair in R is called a *net*, and each vertex covered by R is called a *terminal*.

For $s, t \in V$, a function $f : E \to \mathbb{R}_+$ is called an $s - t$ *flow* if there exists an orientation (V, A) of G such that f is an $s - t$ flow in D.

A *multiflow* is a function f on R such that f_r is an $s - t$ flow for each $r = st \in R$. For $k := |R|$, the multiflow is also called a k-*commodity flow*. Again, occasionally we will list the nets as $\{s_1, t_1\}, \ldots, \{s_k, t_k\}$.

The *value* of a multiflow f is the function $\phi : R \to \mathbb{R}_+$ where ϕ_r is the value of f_r. The *total value*, or just the *value*, is $\sum_{r \in R} \phi_r$.

Given a capacity function $c : E \to \mathbb{R}_+$, we say that a multiflow f is *subject to c* if

(70.4)
$$\sum_{r \in R} f_r(e) \leq c(e)$$

for each edge e. Note that generally for each $r = st \in R$, there is a different orientation D_r of G that makes f_r into an $s - t$ flow in D_r. So in (70.4), the sum of the flows through both orientations of a given edge e are bounded above by $c(e)$.

In this way we obtain the *undirected multiflow problem* or *undirected k-commodity flow problem*, and the *undirected maximum-value multiflow problem* or *undirected maximum-value k-commodity flow problem*. Again, we add *integer* (*half-integer*, etc.) if we require the f_r to be integer (half-integer, etc.) flows. We skip the adjective 'undirected' if it is clear from the context.

70.3. Disjoint paths problems

If all capacities and demands are equal to 1, the integer multiflow problem is equivalent to the *(k) arc-* or *edge-disjoint paths problem*:

(70.5) given: a directed (or undirected) graph $D = (V, A)$ and pairs $(s_1, t_1), \ldots, (s_k, t_k)$ of vertices of G,
find: arc- (or edge-)disjoint paths $P_1, \ldots, P_k$ where P_i is an $s_i - t_i$ path ($i = 1, \ldots, k$).

For undirected graphs, the pairs s_i, t_i need not be ordered.

A *fractional solution* (*half-integer solution* respectively) of the arc- or edge-disjoint paths problem is a fractional (half-integer respectively) multiflow for all capacities and demands 1.

Related is the *(vertex-)disjoint paths problem* (or k *(vertex-)disjoint paths problem*):

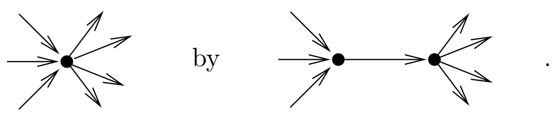
(70.6) given: a (directed or undirected) graph $D = (V, A)$ and pairs $(s_1, t_1), \ldots, (s_k, t_k)$ of vertices of G,
find: vertex-disjoint paths $P_1, \ldots, P_k$ where P_i is an $s_i - t_i$ path ($i = 1, \ldots, k$).

70.4. Reductions

Above we mentioned two versions of the multiflow problem: directed and undirected, and four versions of the disjoint paths problem: directed vertex-disjoint, directed arc-disjoint, undirected vertex-disjoint, and undirected edge-disjoint. There are a number of constructions that reduce versions among them.

First, the *undirected edge-disjoint* paths problem can be reduced to the *undirected vertex-disjoint* paths problem by replacing the graph by its line graph. Similarly, the *directed arc-disjoint* paths problem can be reduced to the *directed vertex-disjoint* paths problem by replacing the digraph by its line digraph.

Conversely, the *directed vertex-disjoint* paths problem can be reduced to the *directed arc-disjoint* paths problem by replacing each vertex

(70.7) 

So far, these reductions do not maintain planarity.

The *undirected vertex-disjoint* paths problem can be reduced to the *directed vertex-disjoint* paths problem by replacing each edge

(70.8)  .

Trivially, this construction maintains planarity.

Finally, there is the following reduction of the *undirected edge-disjoint* paths problem to the *directed arc-disjoint* paths problem: replace each edge

(70.9)  .

This reduction also applies to (integer, half-integer, fractional) multiflow problems. Again, this construction maintains planarity.

We represent these reductions in the following diagram, where a double arrow means a reduction maintaining planarity:

$$(70.10)\qquad \begin{array}{ccc} \boxed{\text{undirected edge-disjoint}} & \longrightarrow & \boxed{\text{undirected vertex-disjoint}} \\ \Downarrow & & \Downarrow \\ \boxed{\text{directed arc-disjoint}} & \longleftrightarrow & \boxed{\text{directed vertex-disjoint}} \end{array}$$

Notes. These reductions maintain the set of nets and the demand values. Even, Itai, and Shamir [1975,1976] gave an interesting construction reducing the *directed arc-disjoint* paths problem to the *undirected edge-disjoint* paths problem. It reduces the directed arc-disjoint paths problem with k commodities of demands $d_1, \ldots, d_k$ in a digraph $D = (V, A)$, to the undirected edge-disjoint paths problem with k commodities of demands $d_1 + |A|, \ldots, d_k + |A|$. The construction does not maintain planarity.

70.5. Complexity of the disjoint paths problem

In Section 70.6 we shall see that the fractional multiflow problem is solvable in strongly polynomial time, since it is a linear programming problem.

The integer multiflow problem is NP-complete, even the disjoint paths problem is NP-complete, in any mode (directed/undirected, vertex/edge-disjoint), even for planar graphs. In some cases, however, the problem is polynomial-time solvable if we fix the number k of commodities. We survey the complexity results in the following table:

	directed		*undirected*	
	arc-disjoint	*vertex-disjoint*	*edge-disjoint*	*vertex-disjoint*
general	NP-complete[2]	NP-complete[2]	NP-complete[3]	NP-complete[2]
planar	NP-complete[4]	NP-complete[5]	NP-complete[4]	NP-complete[5]
for fixed k:				
general	NP-complete[6]	NP-complete[6]	polynomial-time[7]	polynomial-time[7]
planar	?[8]	polynomial-time[9]	polynomial-time[7]	polynomial-time[7]

Complexity of the k disjoint paths problem

By the reduction described at the end of Section 70.1, the NP-completeness of the integer multiflow and disjoint paths problems implies that also the corresponding maximization problems are NP-complete.

70.6. Complexity of the fractional multiflow problem

The fractional multiflow problem can easily be described as one of solving a system of linear inequalities in the variables $f_i(a)$ for $i = 1, \dots, k$ and $a \in A$. The constraints are the flow conservation laws and the demand constraint for each flow f_i separately, together with the capacity constraints (70.1). Therefore, the fractional multiflow problem can be solved in polynomial time with any polynomial-time linear programming algorithm. Tardos [1986] showed that the fractional multiflow problem is solvable in strongly polynomial time,

[2] D.E. Knuth, 1974 (cf. Karp [1975]), who proved the NP-completeness of the undirected vertex-disjoint version. It implies the NP-completeness of the directed vertex-disjoint case (by reduction (70.8)), which in turn implies the NP-completeness of the directed arc-disjoint version (by reduction (70.7)). Even, Itai, and Shamir [1975,1976] showed that the directed arc-disjoint paths problem is NP-complete even if the digraph is acyclic and $s_2 = \cdots = s_k$ and $t_2 = \cdots = t_k$.

[3] Even, Itai, and Shamir [1975,1976] — NP-complete even if $|\{\{s_1, t_1\}, \dots, \{s_k, t_k\}\}| = 2$; equivalently, the integer 2-commodity flow problem is NP-complete even if all capacities are 1.

[4] Kramer and van Leeuwen [1984], who proved the NP-completeness of the planar undirected edge-disjoint paths problem, implying the NP-completeness of the planar directed arc-disjoint paths problem, by reduction (70.9). Kramer and van Leeuwen showed NP-completeness even if the graphs are restricted to rectangular grids.

[5] Lynch [1975], who proved the NP-completeness of the planar undirected vertex-disjoint paths problem. It implies the NP-completeness of the planar directed vertex-disjoint paths problem, by reduction (70.8). The problems remain NP-complete for cubic planar graphs (Richards [1984]), and also if the graph together with the nets is planar and cubic (Middendorf and Pfeiffer [1993]).

[6] Fortune, Hopcroft, and Wyllie [1980] — NP-complete even for $k = 2$ opposite nets (s, t) and (t, s).

[7] Robertson and Seymour [1995], who proved the polynomial-time solvability of the k vertex-disjoint paths problem in undirected graphs, for any fixed k. By replacing a graph by its line graph, it implies the polynomial-time solvability of the k edge-disjoint paths problem in undirected graphs, for any fixed k.

[8] unknown also if $k = 2$ and the two nets are opposite.

[9] Schrijver [1994a].

by proving that any linear programming problem with $\{0, \pm 1\}$ constraint matrix is solvable in strongly polynomial time.

Onaga [1970] gave the following good characterization for the feasibility of the fractional multiflow problem, which can be derived (as Iri [1971] observed) from Farkas' lemma ($\mathrm{dist}_l(s,t)$ denotes the length of a shortest $s-t$ path with respect to a length function l):

Theorem 70.1. *The (directed or undirected) fractional multiflow problem* (70.2) *has a solution if and only if*

$$\sum_{i=1}^{k} d_i \cdot \mathrm{dist}_l(s_i,t_i) \leq \sum_{a \in A} l(a)c(a) \tag{70.11}$$

for each length function $l : A \to \mathbb{Z}_+$.

Proof. For $i = 1, \ldots, k$, let $\mathcal{P}_i$ denote the collection of arc sets of $s_i - t_i$ paths. Then there is a feasible multiflow if and only if there exist $\lambda_{i,P} \geq 0$ (for $i = 1, \ldots, k$ and $P \in \mathcal{P}_i$), such that

$$\begin{aligned} &\sum_{P \in \mathcal{P}_i} \lambda_{i,P} = d_i && \text{for } i = 1, \ldots, k, \\ &\sum_{i=1}^{k} \sum_{P \in \mathcal{P}_i} \lambda_{i,P} \chi^P(a) \leq c(a) && \text{for } a \in A. \end{aligned} \tag{70.12}$$

By Farkas' lemma, this is equivalent to: for all $b_1, \ldots, b_k \in \mathbb{R}$ and $l : A \to \mathbb{R}_+$, if

$$b_i \leq \sum_{a \in P} l(a) \text{ for } i = 1, \ldots, k \text{ and } P \in \mathcal{P}_i, \tag{70.13}$$

then

$$\sum_{i=1}^{k} b_i d_i \leq \sum_{a \in A} l(a)c(a). \tag{70.14}$$

Now we may assume that each b_i is chosen maximal such that it satisfies (70.13). Then b_i is equal to the minimum of $\sum_{a \in P} l(a)$ taken over all $P \in \mathcal{P}_i$, that is, to $\mathrm{dist}_l(s_i,t_i)$. Hence the condition is equivalent to (70.11). ∎

(Onaga and Kakusho [1971] gave an alternative proof. If we restrict l to $\{0,1\}$-valued functions, we obtain a necessary condition (a 'multicut condition'), as was observed by Naniwada [1969], who raised the question if the above theorem may hold.)

A min-max relation for the maximum-value multiflow problem can be derived similarly from LP-duality (cf. Lomonosov [1978a]):

Theorem 70.2. *Let $D = (V,A)$ be a directed or undirected graph, let $(s_1,t_1), \ldots, (s_k,t_k)$ be nets, and let $c : A \to \mathbb{R}_+$ be a capacity function. Then* *the maximum total value of a multiflow subject to c is equal to the minimum value of* $\sum_{a \in A} l(a)c(a)$ *taken over all* $l : A \to \mathbb{R}_+$ *satisfying*

(70.15) $\quad \text{dist}_l(s_i, t_i) \geq 1$ *for each* $i = 1, \ldots, k$.

Proof. Let $\mathcal{P}$ denote the collection of arc sets of paths running from s_i to t_i for some $i = 1, \ldots, k$. Then the maximum total value of a multiflow is equal to the maximum of $\sum_{P \in \mathcal{P}} \lambda_P$, where $\lambda_P \geq 0$ for $P \in \mathcal{P}$, such that

(70.16) $$\sum_{P \in \mathcal{P}} \lambda_P \chi^P(a) \leq c(a) \text{ for } a \in A.$$

By LP-duality, this value is equal to the minimum value of $\sum_{a \in A} l(a)c(a)$ where $l : A \to \mathbb{R}_+$ such that

(70.17) $$\sum_{a \in P} l(a) \geq 1 \text{ for each } P \in \mathcal{P}.$$

As (70.17) is equivalent to (70.15), we have the theorem. ∎

70.7. The cut condition for directed graphs

In Theorem 70.1 we saw a good characterization for the feasibility of the fractional multiflow problem. In some cases, it can be replaced by a weaker condition, the *cut condition*:

(70.18) $$c(\delta^{\text{out}}_A(U)) \geq d(\delta^{\text{out}}_R(U)) \text{ for each } U \subseteq V.$$

The cut condition indeed is a direct consequence of condition (70.14) described in Theorem 70.1. For define $l(a) := 1$ if $a \in \delta^{\text{out}}(U)$, and $l(a) := 0$ otherwise. Then (70.11) implies:

(70.19) $$c(\delta^{\text{out}}_A(U)) = \sum_{a \in A} l(a)c(a) \geq \sum_{i=1}^{k} d_i \cdot \text{dist}_l(s_i, t_i) \geq d(\delta^{\text{out}}_R(U)).$$

However, the cut condition is in general not sufficient, even not in the two simple cases given in Figure 70.1.

For directed graphs, the cut condition is known to be sufficient for the existence of a fractional multiflow only if $s_1 = \cdots = s_k$ or $t_1 = \cdots = t_k$ (this follows from the (one-commodity) max-flow min-cut theorem). In a sense, this is the only case:

Theorem 70.3. *Let* $H = (T, R)$ *be a demand digraph, where* R *contains no loops. Then for each supply digraph* $D = (V, A)$ *with* $V \supseteq T$*, the cut condition* (70.18) *is sufficient for the existence of a fractional multiflow if and only if all arcs of* H *have a common head, or they all have a common tail.*

Proof. Let $R = \{(s_1, t_1), \ldots, (s_k, t_k)\}$.

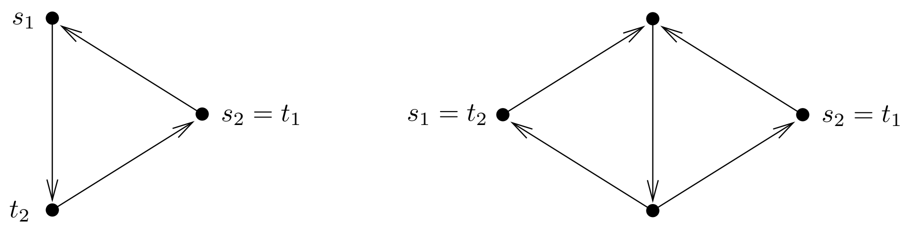


Figure 70.1
Two digraphs where the cut condition holds, but no fractional multiflow exists (taking all capacities and demands equal to 1). The nonexistence of a fractional multiflow can be shown with Theorem 70.1, by taking $l(a) := 1$ for each arc a.

To see sufficiency, by symmetry we can assume that $s_1 = \cdots = s_k$. Let $s := s_1$, and let t be a new vertex. For $i = 1, \ldots, k$, add a new arc (t_i, t), with capacity d_i. Then, by the max-flow min-cut theorem, the cut condition implies that the extended graph has an $s-t$ flow of value $d_1 + \cdots + d_k$, subject to the capacity. Restricted to the original graph, we can decompose the flow into a feasible k-commodity flow of values $d_1, \ldots, d_k$.

To see necessity, if the condition is not met, then there exist nets (s_i, t_i) and (s_j, t_j) with $s_i \neq s_j$ and $t_i \neq t_j$. We can assume that $i = 1$, $j = 2$. Then $\{s_1, t_2\}$ is disjoint from $\{s_2, t_1\}$, and then the second example in Figure 70.1 can be adapted to obtain an example with net set R, and where the cut condition holds but no fractional multiflow exists. ∎

As for maximizing the total value of a multiflow, in a directed triangle, with as nets the opposites of all arcs and all capacities equal to 1, the maximum total value is $\frac{3}{2}$, while the minimum capacity of an arc set disconnecting all nets is 2.

70.8. The cut condition for undirected graphs

Similarly, one can formulate the cut condition in the undirected case:

(70.20) $\quad c(\delta_E(U)) \geq d(\delta_R(U))$ for each $U \subseteq V$.

In the special case of the edge-disjoint paths problem (where all capacities and demands are equal to 1), the cut condition amounts to:

(70.21) $\quad d_E(U) \geq d_R(U)$ for each $U \subseteq V$.

As was observed by Tang [1965], in the undirected case the cut condition is equivalent to the 'disconnecting set condition':

(70.22) $\quad c(F) \geq d(\mathrm{disc}_R(F))$ for each $F \subseteq E$, where $\mathrm{disc}_R(F)$ denotes the family of nets st where s and t are in different components of $G - F$.

Indeed, trivially, the cut condition is implied by (70.22). To see the reverse implication, let $\mathcal{K}$ be the set of components of $G - F$. Then the cut condition implies

$$(70.23) \qquad c(F) \geq \tfrac{1}{2} \sum_{K \in \mathcal{K}} c(\delta_E(K)) \geq \tfrac{1}{2} \sum_{K \in \mathcal{K}} d(\delta_R(K)) = d(\mathrm{disc}_R(F)),$$

which is (70.22).

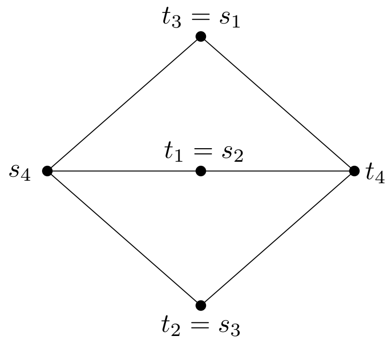


Figure 70.2
An undirected graph where the cut condition holds, but no fractional multiflow exists (taking all capacities and demands equal to 1). This last can be shown with Theorem 70.1, by taking $l(e) := 1$ for each edge e.

Figure 70.2 shows that, also in the undirected case, the cut condition is not sufficient[10]. Hu [1963] showed that, in the undirected case, if $k = 2$, then the cut condition is sufficient for the existence of a fractional multiflow. This is *Hu's* 2-*commodity flow theorem* (Theorem 71.1b). In Section 70.11, we will list more cases where the cut condition is sufficient for the existence of a fractional multiflow.

Hu's 2-commodity flow theorem implies the *max-biflow min-cut theorem* (Corollary 71.1d): in the undirected case with $k = 2$, the maximum value of a 2-commodity flow is equal to the minimum capacity of a cut separating both s_1 and t_1 and s_2 and t_2.

[10] Hakimi [1962b] and Tang [1962] claimed erroneously to give proofs that the cut condition is sufficient for any number k of commodities. According to Hu [1963], a counterexample was first found by L.R. Ford, Jr.

A strengthening of the cut condition that Hu [1964] claimed to be necessary (and conjectured to be sufficient) for the existence of a fractional multiflow, was shown to be not necessary by Tang [1965].

A similar 'maximum-triflow min-cut theorem' does not hold, even not if the three nets form a triangle: take $K_{1,3}$ and all pairs of end vertices as nets, all capacities being 1; then the minimum number of edges disconnecting each commodity is equal to 2, while the maximum total value of a fractional multiflow is equal to $\frac{3}{2}$ (example of Ford and Fulkerson [1954,1956b]).

Anyway, if the nets form a triangle, finding a minimum-size set of edges disconnecting each net, is NP-complete (Dahlhaus, Johnson, Papadimitriou, Seymour, and Yannakakis [1992,1994]).

It will be useful to note that the cut condition only needs to be required for cuts with both sides connected (if G is connected):

Theorem 70.4. *Let $G = (V, E)$ an $H = (T, R)$ be a supply and demand graph, with G connected. Let $c : E \to \mathbb{R}_+$ and $d : R \to \mathbb{R}_+$. If the cut condition* (70.20) *is violated, then it is violated by some $U \subseteq V$ for which both $G[U]$ and $G[V \setminus U]$ are connected.*

Proof. Let U violate the cut condition; that is, $c(\delta_E(U)) < d(\delta_R(U))$. Choose U such that $|\delta_E(U)|$ is as small as possible. We show that $G[U]$ and $G[V \setminus U]$ are connected. By symmetry, it suffices to show that $G[U]$ is connected. Let $K_1, \ldots, K_t$ be the components of $G[U]$. Suppose $t \geq 2$. Then

$$\text{(70.24)} \qquad \sum_{j=1}^{t} c(\delta_E(K_j)) = c(\delta_E(U)) < d(\delta_R(U)) \leq \sum_{j=1}^{t} d(\delta_R(K_j)).$$

So $c(\delta_E(K_j)) < d(\delta_R(K_j))$ for at least one j. As $|\delta_E(K_j)| < |\delta_E(U)|$ (by the connectivity of G), this contradicts the minimality of $|\delta_E(U)|$. ∎

Notes. Călinescu, Fernandes, and Reed [1998] gave a polynomial-time approximation algorithm for finding a minimum multicut in an unweighted graph of bounded degree and bounded 'tree-width'. More on the minimum multicut problem can be found in Klein, Agrawal, Ravi, and Rao [1990], Garg, Vazirani, and Yannakakis [1993a,1993b,1996,1997], Tardos and Vazirani [1993], Klein, Rao, Agrawal, and Ravi [1995], and Naor and Zosin [1997,2001].

70.9. Relations between fractional, half-integer, and integer solutions

There are the following implications for the multiflow problem:

(70.25) $\exists$ integer multiflow $\Longrightarrow$ $\exists$ half-integer multiflow $\Longrightarrow$ $\exists$ fractional multiflow.

As the existence of a fractional multiflow can be tested in strongly polynomial time, it yields a useful necessary condition for the existence of an integer multiflow.

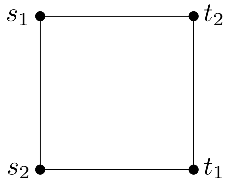


Figure 70.3
There is a half-integer, but no integer multiflow (where all capacities and demands are 1).

As has been discussed in Chapter 10, for 1-commodity flow problems with integer capacities, we can turn all implications around in (70.25). For general multiflow problems, however, this is not the case. For undirected graphs, Figure 70.3 shows that a half-integer multiflow does not imply the existence of an integer multiflow (for integer capacities and demands). Middendorf and Pfeiffer [1993] showed that the half-integer multiflow problem in undirected graphs is NP-complete, even if all capacities and demands are equal to 1.

For undirected 2-commodity flows, Hu [1963] showed that the existence of a fractional multiflow implies the existence of a half-integer multiflow, if all capacities and demands are integer. Figure 70.3 shows that an integer multiflow need not exist. In fact, the undirected integer 2-commodity flow problem is NP-complete (Even, Itai, and Shamir [1975,1976]).

Hu's theorem prompted Jewell [1967] to conjecture that if a k-commodity flow problem with integer capacities and demands has a fractional solution, then it has a $1/p$-integer solution for some $p \leq k$. More strongly, Seymour [1981d] conjectured that a fractional multiflow implies the existence of a half-integer multiflow (for integer capacities and demands).

This was disproved by a series of examples of Lomonosov [1985], which even imply that there is no integer p such that each undirected 3-commodity flow problem has a $1/p$-integer solution when it has a fractional solution (for integer capacities and demands). A simplified version of Lomonosov's example is given in Figure 70.4. It consists of an integer-capacitated 3-commodity flow problem with demands 1, $2k$, and $2k$, such that each feasible multiflow has $\frac{1}{2k}$ among its values.

A simpler counterexample to Seymour's conjecture was given by Pfeiffer [1990] — see Figure 70.5, showing that a quarter-integer multiflow need not imply the existence of a half-integer multiflow (for integer capacities and demands).

With construction (70.9) we obtain similar results for directed graphs.

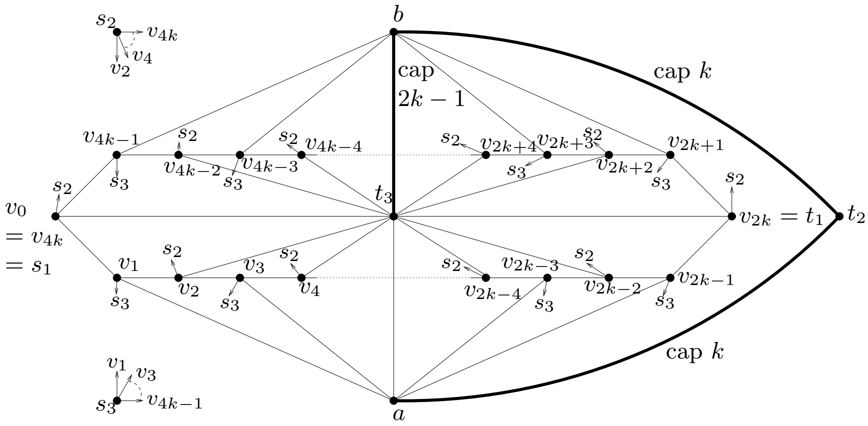


Figure 70.4

A feasible integer-capacitated 3-commodity flow problem with demands 1, $2k$, and $2k$, such that each feasible multiflow has $\frac{1}{2k}$ among its values. The nets are the pairs s_1t_1, s_2t_2, and s_3t_3. The graph consists of a circuit C of length $4k$, with vertices $v_1, \ldots, v_{4k}$ (in order), vertices s_2 and t_3 adjacent to each v_i with i even, a vertex s_3 adjacent to each v_i with i odd, a vertex a adjacent to t_3 and to each v_i with i odd and $0 < i < 2k$, a vertex b adjacent to t_3 and to each v_i with i odd and $2k < i < 4k$, and a vertex t_2 adjacent to a and b. Set $s_1 := v_0 := v_{4k}$ and $t_1 := v_{2k}$. Let P and Q be the paths $v_0, v_1, \ldots, v_{2k}$ and $v_{4k}, v_{4k-1}, \ldots, v_{2k}$, respectively.

Edges t_2a and t_2b have capacity k, and edge bt_3 capacity $2k-1$. All other edges have capacity 1. Let $d(s_1t_1) := 1$ and $d(s_2t_2) := d(s_3t_3) := 2k$.

To see that there exists a feasible multiflow, reset (temporarily) the capacities of at_3 and bt_3 to 0 and $2k$ respectively. Then a feasible multiflow (f_1, f_2, f_3) is given as follows. Flow f_1 consists of the incidence vector of path Q. Flow f_2 takes value 1 on the edges t_3v_i for $i = 2k+2, 2k+4, \ldots, 4k-2$, on av_i for $i = 1, 3, \ldots, 2k-1$, and on s_2v_i for $i = 2, 4, \ldots, 4k$, value $\frac{1}{2}$ on the edges of P, and on t_3v_0 and t_3v_{2k}, value k on t_3b, t_2a, and t_2b, and value 0 on all other edges. Flow f_3 takes value 1 on t_3v_i for $i = 2, 4, \ldots, 2k-2$, on s_3v_i for $i = 1, 3, \ldots, 4k-1$, and on bv_i for $i = 2k+1, 2k+3, \ldots, 4k-1$, value $\frac{1}{2}$ on the edges of P, and on t_3v_0 and t_3v_{2k}, and value k on bt_3. By symmetry, also after resetting the capacities of at_3 and bt_3 to $2k$ and 0 respectively, there exists a feasible solution. Hence also the original capacity function (which is a convex combination of the modified capacity functions) has a feasible solution.

To see that any feasible multiflow contains a value $\frac{1}{2k}$, note that the $s_1 - t_1$ flow can only use edges on the circuit C: each edge leaving C is in a tight cut (= a cut having equality in the cut condition) not separating s_1 and t_1 (consider the cuts determined by $\{s_2\}$, $\{s_3\}$, and $\{t_2, t_3, a, b\}$). So in any feasible multiflow, the $s_1 - t_1$ flow f_1 is a convex combination of the incidence vectors of P and Q. Consider now the cut determined by $U := \{t_2, s_3, a, v_1, v_3, \ldots, v_{2k-1}\}$. It has capacity $4k+1$ and demand $4k$, it does not split s_1 and t_1, and contains all edges of P. Hence the capacity left for f_1 is at most 1. As P has length $2k$, it implies that f_1 can send a flow of value at most $\frac{1}{2k}$ along P. Similarly, the cut determined by

$U := \{t_2, s_3, b, v_{2k+1}, v_{2k+3}, \ldots, v_{4k-1}\}$ has capacity $6k - 1$ and demand $4k$, it does not split s_1 and t_1, and contains all edges of Q. Hence the capacity left for f_1 is at most $2k - 1$. As Q has length $2k$, it implies that f_1 can send a flow of value at most $1 - \frac{1}{2k}$ along Q. Concluding, f_1 sends $\frac{1}{2k}$ flow along P and $1 - \frac{1}{2k}$ flow along Q.

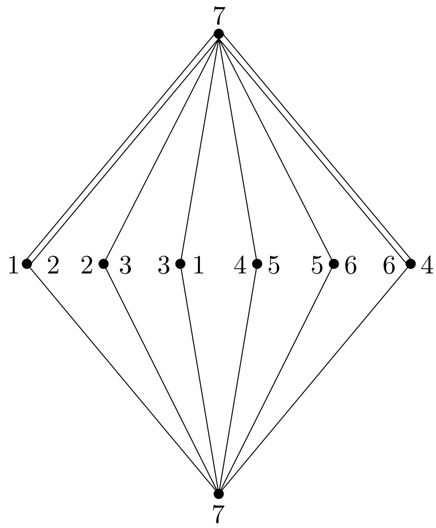


Figure 70.5

There is a quarter-integer, but no half-integer multiflow. The 7 nets are indicated by indices $1, \ldots, 7$ at the terminals. All capacities and demands are equal to 1.

In fact, there is a unique fractional multiflow. Since the distance between the terminals in any net is 2 and since there are 14 edges, any multiflow uses the capacity of each edge fully, and each of the flows is a convex combination of incidence vectors of paths of length 2. Also, the edges incident with any vertex v of degree 2 can only be used by the nets that have a terminal at v.

Let β be the fraction of flow for net 7 that traverses the leftmost vertex. For $i = 1, \ldots, 6$, let α_i be the fraction of flow for net i that traverses the topmost vertex. Then $\alpha_2 + \alpha_3 = 1$ and $\alpha_1 + \alpha_3 = 1$, and hence $\alpha_1 = \alpha_2$. So $\beta + \alpha_1 + \alpha_2 = 2$ and $\beta + (1 - \alpha_1) + (1 - \alpha_2) = 1$, hence $\alpha_1 + \alpha_2 = 1 + \beta = 2 - \beta$. So $\beta = \frac{1}{2}$ and $\alpha_1 = \alpha_2 = \frac{3}{4}$.

70.10. The Euler condition

In some cases adding the following *Euler condition* turns out to be of help:

(70.26) $\quad c(\delta_E(v)) + d(\delta_R(v))$ is even, for each vertex v.

In case all capacities and demands are equal to 1, that is, for the edge-disjoint paths problem, the Euler condition is equivalent to

(70.27) the graph $G + H = (V, E \cup R)$ is Eulerian

(taking multiplicities into account).

If $k = 2$ and the capacities and demands are integer and satisfy the Euler condition, then the cut condition implies the existence of an integer multiflow. This result, also due to Rothschild and Whinston [1966a], implies Hu's 2-commodity flow theorem, as mentioned in Section 70.8 (by multiplying all capacities and demands by 2, so as to achieve the Euler condition).

We will see several other cases where the existence of a half-integer multiflow, together with the Euler condition, implies the existence of an integer multiflow. But it is not sufficient in general, as otherwise a quarter-integer multiflow would always imply the existence of a half-integer multiflow (by multiplying all capacities and demands by 2), and to this we saw the counterexample of Pfeiffer [1990] in Figure 70.5[11]. The NP-completeness of the half-integer multiflow problem, with all capacities and demands equal to 1 (Middendorf and Pfeiffer [1993]), implies that the edge-disjoint paths problem is NP-complete even if the Euler condition holds.

Fractional and integer multiflows for digraphs. As for the *directed* case, Figure 70.5 implies with construction (70.9) that a quarter-integer multiflow does not imply the existence of a half-integer multiflow. The graph in Figure 70.6 (Hurkens, Schrijver, and Tardos [1988]) shows that a half-integer multiflow does not imply the existence of an integer multiflow, even if the directed analogue of the Euler condition holds (the graph obtained from the supply digraph and the reverse of the demand digraph is an Eulerian digraph). Note that in Figure 70.6 the union $D + H$ of D and H is planar.

70.11. Survey of cases where a good characterization has been found

Let $G = (V, E)$ be an undirected graph and let $R = \{s_1t_1, \ldots, s_kt_k\}$ be a family of nets. Let $c : E \to \mathbb{R}_+$ be a capacity function and let $d_1, \ldots, d_k$ be demands (so $d(s_it_i) := d_i$).

In the following cases of the undirected multiflow problem, the cut condition has been proved to imply the existence of a fractional multiflow; if moreover the capacities and demands are integer, there is a half-integer multiflow; if moreover the Euler condition holds, there is an integer solution[12]:

(70.28) (i) if there exist two vertices u, v such that each s_it_i intersects uv (Hu [1963], E.A. Dinits — see Corollary 71.1b),

[11] A more complicated (planar) example satisfying the Euler condition and where a half-integer but no integer multiflow exists, was given by Hurkens, Schrijver, and Tardos [1988]. Earlier, a nonplanar example with these properties was given by P.D. Seymour (personal communication).

[12] For graphs $G = (V, E)$ and $H = (T, R)$, $G + H$ is the graph $(V \cup T, E \cup R)$, where $E \cup R$ is the disjoint union (as families).

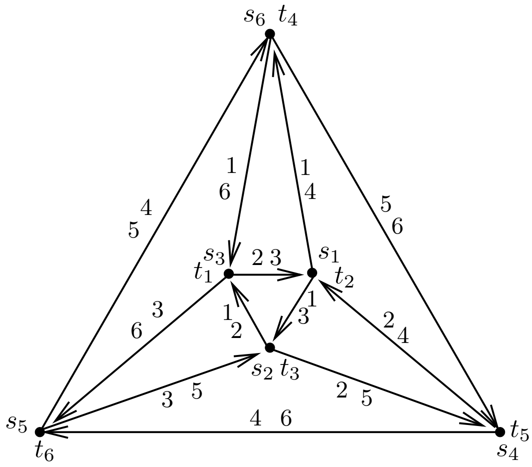


Figure 70.6

A directed example where the Euler condition holds, with $D + H$ is planar, and where a half-integer, but no integer multiflow exists. All capacities and demands are 1. The half-integer multiflow is indicated by the indices of the nets: index i at arc a means $f_i(a) = \frac{1}{2}$.

(ii) if $|\{s_1, t_1, \ldots, s_k, t_k\}| \leq 4$ or $s_1t_1, \ldots, s_kt_k$ form a five-circuit (Papernov [1976], Lomonosov [1976,1985], Seymour [1980c] — see Section 72.1),

(iii) if $G + H$ has no K_5 minor, in particular if $G + H$ is planar (Seymour [1981a] — see Sections 74.2 and 75.6),

(iv) if G is planar and there exist two faces F_1 and F_2 such that for each $i = 1, \ldots, k$: $s_i, t_i \in \mathrm{bd}(F_1)$ or $s_i, t_i \in \mathrm{bd}(F_2)$ (Okamura and Seymour [1981], Okamura [1983] — see Theorems 74.1 and 74.4),

(v) if G is planar and has a vertex r on the outer boundary such that for each i either both s_i and t_i are on the outer boundary, or $r \in \{s_i, t_i\}$ (Okamura [1983] — Theorem 74.5).

(vi) if G is planar and has two bounded faces F_1 and F_2 such that $s_1, \ldots, s_k$ occur clockwise around $\mathrm{bd}(F_1)$ and $t_1, \ldots, t_k$ occur clockwise around $\mathrm{bd}(F_2)$ (Schrijver [1989b] — cf. Section 74.3b).

Here $\mathrm{bd}(F)$ denotes the boundary of F.

In particular, in each of these cases, if the capacities and demands are integer and satisfy the Euler condition, the existence of a fractional multiflow implies the existence of an integer multiflow. Next to the cases listed in (70.28), this property has been proved in the following cases (extending (70.28)(ii) and (iv)):

(70.29) (i) if $|\{s_1, t_1, \ldots, s_k, t_k\}| \leq 5$ (Karzanov [1987a] — see Section 72.2a),
(ii) if G is planar and there exist faces F_1, F_2, F_3 such that for each $i = 1, \ldots, k$ there is a $j \in \{1, 2, 3\}$ such that $s_i, t_i \in \mathrm{bd}(F_j)$ (Karzanov [1994b] — see Section 74.3c).

This in particular implies that if the capacities and demands are integer and there exists a fractional multiflow, then there exists a half-integer multiflow.

In the following case (extending (70.29)(ii)), it has been proved that if c and d are integer and a fractional multiflow exists, then a quarter-integer multiflow exists; if moreover the Euler condition holds, then a half-integer solution exists:

(70.30) if G is planar and there are four faces such that each net is spanned by one of these faces (Karzanov [1995] — see Section 74.3c).

(We say that a pair of vertices is *spanned* by a face F if it is spanned by the boundary of F.)

In Section 73.1c we shall see that an integer multiflow can be found in polynomial time also if the nets form a triangle (no Euler condition is required).

70.12. Relation between the cut condition and fractional cut packing

As was noted by Karzanov [1984] and Seymour [1979b], if the cut condition is sufficient for the existence of a fractional multiflow, one can derive an interesting polarity relation between multiflows and fractional packing of cuts.

Let $G = (V, E)$ and $H = (V, R)$ be graphs. Consider the cone K in $\mathbb{R}^R \times \mathbb{R}^E$ generated by the vectors[13]

(70.31) $(\chi^r; \chi^{EP})$ for $r \in R$ and r-path P in G,
$(\mathbf{0}; \chi^e)$ for $e \in E$.

Here EP denotes the set of edges of P. For any $r = st \in R$, an *r-path* is a path connecting s and t. χ^r and χ^e denote the rth and eth unit base vectors in $\mathbb{R}^R$ and $\mathbb{R}^E$, respectively.

For any $c : E \to \mathbb{R}_+$ and $d : R \to \mathbb{R}_+$, the existence of a feasible multiflow subject to c and of value d is equivalent to the fact that $(d; c)$ belongs to K. So we have that the property:

[13] We write $(x; y)$ for $\begin{pmatrix} x \\ y \end{pmatrix}$.

(70.32) for each $c : E \to \mathbb{R}_+$ and $d : R \to \mathbb{R}_+$, the cut condition implies the existence of a feasible multiflow,

is equivalent to the fact that K consists of all vectors $(d; c) \in \mathbb{R}^R \times \mathbb{R}^E$ satisfying:

(70.33)
$$\begin{array}{ll} d(\delta_R(U)) \leq c(\delta_E(U)) & \text{for } U \subseteq V, \\ d_r \geq 0 & \text{for } r \in R, \\ c_e \geq 0 & \text{for } e \in E. \end{array}$$

Let K^* be the polar cone of K (cf. Section 5.7). Then (70.32) is equivalent to $-K^*$ being generated by the vectors:

(70.34)
$$\begin{array}{ll} (-\chi^{\delta_R(U)}; \chi^{\delta_E(U)}) & \text{for } U \subseteq V, \\ (\chi^r; \mathbf{0}) & \text{for } r \in R, \\ (\mathbf{0}; \chi^e) & \text{for } e \in E. \end{array}$$

Also, by definition of K, $-K^*$ consists of all vectors $(m; l) \in \mathbb{R}^R \times \mathbb{R}^E$ satisfying:

(70.35)
$$\begin{array}{ll} m_r + l(EP) \geq 0 & \text{for } r \in R \text{ and } r\text{-path } P \text{ in } G, \\ l_e \geq 0 & \text{for } e \in E. \end{array}$$

This implies the following theorem relating the cut condition to distances and fractional packings of cuts:

Theorem 70.5. *Let $G = (V, E)$ and $H = (V, R)$ be supply and demand graphs. Then for each $c : E \to \mathbb{R}_+$ and $d : R \to \mathbb{R}_+$, the cut condition implies the existence of a feasible fractional multiflow if and only if for each length function $l : E \to \mathbb{R}_+$ there exist $\lambda_U \geq 0$ for $U \subseteq V$ such that*

(70.36)
$$\sum_U \lambda_U \chi^{\delta_E(U)} \leq l$$

and

(70.37)
$$\text{dist}_l(s, t) = \sum_U \lambda_U \chi^{\delta_R(U)}(r),$$

for each $r = st \in R$. Here $\text{dist}_l(s, t)$ denotes the minimum length of an $s - t$ path in G, with respect to l.

Proof. As we saw above, (70.32) is equivalent to the fact that $-K^*$ is generated by the vectors (70.34). It is equivalent to: each $(m; l) \in \mathbb{R}^R \times \mathbb{R}^E$ satisfying (70.35) is a nonnegative combination of vectors (70.34). Since $(\chi^r; \mathbf{0})$ is one of the vectors (70.34), we can restrict the $(m; l)$ to those for which m_r is smallest so as to satisfy (70.35). That is, we can assume that $m_r = -\text{dist}_l(s, t)$ where $r = st$. Hence (70.32) is equivalent to: for each $l : E \to \mathbb{R}_+$, the vector $(-\text{dist}_l; l)$ is a nonnegative combination of vectors (70.34). This is equivalent to the condition stated in the theorem. ∎

This is based on interpreting *feasibility* of multiflows in terms of *cones*. We next consider an interpretation of the *maximization* of multiflows in terms of *polyhedra*.

Let $\mathcal{P}$ be the collection of r-paths for all $r \in R$. Let $\mathcal{B}$ be the collection of subsets of E that intersect each path in $\mathcal{P}$.

Consider the inequality system:

$$(70.38) \qquad \begin{array}{ll} x_e \geq 0 & \text{for } e \in E, \\ x(EP) \geq 1 & \text{for } P \in \mathcal{P}. \end{array}$$

Then by the theory of blocking polyhedra:

Theorem 70.6. *The up hull of the incidence vectors of the sets in $\mathcal{B}$ is determined by* (70.38) *if and only if the up hull of the incidence vectors of paths in $\mathcal{P}$ is determined by*

$$(70.39) \qquad \begin{array}{ll} x_e \geq 0 & \text{for } e \in E, \\ x(B) \geq 1 & \text{for } B \in \mathcal{B}. \end{array}$$

Proof. Directly from the theory of blocking polyhedra. ∎

In terms of flows this is equivalent to:

Corollary 70.6a. *Let $G = (V, E)$ and $H = (V, R)$ be supply and demand graphs. For each $c : E \to \mathbb{R}_+$, the maximum total value of a multiflow subject to c is equal to the minimum capacity of a set in $\mathcal{B}$ if and only if for each length function $l : E \to \mathbb{R}_+$ satisfying* $\text{dist}_l(s,t) \geq 1$ *for each $r = st \in R$, there exist $\lambda_B \geq 0$ for $B \in \mathcal{B}$ such that*

$$(70.40) \qquad \sum_{B \in \mathcal{B}} \lambda_B = 1 \textit{ and } \sum_{B \in \mathcal{B}} \lambda_B \chi^B \leq l.$$

Proof. The first statement is equivalent to the fact that the up hull of the incidence vectors of sets in $\mathcal{B}$ is determined by (70.38). The second statement is equivalent to the fact that the up hull of the incidence vectors of paths in $\mathcal{P}$ is determined by (70.39). The equivalence is stated by Theorem 70.6. ∎

70.12a. Sufficiency of the cut condition sometimes implies an integer multiflow

As was also noted by Karzanov [1984,1987a] and Seymour [1979b], in certain collections of graphs+nets, if the cut condition implies the existence of a fractional multiflow, we can derive integrality of solutions. This can be made explicit as follows.

Consider an Eulerian graph $G = (V, E)$, and let e and f be distinct edges incident with a vertex v of degree ≥ 4. We describe the operation of separating e and f at v: introduce a new vertex v', rejoin half of the edges incident with v to

v', such that e remains incident with v and f becomes incident with v', and add $\frac{1}{2}\deg_E(v) - 2$ parallel edges connecting v and v'.

Call any graph G' arising in this way a *splitting* of G *separating* e *and* f *at* v. Note that, if $G' = (V', E')$ denotes the new graph, then $\deg_{E'}(v) = \deg_{E'}(v') = \deg_E(v) - 2$.

Let $\mathcal{I}$ be a collection of pairs (G, R) of an Eulerian graph $G = (V, E)$ and a subset R of E, with the following property:

(70.41) for each $(G, R) \in \mathcal{I}$, for each vertex v of G of degree at least 4 with $\deg_{E\setminus R}(v) > \deg_R(v)$, and for each two edges e and f of G incident with v, not both in R, $\mathcal{I}$ contains a pair (G', R') where G' is a splitting of G separating e and f at v, and where R' is the set of edges arising from R by this splitting.

As examples we can take for $\mathcal{I}$ the set of all pairs (G, R) consisting of an Eulerian graph $G = (V, E)$ and $R \subseteq E$ such that one of the following holds (the first four examples follow from the fact that for each fixed graph $H = (T, R)$, the class of pairs (G, R) with $G = (V, E)$ Eulerian and $R \subseteq E$ satisfies (70.41)):

(70.42)
(i) R consists of two parallel classes of edges;
(ii) there are two vertices intersecting all edges in R;
(iii) R covers at most four vertices;
(iv) the edges in R form a pentagon, with parallel edges added;
(v) (V, E) is planar;
(vi) $(V, E \setminus R)$ is planar, such that all vertices covered by R are on the outer boundary of $(V, E \setminus R)$;
(vii) $(V, E \setminus R)$ is planar, such that it has two faces with the property that each edge in R is spanned by one of these faces.

As we shall see in later chapters, in each of these cases the premise, and hence the conclusion, of the following theorem hold. The theorem applies to the multiflow problem with supply graph $(V, E \setminus R)$ and demand graph (V, R), with all capacities and demands equal to 1 (so to the edge-disjoint paths problem):

Theorem 70.7. *Let $\mathcal{I}$ satisfy* (70.41) *and have the property that for each $(G, R) \in \mathcal{I}$, the cut condition implies the existence of a fractional multiflow. Then, for each $(G, R) \in \mathcal{I}$, the cut condition implies the existence of an integer multiflow.*

Proof. Consider a counterexample $(G, R) \in \mathcal{I}$ with

(70.43) $$\sum_{v \in V} 2^{\deg_G(v)}$$

minimal. So the cut condition holds, and hence there is a fractional multiflow. It implies that there is a collection $\mathcal{C}$ of circuits in G, each intersecting R in exactly one edge, and, for each $C \in \mathcal{C}$, there is a $\lambda_C > 0$ such that for each edge e:

(70.44) $$\sum_{C \in \mathcal{C}} \lambda_C \chi^C(e) \leq 1,$$

with equality if $e \in R$. Here we consider circuits as edge sets.

Note that:

(70.45) for each $C \in \mathcal{C}$ and each $U \subseteq V$ with $|\delta_R(U)| = |\delta_{E\setminus R}(U)|$, if U splits at least one edge of C, then U splits the edge in $C \cap R$ and exactly one edge in $C \setminus R$.

(Here U *splits* e if $e \in \delta(U)$.) This follows from the fact that if U splits an edge in C, then it splits at least one edge in $C \setminus R$, and hence

(70.46) $$|\delta_R(U)| = \sum_C \lambda_C \chi^C(\delta_R(U)) \leq \sum_C \lambda_C \chi^C(\delta_{E\setminus R}(U)) \leq |\delta_{E\setminus R}(U)|.$$

Here we use $|C \cap \delta_R(U)| \leq |C \cap \delta_{E\setminus R}(U)|$, since $|C \cap \delta_R(U)| \leq 1$ and $|C \cap \delta_E(U)|$ is even. Equality throughout in (70.46) implies (70.45).

Now it suffices to show that

(70.47) for any two $C, D \in \mathcal{C}$, if $(C \setminus R) \cap (D \setminus R) \neq \emptyset$, then $C \setminus R = D \setminus R$.

That this is sufficient follows from the following. (70.47) implies that for each parallel class in R consisting of (say) μ edges connecting s and t, there are at least μ different $s - t$ paths among the $C \setminus R$ for $C \in \mathcal{C}$. Since they are edge-disjoint (by (70.47)), there exists an obvious integer solution.

To prove (70.47), suppose to the contrary that $C \setminus R$ and $D \setminus R$ have an edge in common and that $C \setminus R \neq D \setminus R$. Then (possibly after exchanging C and D), there is a vertex v on the paths made by $C \setminus R$ and $D \setminus R$ such that $C \setminus R$ and $D \setminus R$ have an edge f incident with v in common and such that $C \setminus R$ contains another edge, e say, incident with v with $e \notin D$. Let g be the edge in D incident with v and satisfying $g \neq f$. So $g \neq e$. Possibly $g \in R$.

So v has degree at least 4. Moreover, $\deg_{E\setminus R}(v) > \deg_R(v)$, by (70.45), since C contains two edges in $E \setminus R$ incident with v. Let $G' = (V', E')$ be a splitting of e and g at v with $(G', R') \in \mathcal{I}$, where R' is the set of edges arising from R by this splitting. By symmetry, we can assume that, in G', edge f is incident with v'. (We leave open which of e and g is incident with v'.) Then (G', R') has no integer multiflow, as it would give an integer multiflow in (G, R) (by contracting the new edges). Hence, as for G' the sum (70.43) is reduced, the cut condition is violated for (G', R'). Let $U \subseteq V'$ violate the cut condition. That is, $|\delta_{R'}(U)| > |\delta_{E'\setminus R'}(U)|$. Then, as G' is Eulerian, $|\delta_{R'}(U)| \geq |\delta_{E'\setminus R'}(U)| + 2$. Also, U separates v and v', since otherwise it would give a cut violating the cut condition for G, R. So we can assume that $v \in U$ and $v' \notin U$. Hence $U \subseteq V$.

Let G' have γ parallel edges connecting v and v'. So $\deg_{E'}(v') = 2\gamma + 2$. Let $\alpha := \deg_{R'}(v')$. Then:

(70.48) $$\begin{aligned}&|\delta_R(U)| \leq |\delta_{E\setminus R}(U)| = |\delta_{E'\setminus R'}(U \cup \{v'\})|\\ &\leq |\delta_{E'\setminus R'}(U)| + \deg_{E'\setminus R'}(v') - 2\gamma\\ &= |\delta_{E'\setminus R'}(U)| + \deg_{E'}(v') - 2\gamma - \deg_{R'}(v') = |\delta_{E'\setminus R'}(U)| + 2 - \alpha\\ &\leq |\delta_{R'}(U)| - \alpha \leq |\delta_R(U)|.\end{aligned}$$

Hence we have equality throughout. In particular, $|\delta_R(U)| = |\delta_{E\setminus R}(U)|$ (as the first inequality is an equality), U splits all edges in $E \setminus R$ that become incident in G' with v' (as the second inequality becomes equality), and U splits no edge in R that becomes incident in G' with v' (as the last inequality becomes equality). In particular, U splits f.

Now one of e and g is (in G') incident with v'. If e is incident with v', then U splits e, and we have a contradiction with (70.45) for circuit C. If g is incident with v', then if $g \in E \setminus R$, U splits g, and if $g \in R$, U does not split g; in both cases we have a contradiction with (70.45) for circuit D. ∎

70.12b. The cut condition and integer multiflows in directed graphs

Nagamochi and Ibaraki [1989] showed that for directed graphs, if the cut condition implies the existence of a fractional multiflow, and if this holds in a certain hereditary way, then it implies the existence of an integer multiflow:

Theorem 70.8. *Let $D = (V, A)$ and $H = (V, R)$ be a supply and demand digraph, respectively. Suppose that for each $c : A \to \mathbb{Z}_+$ and $d : R \to \mathbb{Z}_+$, the cut condition implies the existence of a fractional multiflow. Then it implies the existence of an integer multiflow.*

Proof. Let c and d be such that the cut condition holds, but no integer multiflow exists. Choose such c, d with $c(A) + d(R)$ minimal. By assumption, there exists a fractional multiflow $(f_r : A \to \mathbb{R}_+ \mid r \in R)$. Then for each $a \in A$ we have

(70.49) $$c(a) = \sum_{r \in R} f_r(a),$$

for otherwise we have, for any $U \subseteq V$ with $a \in \delta^{\text{out}}_A(U)$:

(70.50) $$c(\delta^{\text{out}}_A(U)) = \sum_{a \in \delta^{\text{out}}_A(U)} c(a) > \sum_{a \in \delta^{\text{out}}_A(U)} \sum_{r \in R} f_r(a) \geq \sum_{r \in \delta^{\text{out}}_R(U)} d(r) = d(\delta^{\text{out}}_R(U)).$$

Hence (by integrality of c and d), we can replace $c(a)$ by $c(a) - 1$ without violating the cut condition, and obtain a smaller counterexample — a contradiction.

This proves (70.49). It implies that the directed analogue of the Euler condition holds, since for any vertex v:

(70.51) $$c(\delta^{\text{out}}_A(v)) - c(\delta^{\text{in}}_A(v)) = \sum_{r \in R} (f_r(\delta^{\text{out}}_A(v)) - f_r(\delta^{\text{in}}_A(v))) = d(\delta^{\text{out}}_R(v)) - d(\delta^{\text{in}}_R(v)).$$

The latter equality holds as each f_r is a flow.

Now consider any $r' \in R$ with $d(r') \geq 1$, say $r' = (s, t)$. Replacing $d(r')$ by $d(r') - 1$, the cut condition is maintained. Hence (by the minimality of c, d) there is an integer multiflow $(f'_r \mid r \in R)$ satisfying the new demands. Consider the capacity function

(70.52) $$c' := c - \sum_{r \in R} f'_r.$$

By (70.51), there is at least one $s - t$ path traversing only arcs a with $c'(a) \geq 1$. Hence we can increase $f'_{r'}$ along this path by 1, to obtain an integer multiflow satisfying the original demands. ∎

70.13. Further results and notes

70.13a. Fixing the number of commodities in undirected graphs

Robertson and Seymour [1995] showed that for each fixed k, the k vertex-disjoint paths problem in undirected graphs is polynomial-time solvable.[14] Describing the algorithm would require more space than fits within the limits of this book. The methods are quite different in nature from the more polyhedral methods discussed here, and are based on the deep graph minors techniques developed by Robertson and Seymour. For an outline of the disjoint paths algorithm, see Robertson and Seymour [1990].

The running time of Robertson and Seymour's algorithm is bounded by $O(n^3)$ (where the constant depends (heavily) on k). It implies a polynomial-time algorithm for the k edge-disjoint paths problem for fixed k, by considering the line graph.

More generally, Robertson and Seymour gave for each fixed k an $O(n^3)$-time algorithm for the *vertex-disjoint trees problem*:

(70.53) given: a graph $G = (V, E)$ and subsets $W_1, \ldots, W_p$ of V,
find: vertex-disjoint subtrees $T_1, \ldots, T_p$ in G such that T_i spans W_i, for $i = 1, \ldots, p$,

taking $k := |W_1 \cup \cdots \cup W_p|$.

For *planar* graphs, Reed, Robertson, Schrijver, and Seymour [1993] gave a linear-time algorithm for the disjoint trees problem, fixing $|W_1 \cup \cdots \cup W_p|$. Moreover, Schrijver [1991c] showed that for each fixed q, there is a polynomial-time algorithm for the disjoint trees problem in planar graphs such that $W_1 \cup \cdots \cup W_p$ can be covered by the boundary of at most q faces. The method is based on enumerating homotopy classes (see Section 76.7a), and here the degree of the polynomial depends on q.

Sebő [1993c] showed that for each fixed k, if $G + H$ is planar and $|VH| \leq k$, then the integer multiflow problem is polynomial-time solvable. (The demands and capacities can be arbitrarily large, so there is no reduction to the edge-disjoint paths problem for a fixed number of paths.) Sebő showed this by proving a more general result on the complexity of packing T-cuts for fixed $|T|$.

Related is the following. For any k, let $f(k)$ be the smallest number such that in any $f(k)$-connected graph, any instance of the k vertex-disjoint paths problem has a solution. Jung [1970] showed that $f(k) \leq 2^{3k}$ (a larger bound was shown by Larman and Mani [1970]). Thomassen [1980] proved $f(2) = 6$ and conjectures that $f(k) = 2k + 2$ for $k \geq 2$.

For any k, let $g(k)$ be the smallest number such that in each $g(k)$-edge-connected graph, any instance of the k edge-disjoint paths problem has a solution. This value is finite — in fact, $g(k) \leq 2k$, since a $2k$-edge-connected graph has k edge-disjoint spanning trees (by the Tutte-Nash-Williams disjoint trees theorem (Corollary 51.1a)). These trees contain k edge-disjoint paths as required.

Trivially $g(k) \geq k$. Moreover, for even k one has $g(k) \geq k + 1$, as is shown by replacing each edge of the circuit C_{2k} by $\frac{1}{2}k$ parallel edges, taking as nets the

[14] The correctness of the algorithm depends on a lemma proved in the preprint Robertson and Seymour [1992], which did not appear yet.

k pairs of opposite vertices. Cypher [1980] and Thomassen [1980] conjecture that $g(k) = k$ if k is odd and $g(k) = k + 1$ if k is even.

It is known that $g(2) = 3$ (as follows from a result of Dinits and Karzanov [1979] and Seymour [1980b] (see Section 71.4a)), $g(3) = 3$ (Okamura [1984a]), $g(4) = 5$ (Mader [1985], Hirata, Kubota, and Saito [1984], H. Enomoto and A. Saito (cf. Hirata, Kubota, and Saito [1984])), $g(k) \leq k + 1$ if k is odd, and $g(k) \leq k + 2$ if k is even (Huck [1991]).

Earlier, partial results were obtained by Cypher [1980] showing that $g(k) \leq k+2$ for $k \leq 5$, Hirata, Kubota, and Saito [1984] showing that $g(k) \leq 2k - 3$ if $k \geq 4$, and Okamura [1987,1988,1990]. Related results can be found in Enomoto and Saito [1984] and Huck [1992]. See also the notes in Section 72.2b.

The corresponding result for *directed* graphs has been shown for any k — see Section 70.13b.

70.13b. Fixing the number of commodities in directed graphs

For directed graphs, Fortune, Hopcroft, and Wyllie [1980] showed that deciding if two given vertices of a digraph belong to a directed circuit, is NP-complete. It implies that the arc-disjoint paths problem is NP-complete for $k = 2$ commodities, even if the nets are 'opposite' (that is, $s_2 = t_1$ and $t_2 = s_1$). It also implies that the directed vertex-disjoint paths problem is NP-complete (as the arc-disjoint problem can be reduced to vertex-disjoint by considering the line digraph).

Shiloach [1979a] observed that Edmonds' disjoint arborescences theorem implies that in any k-arc-connected digraph the k arc-disjoint problem always has a solution. (This can be shown by adding a new vertex r and new arcs (r, s_i) for each beginning terminal s_i. As the original digraph is k-arc-connected, by Edmonds' disjoint arborescences theorem (Corollary 53.1b) the new digraph has k arc-disjoint r-arborescences. They contain paths as required.)

If we restrict ourselves to *planar* digraphs, then for each fixed k, the k vertex-disjoint paths problem is polynomial-time solvable (Schrijver [1994a]). The method is based again on enumerating homotopy types of paths. (The polynomial-time solvability for $k = 2$ opposite nets (requiring only *internally* vertex-disjoint paths), was shown by Seymour [1991].)

It can be extended to the polynomial-time solvability, for any fixed q, of the problem of finding vertex-disjoint rooted subarborescences in a planar graph, with prescribed roots and terminals to be covered, provided that these roots and terminals can be covered by the boundaries of at most q faces.

An open problem is the complexity of the k *arc*-disjoint paths problem in directed planar graphs, for any fixed $k \geq 2$. This is even unknown for $k = 2$, also if we restrict ourselves to two opposite nets.

For *acyclic* digraphs, the k vertex-disjoint paths problem is polynomial-time solvable for each fixed k. This was shown by Fortune, Hopcroft, and Wyllie [1980] (extending an earlier result for $k = 2$ of Perl and Shiloach [1978]) — see Section 70.13c. By considering line digraphs, it implies the polynomial-time solvability of the k arc-disjoint paths problem in acyclic digraphs for each fixed k.

70.13c. Disjoint paths in acyclic digraphs

Fortune, Hopcroft, and Wyllie [1980] showed that the vertex-disjoint paths problem is NP-complete for digraphs, even when fixing the number of paths to $k = 2$. Moreover, Even, Itai, and Shamir [1975,1976] showed that the arc-disjoint paths problem in acyclic digraphs is NP-complete, even if the nets form two parallel classes. By taking the line digraph, it implies that the vertex-disjoint paths problem is NP-complete for acyclic digraphs. Vygen [1995] showed that the arc-disjoint paths problem in acyclic digraphs remains NP-complete, even if the nets form three parallel classes and the Euler condition holds; and also if the digraphs are restricted to acyclic and planar.

On the other hand, Fortune, Hopcroft, and Wyllie [1980] proved that for each fixed k, the k vertex-disjoint paths problem in acyclic digraphs can be solved in polynomial time. (This was proved for $k = 2$ by Perl and Shiloach [1978].)

Theorem 70.9. *For each fixed k, there exists a polynomial-time algorithm for the k vertex-disjoint paths problem for acyclic digraphs.*

Proof. Let $D = (V, A)$ be an acyclic digraph and let $(s_1, t_1), \ldots, (s_k, t_k)$ be pairs of vertices of D (the nets), all distinct. To solve the disjoint paths problem we may assume that each s_i is a source of D and each t_i is a sink of D.

Make an auxiliary digraph $D' = (V', A')$ as follows. The vertex set V' consists of all k-tuples $(v_1, \ldots, v_k)$ of distinct vertices of D. In D' there is an arc from $(v_1, \ldots, v_k)$ to $(w_1, \ldots, w_k)$ if and only if there exists an $i \in \{1, \ldots, k\}$ such that:

(70.54)
(i) $v_j = w_j$ for all $j \neq i$;
(ii) (v_i, w_i) is an arc of D;
(iii) for each $j \neq i$ there is no directed path in D from v_j to v_i.

Now the following holds:

(70.55) D contains vertex-disjoint directed paths $P_1, \ldots, P_k$ such that P_i runs from s_i to t_i $(i = 1, \ldots, k)$ $\iff$ D' contains a directed path P from $(s_1, \ldots, s_k)$ to $(t_1, \ldots, t_k)$.

To see $\Longrightarrow$, let P_i follow the vertices $v_{i,0}, v_{i,1}, \ldots, v_{i,p_i}$ for $i = 1, \ldots, k$. So $v_{i,0} = s_i$ and $v_{i,p_i} = t_i$ for each i. Choose $j_1, \ldots, j_k$ such that $0 \leq j_i \leq p_i$ for each i and such that:

(70.56)
(i) D' contains a directed path from $(s_1, \ldots, s_k)$ to $(v_{1,j_1}, \ldots, v_{k,j_k})$,
(ii) $j_1 + \cdots + j_k$ is as large as possible.

Let $I := \{i \mid j_i < p_i\}$. If $I = \emptyset$ we are done, so assume $I \neq \emptyset$. Then by the definition of D' and the maximality of $j_1 + \cdots + j_k$ there exists for each $i \in I$ an $i' \neq i$ such that there is a directed path in D from $v_{i',j_{i'}}$ to v_{i,j_i}. Since $t_{i'}$ is a sink we know that $v_{i',j_{i'}} \neq t_{i'}$ and that hence i' belongs to I. So each vertex in $\{v_{i,j_i} \mid i \in I\}$ is end vertex of a directed path in D starting at another vertex in $\{v_{i,j_i} \mid i \in I\}$. This contradicts the fact that D is acyclic.

To see $\Longleftarrow$ in (70.55), let P be a directed path from $(s_1, \ldots, s_k)$ to $(t_1, \ldots, t_k)$ in D'. Let P follow the vertices $(v_{1,j}, \ldots, v_{k,j})$ for $j = 0, \ldots, p$. So $v_{i,0} = s_i$ and $v_{i,p} = t_i$ for $i = 1, \ldots, k$. For each $i = 1, \ldots, k$, let P_i be the path in D following $v_{i,j}$ for $j = 0, \ldots, p$, taking repeated vertices only once. So P_i is a directed path from s_i to t_i.

Then $P_1, \ldots, P_k$ are vertex-disjoint. For suppose that P_1 and P_2 (say) have a vertex in common. That is $v_{1,j} = v_{2,j'}$ for some $j \neq j'$. Without loss of generality, $j < j'$ and $v_{1,j} \neq v_{1,j+1}$. By definition of D', there is no directed path in D from $v_{2,j}$ to $v_{1,j}$. However, this contradicts the facts that $v_{1,j} = v_{2,j'}$ and that there exists a directed path in D from $v_{2,j}$ to $v_{2,j'}$. ∎

One can derive from Theorem 70.9 that for fixed k also the k *arc*-disjoint paths problem is solvable in polynomial time for acyclic digraphs (by considering the line digraph).

Similarly to the proof of Theorem 70.9, one can prove that for each fixed k, the following problem is solvable in polynomial time: given an acyclic digraph $D = (V, A)$, pairs $(s_1, t_1), \ldots, (s_k, t_k)$ of vertices, and subsets $A_1, \ldots, A_k$ of A, find arc-disjoint directed paths $P_1, \ldots, P_k$, where P_i runs from s_i to t_i and traverses only arcs in A_i $(i = 1, \ldots, k)$.

Thomassen [1985] characterized the solvability of the 2 vertex-disjoint paths problem for acyclic digraphs, similarly to characterization (71.26). (Metzlar [1993] gave a generalization.)

70.13d. A column generation technique for multiflows

The (fractional) multiflow problem is a linear programming problem, and hence can be solved with linear programming techniques (in strongly polynomial time). Ford and Fulkerson [1958a] suggested a different LP-formulation of the multiflow problem, and a column generation technique to solve it with the simplex method.

As we saw in Section 70.1, the multiflow (feasibility) problem can be reduced to the maximum-value multiflow problem. This is equivalent to the following LP-problem. Let $D = (V, A)$ be a digraph, let $(s_1, t_1), \ldots, (s_k, t_k)$ be nets, and let $c : A \to \mathbb{R}_+$ be a capacity function. Let $\mathcal{P}$ denote the collection of all $s_i - t_i$ paths for all $i = 1, \ldots, k$ (taken as arc sets). Then the maximum-value multiflow problem can be formulated as:

$$
\text{(70.57)} \qquad \begin{array}{lll} \text{maximize} & \displaystyle\sum_{P \in \mathcal{P}} z_P & \\ \text{subject to} & \text{(i)} \quad z_P \geq 0 & (P \in \mathcal{P}). \\ & \text{(ii)} \quad \displaystyle\sum_{P \in \mathcal{P}} z_P \chi^P(a) \leq c(a) & (a \in A). \end{array}
$$

This is a linear programming problem with an exponential number of variables. Ford and Fulkerson [1958a] showed that this large number of variables can be avoided when solving the problem with the simplex method. The variables can be handled implicitly by using a *column generation technique* as follows.

When solving (70.57) with the simplex method we first should add a slack variable z_a for each $a \in A$. Let M denote the $A \times \mathcal{P}$ matrix with the incidence vectors of all paths in $\mathcal{P}$ as its columns and let w be the vector in $\mathbb{R}^{\mathcal{P}} \times \mathbb{R}^E$ with $w_P := 1$ for $P \in \mathcal{P}$ and $w_a := 0$ for $a \in A$. Then (70.57) is equivalent to:

$$
\text{(70.58)} \qquad \begin{array}{ll} \text{maximize} & w^{\mathsf{T}} z \\ \text{subject to} & [M\ I] z = c, \\ & z \geq \mathbf{0}. \end{array}
$$

If we solve (70.58) with the simplex method, each simplex tableau is completely determined by the set of variables in the current base. So it is determined by subsets $\mathcal{P}'$ of $\mathcal{P}$ and A' of A, giving the indices of variables in the base. This is enough to know implicitly the whole tableau. Note that $|\mathcal{P}'| + |A'| = |A|$. So although the tableau is exponentially large, it can be represented in a concise way.

Let B be the matrix consisting of those columns of $[M\ I]$ corresponding to $\mathcal{P}'$ and A'. So the rows of B are indexed by A and the columns by $\mathcal{P}' \cup A'$. The basic solution corresponding to B is easily computed: the vector $B^{-1}c$ gives the values for z_P if $P \in \mathcal{P}'$ and for z_a if $a \in A'$, while we set $z_P := 0$ if $P \notin \mathcal{P}'$ and $z_a := 0$ if $a \notin A'$. Initially, $B = I$, that is $\mathcal{P}' = \emptyset$ and $A' = A$, implying $z_P = 0$ for all $P \in \mathcal{P}$ and $z_a = c(a)$ for all $a \in A$.

Now we describe pivoting (that is, finding variables leaving and entering the base) and checking optimality. Interestingly, it turns out that this can be done by solving a set of shortest path problems.

First consider the dual variable corresponding to an arc a. It has value (in the current tableau):

$$(70.59) \qquad w_B^{\mathsf{T}} B^{-1} \chi^a - w_a = w_B^{\mathsf{T}} (B^{-1})_a,$$

where, as usual, w_B denotes the part of vector w corresponding to B (that is, corresponding to $\mathcal{P}'$ and A') and where χ^a denotes the ath unit base vector in $\mathbb{R}^A$ (which is the column corresponding to a in $[M\ I]$). Note that the columns of B^{-1} are indexed by A; then $(B^{-1})_a$ is the column corresponding to a. Note also that $w_a = 0$ by definition.

Similarly, the dual variable corresponding to a path P in $\mathcal{P}$ has value:

$$(70.60) \qquad w_B^{\mathsf{T}} B^{-1} \chi^P - w_P = \big(\sum_{a \in P} w_B^{\mathsf{T}} (B^{-1})_a\big) - 1.$$

In order to pivot, we should find a negative dual variable. To this end, we first check if (70.59) is negative for some arc a. If so, we choose such an arc a and take z_a as the variable entering the base. Selecting the variable leaving the base now belongs to the standard simplex routine. For that, we only have to consider that part of the tableau corresponding to $\mathcal{P}', A'$, and a. We select an element f in $\mathcal{P}' \cup A'$ for which the quotient $z_f/(B^{-1})_{f,a}$ has positive denominator and is as small as possible. Then z_f is the variable leaving the base.

Suppose next that (70.59) is nonnegative for each arc a. We consider $w_B^{\mathsf{T}}(B^{-1})_a$ as the length $l(a)$ of a. Then for any path P,

$$(70.61) \qquad \sum_{a \in P} w_B^{\mathsf{T}} (B^{-1})_a$$

is equal to the length $\sum_{a \in P} l(a)$ of P. Hence, finding a dual variable (70.60) of negative value is the same as finding a path in $\mathcal{P}$ of length less than 1.

Such a path can be found by applying a shortest path algorithm: for each $i = 1, \ldots, k$, we find a shortest s_i–t_i path (with respect to l). If each of these shortest paths has length at least 1, we know that all dual variables have nonnegative value, and hence the current basic solution is optimum.

If we find some $s_i - t_i$ path P of length less than 1, we choose z_P as variable entering the base. Again selecting a variable leaving the base is standard: we select an element f in $\mathcal{P}' \cup A'$ for which the quotient $z_f/(B^{-1}\chi^P)_f$ has positive denominator and is as small as possible.

This describes pivoting. In order to avoid cycling and to guarantee termination, a lexicographic rule can be incorporated for selecting the variable leaving the base as usual. (This only requires ordering A.)

The length function l in the final tableau has the properties described in Theorem 70.2.

70.13e. Approximate max-flow min-cut theorems for multiflows

In general, the cut condition is not sufficient for the existence of a feasible multiflow. Leighton and Rao [1988,1999] gave an upper bound (only depending on the number of vertices) on the relative error in case each pair of vertices forms a net, with all demands equal.

Let $G = (V, E)$ and $H = (V, R)$ be a supply and a demand graph, and let $c : E \to \mathbb{Z}_+$ and $d : R \to \mathbb{Z}_+$ be a capacity and a demand function. Let λ be the maximum value for which there exists a multiflow subject to c of demand $\lambda \cdot d$. By the cut condition,

(70.62) $$\lambda \leq \mu := \min_U \frac{c(\delta_E(U))}{d(\delta_R(U))},$$

where the minimum is taken over all subsets U of V with $d(\delta_R(U)) > 0$.

Leighton and Rao proved that if R is the collection of *all* pairs from V and d is constant, then $\mu/\lambda = O(\log n)$ where $n := |V|$. They also showed that $O(\log n)$ is best possible, and that a set U attaining the minimum in (70.62) up to a factor $O(\log n)$ can be found in polynomial time.

Klein, Agrawal, Ravi, and Rao [1990] (cf. Klein, Rao, Agrawal, and Ravi [1995]) showed that if H is any demand graph, then $\mu/\lambda = O(\log C \log D)$, where C and D denote the sum of the capacities and demands, respectively. This was improved to $O(\log n \log D)$ by Tragoudas [1996], to $O(\log |R| \log D)$ by Garg, Vazirani, and Yannakakis [1993a,1996], and to $O(\log^2 |R|)$ by Plotkin and Tardos [1993,1995]. These papers also give polynomial-time algorithms to find a subset U attaining the minimum (70.62) up to the corresponding factor.

For planar graphs, Klein, Plotkin, and Rao [1993] gave a bound of $O(\log D)$, improved to $O(\log |R|)$ by Plotkin and Tardos [1993,1995], and of $O(1)$ if R consists of all pairs of vertices.

More results on approximate multiflows are given by Raghavan and Thompson [1987], Klein, Stein, and Tardos [1990], Shahrokhi and Matula [1990], Leighton, Makedon, Plotkin, Stein, Tardos, and Tragoudas [1991,1995], Goldberg [1992], Klein, Plotkin, and Rao [1993], Leong, Shor, and Stein [1993], Tardos and Vazirani [1993], Awerbuch and Leighton [1994], Klein, Plotkin, Stein, and Tardos [1994], Kamath and Palmon [1995], Linial, London, and Rabinovich [1995], Radzik [1995,1997], Aumann and Rabani [1998], Garg and Könemann [1998], Fleischer [1999a,2000a], Guruswami, Khanna, Rajaraman, Shepherd, and Yannakakis [1999], Leighton and Rao [1999], Baveja and Srinivasan [2000], Srivastav and Stangier [2000], Cheriyan, Karloff, and Rabani [2001], Fleischer and Wayne [2002], Günlük [2002], Karakostas [2002], and Kolman and Scheideler [2002]. A survey is given by Shmoys [1997]. Approximation algorithms for Steiner and directed multicuts are given by Klein, Plotkin, Rao, and Tardos [1997].

For approximating minimum-cost multiflows, see Plotkin, Shmoys, and Tardos [1991,1995], Kamath, Palmon, and Plotkin [1995], Karger and Plotkin [1995], Grigoriadis and Khachiyan [1996b,1996a], Garg and Könemann [1998], Goldberg, Oldham, Plotkin, and Stein [1998], and Karakostas [2002].

The 'quickest multicommodity flow problem' was investigated by Fleischer and Skutella [2002].

For surveys on approximation algorithms, see Shmoys [1995] and the book by Vazirani [2001].

70.13f. Further notes

Ford and Fulkerson [1958a] designed a (non-polynomial-time) algorithm for the fractional multiflow problem, based on the simplex method, with column generation — see Section 70.13d. Jewell [1958,1966] described a primal-dual simplex method, Sakarovitch [1966] gave a labeling algorithm solving a sequence of one-commodity flow problems after allocating the total capacity of each arc to each net, and Saigal [1967] developed an algorithm based on an arc-circuit formulation, using a column generation technique to handle the circuits. Dantzig-Wolfe decomposition was applied to multiflow problems by Chen and DeWald [1974]. Kapoor and Vaidya [1986,1996] and Kamath and Palmon [1995] study the complexity of applying interior point algorithms to multiflows.

Grinold [1968,1969] described a primal-dual algorithm for the maximum-value multiflow problem, based on allocating capacities to commodities and iteratively adapt the allocation. A simplex-based algorithm for minimum-cost and maximum-value multiflow problems was given by Hartman and Lasdon [1972]. Also Tomlin [1966], Wollmer [1972], Dragan [1974], and Nagamochi, Fukushima, and Ibaraki [1990] studied minimum-cost multiflows. A 'partitioning' algorithm for the multiflow problem was given by Grigoriadis and White [1972]. Related work was done by Kennington [1977], Farvolden, Powell, and Lustig [1993], and Hadjiat, Maurras, and Vaxes [2000]. Jarvis [1969] noticed the equivalence of vertex-arc and arc-chain formulations of the multiflow problem.

Bellmore, Greenberg, and Jarvis [1970] and Jarvis and Tindall [1972] described algorithms to find a minimum-capacity set disconnecting all nets in a directed multiflow problem.

Swoveland [1973] studied a generalization of the multiflow problem, where upper bounds can be prescribed for the sum of the flows of subsets of the nets on arcs. Ferland [1974] and Klessig [1974] studied nonlinear costs.

Computational work on multiflows is reported by Minoux [1975], Ulrich [1975], Helgason and Kennington [1977a], Kennington [1977,1978] (also minimum-cost), Kennington and Shalaby [1977], Ali, Helgason, Kennington, and Lall [1980], Kennington and Helgason [1980], Ali, Barnett, Farhangian, Kennington, Patty, Shetty, McCarl, and Wong [1984], Saviozzi [1986], Boland and Mees [1990], Nagamochi, Fukushima, and Ibaraki [1990], Barnhart [1993], Leong, Shor, and Stein [1993], Bienstock and Günlük [1995], Barnhart, Hane, and Vance [1996], Castro and Nabona [1996], Barnhart, Hane, and Vance [1997], McBride and Mamer [1997], McBride [1998], and Frangioni and Gallo [1999].

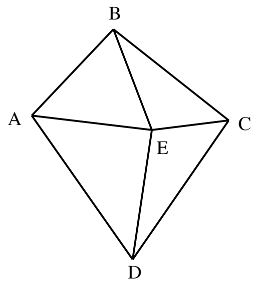
Surveys on multiflows were given by Hu [1969], Frank and Frisch [1971], Assad [1978], Kennington [1978], Phillips and Garcia-Diaz [1981], Gondran and Minoux [1984], Bazaraa, Jarvis, and Sherali [1990], Ahuja, Magnanti, and Orlin [1993], and Korte and Vygen [2000], on disjoint paths by Frank [1990e,1993a,1995], and on maximum-value multiflows by Karzanov [1991]. A bibliography on network optimization, including multicommodity flows, was compiled by Golden and Magnanti [1977].

70.13g. Historical notes on multicommodity flows

We review a few papers on multicommodity flows that are of historical interest.

In his monograph *Mathematical Methods of Organizing and Planning Production*, Kantorovich [1939] introduced linear programming methods for the multicommodity flow problem, giving as example the problem of a railroad network on which several connections have to be made simultaneously:

> Let us mention still another problem of different character which, although it does not lead directly to questions A, B, and C, can still be solved by our methods. That is the choice of transportation routes.



> Let there be several points A, B, C, D, E (Fig. 1) which are connected to one another by a railroad network. It is possible to make the shipments from B to D by the shortest route BED, but it is also possible to use other routes as well: namely BCD, BAD. Let there also be given a schedule of freight shipments; that is, it is necessary to ship from A to B a certain number of carloads, from D to C a certain number, and so on. The problem consists of the following. There is given a maximum capacity for each route under the given conditions (it can of course change under new methods of operation in transportation). It is necessary to distribute the freight flows among the different routes in such a way as to complete the necessary shipments with a minimum expenditure of fuel, under the condition of minimizing the empty runs of freight cars and taking account of the maximum capacities of the routes. As was already shown, this problem can also be solved by our methods.

A problem analogous to the multicommodity flow problem, the multi-index transportation problem, was considered by Motzkin [1952] and Schell [1955].

It was noted by Ford and Fulkerson [1954,1956b] that the max-flow min-cut theorem does not extend to maximum multiflows:

> It is worth pointing out that the minimal cut theorem is not true for networks with several sources and corresponding sinks, where shipment is restricted to be from a source to its sink.

Ford and Fulkerson give the example of the graph $K_{1,3}$, with nets all pairs of vertices of degree 1.

Robacker [1956a] observed that the following 'decomposition theorem' applies: for a graph $G = (V, E)$, nets $\{s_1, t_1\}, \ldots, \{s_k, t_k\}$, and a capacity function $c : E \to \mathbb{R}_+$, the maximum total value of a multiflow subject to c is equal to

(70.63) $$\max_{c_1,\ldots,c_k} \sum_{i=1}^{k} \min_{C \in \mathcal{C}_i} c_i(C).$$

Here the maximum ranges over all k-tuples of vectors $c_1, \ldots, c_k$ in $\mathbb{R}_+^E$ with $c_1 + \cdots + c_k = c$. Moreover, $\mathcal{C}_i$ denotes the set of all $s_i - t_i$ cuts and $c_i(C)$ denotes the capacity of cut C with respect to the capacity function c_i.

So the theorem decomposes the maximum multicommodity flow problem into k maximum single-commodity flow problems. The problem is reduced to finding the optimum decomposition of the capacity function c into k functions $c_1, \ldots, c_k$. Robacker [1956a] remarked:

> At present there are no computational techniques other than those of linear programming for determining maximal flow through multicommodity networks. It is hoped, however, that the decomposition theorem may lead to new methods as did the minimum-cut, maximum-flow theorem for single-commodity networks.

Kalaba and Juncosa [1956] described applications of the multicommodity flow problem to telecommunication networks. In particular they mention:

> In a system such as the Western Union System, which has some 15 regional switching centers all connected to each other, an optimal routing problem of this type would have about 450 conditions and involve around 3000 variables. If solved using the simplex method in its most general form, this would be at the threshhold of the capacity of modern large-scale computers and would require several hours for solution.

They express the expectation that developments in computer technology and possible extensions of the combinatorial methods for one-commodity flows, will improve the situation greatly.

It turned out, however, that the combinatorial techniques that made the single-commodity flow problem so tractable, do not extend to multicommodity flows. Ford and Fulkerson [1958a] suggested a variant of the simplex method based on a column-generation technique, where each simplex step consists of determining a shortest path. Although they did not carry out computations, they expected that their method is more practicable than the direct simplex method, at least in space required. A primal-dual algorithm for multiflows was designed by Jewell [1958] (cf. Jewell [1966]).

Hu [1963] gave a combinatorial algorithm for the 2-commodity flow problem, but doubted whether it could be extended to general multicommodity flows:

> Although the algorithm for constructing maximum bi-flow is very simple, it is unlikely that similar techniques can be developed for constructing multicommodity flows. The linear programming approach used by Ford and Fulkerson to construct maximum multicommodity flows in a network is the only tool now available.

For remarks on the early history of multicommodity flows, see Jewell [1966].

Chapter 71

Two commodities

The integer 2-commodity flow problem is NP-complete, even if all capacities are 1 (Even, Itai, and Shamir [1975,1976]). Equivalently, the edge-disjoint paths problem in undirected graphs is NP-complete, even if the nets form two parallel classes.
However, if we add the Euler condition, the problem has a good characterization and can be solved in polynomial time. It is a generalization of Hu's 2-commodity flow theorem, stating that the cut condition implies the existence of a half-integer multiflow (for integer capacities and demands). This and related results form the topic of this chapter.
Except if stated otherwise, throughout this chapter $G = (V, E)$ and $H = (T, R)$ denote the supply and demand graph, in the sense of Chapter 70. The pairs in R are called the *nets*. If $s_1, t_1, \ldots, s_k, t_k$ are given, then $R := \{s_1t_1, \ldots, s_kt_k\}$. In fact, often in this chapter, $k = 2$, so $R = \{s_1t_1, s_2t_2\}$. If demands $d_1, \ldots, d_k$ are given, then $d(s_it_i) = d_i$. We denote $G+H = (V, E \cup R)$, where the disjoint union of E and R is taken, respecting multiplicities.

71.1. The Rothschild-Whinston theorem and Hu's 2-commodity flow theorem

It is a basic theorem of Hu [1963], that for 2-commodity flow problems in undirected graphs, the cut condition implies the existence of a feasible 2-commodity flow. Recall that the cut condition (in the undirected case) states that

(71.1) $\quad c(\delta_E(U)) \geq d(\delta_R(U))$

for each $U \subseteq V$. This theorem, 'Hu's 2-commodity flow theorem', will be shown below as Corollary 71.1b.

Hu also showed that if moreover all capacities are integer, there is a half-integer 2-commodity flow. Generally, an integer multiflow need not exist, as is shown by Figure 70.3. In fact, the undirected integer 2-commodity flow problem is NP-complete (Even, Itai, and Shamir [1975,1976]).

Rothschild and Whinston [1966a] extended Hu's theorem by showing that adding the Euler condition guarantees the existence of an integer 2-commodity flow. We recall that the Euler condition states that

(71.2) $c(\delta_E(v)) + d(\delta_R(v))$ is even for each $v \in V$.

Theorem 71.1 (Rothschild-Whinston theorem)**.** *Let $G = (V, E)$ be a graph, let $\{s_1, t_1\}$ and $\{s_2, t_2\}$ be pairs of vertices of G, and let $c : E \to \mathbb{Z}_+$ and $d_1, d_2 \in \mathbb{Z}_+$ satisfy the Euler condition. Then there exists an integer 2-commodity flow subject to c and with value d_1, d_2 if and only if the cut condition is satisfied.*

Proof. Necessity being trivial, we show sufficiency. Suppose that the cut condition holds. Orient the edges of G arbitrarily, yielding the digraph $D = (V, A)$. For any $a \in A$, we denote by $c(a)$ the capacity of the underlying undirected edge. For $i = 1, 2$, define $p_i : V \to \mathbb{Z}$ by

(71.3) $p_i := d_i \cdot (\chi^{t_i} - \chi^{s_i})$.

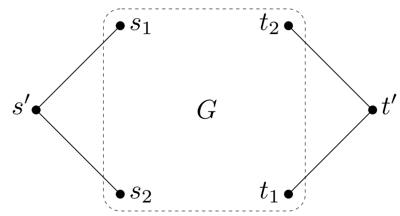


Figure 71.1

Extend G by two new vertices, s' and t', and new edges $s's_1$ and t_1t', each of capacity d_1, and new edges $s's_2$ and t_2t', each of capacity d_2 (Figure 71.1). This gives the graph G'.

By the max-flow min-cut theorem, G' contains an integer $s' - t'$ flow g of value $d_1 + d_2$, since by the cut condition the minimum capacity of an $s' - t'$ cut in G' is equal to $d_1 + d_2$. By the Euler condition we can assume that $g(e) \equiv c(e) \pmod 2$ for each $e \in E$: the edges e with $g(e) \not\equiv c(e) \pmod 2$ form an Eulerian graph; that is, each vertex is incident with an even number of such edges. Hence we can add a unit flow along a circuit, so as to decrease the number of such edges e.

Now in D, g gives a function $g' : A \to \mathbb{Z}$ satisfying

(71.4) $g'(a) \equiv c(a) \pmod 2$ and $|g'(a)| \leq c(a)$ for each $a \in A$, and $\text{excess}_{g'} = p_1 + p_2$.

(Here $\text{excess}_{g'}(v) := g'(\delta^{\text{in}}(v)) - g'(\delta^{\text{out}}(v))$ for $v \in V$.)

Similarly, by extending G by two new vertices, s'' and t'', and new edges $s''s_1$ and t_1t'', each of capacity d_1, and $s''t_2$ and s_2t'', each of capacity d_2 (Figure 71.2), we obtain a function $g'' : A \to \mathbb{Z}$ satisfying

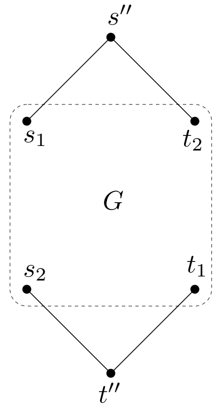


Figure 71.2

(71.5) $g''(a) \equiv c(a) \pmod 2$ and $|g''(a)| \leq c(a)$ for each $a \in A$, and $\text{excess}_{g''} = p_1 - p_2$.

Now define $f_1 := \frac{1}{2}(g' + g'')$ and $f_2 := \frac{1}{2}(g' - g'')$. Then f_1 and f_2 form a 2-commodity flow as required. Indeed, since $g' \equiv c \equiv g'' \pmod 2$, we know that f_1 and f_2 are integer. Moreover, $|f_1(a)| + |f_2(a)| = \frac{1}{2}(|g'(a)| + |g''(a)|) \leq c(a)$ for each $a \in A$. Finally, $\text{excess}_{f_i} = p_i$ for $i = 1, 2$, as follows directly from (71.4) and (71.5). ∎

This method of proof was given by Rothschild and Whinston [1966a] (similar proofs were given by Sakarovitch [1973] and Seymour [1978]).

A combinatorial form of Theorem 71.1 is:

Corollary 71.1a. *Let $G = (V, E)$ be a graph, let $s_1, t_1, s_2, t_2 \in V$, and let $d_1, d_2 \in \mathbb{Z}_+$, such that each vertex $v \neq s_1, t_1, s_2, t_2$ has even degree, while $\deg_G(s_i) \equiv \deg_G(t_i) \equiv d_i$ (mod 2) for $i = 1, 2$. Then there exist d_1 $s_1 - t_1$ paths and d_2 $s_2 - t_2$ paths, all edge-disjoint if and only if the cut condition* (70.21) *is satisfied.*

Proof. Directly from Theorem 71.1 by taking all capacities equal to 1. ∎

Conversely, Theorem 71.1 follows from Corollary 71.1a by replacing each edge e by $c(e)$ parallel edges.

Theorem 71.1 also implies a half-integer 2-commodity flow theorem, given by Hu [1963][15]:

Corollary 71.1b (Hu's 2-commodity flow theorem)**.** *Let $G = (V, E)$ be a graph, let s_1, t_1 and s_2, t_2 be pairs of vertices of G, let $c : E \to \mathbb{R}_+$, and*

[15] Hakimi [1962b] gave an erroneous proof of this theorem.

let $d_1, d_2 \in \mathbb{R}_+$. Then there exists a 2-commodity flow subject to c and with value d_1, d_2 if and only if the cut condition is satisfied. If all capacities and demands are integer, then we can take the flow half-integer.

Proof. By continuity and compactness, we can assume that c and the d_i are rational-valued, and hence, by scaling, even-integer-valued. So the Euler condition holds. Let the cut condition be satisfied. Then Theorem 71.1 gives the existence of a 2-commodity flow.

If c and the d_i are integer-valued, multiplying them by 2 and applying Theorem 71.1 gives an integer 2-commodity flow, and hence a half-integer multiflow for the original c and d_i. ∎

Notes. The proof of Theorem 71.1 yields a strongly polynomial-time algorithm to find a feasible integer 2-commodity flow if the Euler condition holds, of the same time order as that of finding a maximum one-commodity integer flow. It implies a strongly polynomial-time algorithm to find a half-integer 2-commodity flow, for integer capacities and demands.

Also Cherkasskiĭ [1973] gave a strongly polynomial-time ($O(n^2m)$) algorithm to find a feasible half-integer 2-commodity flow. Hu [1963] gave a combinatorial algorithm, which Itai [1978] showed to have a strongly polynomial-time implementation ($O(n^3)$). A similar algorithm was described by Arinal [1969].

71.1a. Nash-Williams' proof of the Rothschild-Whinston theorem

An alternative simple proof of the Rothschild-Whinston theorem was given by C.St.J.A. Nash-Williams (cf. Lovász [1979a] p. 289). We give the proof for the equivalent Corollary 71.1a. As necessity is easy, we show sufficiency.

By Menger's theorem (undirected version), G has d_1+d_2 edge-disjoint $\{s_1, s_2\}-\{t_1, t_2\}$ paths such that d_i of them start at s_i, and d_i of them end at t_i, for $i = 1, 2$. (But the paths starting at s_1 may end at t_2, and those starting at s_2 may end at t_1.) Hence G has an orientation $D = (V, A)$ with $(V, A \cup B)$ Eulerian, where B consists of d_i parallel arcs from t_i to s_i, for $i = 1, 2$.

Then Menger's theorem (directed version) implies that D has d_1 arc-disjoint directed $s_1 - t_1$ paths. Indeed, consider any $U \subseteq V$ with $s_1 \in U$, $t_1 \notin U$. We show $d_A^{\text{out}}(U) \geq d_1$. As $(V, A \cup B)$ is Eulerian, we have

$$(71.6) \qquad d_A^{\text{out}}(U) + d_B^{\text{out}}(U) = d_{A\cup B}^{\text{out}}(U) = d_{A\cup B}^{\text{in}}(U) = d_A^{\text{in}}(U) + d_B^{\text{in}}(U).$$

If $d_B^{\text{out}}(U) = 0$, this gives $d_A^{\text{out}}(U) \geq d_B^{\text{in}}(U) \geq d_1$. If $d_B^{\text{out}}(U) > 0$, then $t_2 \in U$, $s_2 \notin U$, hence $d_B^{\text{in}}(U) = d_1$ and $d_B^{\text{out}}(U) = d_2$. So

$$(71.7) \qquad d_A^{\text{out}}(U) = \tfrac{1}{2}(d_A^{\text{out}}(U) + d_A^{\text{in}}(U) + d_1 - d_2) = \tfrac{1}{2}(d_E(U) + d_1 - d_2) \geq d_1,$$

since $d_E(U) \geq d_1 + d_2$.

So D contains d_1 arc-disjoint $s_1 - t_1$ paths. Now delete from $(V, A \cup B)$ all arcs occurring in these paths, and delete the d_1 parallel arcs from t_1 to s_1. We are left with an Eulerian digraph, and hence the d_2 parallel arcs from t_2 to s_2 belong to d_2 arc-disjoint directed circuits. This gives the d_2 paths from s_2 to t_2 as required.

71.2. Consequences

E.A. Dinits (cf. Adel'son-Vel'skiĭ, Dinits, and Karzanov [1975]) observed that Hu's 2-commodity flow theorem and the Rothschild-Whinston theorem imply:

Corollary 71.1c. *Let $G = (V, E)$ be an undirected graph and let $\{s_1, t_1\}, \ldots, \{s_k, t_k\}$ be pairs of vertices, such that there exist a two vertices intersecting each $\{s_i, t_i\}$. Let $c : E \to \mathbb{R}_+$ and $d_1, \ldots, d_k \in \mathbb{R}_+$. Then the cut condition implies the existence of a feasible multiflow. If c and the d_i are integer, there exists a half-integer multiflow. If moreover the Euler condition holds, there exists an integer multiflow.*

Proof. We can assume that $s_i = s$ for $i = 1, \ldots, l$, and that $t_i = t'$ for $i = l+1, \ldots, k$. Let t and s' be two new vertices. For each $i = 1, \ldots, l$, add a new edge connecting t_i and t, of capacity d_i. For each $i = l+1, \ldots, k$, add a new edge connecting s' and s_i, of capacity d_i. This makes the graph H. Define $d := d_1 + \cdots + d_l$ and $d' := d_{l+1} + \cdots + d_k$.

Then the cut condition for G implies that each cut $\delta_H(U)$ in H has capacity at least $d + d'$ if it is both $s - t$ and $s' - t'$ separating; at least d if it separates s and t; and at least d' if it separates s' and t'. Hence, by Hu's 2-commodity flow theorem, H has a feasible 2-commodity flow. Restriction to G gives a feasible multiflow.

The last two statement of this corollary follow similarly. ∎

Another consequence of Theorem 71.1 is what Hu called the *max-biflow min-cut theorem*[16]:

Corollary 71.1d (max-biflow min-cut theorem). *Let $G = (V, E)$ be a graph, let $\{s_1, t_1\}$ and $\{s_2, t_2\}$ be pairs of vertices, and let $c : E \to \mathbb{R}_+$. Then the maximum total value M of a 2-commodity flow subject to c is equal to the minimum capacity m of a cut which is both $s_1 - t_1$ and $s_2 - t_2$ separating. If c is integer, the maximum is attained by a half-integer multiflow. If c is integer and $c(\delta(v))$ is even for each vertex $v \neq s_1, t_1, s_2, t_2$, the maximum is attained by an integer multiflow.*

Proof. By continuity, compactness, and scaling, we can assume that c is integer and that $c(\delta(v))$ is even for each $v \neq s_1, t_1, s_2, t_2$. By replacing edges by parallel edges, we can assume that $c(e) = 1$ for each $e \in E$. So M is equal to the maximum number of edge-disjoint paths, each connecting either s_1 and t_1, or s_2 and t_2. As trivially $M \leq m$, it suffices to prove $M \geq m$. We can assume that $m > 0$.

[16] due to Hu [1963] and (the last statement) to Rajagopalan [1994] (who also showed a hole in the proof by Sakarovitch [1973] of this); it sharpens a result of Rothschild and Whinston [1966b], who required that $c(\delta(v))$ is even for *all* vertices v.

First assume that $\deg_G(s_1) \equiv \deg_G(t_1) \pmod 2$ and (hence) $\deg_G(s_2) \equiv \deg_G(t_2) \pmod 2$. For $i = 1, 2$, let m_i be the minimum size of an $s_i - t_i$ cut. We show that

(71.8) there exists $d_1, d_2 \in \mathbb{Z}_+$ such that $d_1 \leq m_1$, $d_2 \leq m_2$, $d_1 + d_2 = m$, $d_1 \equiv \deg_G(s_1) \pmod 2$, and $d_2 \equiv \deg_G(s_2) \pmod 2$.

To see this, note that $m \leq m_1 + m_2$ (since the union of an $s_1 - t_1$ cut and an $s_2 - t_2$ cut separates both s_1 and t_1, and s_2 and t_2), $m_1 \leq m$, and $m \equiv \deg_G(s_1) + \deg_G(s_2) \pmod 2$. As $m > 0$, by symmetry we may assume that $m_2 > 0$. If $m_1 \equiv \deg_G(s_1) \pmod 2$, then we can take $d_1 := m_1$ and $d_2 := m - m_1$. If $m_1 \not\equiv \deg_G(s_1) \pmod 2$, then we can take $d_1 := m_1 - 1$ and $d_2 := m - m_1 + 1$. Indeed, as $m_1 \not\equiv \deg_G(s_1) \pmod 2$, any minimum-size $s_1 - t_1$ cut also separates s_2 and t_2. So $m = m_1$. Hence $d_1 \geq 0$ (as $m > 0$) and $d_2 = 1 \leq m_2$ (as $m_2 > 0$).

This shows (71.8). By Corollary 71.1a, there exist d_1 $s_1 - t_1$ paths and d_2 $s_2 - t_2$ paths, any two of which are edge-disjoint. So $M \geq d_1 + d_2 = m$.

Next assume that $\deg_G(s_1) \not\equiv \deg_G(t_1) \pmod 2$ and (hence) $\deg_G(s_2) \not\equiv \deg_G(t_2) \pmod 2$. By symmetry, we may assume that m is attained by a cut with s_1, s_2 at one side and t_1, t_2 at the other side. So the size of any cut with s_1, t_2 at one side and t_1, s_2 at the other side, has parity different from that of m; hence its size is at least $m+1$. Therefore, adding a new edge connecting s_2 and t_1 increases the minimum m by 1. Moreover, the maximum M increases by at most 1. In the new situation, the degrees of s_1 and t_1 have the same parity, and similarly for s_2 and t_2. Hence the first part of this proof applies, showing $M \geq m$. ∎

(An alternative proof was given by Lovász [1976b].)

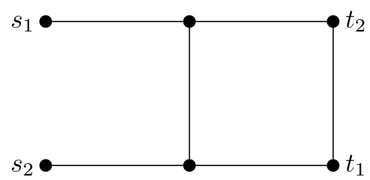


Figure 71.3
The maximum total value of a 2-commodity flow (subject to capacity **1**) is equal to 2, but the maximum total value of an integer 2-commodity flow is equal to 1.

The graph in Figure 71.3 shows that in the max-biflow min-cut theorem (Corollary 71.1d) we cannot delete the parity conditions (example of Rothschild and Whinston [1966b]). This example is critical, as is shown by the following result, which is a special case of a general hypergraph theorem of Seymour [1977b] (Theorem 80.1).

Theorem 71.2. *Let $G = (V, E)$ be a graph and $s_1, t_1, s_2, t_2 \in V$. Then for each capacity function $c : E \to \mathbb{Z}_+$, the maximum total value of an integer 2-commodity flow is equal to the minimum capacity of a cut separating both s_1 and t_1, and s_2 and t_2, if and only if G has no subgraph contractible to the graph of Figure 71.3, up to exchanging s_1 and t_1, and s_2 and t_2.*

Here we assume that the subgraph contains the s_i and t_i, and that these vertices are contracted to the vertices indicated by s_i and t_i in the figure. For a proof, we refer to Section 80.5a.

A similar result for feasibility can be derived (Seymour [1981a]):

(71.9) Let $G = (V, E)$ be a graph and $s_1, t_1, s_2, t_2 \in V$. Then for each capacity function $c : E \to \mathbb{Z}_+$ and each demands $d_1, d_2 \in \mathbb{Z}_+$, the cut condition implies the existence of an integer multiflow if and only if the graph of Figure 70.3 is not a minor of G.

We derive this result from Theorem 71.2. By taking $c(e)$ large one can see that the property described is closed under contractions of edges. As Figure 70.3 satisfies the cut condition but has no integer multiflow (for $c = \mathbf{1}$, $d = \mathbf{1}$), we have necessity of the condition in (71.9).

To derive sufficiency from Theorem 71.2, let G have no subgraph contractible to Figure 70.3 and let $c : E \to \mathbb{Z}_+$ and $d_1, d_2 \in \mathbb{Z}_+$ satisfy the cut condition. Let s'_1 and s'_2 be two new vertices, and let $s'_1 s_1$ and $s'_2 s_2$ be two new edges, of capacity d_1 and d_2 respectively. Then the extended graph G' has no subgraph contractible to the graph of Figure 71.3, with s_i replaced by s'_i $(i = 1, 2)$, up to exchanging s'_1 and t_1, and s'_2 and t_2. Also, the minimum capacity of a cut in G' separating both s'_1 and t_1, and s'_2 and t_2, is equal to $d_1 + d_2$. Hence by Theorem 71.2, G' has an integer multiflow of total value $d_1 + d_2$. Restricted to G this gives a multiflow satisfying d_1, d_2.

Notes. For the case where $G + H$ is planar, Lomonosov [1983] characterized for *fixed* integer capacity function c, when the maximum and minimum in Theorem 71.2 are equal. He also showed that if $G + H$ is planar, the maximum and minimum differ by at most 1.

71.3. 2-commodity cut packing

By Theorem 70.5, Hu's 2-commodity flow theorem implies that if $G = (V, E)$ is an undirected graph, $s_1, t_1, s_2, t_2 \in V$, and $l : E \to \mathbb{R}_+$, then there exist $\lambda_U \geq 0$ for $U \subseteq V$ such that

(71.10) $$\sum_U \lambda_U \chi^{\delta_E(U)} \leq l$$

and

(71.11) $$\mathrm{dist}_l(s_i, t_i) = \sum_U \lambda_U \chi^{\delta_R(U)}(s_i t_i),$$

for $i = 1, 2$, where $R := \{s_1t_1, s_2t_2\}$. (Here $\mathrm{dist}_l(s,t)$ is the distance of s and t, taking l as length function.)

We shall see that if l is integer, we can take the λ_U half-integer. More precisely, and more strongly:

(71.12) if l is integer such that each circuit in G has even length, then we can take the λ_U integer.

This was proved by Seymour [1978]. Equivalently (by replacing each edge e by a path of length $l(e)$; $\mathrm{dist}_G(s,t)$ denotes the distance of s and t in G (for length function $\mathbf{1}$)):

Theorem 71.3. *Let $G = (V,E)$ be a bipartite graph and let $s_1, t_1, s_2, t_2 \in V$. Then there exist disjoint cuts such that s_i and t_i are separated by $\mathrm{dist}_G(s_i,t_i)$ of these cuts, for $i = 1, 2$.*

Proof. We may assume that G is connected. Denote $d(u,v) := \mathrm{dist}_G(u,v)$ for $u, v \in V$. Define for each vertex v:

(71.13)
$$\varphi(v) := \tfrac{1}{2}(d(s_1,v) + d(s_2,v) - d(s_1,s_2)),$$
$$\psi(v) := \tfrac{1}{2}(d(s_1,v) - d(s_2,v) + d(s_1,s_2)).$$

These numbers are nonnegative and integer, by the triangle inequality and by the fact that each circuit in G has even length.

If u and v are adjacent vertices of G, then either $\varphi(u) = \varphi(v)$ and $|\psi(v) - \psi(u)| = 1$, or $\psi(u) = \psi(v)$ and $|\varphi(v) - \varphi(u)| = 1$, since $d(s_1,v) - d(s_1,u) = \pm 1$ and $d(s_2,v) - d(s_2,u) = \pm 1$. Let A_i be the set of edges uv with $\varphi(v) = i - 1$ and $\varphi(u) = i$. Let B_i be the set of edges uv with $\psi(v) = i - 1$ and $\psi(u) = i$. So the sets $A_1, A_2, \ldots, B_1, B_2, \ldots$ are cuts partitioning E.

Now $\varphi(s_1) = \psi(s_1) = 0$ and $\varphi(t_1) + \psi(t_1) = d(s_1,t_1)$. So there exist $d(s_1,t_1)$ cuts among $A_1, A_2, \ldots, B_1, B_2, \ldots$ that separate s_1 and t_1.

Moreover

(71.14)
$$|\varphi(t_2) - \varphi(s_2)| + |\psi(t_2) - \psi(s_2)| = \tfrac{1}{2}|d(s_1,t_2) + d(s_2,t_2) - d(s_1,s_2)| + \tfrac{1}{2}|d(s_1,t_2) - d(s_2,t_2) - d(s_1,s_2)| = d(s_2,t_2).$$

This implies that there exist $d(s_2,t_2)$ cuts among $A_1, A_2, \ldots, B_1, B_2, \ldots$ that separate s_2 and t_2. ∎

A consequence of Theorem 71.3 is a min-max relation for the maximum number of disjoint cuts that are both $s_1 - t_1$ and $s_2 - t_2$ separating, in a bipartite graph (Seymour [1978]):

Corollary 71.3a. *Let $G = (V,E)$ be a bipartite graph and let $s_1, t_1, s_2, t_2 \in V$. Then the maximum number of disjoint cuts each separating both s_1 and t_1, and s_2 and t_2, is equal to the minimum of $\mathrm{dist}_G(s_1,t_1)$ and $\mathrm{dist}_G(s_2,t_2)$.*

Proof. We may assume that G is connected. Let $d(u,v) := \text{dist}_G(u,v)$ for $u, v \in V$. Let $k := \min\{d(s_1,t_1), d(s_2,t_2)\}$. Let $C_1, \ldots, C_t$ be the cuts described in Theorem 71.3. At least k of these cuts separate s_1 and t_1, and at least k of these cuts separate s_2 and t_2. If C_i separates s_1 and t_1 and C_j separates s_2 and t_2, then $C_i \cup C_j$ contains a cut separating both s_1 and t_1, and s_2 and t_2. Thus by properly combining the C_i, we obtain k disjoint cuts as required.

(To see this, we can assume that $C_1, \ldots, C_k$ separate s_1 and t_1, and that $C_{l+1}, \ldots, C_{l+k}$ separate s_2 and t_2, where $0 \leq l \leq k$. Then each of the (disjoint) sets $C_1 \cup C_{k+1}, \ldots, C_l \cup C_{l+k}, C_{l+1}, \ldots, C_k$ contains a cut separating both s_1 and t_1, and s_2 and t_2.) ∎

Let $G = (V, E)$ be a graph and let $s_1, t_1, s_2, t_2 \in V$. Let $\mathcal{C}$ be the collection of all cuts that are both $s_1 - t_1$ and $s_2 - t_2$ separating. Consider a length function $l : E \to \mathbb{R}_+$. Corollary 70.6a applied to the max-biflow min-cut theorem gives:

(71.15) $\min\{\text{dist}_l(s_1,t_1), \text{dist}_l(s_2,t_2)\}$ is equal to the maximum value of $\sum_{C\in\mathcal{C}} y(C)$, where $y : \mathcal{C} \to \mathbb{R}_+$ is such that $\sum_{C\in\mathcal{C}} y(C)\chi^C \leq l$.

Then Corollary 71.3a implies (Seymour [1978], Pevzner [1979b]):

Corollary 71.3b. *If l is integer-valued, we can take y half-integer valued.*

Proof. Replace each edge e by a path of length $2l(e)$. This makes the bipartite graph H. Applying Corollary 71.3a to H does the rest. ∎

Bipartiteness is necessary in Corollary 71.3a, since otherwise the graph in Figure 71.4 (Seymour [1977b], cf. Hu [1973]) would yield a contradiction. (This answers a question of Fulkerson [1971a].)

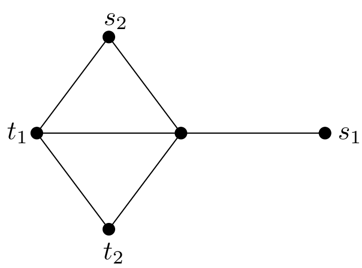


Figure 71.4
The minimum of the distances of s_1 and t_1 and of s_2 and t_2 is equal to 2, but there exist no two disjoint cuts each separating both s_1 and t_1, and s_2 and t_2.

From a more general hypergraph result of Seymour [1977b] (Theorem 80.1), it follows that Figure 71.4 is critical for the existence of an integer-valued packing of cuts:

Theorem 71.4. *Let $G = (V, E)$ be a graph and let $s_1, t_1, s_2, t_2 \in V$. Then for each $l : E \to \mathbb{Z}_+$, the maximum in* (71.15) *is attained by an integer-valued y if and only if G has no subgraph contractible to the graph in Figure* 71.4*, up to permuting indices and permuting s_1 and t_1.*

Again, we assume that the subgraph contains the s_i and t_i, and that these vertices are contracted to the vertices indicated by s_i and t_i in the figure. For a proof, we refer to Section 80.5a.

Notes. Seymour [1981a] showed the following related result: let $G = (V, E)$ be a bipartite graph, and choose $s_1, t_1, s_2, t_2 \in V$, with s_1, s_2 in one colour class and t_1, t_2 in the other. Choose odd integers $d_1 \leq \text{dist}_G(s_1, t_1)$ and $d_2 \leq \text{dist}_G(s_2, t_2)$ such that $d_1 + d_2 \leq \text{dist}_G(s_1, s_2) + \text{dist}_G(t_1, t_2)$ and $d_1 + d_2 \leq \text{dist}_G(s_1, t_2) + \text{dist}_G(t_1, s_2)$. Then there exist disjoint cuts, d_1 of which separate s_1 and t_1 and not s_2 and t_2, and d_2 of which separate s_2 and t_2 and not s_1 and t_1.

Let $\mathcal{S}$ be a collection of nonempty proper subsets of a finite set T. Let $G = (V, E)$ be a graph with $V \supseteq T$. Let $\mathcal{A}$ be the collection of subsets U of V with $U \cap T \in \mathcal{S}$.

Consider any length function $l : E \to \mathbb{R}_+$ and any $d : \mathcal{S} \to \mathbb{R}_+$. The multicut analogue of the multiflow problem asks for a function $y : \mathcal{A} \to \mathbb{R}_+$ such that

$$(71.16) \qquad \sum_{U \in \mathcal{A}} y(U) \chi^{\delta_E(U)} \leq l$$

and such that

$$(71.17) \qquad \sum (y(U) \mid U \in \mathcal{A}, U \cap T = X) = d(X)$$

for each $X \in \mathcal{S}$. A necessary condition for the existence of y is that

$$(71.18) \qquad \text{dist}_l(s, t) \geq \sum (d(X) \mid X \in \mathcal{S}, X \text{ splits } s, t)$$

for all distinct $s, t \in T$. (Here X *splits* s, t if X contains precisely one of s, t.) Karzanov [1984] showed that this condition is sufficient for each graph G and each l if and only if $\mathcal{S}$ contains no three pairwise crossing sets. If moreover l and d are integer, there is a half-integer y. If moreover $l(C)$ is even for each circuit C and both sides of (71.18) have the same parity for all s, t, then there is an integer y. Karzanov [1984] also gave a polynomial-time greedy-type algorithm to find y.

As for the corresponding maximization problem, consider any length function $l : E \to \mathbb{R}_+$ and the problem

$$(71.19) \qquad \min\{l^{\mathsf{T}} x \mid x \in \mathbb{R}_+^E : x(\delta_E(U)) \geq 1 \text{ for each } U \in \mathcal{A}\}.$$

By linear programming duality, this minimum is equal to the maximum value of

$$(71.20) \qquad \sum_{U \in \mathcal{A}} y(U),$$

where $y : \mathcal{A} \to \mathbb{R}_+$ satisfies (71.16). Karzanov [1984] showed the following. Let $\mathcal{S}$ have the following property: for any three pairwise crossing sets A_1, A_2, A_3 in $\mathcal{S}$, there exist $\gamma_1, \gamma_2, \gamma_3 \geq 0$ and $z : \mathcal{S} \to \mathbb{R}_+$ such that

(71.21) $$\sum_{i=1}^{3} \gamma_i \leq \sum_{U \in \mathcal{S}} z_U \text{ and } \sum_{i=1}^{3} \gamma_i \chi^{\delta_R(A_i)} > \sum_{U \in \mathcal{S}} z_U \chi^{\delta_R(U)},$$

where $R := \{st \mid s, t \in T, s \neq t\}$. Then the maximum value of (71.20) is equal to the minimum value of

(71.22) $$\sum_{r \in R} \beta(r) \mathrm{dist}_l(r),$$

where $\beta : R \to \mathbb{R}_+$ satisfies $\beta(\delta_R(U)) \geq 1$ for each $U \in \mathcal{S}$. (We write $\mathrm{dist}_l(r)$ for $\mathrm{dist}_l(s,t)$ if $r = st$.) This specifies the above $x : E \to \mathbb{R}_+$ as a function

(71.23) $$x = \sum_{r \in R} \beta(r) \chi^{P_r},$$

where P_r is a shortest r-path with respect to l, since

(71.24) $$l^{\mathsf{T}} x = \sum_{r \in R} \beta(r) l(P_r) \text{ and } x(\delta_E(U)) \geq \sum_{r \in \delta_R(U)} \beta(r) = \beta(\delta_R(U)) \geq 1$$

for each $U \in \mathcal{S}$. (An *r-path* is a path connecting the vertices in r.) Again this characterization is tight.

This has as special cases theorems on packing $s-t$ cuts (Theorem 6.1), 2-commodity cuts (Theorem 71.3), and T-cuts (Corollary 29.9a).

These cases are further characterized by the following result of Karzanov [1985a]. Let T be a finite set and let $\mathcal{S}$ be a collection of nonempty proper subsets of T such that (i) if $U \in \mathcal{S}$, then $T \setminus U \in \mathcal{S}$, (ii) for each $t \in T$ there is a $U \in \mathcal{S}$ with $t \in U$ and $U \setminus \{t\} \notin \mathcal{S}$, (iii) for all distinct $s, t \in T$ there is a $U \in \mathcal{S}$ separating s and t. Let G be the complete graph on vertex set V with $V \supset T$ and $|V| \geq |T| + 2$. Then minimum (71.19) is attained by an integer optimum solution x for each $l : E \to \mathbb{R}_+$ if and only if:

(71.25)
(i) there exist $s, t \in T$ such that each set in $\mathcal{S}$ contains exactly one of s and t, and such that the collection of sets in $\mathcal{S}$ containing s, is closed under unions and intersections,
or (ii) $T = \{s_1, t_1, s_2, t_2\}$ and $\mathcal{S} = \{\{s_1, s_2\}, \{s_1, t_2\}, \{t_1, s_2\}, \{t_1, t_2\}\}$,
or (iii) $\mathcal{S}$ is equal to the collection of odd-size subsets of T, where $|T|$ is even.

71.4. Further results and notes

71.4a. Two disjoint paths in undirected graphs

The polynomial-time solvability of the 2 vertex-disjoint paths problem in undirected graphs was shown by Seymour [1980b], Shiloach [1980b], and Thomassen [1980]. As was observed by Seymour [1980b], this can be derived from the following characterization of Seymour [1980b] and Thomassen [1980] (as usual, $N(K)$ denotes the set of vertices not in K adjacent to at least one vertex in K):

Theorem 71.5. *Let $G = (V, E)$ be a graph and let s_1, t_1, s_2, t_2 be distinct vertices. Then G has disjoint paths P_1 and P_2, where P_i connects s_i and t_i ($i = 1, 2$), if and only if there is no subset U of V such that:*

(71.26) (i) $s_1, t_1, s_2, t_2 \in U$,
(ii) $|N(K)| \leq 3$ *for each component* K *of* $G - U$*;*
(iii) *the graph* H *obtained from* $G[U]$ *by adding, for each component* K *of* $G - U$ *and each distinct* $u, v \in N(K)$*, an edge connecting* u *and* v*, is planar, with* s_1, s_2, t_1, t_2 *in this order cyclically on the outer boundary of* H*.*

In fact, condition (ii) is superfluous. (Theorem 71.5 was proved for 4-connected graphs by Jung [1970], generalizing Watkins [1968] who proved it for 4-connected graph containing a subdivision of K_5.)

The polynomial-time solvability of the 2 vertex-disjoint paths problem can be derived by observing that we can reduce the problem if there is a $K \subseteq V \setminus \{s_1, t_1, s_2, t_2\}$ with $|N(K)| \leq 3$ (remove K and add edges as in (71.26)(iii)). So we can assume that no such K exists. Hence, by the characterization, if no paths as required exist, the graph should be planar with the terminals in the cyclic order s_1, s_2, t_1, t_2 along the outer boundary — this can be tested in polynomial time. (Khuller, Mitchell, and Vazirani [1992] gave a parallel implementation.)

A related result for the 2 edge-disjoint paths problem was given by Dinits and Karzanov [1979] and Seymour [1980b]:

(71.27) Let $G = (V, E)$ be a connected graph and let $s_1, t_1, s_2, t_2 \in V$. Then G has edge-disjoint paths P_1 and P_2, where P_i connects s_i and t_i ($i = 1, 2$) if and only if the cut condition holds and there is no $F \subseteq E$ such that the graph G/F, obtained from G by contracting all edges in F, is connected and planar and has maximum degree ≤ 3, while s_1, s_2, t_1, t_2 are distinct, all have degree at most 2, and occur in this order around the outer boundary of G/F.

This implies in particular that if G is 3-edge-connected, then the 2 edge-disjoint paths problem has a solution, for any choice of two nets.

71.4b. A directed 2-commodity flow theorem

Frank [1989] observed that a directed version of the 2-commodity flow theorem holds:

Theorem 71.6. *Let* $D = (V, A)$ *be as digraph, and let* R *consist of two parallel classes of arcs, with* $(V, A \cup R^{-1})$ *Eulerian. Then the cut condition is necessary and sufficient for the solvability of the arc-disjoint paths problem.*

Proof. Let R consist of k_i parallel arcs from s_i to t_i, for $i = 1, 2$. With Menger's theorem, the cut condition implies that there exist k_1 arc-disjoint $s_1 - t_1$ paths in D. After deleting the arcs of these paths from D, the remainder has k_2 arc-disjoint $s_2 - t_2$ paths, as adding k_2 parallel $t_2 - s_2$ arcs makes the remainder Eulerian. ∎

This proof also gives a polynomial-time algorithm. We should note that in the directed case, Eulericity is rather prohibitive: unlike in the undirected case we cannot make a digraph Eulerian by some simple doubling argument.

Frank, Ibaraki, and Nagamochi [1995,1998] gave a characterization and polynomial-time algorithm for the problem: given an Eulerian digraph $D = (V, A)$ and

$s_1, t_1, s_2, t_2 \in V$, find two arc-disjoint directed paths P_1 and P_2, where P_i connects s_i and t_i, in one way or the other ($i = 1, 2$). The characterization is analogous to Theorem 71.5.

It implies a characterization and algorithm of Ibaraki and Poljak [1991] for the 3 arc-disjoint paths problem if the Euler condition holds. For let $D = (V, A)$ be a digraph, and let $s_1, t_1, \ldots, s_3, t_3 \in V$, such that for $R := \{(s_1, t_1), (s_2, t_2), (s_3, t_3)\}$, the digraph $(V, A \cup R^{-1})$ is Eulerian. Extend D by four new vertices x_1, y_1, x_2, y_2 and arcs (t_1, x_2), (x_2, s_2), (t_3, y_1), (y_1, s_1), (t_2, y_2), (y_2, x_1), (x_1, s_3). Then the new digraph is Eulerian. Moreover, it has arc-disjoint directed $x_i - y_i$ paths (for $i = 1, 2$) if and only if D has arc-disjoint $s_i - t_i$ paths (for $i = 1, 2, 3$).

71.4c. Kleitman, Martin-Löf, Rothschild, and Whinston's theorem

Let G be an undirected graph. Suppose that we have four disjoint sets S_1, T_1, S_2, T_2 of vertices, and that we want to know the maximum number of edge-disjoint paths, each connecting either S_1 and T_1, or S_2 and T_2. Generally it is *not true* that the maximum number of such paths is equal to the minimum number of edges intersecting each such path. This even is not the case if the graph is Eulerian, as is shown by the graph in Figure 71.5 (cf. Rothschild and Whinston [1966b]). (One

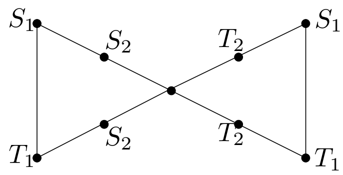


Figure 71.5

The maximum number of edge-disjoint paths each connecting vertices labeled S_i and T_i for some i, is equal to 4, whereas the minimum size of an edge set intersecting each such path is equal to 5. Note that the graph is Eulerian.

could think of a proof method based on adding 4 new vertices s_1, t_1, s_2, t_2, adjacent, by a large number of parallel edges, to all vertices in S_1, T_1, S_2, T_2 respectively, and then applying Corollary 71.1d. But this procedure can create new paths, for instance, from S_1 to T_1 via s_2.)

However, if S_1, T_1, S_2, T_2 partition the vertex set, such a generalization holds, as was shown by Kleitman, Martin-Löf, Rothschild, and Whinston [1970]. In fact, they showed a more general result, that can be proved with the help of the following theorem equivalent to (the edge-disjoint undirected version of) Menger's theorem (which is the special case where A and B are stars).

If $G = (V, E)$ a graph and $A, B \subseteq E$, we say that a path *connects* A and B if it traverses at least one edge in A and at least one edge in B.

Theorem 71.7. *Let $G = (V, E)$ be a graph and let $A, B \subseteq E$. Then the maximum number of edge-disjoint paths each connecting A and B is equal to the minimum number of edges intersecting each such path.*

Proof. We can assume that $A \cap B = \emptyset$, since deleting any edge in $A \cap B$ reduces both optima by 1.

Construct a new graph H as follows. Add two new vertices s and t. For each edge $e \in A \cup B$, put a new vertex v_e on e, and connect it to s if $e \in A$ and to t if $e \in B$.

We apply Menger's theorem to the $s - t$ paths in H. Let $Q_1, \ldots, Q_k$ be a maximum number of edge-disjoint $s - t$ paths in H. Consider any of these paths Q_j. We can assume that the second vertex, v_a say, and the one but last vertex, v_b say, of Q_j are the only two vertices on Q_j that belong to $\{v_e \mid e \in A \cup B\}$ (otherwise we can shortcut Q_j, since each vertex v_e has degree 3). Replacing the first two edges of Q_j by edge a of G, and the last two edges of Q_j by edge b of G, we obtain a path P_j in G connecting A and B.

This gives k edge-disjoint paths in G each connecting A and B. By Menger's theorem, there exists a set D of k edges of H intersecting each $s - t$ path. For $e \in A \cup B$, replacing (in D) any edge sv_e and any of the split-offs of e, by e, we obtain a set C of at most k edges in G that intersects each path connecting A and B. Indeed, consider any path P in G connecting A and B. We can assume that it intersects A and B only at its ends. So we can transform P to an $s - t$ path Q in H, by deviating the end edges towards s and t. Then Q intersects D, implying that P intersects C. This shows the theorem. ∎

This implies the theorem of Kleitman, Martin-Löf, Rothschild, and Whinston [1970]:

Corollary 71.7a. *Let $G = (V, E)$ be a graph, let $S_1, T_1, \ldots, S_k, T_k$ be subsets of V, with $S_i \cap T_i = \emptyset$ for $i = 1, \ldots, k$, and define $U_i := V - S_i - T_i$ for $i = 1, \ldots, k$. If $U_1, \ldots, U_k$ are disjoint, then the maximum number of edge-disjoint paths among $\{P \mid \exists i : P$ is an $S_i - T_i$ path$\}$ is equal to the minimum number of edges intersecting each such path.*

Proof. We can assume that the U_i partition V, since we can add an extra pair S_0, T_0 with $S_0 := V - U_1 - \cdots - U_k$ and $T_0 := \emptyset$. We can also assume that, for any i, no edge connects S_i and T_i, since deleting it reduces both optima by 1.

Let R be the set of edges connecting distinct sets among $U_1, \ldots, U_k$. Then for each i, any inclusionwise minimal $S_i - T_i$ path has its end edges in R and has no other edges in R (since all internal vertices belong to U_i). Let A be the set of edges e in R such that e is disjoint from an even number of $S_1, \ldots, S_k$, and let $B := R - A$.

The sets A and B have the following property. Let P be a path with only its end edges in R. Then:

(71.28) P connects S_i and T_i for some i if and only if P connects A and B.

With Theorem 71.7, this immediately proves the present corollary.

To prove (71.28), let u and w be the first and last vertex of P, let I be the set of internal vertices of P, and let c and d be the first and last edge of P. Since only the end edges of P are in R, we know by definition of R that there exists an i such that each internal edge of P only meets U_i and such that $u, v \notin U_i$. In other words, $I \subseteq U_i$ and $u, w \in S_i \cup T_i$.

Consider any $j \neq i$. As $I \subseteq U_i$ and $U_i \cap U_j = \emptyset$, we know $I \subseteq S_j \cup T_j$. As no edge connects S_j and T_j, either $I \subseteq S_j$ and $u, w \in U_j \cup S_j$, or $I \subseteq T_j$ and $u, w \in U_j \cup T_j$. So $c \cap S_j = \emptyset$ if and only if $d \cap S_j = \emptyset$. Hence, by definition of A and B:

(71.29) P connects A and B $\iff$ precisely one of $c \cap S_i$ and $d \cap S_i$ is nonempty $\iff$ precisely one of u, w belongs to S_i, the other belongs to T_i $\iff$ P connects S_i and T_i.

So we have (71.28). ∎

The proof method directly gives an algorithmic reduction to the (one-commodity) disjoint paths problem. (Kleitman [1971] and Kant [1974] describe other methods.)

71.4d. Further notes

Itai and Zehavi [1984] showed that if $G = (V, E)$ is a graph and $s_1, t_1, s_2, t_2 \in V$ are such that for $i = 1, 2$, there exist k edge-disjoint $s_i - t_i$ paths, then for each choice of d_1, d_2 with $d_1 + d_2 = k$, there exist d'_1 and d'_2 with $d'_1 + d'_2 = k$, $d_1 \leq d'_1 \leq d_1 + 1$, and a collection of edge-disjoint paths such that d'_i of them connect s_i and t_i $(i = 1, 2)$.

The integer 2-commodity flow problem is solvable in polynomial time if $G + H$ is planar — see Section 74.2b.

Rebman [1974] studied a generalization of totally unimodular matrices appropriate for 2-commodity flows.

Chapter 72

Three or more commodities

Hu's 2-commodity theorem concerns multiflows where the demand graph H consists of two edges — whatever the supply graph is. In this chapter we consider to which extent Hu's theorem can be generalized to other demand graphs. That is, we study for which graphs $H = (T, R)$ it is true that for *each* graph $G = (V, E)$ with $V \supseteq T$ and each capacity and demand functions the phenomena described in the previous chapter are maintained (sufficiency of the cut condition, existence of a half-integer multiflow, sufficiency of the Euler condition to obtain an integer multiflow).

Results of Papernov [1976], Lomonosov [1976,1985], and Seymour [1980c] give an answer to this question: the graphs H are those containing neither of the two graphs in Figure 72.1 below as a subgraph. These are exactly the graphs H that are the union of two stars or are equal to K_4 or C_5 (up to adding isolated vertices, loops, and parallel edges).

Except if stated otherwise, throughout this chapter $G = (V, E)$ and $H = (T, R)$ denote the supply and demand graph, in the sense of Chapter 70. The pairs in R are called the *nets*. If $s_1, t_1, \ldots, s_k, t_k$ are given, then $R := \{s_1t_1, \ldots, s_kt_k\}$. If demands $d_1, \ldots, d_k$ are given, then $d(s_it_i) = d_i$. We denote $G + H = (V, E \cup R)$, where the disjoint union of E and R is taken, respecting multiplicities.

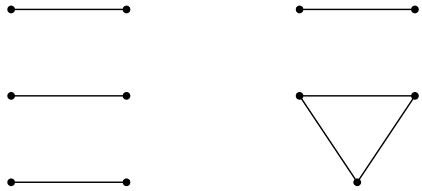
72.1. Demand graphs for which the cut condition is sufficient

Consider for any graph $H = (T, R)$ the following property:

(72.1) H has neither of the two graphs of Figure 72.1 as a subgraph.

Theorem 72.1. *Let $H = (T, R)$ be a simple graph without isolated vertices. Then H satisfies* (72.1) *if and only if $H = K_4$, or $H = C_5$, or H is the union of two stars.*

Proof. Sufficiency is direct. Necessity is shown by induction on $|R|$. If all degrees of H are at most 2, the theorem is easy. Assume now that H has a vertex u of degree at least 3. For any edge $e = uw$ incident with u, if $H - e$ is K_4 or C_5 (after deleting any isolated vertex), then H contains one of the



(a) (b)

Figure 72.1

graphs in Figure 72.1. So $H - e \neq K_4$ and $H - e \neq C_5$. Hence, by induction, there exist two vertices s, t such that each edge of $H - e$ intersects $\{s, t\}$. If $u \in \{s, t\}$, then H is the union of two stars. So we can assume that $u \notin \{s, t\}$. Hence each neighbour v of u with $v \neq w$ belongs to $\{s, t\}$. So u has degree 3, and each edge f of H not incident with u connects two neighbours of u (as any neighbour of u can serve as w). So $H = K_4$. ∎

The following theorem extends Theorem 71.1 and Corollary 71.1c, and was proved by Lomonosov [1976,1985] and Seymour [1980c] for $H = K_4$, and by Lomonosov [1976,1985] for $H = C_5$. The proof below is inspired by the direct proof given by Frank [1990e]. (Here $G + H$ is the graph $(V, E \cup R)$, taking multiplicities of edges into account.)

Theorem 72.2. *Let $G = (V, E)$ and $H = (T, R)$ be supply and demand graphs, with $T \subseteq V$. Let $H = (T, R)$ satisfy (72.1), with $G + H$ Eulerian. Then there exist edge-disjoint paths P_r (for $r \in R$) such that P_r connects the vertices in r if and only if the cut condition holds.*

Proof. Necessity being easy, we show sufficiency. Let G, H form a counterexample with $|E| + |R|$ minimal. Then G is connected. Also, there is no net $r \in R$ parallel to an edge $e \in E$, since otherwise deleting r and e would give a smaller counterexample.

Call a subset U of V *tight* if $d_E(U) = d_R(U)$.[17] By the minimality of the counterexample we have[18]

(72.2) for each pair of edges e and f incident with a vertex v there is a tight set splitting both e and f.

Otherwise we can replace $e = uv$ and $f = wv$ by a new edge uw to obtain a smaller counterexample.

Another observation is:[19]

[17] As usual, $d_E(U) = |\delta_E(U)|$ and $d_R(U) = |\delta_R(U)|$.

[18] A set X *splits* a pair uv if X contains exactly one of u and v.

[19] $F[X, Y]$ denotes the set of pairs xy in F with $x \in X$ and $y \in Y$.

(72.3) for each tight set X and each $v \in V \setminus X$ we have $|E[X,v]| - |R[X,v]| \leq \frac{1}{2}(\deg_E(v) - \deg_R(v))$,

since, setting $X' := X \cup \{v\}$ we have $d_E(X') = d_E(X) + \deg_E(v) - 2|E[X,v]|$ and $d_R(X') = d_R(X) + \deg_R(v) - 2|R[X,v]|$. Then $d_E(X) = d_R(X)$ and $d_E(X') \geq d_R(X')$ give (72.3).

The following is also useful to observe:

(72.4) if X and Y are tight, and no net connects $X \setminus Y$ and $Y \setminus X$, then $X \cap Y$ and $X \cup Y$ are tight again, and no edge connects $X \setminus Y$ and $Y \setminus X$.

To see this, consider:

(72.5)
$$\begin{aligned}
&d_R(X) + d_R(Y) = d_E(X) + d_E(Y)\\
&= d_E(X \cap Y) + d_E(X \cup Y) + 2|E[X \setminus Y, Y \setminus X]|\\
&\geq d_E(X \cap Y) + d_E(X \cup Y) \geq d_R(X \cap Y) + d_R(X \cup Y)\\
&= d_R(X) + d_R(Y).
\end{aligned}$$

So we have equality throughout, proving (72.4).

This implies:

(72.6) let B be a set of vertices intersecting all nets, and let X and Y be tight sets with $X \cap B = Y \cap B$. Then $X \cap Y$ and $X \cup Y$ are tight.

Otherwise, by (72.4) there is a net connecting $X \setminus Y$ and $Y \setminus X$, and hence not intersecting B, a contradiction.

We next show that for each terminal t:[20]

(72.7) $\deg_E(t) = \deg_R(t)$.

Assume $\deg_E(t) > \deg_R(t)$. Let $\mathcal{X}$ be the collection of inclusionwise maximal tight subsets of $V \setminus \{t\}$. For each edge or net p, let $\mathcal{X}_p$ denote the set of $U \in \mathcal{X}$ splitting p.

We have $|\mathcal{X}_e| \geq 2$ for each edge $e \in \delta_E(t)$, since by (72.2), each pair of edges incident with t is split by some $U \in \mathcal{X}$, and since no tight set X splits all edges incident with t simultaneously, as it would imply

(72.8)
$$\begin{aligned}
&d_E(X \cup \{t\}) = d_E(X) - \deg_E(t) = d_R(X) - \deg_E(t)\\
&< d_R(X) - \deg_R(t) \leq d_R(X \cup \{t\}).
\end{aligned}$$

Also we have $|\mathcal{X}_r| \leq 2$ for each $r \in \delta_R(t)$, and we have $|\mathcal{X}| \leq 4$. Indeed, let $r = st$. By (72.1), there exists a vertex u such that each net intersects $B := \{s, t, u\}$. Therefore, by (72.6), any two sets in $\mathcal{X}$ have a different intersection with B (as otherwise their union is tight, contradicting their maximality). As no set in $\mathcal{X}$ contains t, we have the required inequalities.

This gives with (72.3):

[20] A *terminal* is a vertex covered by at least one net.

(72.9)
$$2(\deg_E(t) - \deg_R(t)) \leq \sum_{e\in\delta_E(t)} |\mathcal{X}_e| - \sum_{r\in\delta_R(t)} |\mathcal{X}_r|$$
$$= \sum_{U\in\mathcal{X}} (|E[U,t]| - |R[U,t]|) \leq \tfrac{1}{2}|\mathcal{X}|(\deg_E(t) - \deg_R(t))$$
$$\leq 2(\deg_E(t) - \deg_R(t)).$$

Hence equality holds throughout in (72.9). So $|\mathcal{X}| = 4$, $|\mathcal{X}_e| = 2$ for each $e \in \delta_E(t)$, and $|\mathcal{X}_r| = 2$ for each $r \in \delta_R(t)$. Hence the $\mathcal{X}_e$ form a graph on the vertex set $\mathcal{X}$, such that any two of its edges intersect (by (72.2)), but no vertex is in all edges (since no tight set splits all edges in $\delta_E(t)$). So there is a $U \in \mathcal{X}$ contained in no $\mathcal{X}_e$; that is, $E[U,t] = \emptyset$. Since we have equality in (72.3) (as we have equality throughout in (72.9)), it follows that $\deg_E(t) - \deg_R(t) = 0$, that is, we have (72.7).

(72.7) implies that

(72.10) no two terminals s and t are adjacent,

since otherwise $d_E(\{s,t\}) < \deg_E(s) + \deg_E(t) = \deg_R(s) + \deg_R(t) = d_R(\{s,t\})$, contradicting the cut condition.

Now choose $st \in R$. By (72.1), there is a vertex $u \notin \{s,t\}$ such that each commodity disjoint from st intersects u. We can assume that u is a terminal, as otherwise s and t are the only terminals, in which case the theorem follows from Menger's theorem. Since, by (72.7), $V \setminus \{u\}$ is tight, there exists a tight subset Z which is inclusionwise minimal under the conditions that $s,t \in Z$ and $u \notin Z$.

Then s has a neighbour $v \in Z$. Otherwise we have

(72.11)
$$d_E(Z) = \deg_E(s) + d_E(Z \setminus \{s\}) \geq \deg_R(s) + d_R(Z \setminus \{s\}) > d_R(Z)$$

(as $s,t \in Z$), contradicting the tightness of Z.

Let $\mathcal{Y}$ be the collection of all inclusionwise maximal tight subsets of $V \setminus \{v\}$ containing s. By (72.10), v is not a terminal. Hence, by (72.3), $|E[Y,v]| \leq \frac{1}{2}\deg_E(v)$ for each $Y \in \mathcal{Y}$. Therefore, since (by (72.2)) each edge incident with v is split by at least one $Y \in \mathcal{Y}$, we have $|\mathcal{Y}| \geq 3$.

Then by (72.6), the sets in $\mathcal{Y}$ all have different intersections with $\{t,u\}$. (By definition, each set in $\mathcal{Y}$ contains s.) Moreover, $Y \cap \{t,u\} \neq \{t\}$ for each $Y \in \mathcal{Y}$, since otherwise also $Y \cap Z$ is tight (by (72.6), as $Z \cap \{t,u\} = \{t\}$), contradicting the minimality of Z (note that $v \notin Y \cap Z$).

So $|\mathcal{Y}| = 3$ and the sets in $\mathcal{Y}$ intersect $\{t,u\}$ in $\emptyset$, $\{u\}$, and $\{t,u\}$ — denote these sets by S, U, and W, respectively (cf. Figure 72.2).

By the maximality of S, U, and W, $S \cup U$ and $U \cup W$ are not tight. Hence, by (72.4), there is a net γ connecting $S \setminus U$ and $U \setminus S$, and a net δ connecting $W \setminus U$ and $U \setminus W$. Then γ and δ intersect $\{s,t,u\}$. As $s,t \notin \gamma$ (since $s,t \in S \cap U$) and $u \notin S \setminus U$, we know $\gamma = uw$ for some $w \in S \setminus U$. As $s,u \notin \delta$ (since $s \in S \cap W$ and $u \notin S \cup W$) and $t \notin U \setminus W$, we know $\delta = tx$ for some $x \in U \setminus W$. As st and tx are disjoint from uw, each net disjoint from uw contains t.

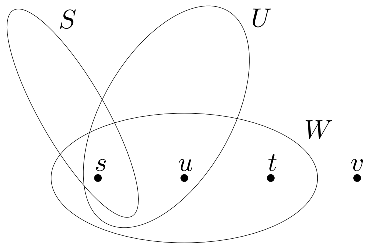
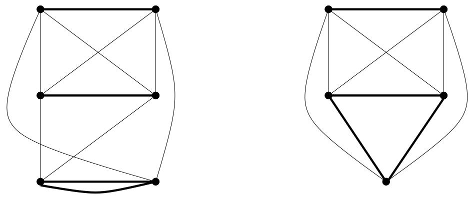


Figure 72.2

However, as edge sv connects $W \cap S$ and $V \setminus (W \cup S)$, by (72.4) (applied to $X := W$ and $Y := V \setminus S$), there is a net sa connecting these two sets. Then $a \neq u, w, t$, and therefore sa is disjoint from u, w, t, a contradiction. ∎



(a) (b)

Figure 72.3
Examples where the cut and Euler conditions hold, but no fractional multiflow exists. The heavy lines are the nets and the other lines the edges. All capacities and demands are equal to 1.

Theorem 72.2 also holds if H consists of three disjoint edges — see Theorem 72.3. The examples in Figure 72.3 (from Papernov [1976]) show that the condition on the demand graph H in Theorem 72.2 is close to tight. This is made more precise in the following characterization implied by Theorem 72.2 (the equivalence (i)⇔(iv)⇔(v) is due to Papernov [1976], the other equivalences to Lomonosov [1976,1985] and (for K_4) to Seymour [1980c]):

Corollary 72.2a. *For each simple graph $H = (T, R)$ without isolated vertices, the following are equivalent:*

(72.12) (i) *for each graph $G = (V, E)$ with $V \supseteq T$, and each $c : E \to \mathbb{R}_+$ and $d : R \to \mathbb{R}_+$, the cut condition implies the existence of a fractional multiflow;*

(ii) *for each graph $G = (V, E)$ with $V \supseteq T$, and each $c : E \to \mathbb{Z}_+$ and $d : R \to \mathbb{Z}_+$, the cut condition implies the existence of a half-integer multiflow;*
(iii) *for each graph $G = (V, E)$ with $V \supseteq T$, and each $c : E \to \mathbb{Z}_+$ and $d : R \to \mathbb{Z}_+$, the cut and Euler conditions imply the existence of an integer multiflow;*
(iv) *H contains none of the graphs in Figure* 72.1 *as subgraph;*
(v) *$H = K_4$, or $H = C_5$, or H is the union of two stars.*

Proof. The equivalence (iv)⇔(v) was shown in Theorem 72.1. The implications (iii)⇒(ii)⇒(i) are general multiflow theory, while (i)⇒(iv) follows from the examples of Figure 72.3. The implication (iv)⇒(iii) follows from Theorem 72.2, by replacing each edge of G by $c(e)$ parallel edges and each edge of H by $d(e)$ parallel edges. ∎

Karzanov [1979b] gave a strongly polynomial-time algorithm finding a half-integer multiflow as required if H satisfies (72.12)(iv) (or finding a cut violating the cut condition).

72.2. Three commodities

An important case excluded by the theorems in the previous sections is that of a demand graph consisting of three disjoint edges, with $d = \mathbf{1}$.

Theorem 72.3. *Let $G = (V, E)$ and $H = (T, R)$ be graphs, with $T \subseteq V$, such that $G + H$ is Eulerian and such that R consist of three disjoint edges. Then there exist edge-disjoint paths P_r (for $r \in R$) such that P_r connects the vertices in r if and only if the cut condition holds.*

Proof. Let $R = \{r_1, r_2, r_3\}$. Let G be a counterexample with a minimum number of edges. Then G is connected, and each vertex of G has degree at least two. Call $U \subseteq V$ *tight* if $d_E(U) = d_R(U)$. Also:

(72.13) $d_R(U) = 3$ for each tight nonempty proper subset U of V.

To see this, let U be a counterexample with $d_E(U)$ smallest. Then $G[U]$ and $G-U$ are connected. (Otherwise we could replace U by one of the components K of $G[U]$ or $G - U$, while $d_E(K) < d_E(U)$.) Also, $d_R(U) = d_E(U) \geq 1$ as G is connected. So we can assume that $r_1 \in \delta_R(U)$ and that $V \setminus U$ spans r_2. Contract U to obtain graph G/U. As the cut condition remains to hold, and as G/U is smaller than G (since $|U| \geq 2$, as $d_E(v) \geq 2 > d_R(v)$ for each $v \in V$), G/U contains edge-disjoint paths Q_1 and Q_2 where Q_i connects (the contractions of) the vertices in r_i ($i = 1, 2$). As $G[U]$ is connected, $G[U]$ contains a path connecting the vertex in $r_1 \cap U$ and the end of the edge in $\delta_E(U)$ that is traversed by Q_1. It follows that G contains two edge-disjoint paths P_1 and P_2 where P_i connects the vertices in r_i $(i = 1, 2)$. Removing the edges of P_1 and P_2 from G, we are left with a graph with exactly two vertices of odd degree, namely the vertices in the pair r_3. Hence this graph contains a path P_3 connecting the vertices in r_3. Then P_1, P_2, and P_3 are as required. This is a contradiction, proving (72.13).

Consider now any $r = st \in R$ and any edge tu of G incident with t. Let $R' := (R \setminus \{st\}) \cup \{su\}$. Let $G' = (V, E')$ be the graph obtained from G by deleting edge tu. If the cut condition holds for G', R', we obtain (by induction) three paths in G' that directly yields paths as required in G. So we can assume that there is a subset U of V with $d_{R'}(U) > d_{E'}(U)$ and $t \notin U$. As $G' + H'$ is Eulerian, we know $d_{R'}(U) \geq d_{E'}(U) + 2$. Then

(72.14) $$d_R(U) \geq d_{R'}(U) - 1 \geq d_{E'}(U) + 1 \geq d_E(U) \geq d_R(U).$$

So we have equality throughout, and hence $d_E(U) = d_R(U)$, and $s \notin U$, $u \in U$ (otherwise $d_{R'}(U) \geq d_R(U)$). So $d_R(U) \leq 2$, contradicting (72.13). ∎

With Theorem 72.2, this implies the following characterization:

Corollary 72.3a. *For any loopless graph $H = (T, R)$ without isolated vertices, the following are equivalent:*

(72.15) (i) *for each graph $G = (V, E)$ with $V \supseteq T$ satisfying the cut and Euler condition (with respect to H), the edge-disjoint paths problem has a solution;*

(ii) *T has two vertices intersecting all pairs in R, or $|T| \leq 4$, or H is C_5 with parallel edges added, or R consists of three disjoint edges;*

(iii) *H has no subgraph equal to , , , , or .*

Proof. The implication (iii)⇒(ii) follows from Theorem 72.1. The implication (ii)⇒(i) follows from Theorems 72.2 and 72.3. The implication (i)⇒(iii) follows from the examples in Figure 72.3, since from each of graphs given in (iii) we can obtain or , by identifying some vertices. Then from the examples in Figure 72.3 we can obtain examples for the graphs in (iii) by adding two parallel edges between any pair of identified vertices. ∎

Notes. For $|R| = 3$, Okamura [1984a] showed that the cut condition implies the existence of a half-integer solution for the edge-disjoint paths problem. (This seems not to follow from Theorem 72.3. On the other hand, having Okamura's result, to prove Theorem 72.3 it suffices to show that if the Euler condition holds and a half-integer solution exists, there is an integer solution.)

This implies the following characterization, extending Corollary 72.3a.

Theorem 72.4. *For any loopless graph $H = (T, R)$ without isolated vertices, the following are equivalent:*

(72.16) (i) *for each graph $G = (V, E)$ with $V \supseteq T$ satisfying the cut condition, the edge-disjoint paths problem has a fractional solution;*

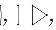
(ii) *for each graph $G = (V, E)$ with $V \supseteq T$ satisfying the cut condition, the edge-disjoint paths problem has a half-integer solution;*
(iii) *for each graph $G = (V, E)$ with $V \supseteq T$ satisfying the cut and Euler condition, the edge-disjoint paths problem has a solution;*
(iv) *T has two vertices intersecting all pairs in R, or $|T| \leq 4$, or H is C_5 with parallel edges added, or R consists of three disjoint pairs;*
(v) *H has no subgraph equal to* , , , , *or* .

Figure 70.4 shows that there is no integer p such that if a 3-commodity problem, with integer capacities and demands, has a fractional solution, then it has a $1/p$-integer solution. More precisely, for each integer $k \geq 2$, there is a graph $G = (V, E)$ and a collection R of three disjoint pairs from V, such that for $c : E \to \mathbb{Z}_+$ defined by $c(e) = 1$ for each edge e and $d : R \to \mathbb{Z}_+$ with values $1, 2k, 2k$ respectively, there is a fractional multiflow, but each feasible solution has some of its values equal to $1/2k$.

By doubling capacities and demands, one obtains an example of a 3-commodity flow problem satisfying the Euler condition, where a fractional but no half-integer multiflow exists. A variant of the example gives a 3-commodity flow problem satisfying the Euler condition, where a half-integer but no integer solution exists.

M. Middendorf and F. Pfeiffer (cf. Pfeiffer [1990]) showed that it is NP-complete to decide if the edge-disjoint paths problem has a half-integer solution, even if the nets consist of three disjoint parallel classes of edges. This implies a result of Vygen [1995] that it is NP-complete to decide if the edge-disjoint paths problem has a solution, even if the nets consist of three disjoint parallel classes of edges and the Euler condition holds.

Let H_6 be the graph obtained from $K_{3,3}$ by adding in each of the two colour class one new edge (cf. Figure 72.3(a)). Seymour [1981a] showed for each graph $G = (V, E)$:

(72.17) G has no H_6 minor if and only if for each $R \subseteq E$ with $|R| \leq 3$ and each $c : E \setminus R \to \mathbb{Z}_+$ and $d : R \to \mathbb{Z}_+$ satisfying the Euler condition, the cut condition implies the existence of an integer multiflow (where $(V, E \setminus R)$ is the supply graph).

The proof is by showing that each 3-connected graph without H_6 minor is K_5 or has no K_5 minor, and hence can be decomposed into planar graphs and copies of V_8 (Wagner's theorem (Theorem 3.3)).

72.2a. The $K_{2,3}$-metric condition

Karzanov [1987a] showed that a strengthened form of the cut condition, the '$K_{2,3}$-metric condition', is sufficient for having a fractional multiflow for a class of demand graphs larger than described in Section 72.1.

This is described as follows. Let Γ be a graph and let V be a finite set. A metric μ on V is called a *Γ-metric* if there is a function $\phi : V \to V\Gamma$ with

(72.18) $$\mu(u, v) = \mathrm{dist}_\Gamma(\phi(u), \phi(v))$$

for all $u, v \in V$. (Here $\mathrm{dist}_\Gamma(x, y)$ denotes the distance of x and y in Γ.)

Γ-metrics give rise to the following necessary condition, the *Γ-metric condition*, for the existence of a feasible fractional multiflow:

(72.19) $$\sum_{r=st\in R} d(r)\mu(s,t) \leq \sum_{e=uv\in E} c(e)\mu(u,v) \text{ for each } \Gamma\text{-metric } \mu \text{ on } V.$$

This is a specialization of condition (70.11). Since each cut gives a K_2-metric, and hence a $K_{2,3}$-metric, condition (72.19) includes the cut condition.

Karzanov [1987a] showed:

Theorem 72.5. *Let $G = (V, E)$ be a graph and let $H = (T, R)$ be a complete graph with $|T| = 5$ and $T \subseteq V$. Let $c : E \to \mathbb{R}_+$ and $d : R \to \mathbb{R}_+$. Then there exists a fractional multiflow if and only if the $K_{2,3}$-metric condition holds. If moreover c and d are integer, there is a half-integer multiflow. If moreover the Euler condition holds, there is an integer multiflow.*

Theorem 72.5 implies the following characterization:

Corollary 72.5a. *For each simple graph $H = (T, R)$ without isolated vertices, the following are equivalent:*

(72.20)
- (i) *for each graph $G = (V, E)$ with $V \supseteq T$, and each $c : E \to \mathbb{Z}_+$ and $d : R \to \mathbb{Z}_+$, the existence of a fractional solution implies the existence of a half-integer solution;*
- (ii) *for each graph $G = (V, E)$ with $V \supseteq T$, and each $c : E \to \mathbb{Z}_+$ and $d : R \to \mathbb{Z}_+$, the Euler condition and the existence of a fractional solution imply the existence of an integer solution;*
- (iii) *H has no three disjoint edges and no two disjoint triangles;*
- (iv) *$|VH| = 5$, or H is the union of a triangle and a star, or H is the union of two stars.*

That (72.20)(iv) implies (72.20)(i) follows from Theorem 72.5, as we can replace a star with center s by an edge sw, where w is a new vertex, with the construction of Dinits given in the proof of Corollary 71.1c. Conversely, (72.20)(i) implies (72.20)(iii). It requires giving a counterexample if H consists of three disjoint edges, and one if H consists of two disjoint triangles. If H consists of three disjoint edges, a counterexample was given in Figure 70.4. If H consists of two disjoint triangles, a counterexample follows (by doubling all capacities and demands) from Figure 72.4 (A.V. Karzanov, personal communication 2000), where c and d are integer, and where a quarter-integer, but no half-integer solution exists.

Karzanov [1991] conjectures that if R consists of two disjoint triangles and c and d are integer and satisfy the Euler condition, then the existence of a fractional solution implies the existence of a half-integer solution[21]. This would imply that for each fixed graph $H = (T, R)$ the following equivalences holds:

(72.21) (?) there is an integer k such that for each graph $G = (V, E)$ with $V \supseteq T$ and each $c : E \to \mathbb{Z}_+$ and $d : R \to \mathbb{Z}_+$, if there is a feasible multiflow, then there exists a $\frac{1}{k}$-integer multiflow
$\iff$ for each graph $G = (V, E)$ with $V \supseteq T$ and each $c : E \to \mathbb{Z}_+$ and $d : R \to \mathbb{Z}_+$, if there is a feasible multiflow, then there exists a $\frac{1}{4}$-integer multiflow
$\iff$ H has no three disjoint edges. (?)

[21] A proof of this was announced in Karzanov [1987a], but A.V. Karzanov communicated to me that the proof failed.

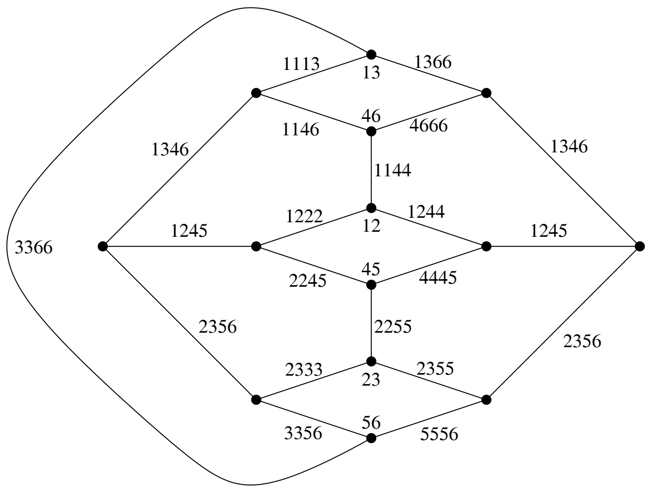


Figure 72.4

A quarter-integer multiflow exists, but no half-integer multiflow. The nets (indexed by $1, \ldots, 6$) are indicated by indices at vertices. All capacities and demands are 1. The quarter-integer multiflow is indicated by indices at the edges: k times index i at edge e means $f_i(e) = \frac{k}{4}$.

The nonexistence of a half-integer multiflow can be seen as follows. Give each of the (three) edges that connect terminals length 2, and any other edge length 1. Then the distance between the two terminals in any net is 4. Also, the sum of the lengths of the edges equals 24. So any flow f_i in a half-integer multiflow, can be decomposed as half of the sum of two flows following $s_i - t_i$ paths $P_{i,1}$ and $P_{i,2}$ of length 4. Moreover, on each edge, the capacity is fully used. Hence, each vertex v of degree 3 not being a terminal, is traversed by three paths $P_{i,j}$, each using a different pair of edges incident with v. One easily checks that this is not possible.

Karzanov [1998d] studied the existence of an integer multiflow if the nets form a disjoint union of a triangle and an edge.

72.2b. Six terminals

Okamura [1987] showed the following. Let $G = (V, E)$ and $H = (T, R)$ be a supply and a demand graph. If $|T| \leq 6$ and $k := |R|$ is odd, and if moreover G has k edge-disjoint $s - t$ paths, for each $st \in R$, then there exists a family $(P_r \mid r \in R)$ of edge-disjoint paths in G, where P_r connects the ends of r (for $r \in R$). (For $|T| \leq 5$ this was proved in Okamura [1984b].)

Okamura [1998] showed that if $|T| \leq 6$ and G is l-edge-connected, where

$$l := \max_{U \subseteq V} d_R(U), \tag{72.22}$$

then the edge-disjoint paths problem has a half-integer solution (that is, for each $r \in R$ there exist paths P'_r and P''_r connecting the ends of r, such that each edge of G is in at most two of the paths P'_r, P''_r (over all $r \in R$)). She conjectures that here the condition $|T| \leq 6$ can be deleted.

72.3. Cut packing

By Theorem 70.5, Corollary 72.2a implies a fractional cut packing theorem. A stronger (integer) version of it was proved by Karzanov [1985b], which generalizes Theorem 71.3 (we follow the proof given in Schrijver [1991e]):

Theorem 72.6. *Let $G = (V, E)$ be a connected bipartite graph and let $H = (T, R)$ be a simple graph satisfying* (72.1)*, with $T \subseteq V$. Then G has disjoint cuts such that for each $st \in R$, $\mathrm{dist}_G(s, t)$ is equal to the number of these cuts separating s and t.*

Proof. Let $G = (V, E)$ be a counterexample with $|E|$ as small as possible. Define $d(u, v) := \mathrm{dist}_G(u, v)$ for $u, v \in V$. We first show:

(72.23) for each nonempty cut C there exist a pair $st \in R$ and an $s-t$ path P with $|EP \setminus C| \leq d(s, t) - 2$.

If not, contract all edges in C, giving graph G'. Then for all $st \in R$ we have

$$(72.24) \qquad d'(s', t') = \begin{cases} d(s, t) - 1 & \text{if } C \text{ separates } \{s, t\}, \\ d(s, t) & \text{if } C \text{ does not separate } \{s, t\}. \end{cases}$$

(Here and below, v' denotes the image of v in G, and d' denotes the distance function of G'.) As G is a smallest counterexample, G' has disjoint cuts $C_1, \ldots, C_t$ such that $d'(s', t')$ is equal to the number of cuts separating s' and t', for each $st \in R$. Together with C this gives, in the original graph G, cuts as required, by (72.24). This proves (72.23).

From (72.23) we derive:

(72.25) for all $u, w \in V$, there exists a pair $st \in R$ such that $\{s, t\} \cap \{u, w\} = \emptyset$ and such that
$$d(s, t) + d(u, w) \geq d(s, w) + d(u, t) \text{ and}$$
$$d(s, t) + d(u, w) \geq d(s, u) + d(w, t).$$

To prove this, let X be the set of vertices that are on at least one shortest $u - w$ path.

First, suppose that $X = V$. By (72.23), there exist $st \in R$ and an $s - t$ path P with $|EP \setminus \delta(u)| \leq d(s, t) - 2$. So P is a shortest $s - t$ path traversing u, and $u \neq s, t$. To see that $w \neq s, t$, suppose $w = t$, say. Then, as $d(u, w) = d(u, s) + d(s, w)$ (since $s \in X$),

$$(72.26) \qquad \begin{aligned} |EP \setminus \delta(u)| &= |EP| - 2 = d(s, u) + d(u, t) - 2 \\ &= d(s, u) + d(u, w) - 2 = 2d(s, u) + d(s, w) - 2 > d(s, w) - 2 \\ &= d(s, t) - 2, \end{aligned}$$

a contradiction. So $\{s,t\} \cap \{u,w\} = \emptyset$. Moreover,

(72.27) $$d(s,t) + d(u,w) = d(s,t) + d(u,s) + d(s,w) \geq d(s,w) + d(u,t).$$

One similarly shows the second inequality in (72.25).

Second, suppose that $X \neq V$. Let $C := \delta(X)$, and let G' be the graph obtained from G by contracting all edges in C. Then for each vertex x:

(72.28) $$d'(u',x') \geq d(u,x) - 1 \text{ and } d'(w',x') \geq d(w,x) - 1.$$

To see the first inequality, let P be a $u-x$ path in G with $|EP \setminus C| = d'(u',x')$. Choose P with $|EP \cap C|$ smallest. If the first inequality does not hold, then $|EP \cap C| \geq 2$. Then we can split P as $P'P''$ such that $|EP' \cap C| = 2$. Let P' connect u and v. As $|EP' \cap C| = 2$ and $u \in X$ we know $v \in X$. Since P' is not fully contained in X, we know that $|EP'| \geq d(u,v) + 2$. Let Q be a shortest $u - v$ path in G. Then $|EQ| = d(u,v) \leq |EP'| - 2$, and Q is fully contained in X. Let R be the concatenation QP''. Then $|ER \setminus C| \leq |EP \setminus C|$ and $|ER \cap C| = |EP \cap C| - 2$, contradicting the minimality of $|EP \cap C|$. This shows the first inequality in (72.28); the second inequality is proved similarly.

By (72.23), there exists $st \in R$ such that $d'(s',t') \leq d(s,t) - 2$. Then (72.28) implies $\{s,t\} \cap \{u,w\} = \emptyset$. Moreover, there exist a $v \in X$ and a shortest $s' - t'$ path in G' traversing v'. Hence

(72.29) $$\begin{aligned} d(s,t) + d(u,w) &\geq d'(s',t') + d(u,w) + 2 \\ &= d'(s',v') + d'(v',t') + d(u,v) + d(v,w) + 2 \\ &\geq d'(s',v') + d'(v',t') + d'(u',v') + d'(v',w') + 2 \\ &\geq d'(s',w') + d'(u',t') + 2 \geq d(s,w) + d(u,t). \end{aligned}$$

This gives the first inequality in (72.25); the second inequality is proved similarly.

(72.25) implies that for each pair $\{u,w\}$ of vertices of G there exists an $st \in R$ disjoint from $\{u,w\}$. So H is not the union of two stars, and hence $H = K_4$ or $H = C_5$ (up to isolated vertices, which we can ignore).

If $H = K_4$, let $T = \{r_1, r_2, r_3, r_4\}$. Then by (72.25):

(72.30) $$\begin{aligned} d(r_1,r_2) + d(r_3,r_4) &\geq d(r_1,r_3) + d(r_2,r_4) \geq d(r_1,r_4) + d(r_2,r_3) \\ &\geq d(r_1,r_2) + d(r_3,r_4). \end{aligned}$$

Hence we have equality throughout, that is

(72.31) $$d(t,u) + d(v,w) = d(t,v) + d(u,w) \text{ for all distinct } t,u,v,w \in T.$$

This implies that there exists a function $\phi : T \to \mathbb{R}_+$ such that $d(u,v) = \phi(u) + \phi(v)$ for each two distinct $u, v \in T$. (Indeed, let $\phi(v) := \frac{1}{2}(d(u,v) + d(v,w) - d(u,w))$ for arbitrary u, w with $v \neq u \neq w \neq v$. That this is independent of the choice of u, w follows from (72.31).)

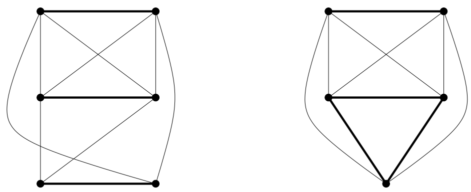
Since all vertices are distinct, $d(u,v) > 0$ for all distinct $u, v \in T$, and so $\phi(v) > 0$ for at least one $v \in T$. By (72.23), there exist $st \in R$ and an $s - t$ path P such that $|EP \setminus \delta(v)| \leq d(s,t) - 2$. So P traverses v, and $|EP| = d(s,t) = \phi(s) + \phi(t)$. However, (72.32) $$|EP| \geq d(s,v) + d(v,t) = \phi(s) + 2\phi(v) + \phi(t) > \phi(s) + \phi(t),$$

a contradiction.

If $H = C_5$, let $T = \{r_1, \ldots, r_5\}$ and $R = \{r_i r_{i+1} \mid i = 1, \ldots, 5\}$, taking indices mod 5. Applying (72.25) to $u := r_i$ and $w := r_{i+2}$, we obtain $st = r_{i+3}r_{i+4}$ (as it is the unique pair in R disjoint from $\{u, w\}$), and hence

(72.33)
$$d(r_i, r_{i+2}) + d(r_{i+3}, r_{i+4}) \geq d(r_i, r_{i+3}) + d(r_{i+2}, r_{i+4}) \quad (i = 1, \ldots, 5),$$
$$d(r_i, r_{i+2}) + d(r_{i+3}, r_{i+4}) \geq d(r_i, r_{i+4}) + d(r_{i+2}, r_{i+3}) \quad (i = 1, \ldots, 5).$$

Adding up these ten inequalities, we obtain the same sum at both sides of the inequality sign. So we have equality in each of (72.33). This is equivalent to (72.31), and we obtain a contradiction in the same way as above. ∎



(a) (b)

Figure 72.5
The heavy lines are the edges of H, the other lines those of G. In both cases, G has no disjoint cuts such that for any edge r in H, the distance in G between the vertices in r is equal to the number of cuts separating them.

We cannot delete condition (72.1) in Theorem 72.6, as is shown by the examples given in Figure 72.5.

Notes. Karzanov [1985b] gave an $O(n^3)$ algorithm to find the cut packings of Theorem 72.6 (also for the weighted case). Theorem 72.2 can also be derived from Theorem 72.6, with the help of Theorems 70.5 and 70.7.

Karzanov [1990b] extended these cut packing results to packing $K_{2,3}$-metrics (cf. Section 72.2a):

(72.34) Let $G = (V, E)$ be a bipartite graph and let $T \subseteq V$ with $|T| = 5$. Then there exist $K_{2,3}$-metrics $\mu_1, \ldots, \mu_k$ such that $\mathrm{dist}_G(u, v) \geq \mu_1(u, v) + \cdots + \mu_k(u, v)$ for all $u, v \in V$, with equality if $u, v \in T$.

Chapter 73

T-paths

We now go over to the problem of finding a maximum number of disjoint paths whose ends are two different vertices in a given set T of vertices — the T-paths. (So the nets are all pairs of distinct vertices in T.) Fundamental theorems of Mader imply min-max relations for this.

73.1. Disjoint T-paths

Let $G = (V, E)$ be a graph and let $T \subseteq V$. A path is called an *T-path* if its ends are distinct vertices in T and no internal vertex belongs to T.

Mader [1978c] gave a min-max formula for the maximum number of internally vertex-disjoint T-paths. It generalizes the undirected, vertex-disjoint version of Menger's theorem (by taking $|T| = 2$) and the Tutte-Berge formula (by adding to each vertex v of a graph G a copy v' of v and an edge vv'; taking for T the set of new vertices, the maximum number of internally vertex-disjoint T-paths is equal to the matching number of G).

As in Schrijver [2001], we derive Mader's theorem from a theorem of Gallai [1961], which we derive (as Gallai did) from matching theory (the Tutte-Berge formula):

Theorem 73.1 (Gallai's disjoint T-paths theorem)**.** *Let $G = (V, E)$ be an undirected graph and let $T \subseteq V$. The maximum number of disjoint T-paths is equal to the minimum value of*

$$|U| + \sum_{K} \lfloor \tfrac{1}{2}|K \cap T| \rfloor \tag{73.1}$$

taken over $U \subseteq V$, where K ranges over the components of $G - U$.

Proof. The maximum is at most the minimum, since for each $U \subseteq V$, each T-path intersects U or has its ends in $K \cap T$ for some component K of $G - U$.

To see equality, let μ be equal to the minimum value of (73.1). Let the graph $\widetilde{G} = (\widetilde{V}, \widetilde{E})$ arise from G by adding a disjoint copy G' of $G - T$, and making the copy v' of each $v \in V \setminus T$ adjacent to v and to all neighbours of v in G. By the Tutte-Berge formula (Theorem 24.1), $\widetilde{G}$ has a matching M of size $\mu + |V \setminus T|$. To see this, we must prove that for any $\widetilde{U} \subseteq \widetilde{V}$:

$$|\widetilde{U}| + \sum_{\widetilde{K}} \lfloor \tfrac{1}{2}|\widetilde{K}| \rfloor \geq \mu + |V \setminus T|, \tag{73.2}$$

where $\widetilde{K}$ ranges over the components of $\widetilde{G} - \widetilde{U}$. Now if for some $v \in V \setminus T$ exactly one of v, v' belongs to $\widetilde{U}$, then we can delete it from $\widetilde{U}$, thereby not increasing the left-hand side of (73.2).

So we can assume that for each $v \in V \setminus T$, either $v, v' \in \widetilde{U}$ or $v, v' \notin \widetilde{U}$. Define $U := \widetilde{U} \cap V$. Then each component K of $G - U$ is equal to $\widetilde{K} \cap V$ for some component $\widetilde{K}$ of $\widetilde{G} - \widetilde{U}$. Hence

$$|\widetilde{U}| + \sum_{\widetilde{K}} \lfloor \tfrac{1}{2}|\widetilde{K}| \rfloor = |U| + \sum_{K} \lfloor \tfrac{1}{2}|K \cap T| \rfloor + |V \setminus T| \geq \mu + |V \setminus T|, \tag{73.3}$$

where K ranges over the components of $G - U$. Thus we have (73.2).

So $\widetilde{G}$ has a matching M of size $\mu + |V \setminus T|$. Let N be the matching $\{vv' \mid v \in V \setminus T\}$ in $\widetilde{G}$. As $|M| = \mu + |V \setminus T| = \mu + |N|$, the union $M \cup N$ has at least μ components with more edges in M than in N. Each such component is a path connecting two vertices in T. Then contracting the edges in N yields μ disjoint T-paths in G. ∎

We now derive Mader's theorem. Let $G = (V, E)$ be a graph and let $\mathcal{S}$ be a collection of disjoint subsets of V. A path in G is called an *$\mathcal{S}$-path* if it connects two different sets in $\mathcal{S}$ and has no internal vertex in any set in $\mathcal{S}$. Denote $T := \bigcup \mathcal{S}$.

Theorem 73.2 (Mader's disjoint $\mathcal{S}$-paths theorem)**.** *The maximum number of disjoint $\mathcal{S}$-paths is equal to the minimum value of*

$$|U_0| + \sum_{i=1}^{n} \lfloor \tfrac{1}{2}|B_i| \rfloor, \tag{73.4}$$

taken over all partitions $U_0, \ldots, U_n$ of V such that each $\mathcal{S}$-path intersects U_0 or traverses some edge spanned by some U_i. Here B_i denotes the set of vertices in U_i that belong to T or have a neighbour in $V \setminus (U_0 \cup U_i)$.

Proof. Let μ be the minimum value of (73.4). Trivially, the maximum number of disjoint $\mathcal{S}$-paths is at most μ, since any $\mathcal{S}$-path disjoint from U_0 and traversing an edge spanned by U_i, traverses at least two vertices in B_i.

Fixing V, choose a counterexample E, $\mathcal{S}$ minimizing

$$|E| - |\{\{x, y\} \mid x, y \in V, \exists X, Y \in \mathcal{S} : x \in X, y \in Y, X \neq Y\}|. \tag{73.5}$$

Then each $X \in \mathcal{S}$ is a stable set of G, since deleting any edge e spanned by X does not change the maximum and minimum value in Mader's theorem (as no $\mathcal{S}$-path traverses e and as deleting e does not change any set B_i), while it decreases (73.5).

If $|X| = 1$ for each $X \in \mathcal{S}$, the theorem reduces to Gallai's disjoint T-paths theorem (Theorem 73.1): we can take for U_0 any set U minimizing (73.1), and for $U_1, \ldots, U_n$ the components of $G - U$.

So $|X| \geq 2$ for some $X \in \mathcal{S}$. Choose $s \in X$. Define

(73.6) $$\mathcal{S}' := (\mathcal{S} \setminus \{X\}) \cup \{X \setminus \{s\}, \{s\}\}.$$

Replacing $\mathcal{S}$ by $\mathcal{S}'$ decreases (73.5), but it does not decrease the minimum in Mader's theorem (as each $\mathcal{S}$-path is an $\mathcal{S}'$-path and as $\bigcup \mathcal{S}' = T$). Hence there exists a collection $\mathcal{P}$ of μ disjoint $\mathcal{S}'$-paths.

Necessarily, there is a path $P_0 \in \mathcal{P}$ connecting s with another vertex in X (otherwise $\mathcal{P}$ forms μ disjoint $\mathcal{S}$-paths). Then all other paths in $\mathcal{P}$ are $\mathcal{S}$-paths. Let u be an internal vertex of P_0 (u exists, since X is a stable set). Define

(73.7) $$\mathcal{S}'' := (\mathcal{S} \setminus \{X\}) \cup \{X \cup \{u\}\}.$$

Replacing $\mathcal{S}$ by $\mathcal{S}''$ decreases (73.5), but it does not decrease the minimum in Mader's theorem (as each $\mathcal{S}$-path is an $\mathcal{S}''$-path and as $\bigcup \mathcal{S}'' \supseteq T$). So there exists a collection $\mathcal{Q}$ of μ disjoint $\mathcal{S}''$-paths. Choose $\mathcal{Q}$ such that $\mathcal{Q}$ uses a minimal number of edges not used by $\mathcal{P}$.

Necessarily, u is an end of some path $Q_0 \in \mathcal{Q}$ (otherwise $\mathcal{Q}$ forms μ disjoint $\mathcal{S}$-paths). Then all other paths in $\mathcal{Q}$ are $\mathcal{S}$-paths. As $|\mathcal{P}| = |\mathcal{Q}|$ and as u is not an end of any path in $\mathcal{P}$, there exists an end r of some path $P \in \mathcal{P}$ that is not an end of any path in $\mathcal{Q}$.

Then P intersects some path in $\mathcal{Q}$ (otherwise $(\mathcal{Q} \setminus \{Q_0\}) \cup \{P\}$ would form μ disjoint $\mathcal{S}$-paths). So when following P starting from r, there is a first vertex w that is on some path in $\mathcal{Q}$, say on $Q \in \mathcal{Q}$. Let Q be split at w into subpaths Q' and Q'' say (possibly of length 0). Let P' be the $r - w$ part of P.

If $EQ' \not\subseteq EP$ and $EQ'' \not\subseteq EP$, we may assume that r is not in the same class of $\mathcal{S}''$ as the end of Q' is. Then after replacing part Q'' of Q by P', Q remains an $\mathcal{S}''$-path disjoint from the other paths in $\mathcal{Q}$. This contradicts our minimality assumption on $\mathcal{Q}$.

So we can assume that $EQ' \subseteq EP$. If $P \neq P_0$, then after resetting Q to P, Q remains an $\mathcal{S}''$-path disjoint from the other paths in $\mathcal{Q}$. Again this contradicts our minimality assumption on $\mathcal{Q}$.

So $P = P_0$, and hence (since $EQ' \subseteq EP$) we have $Q = Q_0$. Then replacing part Q' of Q by P', we obtain μ disjoint $\mathcal{S}$-paths as required. ∎

(The case splitting finishing this proof is due to A. Frank (personal communication 2002).)

Theorem 73.2 is equivalent to the original form of Mader's theorem on internally vertex-disjoint T-paths (instead of fully disjoint $\mathcal{S}$-paths), which reads as follows.

For any graph G, let $B_G(U)$ denote the set of vertices in U having a neighbour that is not in U.

Corollary 73.2a (Mader's internally disjoint T-paths theorem)**.** *Let $G = (V, E)$ be a graph and let T be a stable subset of V. Then the maximum number of internally vertex-disjoint T-paths is equal to the minimum value of*

$$(73.8) \qquad |U_0| + \sum_{i=1}^{n} \lfloor \tfrac{1}{2} |B_{G-U_0}(U_i)| \rfloor,$$

where $U_0, U_1, \ldots, U_n$ partition $V \setminus T$ such that each T-path intersects U_0 or traverses some edge spanned by some U_i.

Proof. Trivially, the maximum is not more than the minimum (since each T-path not intersecting U_0 traverses at least two vertices in some U_i, hence it traverses at least two vertices in U_i that have a neighbour out of $U_i \cup U_0$).

To see equality, we can assume that no two vertices in T have a common neighbour v. Otherwise we can apply induction by deleting v, which reduces both the maximum and the minimum by 1.

Now the present corollary follows from Theorem 73.2 applied to $G - T$ and the collection $\mathcal{S} := \{N(s) \mid s \in T\}$. ∎

Mader's internally disjoint T-paths theorem in turn implies the edge-disjoint version, proved by Mader [1978b]:

Corollary 73.2b (Mader's edge-disjoint T-paths theorem)**.** *Let $G = (V, E)$ be a graph and let $T \subseteq V$. Then the maximum number of edge-disjoint T-paths is equal to the minimum value of*

$$(73.9) \qquad \tfrac{1}{2}\big(\sum_{s \in T} d_E(X_s) - \kappa\big),$$

where the X_s are disjoint sets with $s \in X_s$ (for $s \in T$), and where κ denotes the number of components K of the graph $G - \bigcup_{s \in T} X_s$ with $d_E(K)$ odd.

Proof. Let t be the maximum number of edge-disjoint T-paths. It is easy to see that t cannot exceed the minimum value of (73.9).

To see equality, first observe that, if G has an edge e such that by deleting e, the maximum drops by 1, we can apply induction on $|E|$, since the minimum drops by at most 1.

So we can assume that no such edge exists. We make an auxiliary graph $G' = (V', E')$ as follows. For each $u \in V \setminus T$, let W_u be a stable set of size $3t + 1$. For each edge $e \in E$, let v_e be a new vertex. Let v_e be adjacent to all vertices in W_u if $u \in e$, and to $s \in T$ if $s \in e$. This defines the graph G' (with vertex set $V' = T \cup \{v_e \mid e \in E\} \cup \bigcup_{u \in V \setminus T} W_u$).

Then t is equal to the maximum number of disjoint T-paths in G'. Hence, by Corollary 73.2a, there exist disjoint subsets $U_0, \ldots, U_n$ of $V' \setminus T$ such that

(73.10) each T-path in G' intersects U_0 or traverses an edge spanned by some U_i, and such that

$$(73.11) \qquad t \geq |U_0| + \sum_{i=1}^{n} \lfloor \tfrac{1}{2}|B_i| \rfloor,$$

where B_i is the set of vertices in U_i having a neighbour in $V' \setminus (U_0 \cup U_i)$.

By our assumption that the maximum does not drop by deleting any edge e, we know that U_0 contains no vertex v_e.

We may assume that $|B_i| \geq 2$ for each i, since if $|B_i| \leq 1$, we can delete U_i, as no T-path in G' avoiding U_0 traverses any edge spanned by U_i. This implies that $|B_i| \leq 3\lfloor \frac{1}{2}|B_i| \rfloor$, and hence

$$(73.12) \qquad |U_0| + \sum_{i=1}^{n} |B_i| \leq 3(|U_0| + \sum_{i=1}^{n} \lfloor \tfrac{1}{2}|B_i| \rfloor) < 3t + 1.$$

So for each $u \in V \setminus T$, there exists a $w_u \in W_u$ with $w_u \notin U_0 \cup B_1 \cup \cdots \cup B_n$. For $u \in T$, let $w_u := u$.

For each $i = 1, \ldots, n$, let $Y_i := \{u \in V \setminus T \mid w_u \in U_i\}$ and let E_i be the set of edges $e \in E$ with $v_e \in B_i$. Then

$$(73.13) \qquad \delta_E(Y_i) \subseteq E_i$$

for each $i = 1, \ldots, n$. To see this, let $e \in \delta_E(Y_i)$, with $e = uv$ and $u \in Y_i$ and $v \notin Y_i$. Then $u \in Y_i$ implies $w_u \in U_i$. Hence $w_u \in U_i \setminus B_i$, implying $v_e \in U_i$. As $v \notin Y_i$, we know $w_v \notin U_i$. Hence v_e has a neighbour out of $U_0 \cup U_i$, and so $v_e \in B_i$. Therefore, $e \in E_i$, proving (73.13).

Hence no edge of G connects two different Y_i and Y_j (since $E_i \cap E_j = \emptyset$). Suppose now that $G - Y_1 - \cdots - Y_n$ contains a T-path P. Route P as a T-path P' in G', by replacing each edge of P by v_e and any $u \in V \setminus T$ by w_u. Then P' is disjoint from U_0. So P' traverses an edge spanned by some U_i. Then P' traverses a vertex $w_u \in U_i$ for some $u \in V \setminus T$. Hence P traverses Y_i, a contradiction.

For $s \in T$, let X_s be the set of vertices of G reachable in $G - Y_1 - \cdots - Y_n$ from s. Then we have

$$(73.14) \qquad t \geq \sum_{i=1}^{n} \lfloor \tfrac{1}{2}|E_i| \rfloor \geq \tfrac{1}{2}\big(\sum_{s \in T} d_E(X_s) - \kappa\big),$$

where κ is the number of components K of $G - \bigcup_{s \in T} X_s$ with $d_E(K)$ odd. ∎

(This min-max formula was proved also in an unpublished manuscript of Lomonosov [1978b].)

73.1a. Disjoint T-paths with the matroid matching algorithm

As Lovász [1980a] showed, Mader's theorem can be derived from matroid matching theory, and also a polynomial-time algorithm to find a maximum packing of T-paths follows from it. We restrict ourselves to deriving polynomial-time solvability, and consider the equivalent problem of finding a maximum packing of $\mathcal{S}$-paths.

Let $G = (V, E)$ be a graph and let $S_1, \ldots, S_k$ be disjoint subsets of V. Let $\mathcal{S} := \{S_1, \ldots, S_k\}$ and $T := S_1 \cup \cdots \cup S_k$.

We can assume that each S_i is a stable set. Consider the linear space $(\mathbb{R}^2)^V$, considered as the set of functions $x : V \to \mathbb{R}^2$. For each edge $e = uw$ of G, let L_e be the linear subspace of $(\mathbb{R}^2)^V$ given by:

(73.15) $$L_e := \{x \in (\mathbb{R}^2)^V \mid x(v) = \mathbf{0} \text{ for each } v \in V \setminus \{u, w\}, x(u) + x(w) = \mathbf{0}\}.$$

So $\dim L_e = 2$.

Choose distinct 1-dimensional subspaces $l_1, \ldots, l_k$ of $\mathbb{R}^2$. For each $v \in V$, let $L_v := l_i$ if $v \in S_i$ for some i, and $L_v := \{\mathbf{0}\}$ otherwise. Define

(73.16) $$Q := \{x \in (\mathbb{R}^2)^V \mid \forall v \in V : x(v) \in L_v\}.$$

Let $\mathcal{E}$ be the collection of subspaces L_e/Q (for $e \in E$) of $(\mathbb{R}^2)^V/Q$. Then $\dim(L_e/Q) = 2$ for each edge e, since e connects no two vertices in the same S_i (so $L_e \cap Q = \{\mathbf{0}\}$).

For any $F \subseteq E$, let $\mathcal{L}_F$ denote the corresponding collection of lines in $\mathcal{E}$:

(73.17) $$\mathcal{L}_F := \{L_e/Q \mid e \in F\}.$$

We show that for each $F \subseteq E$:

(73.18) $\mathcal{L}_F$ is a matching if and only if F is a forest such that each component of (V, F) has at most two vertices in common with T, and at most one with each S_i.

Let $X := \sum(L_e \mid e \in F)$. Then one easily checks that X consists of all $x : V \to \mathbb{R}^2$ with $\sum_{v \in K} x(v) = \mathbf{0}$ for each component K of (V, F). So $\dim(X) = 2(|V| - \kappa)$, where κ is the number of components of (V, F). Also, $\dim(X \cap Q) = 0$ if and only if each component of (V, F) has at most two vertices in common with T, and at most one with each S_i. Now

(73.19) $$\dim(\mathcal{L}_F) = \dim(X/Q) = \dim(X) - \dim(X \cap Q) \leq \dim(X) \leq 2|F|.$$

Hence $\mathcal{L}_F$ is a matching if and only if $\dim(X) = 2|F|$ and $\dim(X \cap Q) = 0$. By the previous this gives (73.18).

(73.18) then implies the following relation to $\mathcal{S}$-paths:

(73.20) if G is connected, the maximum number of disjoint $\mathcal{S}$-paths is equal to $\nu(\mathcal{E}) - |V| + |T|$.

To see this, let t be the maximum number of disjoint $\mathcal{S}$-paths. Let Π form a packing of t $\mathcal{S}$-paths and let F' be the set of edges contained in these paths. Extend F' to a forest F such that each component of (V, F) contains either a unique path in Π or a unique vertex in T. Then F satisfies the condition given in (73.18), and $|F| = t + |V| - |T|$. So $\mathcal{L}_F$ forms a matching of size $t + |V| - |T|$, and hence $\nu(\mathcal{E}) \geq t + |V| - |T|$.

Conversely, let $\mathcal{F} \subseteq \mathcal{E}$ be a matching of size $\nu(\mathcal{E})$. Then $\mathcal{F} = \mathcal{L}_F$ for some forest $F \subseteq E$ satisfying the condition in (73.18). Let t be the number of components of (V, F) intersecting T twice. Then deleting t edges from F, we obtain a forest such that each component intersects T at most once. So $|F| - t \leq |V| - |T|$, and hence $t \geq \nu(\mathcal{E}) - |V| + |T|$. This shows (73.20).

Theorem 43.4 implies with (73.20) the polynomial-time solvability of finding a maximum packing of $\mathcal{S}$-paths:

(73.21) Given a graph $G = (V, E)$ and a collection $\mathcal{S}$ of disjoint subsets of V, a maximum number of disjoint $\mathcal{S}$-paths can be found in polynomial time.

73.1b. Polynomial-time findability of edge-disjoint T-paths

J.C.M. Keijsper, R.A. Pendavingh, and L. Stougie (personal communication 2000) showed that with the ellipsoid method one can derive from Mader's edge-disjoint T-paths theorem (Corollary 73.2b) that a maximum number of edge-disjoint T-paths can be found in polynomial time.

To see this, let $G = (V, E)$ be a graph and let $T \subseteq V$. Consider the polyhedron P in $\mathbb{R}^E$ consisting of all $x \in \mathbb{R}^E$ satisfying

$$\text{(73.22)} \qquad \begin{array}{rll} \text{(i)} & 0 \leq x_e \leq 1 & \text{for each } e \in E, \\ \text{(ii)} & x(\delta(U)) \leq |\delta(U)| - 1 & \text{for each } U \subseteq V \setminus T \text{ with } |\delta(U)| \text{ odd}, \\ \text{(iii)} & x(\delta(s)) \leq x(\delta(X)) & \text{for each } s \in T \text{ and } X \subseteq V \\ & & \text{with } X \cap T = \{s\}. \end{array}$$

These conditions can be tested in polynomial time: (i) is easy (one by one). To test (ii), let G' be the graph obtained from G by contracting T to one vertex. Moreover, define $T' := \{v \in VG' \mid \deg_{G'}(v) \text{ odd}\}$. Define a capacity function c by $c_e := 1 - x_e$ for $e \in E$. Then (ii) is valid if and only if the minimum capacity of a T'-cut in G' is at least 1. This can be tested in polynomial time (Corollary 29.6a). Finally, testing (iii) amounts to finding a cut separating s and $T \setminus \{s\}$ of minimum capacity, taking x as capacity function.

So by the ellipsoid method, we can optimize any linear function over P in polynomial time. Now the maximum value λ of

$$\text{(73.23)} \qquad \tfrac{1}{2} \sum_{s \in T} x(\delta(s))$$

over $x \in P$ is equal to the maximum number μ of edge-disjoint T-paths in G.

The inequality $\lambda \geq \mu$ follows from the fact that the incidence vectors of μ edge-disjoint T-paths sum up to a vector x satisfying (73.22) and having (73.23) equal to μ.

To see equality, by Corollary 73.2b there exist disjoint sets X_s ($s \in T$) such that $s \in X_s$ and such that

$$\text{(73.24)} \qquad \mu = \tfrac{1}{2}\big(\sum_{s \in T} d_E(X_s) - \kappa\big),$$

where κ denotes the number of components K of the graph $G' := G - \bigcup_{s \in T} X_s$ with $d(K)$ odd. This implies a dual solution of the linear program defining λ, of value at most μ. Indeed, let x attain the maximum value of (73.23) over P. Let $W := VG'$, let $\mathcal{K}$ be the collection of components of G', and let F be the set of edges connecting different sets X_s. Then

$$\text{(73.25)} \qquad \begin{aligned} 2\lambda &= \sum_{s \in T} x(\delta(s)) \leq \sum_{s \in T} x(\delta(X_s)) = 2x(F) + x(\delta(W)) \\ &= 2x(F) + \sum_{K \in \mathcal{K}} x(\delta(K)) \leq 2|F| + \sum_{K \in \mathcal{K}} x(\delta(K)) \\ &\leq 2|F| + \sum_{K \in \mathcal{K}} 2\lfloor \tfrac{1}{2} d_E(K) \rfloor = \sum_{s \in T} d_E(X_s) - \kappa = 2\mu. \end{aligned}$$

Concluding, we have $\lambda = \mu$.

This implies that μ can be determined in polynomial time. The paths can be found explicitly by iteratively deleting edges if it does not reduce μ. Similarly, we can replace pairs of adjacent edges uv, vw by one edge uw, if it does not reduce μ. We end up with a graph with μ edges spanned by T. Working our way back, we find the required paths in the original graph.

This approach can be extended to obtain a strongly polynomial-time algorithm for the capacitated case, where each edge e has an integer capacity $c(e)$ and we want to find a maximum number of T-paths such that each edge e is contained in at most $c(e)$ of them.

73.1c. A feasibility characterization for integer K_3-flows

Seymour [1980b] showed that Corollary 73.2b also implies the following feasibility characterization for integer K_3-flows (we follow the formulation and proof given by Frank [1990e]). The cut condition is applied to $R = \{s_1s_2, s_1s_3, s_2s_3\}$ with demand $d(s_is_j) = d_{i,j}$, and capacity $\mathbf{1}$:

Corollary 73.2c. *Let $G = (V, E)$ be a graph, let $s_1, s_2, s_3 \in V$, and let $d_{1,2}, d_{1,3}, d_{2,3} \in \mathbb{Z}_+$. Then there exists a collection of edge-disjoint paths such that $d_{i,j}$ of them connect s_i and s_j ($1 \leq i < j \leq 3$), if and only if the cut condition holds and*

(73.26) $$s(U_1) + s(U_2) + s(U_3) \geq \kappa$$

for each choice of disjoint sets U_1, U_2, U_3 with $s_i \in U_i$ ($i = 1, 2, 3$). Here $s(X) := |\delta_E(X)| - d(\delta_R(X))$ and κ is number of components K of $G - U_1 - U_2 - U_3$ with $|\delta_E(K)|$ odd.

Proof. Necessity is easy: we have

(73.27) $$d(\delta_R(U_1))+d(\delta_R(U_2))+d(\delta_R(U_3)) \leq |\delta_E(U_1)|+|\delta_E(U_2)|+|\delta_E(U_3)|-\kappa,$$

since from any component K of $G - U_1 - U_2 - U_3$ with $|\delta_E(K)|$ odd, we cannot use all edges of $\delta_E(K)$.

Next we show sufficiency. We can assume that G is connected. For $i = 1, 2, 3$, extend G by a new vertex r_i and $k_i := \deg_R(s_i)$ parallel edges connecting r_i and s_i. Let $G' = (V', E')$ be the extended graph, and let $T := \{r_1, r_2, r_3\}$. It suffices to show that G' has $d_{1,2} + d_{1,3} + d_{2,3} = \frac{1}{2}(k_1 + k_2 + k_3)$ edge-disjoint T-paths (since then $\frac{1}{2}(k_1 + k_2 - k_3) = d_{1,2}$ of them connect r_1 and r_2; similarly for $d_{1,3}$ and $d_{2,3}$). For this we can invoke Corollary 73.2b. Hence suppose to the contrary that there exist three disjoint subsets X_1, X_2, X_3 of V' with $r_i \in X_i$ ($i = 1, 2, 3$) such that

(73.28) $$\sum_{i=1}^{3} |\delta_{E'}(X_i)| - \kappa < k_1 + k_2 + k_3,$$

where κ denotes the number of components K of the graph $G' - X_1 - X_2 - X_3$ with $d_{E'}(K)$ odd.

We can assume that each X_i induces a connected subgraph of G'. For suppose that L is a component of $G'[X_1]$ not containing r_1. Let $X_1' := X_1 \setminus L$, and let κ' be the number of components K of $G' - X_1' - X_2 - X_3$ with $|\delta_{E'}(K)|$ odd. Then $\kappa' \geq \kappa - |\delta_{E'}(L)|$, and hence

(73.29) $$|\delta_{E'}(X_1')| = |\delta_{E'}(X_1)| - |\delta_{E'}(L)| \leq |\delta_{E'}(X_1)| - \kappa + \kappa'.$$

So replacing X_1 by X_1' preserves (73.28).

If $s_i \in X_i$ for $i = 1, 2, 3$, then $U_i := X_i \setminus \{r_i\}$ for $i = 1, 2, 3$ would violate (73.26). So we can assume that $s_3 \notin X_3$, and so $X_3 = \{r_3\}$.

Then we can assume that $G - X_1 - X_2 - X_3$ has only one component. Otherwise it has a component L not containing s_3, and so L is not connected to X_3. We can assume that $|E'[L, X_1]| \geq |E'[L, X_2]|$. Let $X_1' := X_1 \cup L$ and let κ' be the number of components K of $G' - X_1' - X_2 - X_3$ with $|\delta_{E'}(K)|$ odd. Then $|\delta_{E'}(X_1')| \leq |\delta_{E'}(X_1)| - \kappa + \kappa'$, and so replacing X_1 by X_1' preserves (73.28).

So we may assume $\kappa \leq 1$, and hence, as by parity the left-hand side in (73.28) is even, we obtain the contradiction

$$k_1 + k_2 + k_3 \geq \sum_{i=1}^{3} |\delta_{E'}(X_i)| - \kappa + 2 > \sum_{i=1}^{3} |\delta_{E'}(X_i)| \geq k_1 + k_2 + k_3, \tag{73.30}$$

where the last inequality follows from the cut condition. ∎

A polynomial-time algorithm to find a circuit traversing three prescribed vertices in an undirected graph, was given by LaPaugh and Rivest [1978,1980].

73.2. Fractional packing of T-paths

If all vertices not in T have even degree, Mader's edge-disjoint T-paths theorem (Corollary 73.2b) reduces to the following result of Cherkasskiĭ [1977b] and Lovász [1976b] (thus answering a question of Kupershtokh [1971]):

Corollary 73.2d. *Let $G = (V, E)$ be a graph and let $T \subseteq V$, with $\deg_G(v)$ even for each $v \in V \setminus T$. Then the maximum number of edge-disjoint T-paths in G is equal to*

$$\tfrac{1}{2} \sum_{s \in T} \gamma_G(s). \tag{73.31}$$

Here $\gamma_G(s)$ denotes the minimum size of a cut in G separating s and $T \setminus \{s\}$.

Proof. Directly from Corollary 73.2b, since $\kappa = 0$. ∎

This corollary has the following consequence on multiflows, also due to Cherkasskiĭ [1977b] (the fractional version was stated, with incorrect proof, by Kupershtokh [1971]):

Corollary 73.2e. *Let $G = (V, E)$ be a graph, let $T \subseteq V$, and let $c : E \to \mathbb{R}_+$ be a capacity function. Then the maximum total value of a multiflow for the nets $\{st \mid s, t \in T, s \neq t\}$ is equal to*

$$\tfrac{1}{2} \sum_{s \in T} \gamma_c(s), \tag{73.32}$$

where $\gamma_c(s)$ denotes the minimum capacity of a cut separating s and $T \setminus \{s\}$. If all capacities are integer there is a half-integer maximum-value multiflow. If moreover $c(\delta(v))$ is even for each $v \in V \setminus T$, there is an integer maximum-value multiflow.

Proof. By continuity and compactness, we can assume that c is integer and that $c(\delta(v))$ is even for each $v \in V \setminus T$.

Replacing each edge e by $c(e)$ parallel edges we obtain a graph to which we can apply Corollary 73.2d. The paths obtained in the new graph give an integer multiflow as required in the original graph. ∎

Notes. Karzanov [1979a] gave an $O(\mathrm{MF}(n,m) \cdot \log |T|)$ algorithm to find a half-integer maximum-value multiflow for integer c. ($\mathrm{MF}(n,m)$ is the time needed to find a maximum flow in a digraph with n vertices and m arcs.) Ibaraki, Karzanov, and Nagamochi [1998] extended this algorithm to obtain an integer solution if $c(\delta_E(v))$ is even for each $v \in V \setminus T$. They also gave an extension to directed graphs.

Lovász [1976b] mentioned the following consequence of Corollary 73.2b:

(73.33) Let $G = (V, E)$ be a graph and let $c : E \to \mathbb{Z}_+$ be a capacity function with $c(\delta(v))$ even for each $v \in V$. Then for each $u \in V$, the maximum number of circuits in G that traverse u, such that no edge e is in more than $c(e)$ of these circuits, is equal to half of the minimum capacity of a family of edges meeting each circuit through u at least twice.

To prove this, let $s_1, \ldots, s_d$ be the neighbours of u. Replace u by d new vertices $u_1, \ldots, u_d$, and for each $i = 1, \ldots, d$, add $c(us_i)$ parallel edges connecting u_i and s_i. Moreover, replace each edge e of G not containing u by $c(e)$ parallel edges. Then the assertion follows from Corollary 73.2b applied to the new graph and to $T := \{u_1, \ldots, u_d\}$.

73.2a. Direct proof of Corollary 73.2d

Let G, T form a counterexample with $|V| + |E|$ as small as possible. Let μ be equal to (73.31). Then:

(73.34) for any $s \in T$ and any minimum-size cut $\delta(U)$ separating s and $T \setminus \{s\}$, with $U \cap T = \{s\}$, one has $U = \{s\}$.

To see this, suppose $U \neq \{s\}$. Contract U to one vertex, s' say, obtaining graph G'. Let $T' := (T \setminus \{s\}) \cup \{s'\}$. By the minimality of G, G' contains μ' edge-disjoint T'-paths, where μ' equals (73.31) for G', T'. Each edge in $\delta(U)$ belongs to one of these T'-paths (as in G' it is a minimum-size cut separating s' and $T' \setminus \{s'\}$). Let G'' be the graph obtained from G by contracting $V \setminus U$ to one new vertex, u say. By the minimality of $\delta(U)$, G'' contains $d_E(U)$ edge-disjoint $s - u$ paths (by Menger's theorem). By concatenation, we find μ' edge-disjoint T-paths in G. As $\mu' \geq \mu$, this contradicts the fact that G is a counterexample. This proves (73.34).

As G, T form a counterexample, there is at least one vertex $v \in V \setminus T$ with at least two different neighbours. Let uv and vw be two of the edges incident with v, with $u \neq w$. Replacing these two edges by one new edge uw, we obtain a graph G'''. As G is a counterexample, G''' has no μ edge-disjoint T-paths. As G''' is smaller than G, it is no counterexample, and so there is an $s \in T$ with $\gamma_{G'''}(s) < \gamma_G(s)$. Hence there is a $U \subseteq V$ with $U \cap T = \{s\}$ and $d_{G'''}(U) < \gamma_G(s)$. Then, by parity, $d_{G'''}(U) \leq \gamma_G(s) - 2$, and hence $d_G(U) \leq \gamma_G(s)$. So by (73.34), $U = \{s\}$. Hence $d_G(U) = d_{G'''}(U) < \gamma_G(s)$, a contradiction.

By similar methods one may prove an analogous result for *directed graphs*, due to M.V. Lomonosov (cf. Karzanov [1979b]) and Frank [1989]: Given a digraph $D = (V, A)$ and $T \subseteq V$, call a directed path P an *T-path* if its end vertices are distinct and belong to T, and no internal vertex of P belongs to T. Then, if D is Eulerian, the maximum number of edge-disjoint T-paths in G is equal to the minimum value of

$$(73.35) \qquad \sum_{s \in T} d_A^{\text{out}}(X_s),$$

taken over disjoint sets $X_s \subseteq V$ with $s \in X_s$ for $s \in T$.

73.3. Further results and notes

73.3a. Further notes on Mader's theorem

In general it is not true that given any subset T of the vertex set of a graph, the maximum number M of edge-disjoint T-paths is equal to the minimum size m of an edge set intersecting each T-path: the complete bipartite graph $K_{t,n}$, with t odd and T the colour class with t vertices, has $M = \frac{1}{2}n(t-1)$ and $m = n(t-1)$. Mader's edge-disjoint T-paths theorem (Corollary 73.2b) implies the conjecture of Gallai [1961] (cf. Lovász [1976b]) that $M \geq \frac{1}{2}m$ for any graph. (Lovász [1976b] showed that $M \geq \frac{1}{4}m$, and P.D. Seymour (personal communication 1977) that $M \geq \frac{1}{3}m$.)

For Eulerian graphs, Corollary 73.2d implies the sharper inequality

$$(73.36) \qquad M \geq \frac{t}{2(t-1)} m,$$

where $t := |T|$. Indeed, for each $s \in T$, let E_s be a minimum-size $s - T \setminus s$ cut. Let E_t have the largest size among them. Then $\bigcup_{s \neq t} E_s$ intersects each T-path. Hence

$$(73.37) \qquad m \leq |\bigcup_{s \neq t} E_s| \leq \sum_{s \neq t} |E_s| \leq (1 - \frac{1}{t}) \sum_{s \in T} |E_s| = \frac{t-1}{t} 2M.$$

This proves Gallai's conjecture that $M \geq \frac{1}{2}m$ for Eulerian graphs.

The graph which arises from the complete bipartite graph $K_{t,n}$ by replacing each edge by two parallel edges, with T the colour class with t elements, has $M = tn$ and $m = 2(t-1)n$. So inequality (73.36) is sharp for Eulerian graphs.

Gallai [1961] derived, from matching theory, the following on edge-disjoint paths with both ends in T (not necessarily distinct). Let $G = (V, E)$ be a graph and $T \subseteq V$. Call a path a *weak T-path* if it has length at least 1, and connects two (not necessarily distinct) vertices in T, while no internal vertex belongs to T. For any $U \subseteq V$, let $\mathcal{K}_U$ denote the set of components of $G - U$. Then the maximum number of edge-disjoint weak T-paths is equal to the minimum value of

$$(73.38) \qquad |E[U]| + \sum_{K \in \mathcal{K}_U} \lfloor \frac{d_E(K)}{2} \rfloor,$$

over U with $T \subseteq U \subseteq V$. The maximum number of internally vertex-disjoint weak T-paths is equal to the minimum value of

$$|E[U]| + |W \setminus U| + \sum_{K \in \mathcal{K}_W} \lfloor \frac{d_E(K)}{2} \rfloor, \tag{73.39}$$

over U, W satisfying $T \subseteq U \subseteq W \subseteq V$.

A min-max relation and a polynomial-time algorithm for the minimum cost of a maximum collection of edge-disjoint T-paths were given by Karzanov [1993,1997]. A corresponding polyhedron was described by Burlet and Karzanov [1998].

Nash-Williams [1961a] gave necessary and sufficient conditions for a graph $G = (V, E)$ and a function $g : V \to \mathbb{Z}_+$ such that the edges of G can be partitioned into (nonclosed) paths such that $g(v)$ of these paths end at v, for each $v \in V$.

More on Mader's theorem can be found in Mader [1989], and on Gallai's theorem in Mader [1980].

73.3b. A generalization of fractionally packing T-paths

The following theorem was announced by Karzanov and Lomonosov [1978] and proved by Karzanov [1985d,1987d] and Lomonosov [1985] (the latter paper does not consider the parity case). Taking H to be a complete graph we obtain Corollary 73.2d.

Theorem 73.3. *Let $G = (V, E)$ and $H = (T, R)$ be graphs, where H is the complement of the line graph of some triangle-free graph H_0. Let $c : E \to \mathbb{Z}_+$ be a capacity function. Then there exists a quarter-integer maximum-value multiflow. If H_0 is bipartite, there exists a half-integer maximum-value multiflow. If $c(\delta(v))$ is even for each $v \in V \setminus T$, there exists a half-integer maximum-value multiflow. If $c(\delta(v))$ is even for each $v \in V$ and H_0 is bipartite, there exists an integer maximum-value multiflow.*

(For the special case where H is the union of two complete bipartite graphs H' and H'' such that $VH' \subseteq VH''$ or such that $H' = K_2$, Cherkasskiĭ [1976] showed that the maximum multiflow is attained by a half-integer multiflow (for integer capacities).)

Related is the following characterization of the maximum value of a multiflow, announced by Karzanov and Lomonosov [1978], and proved by Karzanov [1979b, 1985d,1987d] and Lomonosov [1985]:

Theorem 73.4. *Let $G = (V, E)$ and $H = (T, R)$ be graphs, where H is the complement of the line graph of some triangle-free graph. Let $c : E \to \mathbb{R}_+$ be a capacity function. Let $\mathcal{U}$ denote the collection of subsets U of V such that $U \cap T$ is a stable set of H. Then the maximum total value of a multiflow subject to c is equal to the minimum value of*

$$\sum_{U} \lambda_U c(\delta_G(U)) \tag{73.40}$$

taken over $\lambda : \mathcal{U} \to \mathbb{R}_+$ satisfying

$$\sum_{U} \lambda_U \chi^{\delta_R(U)} \geq \mathbf{1}_R. \tag{73.41}$$

It implies that if H is the complement of the line graph of some triangle-free graph, then in Theorem 70.2 one can restrict the length functions l to nonnegative combinations of cut functions. Karzanov and Pevzner [1979] showed that if H is not the line graph of a triangle-free graph, then Theorem 73.4 does not hold for some G, c.

Karzanov [1987b,1989] proved that for any graph $H = (T, R)$:

(73.42) if there exists an integer $k \geq 1$ such that for any graph $G = (V, E)$ with $T \subseteq V$ and any $c : E \to \mathbb{Z}_+$, there is a $\frac{1}{k}$-integer maximum-value multiflow, then any three pairwise intersecting inclusionwise maximal stable sets A, B, C of H satisfy $A \cap B = A \cap C = B \cap C$.

Karzanov [1991] conjectured that the reverse implication holds and that $k = 4$ will do. (Karzanov [1987a] announced a proof of this, but the proof failed.) The techniques of Karzanov [1987d] yield a strongly polynomial-time algorithm for the problems in Theorems 73.3 and 73.4.

73.3c. Lockable collections

Let T be a set and let $H = (T, R)$ be the complete graph on T. A collection $\mathcal{A}$ of subsets of T is called *lockable* if for each (supply) graph $G = (V, E)$ with $V \supseteq T$ and for each capacity function $c : E \to \mathbb{R}_+$, there is a multiflow for demand graph H such that

(73.43) for each $U \in \mathcal{A}$, the sum of the flow values of those nets split by U is equal to the minimum of $c(\delta_E(X))$ taken over $X \subseteq V$ satisfying $X \cap T = U$.

(Here U *splits* a pair of vertices if precisely one of them is in U.)

The following characterization of lockable collections was proved jointly by Karzanov [1979b,1984] and Lomonosov [1985] (announced in Lomonosov [1979b]). Recall that two subsets X, Y of T are called *crossing* if each of $X \cap Y$, $X \setminus Y$, $Y \setminus X$, and $T \setminus (X \cup Y)$ is nonempty. (The 1979 references did not consider the Euler condition.) A short proof was given by Frank, Karzanov, and Sebő [1992, 1997].

(73.44) A collection $\mathcal{A}$ is lockable if and only if $\mathcal{A}$ contains no three pairwise crossing sets. If $\mathcal{A}$ is lockable and c is integer, there is a half-integer multiflow satisfying (73.43). If moreover $c(\delta(v))$ is even for each $v \in V \setminus T$, there is an integer multiflow satisfying (73.43).

We show that this generalizes two results proved earlier. First we show that Corollary 73.2d can be derived. Let $G = (V, E)$ be a graph and let $T \subseteq V$ be such that each vertex $v \in V \setminus T$ has even degree. Let $\mathcal{A} := \{\{v\} \mid v \in T\}$. Then $\mathcal{A}$ contains no three pairwise crossing sets, and hence (73.44) applies. Let $c := \mathbf{1}$. By (73.44), there exists a collection $\mathcal{P}$ of edge-disjoint T-paths such that for each $v \in T$, there are $\gamma_G(v)$ paths in $\mathcal{P}$ with end vertex v. So $|\mathcal{P}| = \frac{1}{2} \sum_{v \in T} \gamma_G(v)$, and we have Corollary 73.2d.

Second we derive Theorem 72.2 for $H = C_5$. Let $G = (V, E)$ be a graph and let $H = (T, R)$ be the graph C_5, with $T \subseteq V$. Let $c : E \to \mathbb{Z}_+$ and $d : R \to \mathbb{Z}_+$ be a capacity and demand function satisfying the Euler and cut conditions. We show that there is a feasible integer multiflow. To this end we can assume that

(73.45) $\qquad c(\delta_E(v)) = d(\delta_R(v))$ for each $v \in T$.

If this is not the case, add an edge $e = v'v$, where v' is a new vertex, define $c(e) := d(\delta_R(v))$, and replace v by v' in H. This does not violate the Euler and cut conditions.

Define

(73.46) $\qquad \mathcal{A} := \{\{v\} \mid v \in T\} \cup \{\{u,v\} \mid u,v \in T, u \neq v, uv \notin R\}$.

Then no three sets in $\mathcal{A}$ are pairwise crossing, and so (73.44) applies; that is, there is an integer multiflow $(x_r \mid r \in R')$ such that (73.43) holds, where

(73.47) $\qquad R' := \{st \in T \mid s,t \in T, s \neq t\}$.

. (So x_r is an $s-t$ flow in G for $r = st \in R'$.) We show that the value of x_r is equal to d_r for each $r \in R$, and is equal to 0 for each $r \in R' \setminus R$, as required.

Let b_r be the value of x_r, for $r \in R'$. By (73.43) and the cut condition,

(73.48) $\qquad b(\delta_{R'}(U)) \geq d(\delta_R(U))$ for each $U \in \mathcal{A}$,

since there exists an $X \subseteq V$ with $X \cap T = U$ and

(73.49) $\qquad b(\delta_{R'}(U)) = c(\delta_E(X)) \geq d(\delta_R(X)) = d(\delta_R(U))$.

Moreover, equality holds if $|U| = 1$, since for $v \in T$ we have $b(\delta_{R'}(v)) \leq c(\delta_E(v)) = d(\delta_R(v))$, by (73.45). Now add up all inequalities (73.48) for those $U \in \mathcal{A}$ with $|U| = 1$. Similarly, add up all inequalities (73.48) for those $U \in \mathcal{A}$ with $|U| = 2$. Both sums have the same terms at the right-hand sides of the inequality sign. But the first sum has more terms at the left-hand side than the second sum has. As the first one has equality, the second one also has equality, and the terms are equal. That is, equality holds in (73.48) for each $U \in \mathcal{A}$. This implies that $b_r = d_r$ for each $r \in R$ and $b_r = 0$ for each $r \in R' \setminus R$, as (73.48) yields a nonsingular system of equations.

This shows Theorem 72.2 for the case $H = C_5$. Lomonosov [1985] argued how also the case $H = K_4$ can be derived from (73.44).

Pevzner [1987] studied the maximum size of a collection of sets no three of which are pairwise crossing. For a short proof, see Fleiner [2001b]. More on lockable collections and related structures can be found in Ibaraki, Karzanov, and Nagamochi [1998], Ilani, Korach, and Lomonosov [2000], and Ilani and Lomonosov [2000].

73.3d. Mader matroids

The exchange phenomenon for $\mathcal{S}$-paths used in the proof of Mader's disjoint $\mathcal{S}$-paths theorem (Theorem 73.2) gives rise to a matroid as follows.

Let $G = (V,E)$ be an undirected graph and let $\mathcal{S} = \{S_1, \ldots, S_k\}$ be a collection of disjoint subsets of V. Define $T := S_1 \cup \cdots \cup S_k$. Let $\mathcal{I}$ be the collection of all subsets I of T with the property that there exists a collection $\mathcal{P}$ of disjoint $\mathcal{S}$-paths with $I \subseteq \text{ends}(\mathcal{P})$. Here ends$(\mathcal{P})$ denotes the set of ends of the paths in $\mathcal{P}$.

Theorem 73.5. *$M = (T, \mathcal{I})$ is a matroid.*

Proof. $\mathcal{I}$ trivially is nonempty and closed under taking subsets. To see that it gives a matroid, we apply Theorem 39.1. For any collection $\mathcal{R}$ of paths, let $E\mathcal{R}$ denote the set of edges traversed by the paths in $\mathcal{R}$. Choose $I, J \in \mathcal{I}$ with $|I \setminus J| = 1$ and $|J \setminus I| = 2$. We show that $I + j \in \mathcal{I}$ for some $j \in J \setminus I$. The proof of this is by induction on $|E\mathcal{Q} \setminus E\mathcal{P}|$, where $\mathcal{P}$ and $\mathcal{Q}$ are collections of disjoint $\mathcal{S}$-paths with $I \subseteq \text{ends}(\mathcal{P})$ and $J \subseteq \text{ends}(\mathcal{Q})$.

Let $I \setminus J = \{r\}$. Let r be an end of path $P \in \mathcal{P}$, and let r belong to S_i say. If P is disjoint from all paths in $\mathcal{Q}$, then $J + r \subseteq \text{ends}(\mathcal{Q} \cup \{P\})$, and hence $I + j \in \mathcal{I}$ for each $j \in J \setminus I$.

If P intersects some path in $\mathcal{Q}$, follow path P starting at r, until we meet, at vertex v say, a path in $\mathcal{Q}$, Q say. Let Q have ends s and t. Let Q^s and Q^t be the $s - v$ and $t - v$ part of Q. By symmetry, we may assume that

(73.50) $\quad EQ^s \not\subseteq EP$ and $t \notin S_i$.

Indeed, if $EQ^s \not\subseteq EP$ and $EQ^t \not\subseteq EP$, then by symmetry we can assume $t \notin S_i$ (as $s \notin S_i$ or $t \notin S_i$). If $EQ^s \subseteq EP$ or $EQ^t \subseteq EP$, then by symmetry we can assume $EQ^t \subseteq EP$, hence $EQ^s \not\subseteq EP$ (as $Q \neq P$) and $t \notin S_i$ (as t is the other end of P and as $r \in S_i$). So we may assume (73.50).

Let Q' be the path obtained by concatenating Q^t and the $v - r$ part of P. Then Q' is an $\mathcal{S}$-path disjoint from all paths in $\mathcal{Q} \setminus \{Q\}$. Define $\mathcal{Q}' := (\mathcal{Q} \setminus \{Q\}) \cup \{Q'\}$ and $J' := J - s + r$. So $J' \subseteq \text{ends}(\mathcal{Q}')$. Hence $J' \in \mathcal{I}$.

If $s \notin I$, we are done, since then there is a $j \in J \setminus I$ with $I + j \subseteq J - s + r$. If $s \in I$, then $J' \setminus I = J \setminus I$, and we can apply the induction hypothesis, since

(73.51) $\quad |E\mathcal{Q}' \setminus E\mathcal{P}| < |E\mathcal{Q} \setminus E\mathcal{P}|$.

Hence, by induction, there is a $j \in J \setminus I$ with $I + j \in \mathcal{I}$ as required. ∎

We call a matroid $M = (T, \mathcal{I})$ obtained in this way a *Mader matroid*. If $k = 2$, we call the Mader matroid also a *Menger matroid*. The *matching matroids* (cf. Section 39.4a) are the special case of Mader matroids where $\mathcal{S} = \{\{v\} \mid v \in V\}$.

The question is how Mader matroids relate to known classes of matroids. The class of gammoids seems close to Mader matroids. Hence the question:

(73.52) $\quad$ Is each Mader matroid a gammoid?

What can be proved is that each *Menger* matroid is a gammoid. More precisely:

Theorem 73.6. *A matroid is a gammoid if and only if it is a contraction of a Menger matroid.*

Proof. To see necessity, each gammoid is the contraction of a transversal matroid (Corollary 39.5a). Hence it suffices to show that each transversal matroid $M = (T, \mathcal{I})$ is a contraction of a Menger matroid. We can assume that the transversal matroid is obtained from a bipartite graph G with colour classes S and T, such that the independent sets of M are the subsets of T covered by some matching in G, and such that G has a matching of size $|S|$. Let M' be the Menger matroid on $S \cup T$ obtained from G by taking $\mathcal{S} := \{S, T\}$. Then contracting S in M' gives M. So M is the contraction of a Menger matroid.

To prove sufficiency, it suffices to show that each Menger matroid is a gammoid (as the class of gammoids is closed under contractions). Let $G = (V, E)$ be an undirected graph and let S_1 and S_2 be disjoint subsets of V. Define $\mathcal{S} := \{S_1, S_2\}$. Let M be the Menger matroid obtained this way. So a subset B of $S_1 \cup S_2$ is a base of M if and only if there exists a maximum-size collection of disjoint $\mathcal{S}$-paths in G such that B is the set of ends of these paths. We can assume that neither S_1 nor S_2 spans an edge of G (as it is not in any $\mathcal{S}$-path).

Let $D = (V, A)$ be the directed graph obtained from G by orienting each edge incident with S_1 away from S_1 and by orienting each edge incident with S_2 towards S_2, and by replacing each remaining edge e by two oppositely oriented arcs connecting the ends of e. So a subset B of $S_1 \cup S_2$ is a base of M if and only if there exists a maximum-size collection of disjoint directed paths in D from S_1 to S_2, such that B is the set of ends of these paths.

Derive an undirected graph $\widetilde{G}$ from D as follows. Replace each vertex $v \notin S_1 \cup S_2$ by two vertices, v' and v''. For $v \in S_1$ define $v' := v$, and for $v \in S_2$ define $v'' := v$. Replace each arc (u, v) of D by an edge $u'v''$ of $\widetilde{G}$. Moreover, for each $v \in V \setminus (S_1 \cup S_2)$, make an edge $v'v''$ of $\widetilde{G}$. This makes the undirected graph $\widetilde{G}$. Then for any subset I of $S_1 \cup S_2$ one has, by a well-known argument (cf. Theorem 39.5):

(73.53) D contains disjoint directed paths from S_1 to S_2, such that I is the collection of the ends of these paths $\iff$ $\widetilde{G}$ contains a matching N which covers all vertices except those in $(S_1 \cup S_2) \setminus I$.

So a subset B of $S_1 \cup S_2$ is a base of M if and only if $\widetilde{G}$ has a maximum-size matching N which covers all vertices except those in $(S_1 \cup S_2) \setminus B$. So M is the matroid obtained from the matching matroid of $\widetilde{G}$ by contracting all vertices in $V \setminus (S_1 \cup S_2)$. As each matching matroid is a transversal matroid (cf. Section 39.4a), this proves that each Menger matroid is the contraction of a transversal matroid, and hence is a gammoid (Corollary 39.5a). ∎

By the results in Section 39.4a, the class of gammoids is also equal to the class of contractions of matching matroids. So contractions of Menger matroids and those of matching matroids (two special cases of Mader matroids) coincide.

A question related to (73.52) is:

(73.54) Is each Mader matroid linear?

As gammoids are representable over all large enough fields, a positive answer to question (73.52) implies a positive answer to question (73.54). The constructions given in Section 73.1a suggest a positive answer to (73.54).

73.3e. Minimum-cost maximum-value multiflows

Karzanov [1979d] showed that if H is a complete graph and all capacities are integer, there exists a half-integer minimum-cost maximum-value multiflow (and he gave a pseudo-polynomial-time algorithm to find it). This can be directly extended to the case where H is a complete multipartite graph.

A short proof, together with a strongly polynomial-time algorithm, was given by Karzanov [1994a], where also the existence of a half-integer optimum dual solution was shown. Other algorithms (based on scaling) were given by Goldberg and Karzanov [1997].

On the other hand, Karzanov [1987b] showed that if $H = (T, R)$ is not a complete multipartite graph (that is, H contains two intersecting inclusionwise maximal stable sets), then there is no fixed integer k such that for each graph $G = (V, E)$ with $V \supseteq T$ and each integer capacity function and each cost function, there is a $\frac{1}{k}$-integer minimum-cost maximum-value multiflow.

73.3f. Further notes

Lomonosov [1985] (announced in Lomonosov [1979a]) gave a min-max formula for the maximum total value of a multiflow if H is the union of two (not necessarily disjoint) cliques.

Karzanov and Manoussakis [1996] showed: Let $G = (V, E)$ and $H = (T, R)$ be graphs, with $T \subseteq V$, where $H = K_{2,r}$, and where $\deg_G(v)$ is even for each $v \in V \setminus T$. For any T-path P, let $\alpha(P)$ denote the distance in H between the ends of P. Then the maximum value of

(73.55) $$\sum_{P \in \mathcal{P}} \alpha(P),$$

where $\mathcal{P}$ is a collection of edge-disjoint T-paths, is equal to the minimum value of

(73.56) $$\sum_{u,v \in T} |E[X_u, X_v]| \cdot \mathrm{dist}_H(u, v),$$

where $(X_u \mid u \in T)$ is a partition of V with $u \in X_u$ for $u \in T$. (As usual, $E[X, Y]$ is the set of edges connecting X and Y.) Extensions and related results are given by Karzanov [1998a,1998b,1998c].

Rothfarb and Frisch [1969] showed that the maximum total value of a 3-commodity flow equals the minimum capacity of a set of edges disconnecting all nets, if $|V| = 3$.

Dahlhaus, Johnson, Papadimitriou, Seymour, and Yannakakis [1992,1994] showed that it is NP-complete to find a minimal number of edges disconnecting any two vertices among three given vertices in an undirected graph. Chopra and Rao [1991] studied the corresponding polyhedron.

Chapter 74

Planar graphs

Finding disjoint paths in *planar* graphs is of interest not only for planar communication or transportation networks, but especially also for the design of VLSI-circuits. The routing of the wires should follow certain channels on layers of the chip. On each layer, these channels form a planar graph.

Even for planar graphs, disjoint paths problems are in general hard. However, for some special cases, polynomial-time algorithms and good characterizations have been found. In this chapter we discuss some of these cases.

Except if stated otherwise, throughout this chapter $G = (V, E)$ and $H = (T, R)$ denote the supply and demand graph, in the sense of Chapter 70. The pairs in R are called the *nets*. If $s_1, t_1, \ldots, s_k, t_k$ are given, then $R := \{s_1t_1, \ldots, s_kt_k\}$. If demands $d_1, \ldots, d_k$ are given, then $d(s_it_i) := d_i$. We denote $G + H = (V, E \cup R)$, where the disjoint union of E and R is taken, respecting multiplicities.

Recall that the *Euler condition* states that $G + H$ is Eulerian.

74.1. All nets spanned by one face: the Okamura-Seymour theorem

The complexity of the edge-disjoint paths problem for planar graphs with all nets on the outer boundary, is open. However, Okamura and Seymour [1981] showed that if the Euler condition holds, the edge-disjoint paths problem is polynomial-time solvable, and the cut condition is sufficient for solvability. We follow their method of proof.

Theorem 74.1 (Okamura-Seymour theorem)**.** *Let $G = (V, E)$ be a planar graph and let $H = (T, R)$ be a graph where T is the set of vertices of G incident with the unbounded face of G. Let the Euler condition hold. Then the edge-disjoint paths problem has a solution if and only if the cut condition holds.*

Proof. Necessity of the cut condition being trivial, we show sufficiency. The cut condition implies that $|R| \leq |E|$ (assuming that each $r \in R$ consists of two distinct vertices), since

(74.1) $$2|R| = \sum_{v\in V} \deg_R(v) \leq \sum_{v\in V} \deg_E(v) = 2|E|.$$

So we can consider a counterexample with $2|E| - |R|$ minimal. Then

(74.2) G is 2-connected.

Indeed, if G is disconnected, we can deal with the components separately. Suppose next that G is connected and has a cut vertex v. We may assume that for no $r = st \in R$, the vertices s and t belong to different components of $G - v$, since otherwise we can replace r by sv and vt, without violating the Euler or cut condition. For any component K of $G - v$ consider the graph induced by $K \cup \{v\}$. Again, the Euler and cut conditions hold (with respect to those nets contained in $K \cup \{v\}$). So by the minimality of $2|E| - |R|$, we know that paths as required exist in $K \cup \{v\}$. As this is the case for each component of $G - v$, we have paths as required in G. This proves (74.2).

Let C be the circuit formed by the outer boundary of G. If some $r \in R$ has the same ends as some edge e of G, we can delete e from G and r from R, and obtain a smaller counterexample. So no such r exists.

Call a subset X of V *tight* if $d_E(X) = d_R(X)$. Then

(74.3) there exists a tight subset X of V such that $\delta_E(X)$ intersects EC in precisely two edges.

Indeed, if there is no tight set X with $\emptyset \neq X \neq V$, we can choose an edge $e \in EC$, and replace E and R by $E \setminus \{e\}$ and $R \cup \{e\}$. This does not violate the cut condition, and hence would give a smaller counterexample.

So there exists a tight proper nonempty subset X of V. Choose X with $|\delta_E(X)|$ minimal. Then $G[X]$ and $G - X$ are connected. For suppose that (say) $G[X]$ is not connected. Let K be a component of $G[X]$. Then

(74.4) $$|\delta_E(K)| + |\delta_E(X \setminus K)| \geq |\delta_R(K)| + |\delta_R(X \setminus K)| \geq |\delta_R(X)| \\ = |\delta_E(X)| = |\delta_E(K)| + |\delta_E(X \setminus K)|.$$

So K is tight, while $|\delta_E(K)| < |\delta_E(X)|$, contradicting the minimality assumption. Hence $G[X]$ and $G - X$ are connected, implying (74.3).

Choose a set X as in (74.3) with $|X|$ minimal. Let e be one of the two edges in EC that belong to $\delta_E(X)$. Say $e = uw$ with $u \notin X$ and $w \in X$.

Since $d_R(X) = d_E(X) \geq 2$, we know $\delta_R(X) \neq \emptyset$. For each $r \in \delta_R(X)$, let s_r be the vertex in $r \cap X$, and t_r the vertex in $r \setminus X$. Choose $r \in \delta_R(X)$ such that t_r is as close as possible to u in the graph $C - X$.

Since s_r and t_r are nonadjacent, we know that $\{s_r, t_r\} \neq \{u, w\}$. So we can choose $v \in \{u, w\} \setminus \{s_r, t_r\}$. Let $R' := (R \setminus \{r\}) \cup \{s_r v, vt_r\}$. Trivially the Euler condition is maintained. We show that also the cut condition is maintained, yielding a contradiction as $2|E| - |R'| < 2|E| - |R|$ and as a solution for R' yields a solution for R.

To see that the cut condition is maintained, suppose to the contrary that there is a $Y \subseteq V$ satisfying

(74.5) $d_E(Y) < d_{R'}(Y)$.

By Theorem 70.4, we can take Y such that $G[Y]$ and $G - Y$ are connected. So $\delta_E(Y)$ has two edges on C. By symmetry we can assume that $t_r \notin Y$. By the Euler condition, (74.5) implies $d_E(Y) \leq d_{R'}(Y) - 2$. So

(74.6) $d_{R'}(Y) \geq d_E(Y) + 2 \geq d_R(Y) + 2 \geq d_{R'}(Y)$.

Hence we have equality throughout. So $\delta_{R'}(Y)$ contains both $s_r v$ and vt_r, that is, $s_r, t_r \notin Y$ and $v \in Y$. Moreover, $d_E(Y) = d_R(Y)$.

By the choice of r, there is no pair r' in R connecting $X \setminus Y$ and $Y \setminus X$ (since then $t_{r'} \in Y \setminus X$, and hence $t_{r'}$ is closer than t_r to u in $C - X$). So (using Theorem 3.1)

(74.7) $d_R(X \cap Y) + d_R(X \cup Y) = d_R(X) + d_R(Y)$.

Moreover,

(74.8) $d_E(X \cap Y) + d_E(X \cup Y) \leq d_E(X) + d_E(Y)$.

As the cut condition holds for $X \cap Y$ and $X \cup Y$, we have equality in (74.8), and therefore $X \cap Y$ is tight. Since $s_r \in X \setminus Y$, we know $|X \cap Y| < |X|$. So by the minimality of X we have $X \cap Y = \emptyset$. So $w \notin Y$, hence $u = v \in Y$. Then edge $e = uw$ connects $X \setminus Y$ and $Y \setminus X$, contradicting equality in (74.8). ∎

For multiflows, the Okamura-Seymour theorem implies the following result of H. Okamura (cf. note on p. 80 of Okamura and Seymour [1981]):

Corollary 74.1a. *Let $G = (V, E)$ be a planar graph, let R be a set of pairs of vertices on the outer boundary of G, and let $c : E \to \mathbb{R}_+$ and $d : R \to \mathbb{R}_+$. Then there exists a feasible multiflow if and only if the cut condition holds. If moreover c and d are integer, there is a half-integer multiflow.*

Proof. By compactness, continuity, and scaling, we can assume that c and d are integer. Replacing any edge e by $2c(e)$ parallel copies, and any pair $r \in R$ by $2d(r)$ parallel nets, we can apply the Okamura-Seymour theorem. The paths in the new graph give the multiflow in the original graph as required. ∎

Notes. The proof of Theorem 74.1 yields a polynomial-time algorithm for finding the edge-disjoint paths, since we can determine a minimum-size cut containing e' and e'', for any pair of edges e', e'' on the outer boundary of G (by finding a shortest path in the dual graph). Frank [1985] outlined that it in fact leads to an $O(n^3 \log n)$-time algorithm. (As P.D. Seymour observed, also the splitting-off technique used by Lins [1981] to prove Corollary 74.1b below yields a polynomial-time algorithm to find paths as required.)

74.1a. Complexity survey

Complexity survey for the disjoint paths problem in planar graphs with all terminals on the outer boundary and satisfying the Euler condition ($*$ indicates an asymptotically best bound in the table):

	$O(n^4)$	Hassin [1984] (also capacitated case)
	$O(n^3 \log n)$	Frank [1985] (also capacitated case)
	$O(n^2 \log^* n)$	Matsumoto, Nishizeki, and Saito [1985]: feasibility test (also capacitated case)
	$O(tn\sqrt{\log n})$	Matsumoto, Nishizeki, and Saito [1985]: feasibility test (also capacitated case)
	$O(kn + n^2\sqrt{\log n})$	Matsumoto, Nishizeki, and Saito [1985] (also capacitated case)
$*$	$O(tn + n\sqrt{t \log n})$	Frederickson [1987b]: feasibility test (also capacitated case)
	$O(n^2)$	Becker and Mehlhorn [1986]
	$O(tn)$	Becker and Mehlhorn [1986]: feasibility test
	$O(n^{5/3}(\log\log n)^{1/3})$	Kaufmann and Klär [1991]
	$O(kn + n\sqrt{\log n})$	Weihe [1993] (also capacitated case)
$*$	$O(n)$	Wagner and Weihe [1993,1995]
$*$	$O(kn)$	Weihe [1997c] (using Klein, Rao, Rauch, and Subramanian [1994], Henzinger, Klein, Rao, and Subramanian [1997]): capacitated case

Here $k := |R|$, t is the number of vertices that belong to at least one pair in R, and $\log^* n$ is the minimum l such that $\log^{(l)} n \leq 1$, where $\log^{(l)} n$ is obtained from n by taking l times the logarithm.

For sketches of the linear-time method of Wagner and Weihe [1993,1995], see Wagner [1993] or Ripphausen-Lipa, Wagner, and Weihe [1995].

Research problem. Is the undirected edge-disjoint paths problem polynomial-time solvable for planar graphs with all nets on the outer boundary? Is it NP-complete?

74.1b. Graphs on the projective plane

The Okamura-Seymour theorem is equivalent to a theorem of Lins [1981] on Eulerian graphs embedded in the projective plane. A closed curve in the projective plane is called *orientation-reversing* if after one turn the meaning of 'left' and 'right' is flipped. If a graph is embedded in a space S, we identify G with its image in S.

Corollary 74.1b (Lins' theorem)**.** *Let $G = (V, E)$ be an Eulerian graph embedded in the projective plane P^2. Then the maximum number of edge-disjoint orientation-*

reversing circuits in G is equal to the minimum number of intersections with G of any orientation-reversing closed curve in $P^2 \setminus V$.

Proof. Since any two orientation-reversing closed curve in the projective plane intersect, the maximum does not exceed the minimum. To see equality, let D be an orientation-reversing closed curve in $P^2 \setminus V$ having a minimum number, k say, of intersections with G. Necessarily, any intersection of D with G is a crossing of D and an edge of G. Let R be the set of edges of G intersected by D and let $G' := (V, E \setminus R)$. Then G' is a planar graph, embedded in the open sphere obtained from P^2 by deleting D. Each pair in R connects two vertices on the outer boundary of G'. It suffices to show that G' contains edge-disjoint paths P_r for $r \in R$, where P_r connects the vertices in r. Then the $P_r \cup \{r\}$ for $r \in R$ form a set of k edge-disjoint orientation-reversing circuits in G as required.

To show that the paths P_r exist, we can apply the Okamura-Seymour theorem. To this end, we must test the cut condition for G', R. Note that the pairs in R can be ordered as $r_1, \ldots, r_k$ such that when following the boundary of $P^2 \setminus D$, in one round we first meet $r_1, \ldots, r_k$ consecutively, and next we meet again $r_1, \ldots, r_k$ consecutively.

Let $X \subseteq V$, with $G'[X]$ and $G' - X$ connected, and with $d_R(X) > 0$. Then $\delta_{E \setminus R}(X)$ contains exactly two edges on the outer boundary of G'. Hence we can find an orientation-reversing closed curve in P^2 intersecting the edges of G' in $\delta_{E \setminus R}(X)$ and those in $R \setminus \delta_R(X)$. Hence

$$(74.9) \qquad d_{E \setminus R}(X) + |R| - d_R(X) \geq k = |R|,$$

that is, $d_{E \setminus R}(X) \geq d_R(X)$. So the cut condition holds for G' and R. ∎

In turn, the Okamura-Seymour theorem can be derived from Lins' theorem. To this end, we first show that in the Okamura-Seymour theorem one can make a number of assumptions that do not restrict the generality. Let $G = (V, E)$ be a planar graph, and let R be a set of pairs of vertices on the outer boundary of G, such that the Euler condition and the cut condition hold.

First one can assume that the pairs in R are disjoint: if $r = st$ and $r' = st'$ are two pairs in R, we can add a new vertex s' in the outer face, and a new edge $s's$, and reset $r' := s't'$. Second one may assume that any two pairs $r = st$, $r' = s't'$ in R are 'crossing' around the outer boundary of G; that is, s, s', t, t' occur in this order cyclically around the outer boundary. If this is not the case, there exist two pairs $r = st$ and $r' = s't'$ such that s, s', t', t occur in this order cyclically around the outer boundary and such that no vertex between s and s' (along the outer boundary) belongs to any pair in R. Now we can add three new vertices, q, q' and p, and edges qp, $q'p$, ps, ps', and reset $r := qt$ and $r' := q't'$ (Figure 74.1).

Let $G' = (V', E')$ be the new graph, and let R' be the new set of pairs. This construction maintains the cut condition. To see this, let $X \subseteq V \cup \{q, q', p\}$. Without loss of generality, $p \in X$. Suppose $d_{E'}(X) < d_{R'}(X)$. Then (using parity)

$$(74.10) \qquad d_{R'}(X) - 2 \geq d_{E'}(X) \geq d_E(X \cap V) \geq d_R(X \cap V) \geq d_{R'}(X) - 2,$$

and hence we have equality throughout. In particular, none of the new edges belong to $\delta_{E'}(X)$, and so $s, s', q, q' \in X$. But then $d_R(X \cap V) = d_{R'}(X)$, a contradiction.

So the cut condition is maintained. Also, any edge-disjoint pair of a $q - t$ path P and a $q' - t'$ path P' in G' contains an edge-disjoint pair of an $s - t$ path Q and

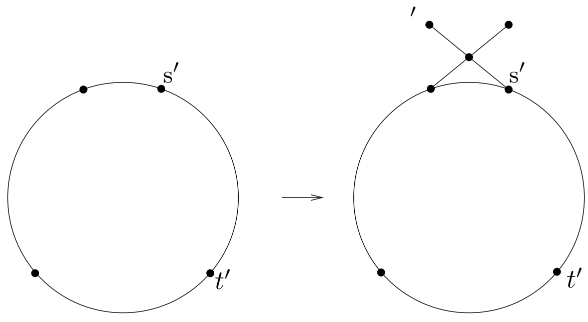


Figure 74.1

an $s'-t'$ path Q': if P traverses s and P' traverses s', this is trivial; if P traverses s' and P' traverses s, then P and P' intersect necessarily in V (as the pairs $s't$ and st' cross), and hence we can exchange P and P' at this intersection to obtain Q and Q' as required.

As we can embed G' such that q, q', t, t' occur in this order cyclically around the outer boundary of G', we have decreased the number of noncrossing pairs in R. Repeating this we can assume that all pairs in R are crossing.

Now, assuming that G is embedded in $\mathbb{R}^2$, we can embed $\mathbb{R}^2$ in the projective plane P^2. Then $P^2\setminus\mathbb{R}^2$ is a 'cross-cap' (Möbius strip). We can extend the embedding of G to an embedding of the Eulerian graph $G+H=(V,E\cup R)$, by embedding any $r\in R$ as an edge over the cross-cap. (Since any two nets cross in $\mathbb{R}^2$, they can be drawn disjoint in $P^2\setminus\mathbb{R}^2$.)

We derive from Lins' theorem that $G+H$ has $|R|$ edge-disjoint orientation-reversing circuits: this gives paths as required for the Okamura-Seymour theorem, as each of the circuits must contain at least one edge traversing the cross-cap, and hence at least one edge in R. As there are $|R|$ circuits, each contains exactly one edge in R, and so deleting the edges in R we obtain paths as required in the Okamura-Seymour theorem.

In order to apply Lins' theorem, we must show that each orientation-reversing closed curve D in $P^2\setminus V$ has at least $|R|$ intersections with $G+H$. To show this, we can assume that D traverses any face of $G+H$ at most once (otherwise we can shortcut D). As D is orientation-reversing, it traverses the cross-cap an odd number of times. Between any two traversals of D over the cross-cap, we can reroute D (in $\mathbb{R}^2$) such that instead of intersecting edges of G, it intersects edges in R, in such a way that the number of new intersections with R is not more than the number of deleted intersections with E (this follows from the cut condition in the Okamura-Seymour theorem). Doing this between any two traversals of the cross-cap, we obtain an orientation-reversing closed curve only intersecting edges in R. As each of the edges in R must be intersected (since D is orientation-reversing), we see that D has at least $|R|$ intersections with $G+H$. This shows that we can apply Lins' theorem.

74.1c. If only inner vertices satisfy the Euler condition

Frank [1985] showed an interesting extension of the Okamura-Seymour theorem, to the case where the parity condition is only required for the vertices not on the outer boundary. The proof amounts to appropriately pairing those vertices v on the outer boundary for which $\deg_E(v) + \deg_R(v)$ is odd. To this end, Frank first showed the following 'pairing lemma'. We say that a pair u, v of vertices of a circuit C *crosses* a pair e, f of edges of C, if u and v are in different components of the graph $C - e - f$.

For any set X let $\binom{X}{2}$ denote the collection of unordered pairs from X. A *pairing* of a set is a partition into pairs.

Lemma 74.2α (pairing lemma)**.** *Let $C = (V, E)$ be a circuit with $|V|$ even, and let $s : \binom{E}{2} \to \mathbb{Z}$ be such that, for each $x \in \binom{E}{2}$, $s(x)$ has the same parity as the size of any of the two components of $C - x$. Then V has a pairing M such that each $x \in \binom{E}{2}$ is crossed by at most $s(x)$ pairs in M if and only if*

(74.11) $$\sum_{x \in \mathcal{B}} s(x) \geq \tfrac{1}{2}q$$

for each collection $\mathcal{B}$ consisting of disjoint pairs in $\binom{E}{2}$. Here q denotes the number of odd components of the graph G obtained from the complete graph on V by deleting all edges crossing at least one pair in $\mathcal{B}$.

Proof. For any $x \in \binom{E}{2}$ and any $R \subseteq \binom{V}{2}$, let $\mathrm{cr}_R(x)$ denote the number of pairs in R crossing x.

Necessity of the condition is easy: if M as required exists, let $N \subseteq M$ be the set of those pairs in M leaving at least one odd component of G. Since each odd component of G is left by at least one pair in N, we have $|N| \geq \frac{1}{2}q$. On the other hand, each pair in N crosses at least one pair in $\mathcal{B}$, and so

(74.12) $$\tfrac{1}{2}q \leq |N| \leq \sum_{x \in \mathcal{B}} \mathrm{cr}_N(x) \leq \sum_{x \in \mathcal{B}} s(x),$$

proving (74.11).

To see sufficiency, first assume that $s(x) > 0$ for each $x \in \binom{E}{2}$. Let M be any of the two perfect matchings in C. Then for any $x \in \binom{E}{2}$, $\mathrm{cr}_M(x)$ is at most 2 and has the same parity as $s(x)$; therefore $\mathrm{cr}_M(x) \leq s(x)$.

Hence we can assume that there is a $y \in \binom{E}{2}$ with $s(y) = 0$. Let K be a component of $C - y$. Among all such y, K, choose y, K such that K is smallest. Let $V' := V \setminus K$, let u and w be the end vertices of the path $C - K$, and let $C' = (V', E')$ be the circuit obtained from $C - K$ by adding the new edge $f = uw$. As $s(y)$ is even, both K and V' have even size. Let N be the unique perfect matching in the path $C[K]$.

For each $x \in \binom{E'}{2}$, define $s'(x)$ by: $s'(x) := s(x)$ if $f \notin x$, and

(74.13) $$s'(x) := \min\{\min\{s(\{e, g\}) \mid g \in E \setminus E', g \notin N\}, \min\{s(\{e, g\}) - 1 \mid g \in N\}\}$$

if $x = \{e, f\}$. Trivially, $s'(x)$ has the same parity as any component of $C' - x$ for each $x \in \binom{E'}{2}$.

We show that condition (74.11) holds for the smaller structure. That is, for any collection $\mathcal{B}'$ of disjoint pairs in $\binom{E'}{2}$ one has

(74.14) $$\sum_{x\in\mathcal{B}'} s'(x) \geq \tfrac{1}{2}q',$$

where q' is the number of odd components of the graph G' obtained from the complete graph on V' by deleting all edges crossing at least one pair in $\mathcal{B}'$.

If $f \notin x$ for each $x \in \mathcal{B}'$, let $\mathcal{B} := \mathcal{B}'$. Then (74.14) follows from (74.11), as $s'(x) = s(x)$ for each $x \in \mathcal{B}$ and as $q' = q$.

If $f \in x$ for some $x \in \mathcal{B}'$, this x is unique. Let $z = \{e, g\} \in \binom{E}{2}$ attain the minimum in (74.13). If $g \notin N$, let $\mathcal{B} := (\mathcal{B}' \setminus \{x\}) \cup \{z\}$. Again (74.14) follows from (74.11), as $q' = q$.

If $g \in N$, let $\mathcal{B} := (\mathcal{B}' \setminus \{x\}) \cup \{y, z\}$. Then $q = q' + 2$ (as each component of G is a component of G' or is one of the odd components of $C[K] - g$). Also $s'(a) = s(a)$ for each $a \in \mathcal{B}' \setminus \{x\}$, while $s(z) = s'(x) - 1$ and $s(y) = 0$. Hence by (74.11) we have (74.14).

Hence, by (74.14), there exists a pairing M' of V' such that for each $x \in \binom{E'}{2}$, $\mathrm{cr}_{M'}(x) \leq s'(x)$. Then $M := N \cup M'$ is a pairing of V. We show that $\mathrm{cr}_M(z) \leq s(z)$ for each $z \in \binom{E}{2}$. If $z \in \binom{E'}{2}$, then $\mathrm{cr}_M(z) = \mathrm{cr}_{M'}(z) \leq s'(z) = s(z)$. If $z \notin \binom{E'}{2}$, let $z = \{e, g\}$ with $g \in E \setminus E'$. If $e \in E'$, let $x := \{e, f\}$. If $g \notin N$, then $\mathrm{cr}_M(z) = \mathrm{cr}_{M'}(x) \leq s'(x) \leq s(z)$. If $g \in N$, then $\mathrm{cr}_M(z) = \mathrm{cr}_{M'}(x) + 1 \leq s'(x) + 1 \leq s(z)$. Finally, if $e \in E \setminus E'$, then $\mathrm{cr}_M(z) = \mathrm{cr}_N(z) \leq s(z)$, since, by the choice of y, $s(z)$ is positive and has the same parity as $\mathrm{cr}_N(z)$, while $\mathrm{cr}_N(z) \leq 2$. ∎

The proof gives a polynomial-time algorithm to find the pairing: iteratively one finds a pair x with $s(x) = 0$ and applies the reduction described in the proof; if no pair x with $s(x) = 0$ exists, one takes any perfect matching in C.

The pairing lemma implies (Frank [1985]):

Theorem 74.2. *Let $G = (V, E)$ be a planar graph such that each vertex not on the outer boundary has even degree. Let R be a set of pairs of vertices on the outer boundary of G. Then there exist edge-disjoint paths P_r for $r \in R$, where P_r connects the vertices in r, if and only if*

(74.15) $$\sum_{j=1}^{l}(d_E(X_j) - d_R(X_j)) \geq \tfrac{1}{2}q$$

for each collection of subsets $X_1, \ldots, X_l$. Here q denotes the number of components K of $G' := G - \delta_E(X_1) - \cdots - \delta_E(X_l)$ with $d_E(K) + d_R(K)$ odd.

Proof. Call a vertex v or a subset X of V *odd* if $\deg_E(v) - \deg_R(v)$ or $d_E(X) - d_R(X)$ is odd.

Necessity of (74.15) is easy: let E' be the set of edges not used by the P_r. Then for any set X, $d_{E'}(X) \leq d_E(X) - d_R(X)$, while on the other hand $d_{E'}(X) \geq 1$ if X is odd. Thus at least $\frac{1}{2}q$ edges from $\bigcup_j \delta_E(X_j)$ belong to E'. So

(74.16) $$\tfrac{1}{2}q \leq \Big|\bigcup_j \delta_{E'}(X_j)\Big| \leq \sum_j d_{E'}(X_j) \leq \sum_j (d_E(X_j) - d_R(X_j));$$

that is, we have (74.15).

Sufficiency follows from the pairing lemma (Lemma 74.2α) and the Okamura-Seymour theorem. Indeed, let $v_1, \ldots, v_{2n}$ be the odd vertices, in cyclic order along the outer boundary. Let C be the circuit with vertices $v_1, \ldots, v_{2n}$ and edges $v_{i-1}v_i$ for $i = 1, \ldots, 2n$, setting $v_0 := v_{2n}$. For each pair x of edges e, e' of C, define

(74.17) $$s(x) := \min\{d_E(U) - d_R(U) \mid U \subseteq V, \delta_F(U) = \{e, e'\}\}.$$

Then s satisfies the conditions described in the pairing lemma. Indeed, the parity condition is easily checked. To see (74.11), let $\mathcal{B}$ be a collection of disjoint pairs from EC. For each $x \in \mathcal{B}$, let U_x attain the minimum (74.17). Let $G' := G - \bigcup_{x \in \mathcal{B}} \delta_E(U_x)$. Let H be the graph obtained from the complete graph on VC by deleting all edges crossed by at least one pair in $\mathcal{B}$. Then for each component K of G' one has: the odd vertices in K are contained in some component of H (since $K \cap VC \subseteq U_x$ or $K \cap VC \subseteq V \setminus U_x$ for each $x \in \mathcal{B}$; so no two vertices in $K \cap VC$ cross any x in $\mathcal{B}$). Hence the number of odd components of G' is at least the number of odd components of H. So the condition in the pairing lemma follows from condition (74.15).

Applying the pairing lemma, we obtain a matching M of the odd vertices with $d_M(U) \leq d_E(U) - d_R(U)$ for each $U \subseteq V$. Also we have that $\deg_E(v) + \deg_R(v) + \deg_M(v)$ is even for each $v \in V$. So for $R' := R \cup M$, we can apply the Okamura-Seymour theorem, to obtain in G for each $r = st \in R'$ an $s - t$ path P_r, such that the P_r are edge-disjoint. Restriction to R gives paths as required. ∎

In the theorem one can assume that $l \leq |E|$, since for each edge e of G we need at most one X_i splitting e. So the theorem is a good characterization.

As the pairing lemma is polynomial-time constructive, one can find edge-disjoint paths as required if the condition is met — similarly for the capacitated case. Frank [1985] showed that under the conditions of Theorem 74.2, the edge-disjoint paths problem, and its capacitated version, can be solved in $O(n^3 \log n)$ time. Also Becker and Mehlhorn [1986] showed that this problem is polynomial-time solvable, and they gave a time bound of $O(tn + T(n))$, where $T(n)$ is the time needed to solve a problem where the Euler condition holds, and where t is the number of vertices on the outer boundary. Weihe [1999] finally gave a linear-time algorithm.

The special case where G is a rectangular grid was solved by Frank [1982c], showing that condition (74.15) can be simplified in this case.

74.1d. Distances and cut packing

With planar duality one may derive another, dual result of the Okamura-Seymour theorem, that relates distances to packings of cuts in planar graphs (Hurkens, Schrijver, and Tardos [1988]):

Corollary 74.2a. *Any planar bipartite graph G contains disjoint cuts such that any two vertices s, t on the outer boundary of G are separated by* $\mathrm{dist}_G(s,t)$ *of these cuts.*

Proof. Let $\mathcal{X}$ be the set of pairs e, e' of edges along the outer boundary of G such that if $e = st$ and $e' = s't'$ where s, t, s', t' occur in this order cyclically around the outer boundary, then

(74.18) $$\mathrm{dist}_G(s, s') + \mathrm{dist}_G(t, t') - \mathrm{dist}_G(s, t') - \mathrm{dist}_G(s', t) = 2.$$

(Note that for any e, e', the left-hand side equals 0 or 2, by the triangle inequality, and by the fact that each $s - s'$ path intersect each $t - t'$ path.)

We say that a pair e, e' of edges along the outer boundary *crosses* a pair u, v of vertices along the outer boundary if any $u-v$ path along the outer boundary traverses exactly one of e and e'. We show that for any two vertices u, v on the outer boundary of G:

(74.19) $\quad \text{dist}_G(u,v) = \text{number of pairs in } \mathcal{X} \text{ that cross } u, v.$

To see this, assume that $v_1, \ldots, v_n$ are the vertices of G cyclically along the outer boundary, and let $u = v_n$ and $v = v_k$. Then (setting $v_0 := v_n$):

(74.20)
$$\begin{aligned}
&\text{number of pairs in } \mathcal{X} \text{ that cross } u, v \\
&= \tfrac{1}{2}\sum_{i=1}^{k}\sum_{j=k+1}^{n}(\text{dist}_G(v_{i-1},v_{j-1}) + \text{dist}_G(v_i,v_j) - \text{dist}_G(v_{i-1},v_j) \\
&-\text{dist}_G(v_i,v_{j-1})) = \\
&\tfrac{1}{2}\sum_{i=1}^{k}\big(\text{dist}_G(v_{i-1},v_k) - \text{dist}_G(v_{i-1},v_n) + \text{dist}_G(v_i,v_n) - \text{dist}_G(v_i,v_k)\big) \\
&= \tfrac{1}{2}\text{dist}_G(v_0,v_k) + \tfrac{1}{2}\text{dist}_G(v_k,v_n) = \text{dist}_G(u,v)
\end{aligned}$$

(by cancellation).

This shows (74.19), which implies that it suffices to show that we can find disjoint cuts C_π for $\pi \in \mathcal{X}$, such that C_π intersects the outer boundary of G in the two edges in π. To show that these cuts exist, we can apply the Okamura-Seymour theorem to a modification of the planar dual graph G^* of G. Indeed, we must show that there exist edge-disjoint circuits D_π in G^*, for $\pi \in \mathcal{X}$, such that D_π traverses the two edges of G^* dual to the edges of G in π. The existence of these circuits follows from the Okamura-Seymour theorem applied to the graph G' obtained from G^* by deleting the vertex of G^* dual to the unbounded face of G, and all edges incident with it. Let R be the set of pairs of vertices of G' that are ends of pairs of edges dual to $\pi \in \mathcal{X}$. Then (74.19) implies that the cut condition holds, and that paths in G', and hence circuits in G^*, as required exist. ∎

This corollary is related to the Okamura-Seymour theorem by two different forms of duality: by planar duality and by polarity. As for planar duality, this is shown in the proof of this corollary. For polarity, this can be seen with Theorem 70.5, which gives that there exist $\lambda_U \in \mathbb{R}_+$ for $U \subseteq V$ such that

(74.21)
$$\begin{aligned}
&\sum_U \lambda_U \chi^{\delta_R(U)}(r) \geq \text{dist}_G(s,t) \text{ for each } r = st \in R \text{ and} \\
&\sum_U \lambda_U \chi^{\delta_E(U)}(e) \leq 1 \text{ for each } e \in E.
\end{aligned}$$

Now Corollary 74.2a asserts that the λ_U can be taken integer if G is bipartite.

74.1e. Linear algebra and distance realizability

As for the results on distances and cut packings discussed in Section 74.1d, the following further observations were made by Hurkens, Schrijver, and Tardos [1988]. Let $C = (V, E)$ be a circuit with n vertices and n edges, say:

(74.22) $\quad V = \{v_1, \ldots, v_n\},\ E = \{e_1 = v_0v_1, \ldots, e_n = v_{n-1}v_n\},$

where $v_0 := v_n$. Again, let $\binom{V}{2}$ and $\binom{E}{2}$ denote the sets of unordered pairs of distinct elements from V and E, respectively. Let M be the $\binom{V}{2} \times \binom{E}{2}$ matrix given by:

(74.23) $$M_{\{v_i,v_j\},\{e_g,e_h\}} := \begin{cases} 1 & \text{if } \{v_i,v_j\} \text{ and } \{e_g,e_h\} \text{ cross,} \\ 0 & \text{otherwise,} \end{cases}$$

where $\{v_i, v_j\}$ and $\{e_g, e_h\}$ are said to *cross* if v_i and v_j belong to different components of the graph $C - e_g - e_h$. Then the matrix M is nonsingular, with $\binom{E}{2} \times \binom{V}{2}$ inverse N given by:

(74.24) $$N_{\{e_g,e_h\},\{v_i,v_j\}} := \begin{cases} +\frac{1}{2} & \text{if } \{i,j\} = \{g,h\} \text{ or } \{i,j\} = \{g-1,h-1\}, \\ -\frac{1}{2} & \text{if } \{i,j\} = \{g,h-1\} \text{ or } \{i,j\} = \{g-1,h\}, \\ 0 & \text{otherwise.} \end{cases}$$

To see

(74.25) $$N = M^{-1},$$

choose $\{e_g, e_h\}, \{e_a, e_b\} \in \binom{E}{2}$. Then

(74.26) $$(NM)_{\{e_g,e_h\},\{e_a,e_b\}} = \tfrac{1}{2}M_{\{v_g,v_h\},\{e_a,e_b\}} + \tfrac{1}{2}M_{\{v_{g-1},v_{h-1}\},\{e_a,e_b\}} - \tfrac{1}{2}M_{\{v_g,v_{h-1}\},\{e_a,e_b\}} - \tfrac{1}{2}M_{\{v_{g-1},v_h\},\{e_a,e_b\}}.$$

If $\{g, h\} = \{a, b\}$, then it is easy to see that this last expression is equal to 1. If $\{g, h\} \neq \{a, b\}$, then without loss of generality $g \notin \{a, b\}$. Then

(74.27) $$M_{\{v_g,v_h\},\{e_a,e_b\}} = M_{\{v_{g-1},v_h\},\{e_a,e_b\}} \text{ and } M_{\{v_g,v_{h-1}\},\{e_a,e_b\}} = M_{\{v_{g-1},v_{h-1}\},\{e_a,e_b\}},$$

which implies that (74.26) equals 0. This proves (74.25). (It can be shown that $|\det M| = 2^{\binom{n-1}{2}}$.)

(74.25) implies that for any function $d : \binom{V}{2} \to \mathbb{R}$ there is a unique $b : \binom{E}{2} \to \mathbb{R}$ such that

(74.28) $$d(\{v_i, v_j\}) = \sum(b(\{e_g, e_h\}) \mid \{e_g, e_h\} \in \tbinom{E}{2} \text{ where } \{e_g, e_h\} \text{ crosses } \{v_i, v_j\}).$$

Indeed, (74.28) is equivalent to: $d = Mb$. Hence $b := Nd$ is the unique b satisfying (74.28).

This can be applied to $d = \mathrm{dist}_G$ for some bipartite planar graph $G = (V', E')$ with $C = (V, E)$ being the outer boundary of G. Consider the collection $\mathcal{X}$ of pairs of edges on the outer boundary of G defined in the proof of Corollary 74.2a. ($\mathcal{X}$ is a partition of E into pairs.) Then the uniqueness of b in (74.28) yields that $\mathcal{X}$ is the unique collection of pairs of edges on the boundary of G with the property that for any two vertices s, t on the outer boundary of G, the distance $\mathrm{dist}_G(s, t)$ is equal to the number of pairs in $\mathcal{X}$ crossing $\{s, t\}$.

Another consequence of (74.25) is as follows. Consider again the circuit $C = (V, E)$ given by (74.22). Call a function $m : \binom{V}{2} \to \mathbb{R}_+$ *realizable as the distance function of a planar graph with boundary* C, or briefly *realizable*, if there exists a planar graph $G = (V', E')$, with $V' \supseteq V$, $E' \supseteq E$ such that $v_1, \ldots, v_n$ occur in this order cyclically around the outer boundary, and a length function $l : E' \to \mathbb{R}_+$ such that for all $s, t \in V$, $m(\{s, t\}) = \mathrm{dist}_G(s, t)$. Then

(74.29) a function $m : \binom{V}{2} \to \mathbb{R}_+$ is realizable if and only if for all $i, j = 1, \ldots, n$ we have

$$m(\{v_i, v_j\}) + m(\{v_{i-1}, v_{j-1}\}) \geq m(\{v_i, v_{j-1}\}) + m(\{v_{i-1}, v_j\}),$$

setting $m(\{v_i, v_i\}) := 0$ for all i.

Necessity of the condition is trivial, since any $v_i - v_j$ path in G crosses any $v_{i-1} - v_{j-1}$ path in G. To see sufficiency, we construct a graph G as follows. Let $w_1, \ldots, w_n$ be points on the unit circle, in this cyclic order. Set $w_0 := w_n$. Add all line-segments $\overline{w_g w_h}$ $(g, h = 1, \ldots, n;\ g \neq h)$. The figure now forms a planar graph H, with vertices the points that are on two or more of these line segments. Let H^* be the dual graph. Put a new point v_i on the edge of H^* dual to the edge $w_i w_{i+1}$ of H $(i = 0, \ldots, n-1)$. Next delete the vertex of H^* dual to the outer face of H and delete all edges incident with it. This makes the graph $G = (V', E')$.

Let $d := Nm$. By the condition given in (74.29), $d \geq \mathbf{0}$. For each edge e of G, define $l(e) := d(\{e_g, e_h\})$ if e is dual to an edge of H which is on the line segment $\overline{w_g w_h}$. Using the fact that $Md = m$ it is easy to see that this gives a realization as required.

Also the 'pairing lemma' (Lemma 74.2α) can be interpreted in terms of the matrix M: it characterizes when there exists an $x : \binom{V}{2} \to \mathbb{Z}_+$ with $x(\delta(v))$ odd for each $v \in V$ and with $x^{\mathsf{T}} M \leq s$ for some given $s : \binom{E}{2} \to \mathbb{Z}_+$.

74.1f. Directed planar graphs with all terminals on the outer boundary

It was observed by Diaz and de Ghellinck [1972] that if the supply graph is directed and planar, and all terminals are on the outer boundary in the order $s_1, \ldots, s_k, t_k, \ldots, t_1$, then the integer multicommodity flow problem is solvable in polynomial time, and the cut condition suffices. This follows by a reduction to a minimum-cost circulation problem: add arcs from t_i to s_i for $i = 1, \ldots, k$.

Related, and more difficult, is the following result of Nagamochi and Ibaraki [1990]. Let the supply digraph $D = (V, A)$ be planar and acyclic, and let the demand digraph $H = (T, R)$ have all terminals on the outer boundary of D. Then for each $c : A \to \mathbb{Z}_+$ and $d : R \to \mathbb{Z}_+$ satisfying the directed analogue of the Euler condition (that is, $(V, A \cup R^{-1})$ is Eulerian), if there is a fractional multiflow, there is an integer multiflow.

Nagamochi and Ibaraki also gave a polynomial-time algorithm to find the integer multiflow. Moreover, they extended the results to the case where the set of vertices that violate the Euler condition, all lie on the outer boundary of D, in such a way that the vertices v with $c(\delta_A^{\text{out}}(v)) - d(\delta_R^{\text{in}}(v)) > c(\delta_A^{\text{in}}(v)) - d(\delta_R^{\text{out}}(v))$ can be separated (on the outer boundary of D) by two vertices from those vertices v where the opposite strict inequality holds.

74.2. $G + H$ planar

Seymour [1981d] gave another tractable case of the planar edge-disjoint paths problem if the Euler condition holds: the case where the graph together with all its nets (taken as edges) is a planar graph; that is, if $G = (V, E)$ and $H = (V, R)$ are graphs with

(74.30) $\qquad G + H := (V, E \cup R)$

planar (where $E \cup R$ is the disjoint union, respecting multiplicities in E and R). This case can be handled with the help of matching theory (more specifically, minimum-size T-joins and disjoint T-cuts).

Theorem 74.3. *Let $G = (V, E)$ and $H = (V, R)$ be supply and demand graphs with $G+H$ planar and Eulerian. Then the edge-disjoint paths problem has a solution if and only if the cut condition holds.*

Proof. Necessity being trivial, we show sufficiency. Let the cut condition be satisfied. Consider the dual graph $(G+H)^*$ of $G+H$. Let R^* be the family of edges of $(G+H)^*$ dual to those in R. Let T be the set of vertices of $(G+H)^*$ which are incident with an odd number of edges in R^*. So R^* is a T-join in $(G+H)^*$.

In fact, R^* is a minimum-size T-join in $(G+H)^*$. For suppose not. Then there exist $E_0 \subseteq E$ and $R_0 \subseteq R$ such that $E_0^* \cup R_0^*$ is a T-join and $|E_0|+|R_0| < |R|$. As $E_0^* \cup R_0^*$ is a T-join, each vertex of $(G+H)^*$ is incident with an even number of edges in

(74.31) $$(E_0^* \cup R_0^*) \triangle R^* = E_0^* \cup (R \setminus R_0)^*.$$

Hence $E_0 \cup (R \setminus R_0)$ forms a cut in $G+H$. Since $|E_0| < |R \setminus R_0|$, this contradicts the cut condition.

So R^* is a minimum-size T-join in $(G+H)^*$. As $(G+H)^*$ is bipartite, by Theorem 29.2, there exist disjoint cuts $D_1, \ldots, D_t$ in $(G+H)^*$ such that (i) each cut D_j intersects R^* in exactly one element and (ii) each edge of R^* is in exactly one of the D_j. Condition (i) implies that the dual C_j of each D_j is a circuit in $G+H$ containing exactly one edge in R. Hence the C_j give edge-disjoint paths in G as required. ∎

Notes. The reduction to matching theory given in this proof implies that feasibility can be tested, and edge-disjoint paths can be found, in strongly polynomial time (also for the capacitated case). Matsumoto, Nishizeki, and Saito [1986] showed that feasibility can be tested in $O(n^{3/2} \log n)$ time, and edge-disjoint paths can be found in $O(n^{5/2} \log n)$ time (also for the capacitated case). The latter bound was improved by Barahona [1990] to $O(n^{3/2} \log n)$.

With the help of Wagner's theorem (Theorem 3.3), Theorem 74.3 can be extended to the case where $G+H$ has no K_5 minor. We derive this result from Guenin's theorem in Section 75.6.

The fractional version of Theorem 74.3 was published in Seymour [1979b].

74.2a. Distances and cut packing

By Theorem 70.5, Theorem 74.3 implies that if $G = (V, E)$ and $H = (V, R)$ are graphs with $G+H$ planar, then there is a fractional packing of cuts in G such that for any $r = st \in R$, s and t are separated by $\mathrm{dist}_G(s, t)$ of these cuts. A.V. Karzanov (personal communication 1986) observed that in fact from a theorem of Seymour [1979b] the existence of a half-integer packing can be derived. More generally:

(74.32) Let $G = (V, E)$ and $H = (V, R)$ be graphs with G bipartite and $G+H$ planar. Then there exist disjoint cuts in G such that for each $r = st \in R$, s and t are separated by $\text{dist}_G(s, t)$ of these cuts.

This can be derived from Theorem 29.3 above (of Seymour [1979b]), saying:

(74.33) Let $G = (V, E)$ be a planar graph and let $p : E \to \mathbb{Z}_+$. Then p is a nonnegative integer sum of incidence vectors of circuits of G if and only if $p(\delta(v))$ is even for each $v \in V$ and $p(e) \leq p(D \setminus \{e\})$ for each cut D of G and each $e \in D$.

Applying planar duality, (74.33) becomes:

(74.34) Let $G = (V, E)$ be a planar graph and let $p : E \to \mathbb{Z}_+$. Then p is a nonnegative integer sum of incidence vectors of cuts of G if and only if $p(C)$ is even for each circuit C of G and $p(e) \leq p(C \setminus \{e\})$ for each circuit C of G and each $e \in C$.

(Here we consider circuits as edge sets.) We apply this to the graph $G+H$, where G is bipartite and $G + H$ is planar. Define $p(e) := 1$ for $e \in E$ and $p(r) := \text{dist}_G(s, t)$ for $r = st \in R$. Then $p(C)$ is even for each circuit C of $G+H$ and $p(e) \leq p(C \setminus \{e\})$ for each circuit of $G + H$ and each $e \in E$. (The latter property is trivial if $e \in E$. If $e = st \in R$, we can replace any occurrence of an r in $C \setminus \{e\}$ with $r = uv \in R$, by a shortest $u - v$ path in G. This does not increase $p(C \setminus \{e\})$. Repeating this, we can assume that $C \cap R = \{e\}$, and so $C \setminus \{e\}$ is an $s - t$ path in G, implying $p(e) = \text{dist}_G(s, t) \leq |C \setminus \{e\}| = p(C \setminus \{e\})$.)

Therefore, by (74.34), p is a nonnegative integer sum of incidence vectors of cuts of $G + H$. By definition of p, this gives edge-disjoint cuts in G as required in (74.32).

74.2b. Deleting the Euler condition if $G + H$ is planar

Middendorf and Pfeiffer [1993] showed that if $G + H$ is planar (but not necessarily Eulerian), then the edge-disjoint paths problem is NP-complete. (With construction (70.9), it implies the same result for the directed case.) In fact they showed that if $G + H$ is planar and cubic, then the edge-disjoint paths problem is NP-complete. Hence, also the vertex-disjoint paths problem is NP-complete if $G+H$ is planar and cubic. (Assuming P≠NP, this disproves a conjecture of Schrijver [1990b].) Middendorf and Pfeiffer [1990b,1993] showed that, on the other hand, if $G + H$ is planar and the edges of H belong to a bounded number of faces of G, then the edge-disjoint paths problem is polynomial-time solvable.

Middendorf and Pfeiffer [1990b,1993] also presented a counterexample to a conjecture of A. Frank (cf. Sebő [1988a]) that if $G+H$ is planar, then the edge-disjoint paths problem has a solution if and only if G contains a fractional packing of paths as required and of a T'-join, where T' is the set of vertices having odd degree in $G + H$.

Korach and Penn [1992] showed that if $G + H$ is planar and the cut condition holds, then there is an 'almost complete' packing of paths as required: there is at most one edge in R on each bounded face of G such that leaving out these edges from R, the problem has a solution. A generalization of this was given by Frank and Szigeti [1995]. (Related work can be found in Granot and Penn [1992,1993,1996].)

Seymour [1981d] also showed the following:

(74.35) Let $G = (V, E)$ and $H = (V, R)$ be supply and demand graphs such that $G + H$ is planar and such that R consists of two classes of parallel edges. Then there exist edge-disjoint paths if and only if the cut condition holds and we cannot contract edges of G to obtain a graph G' with at most four vertices in which the corresponding edge-disjoint paths do not exist.

Frank [1990d] observed that the latter condition can be formulated as:

(74.36) $d_{E \cup R}(X \cap Y)$ is even for any two tight sets $X, Y \subseteq V$,

which Frank called the *intersection criterion*. (A subset X of Y is called *tight* if $d_E(X) = d_R(X)$.)

The intersection criterion is a necessary condition for the existence of edge-disjoint paths: if paths as required exists, then for each tight X all edges in $\delta_E(X)$ are used by these paths; hence if X and Y are tight, all edges in $\delta_E(X \cap Y)$ are used; hence $d_E(X \cap Y) \equiv d_R(X \cap Y) \pmod 2$, that is, $d_{E \cup R}(X \cap Y)$ is even.

In other words, Frank observed that (74.35) is equivalent to:

(74.37) Let $G = (V, E)$ and $H = (V, R)$ be graphs such that $G+H$ is planar and such that R consists of two classes of parallel edges. Then there exist edge-disjoint paths if and only if the cut condition and the intersection criterion hold.

This was extended by Frank [1990d] to:

(74.38) Let $G = (V, E)$ and $H = (V, R)$ be graphs such that $G + H$ is planar and such that the edges of H are on at most two of the faces of G. Then there exist edge-disjoint paths if and only if the cut condition and the intersection criterion hold.

Lomonosov [1983] proved a maximization version of (74.35), which Frank [1990e] showed to follow from (74.35). Korach and Penn [1993] gave an $O(n\sqrt{\log n})$-time algorithm for the edge-disjoint paths problem if $G + H$ is planar and H consists of two parallel classes of nets.

Sebő [1993c] showed that for each fixed k, if $G + H$ is planar and $|VH| \leq k$, then the integer multiflow problem is polynomial-time solvable. (The demands can be arbitrarily large, so there is no reduction to the edge-disjoint paths problem for a fixed number of paths. It was shown for $k = 3$ by Korach [1982].) Sebő showed this by proving a more general result on the complexity of packing T-cuts for fixed $|T|$.

It is an open question if one may relax this condition to H being spanned by a fixed number of faces of G. (For demand $d = \mathbf{1}$ this was shown by Middendorf and Pfeiffer, as mentioned above.)

Pfeiffer [1990] raised the question if the edge-disjoint paths problem has a half-integer solution if $G + H$ is embeddable in the torus and there is a quarter-integer solution. He gave the example of Figure 70.5 with 8 vertices to show that this generally does not hold if $G + H$ is embeddable in the double torus.

Pfeiffer [1994] showed that the half-integer multiflow problem is NP-complete if $G + H$ is apex. (An *apex graph* is a graph having a vertex whose deletion makes the graph planar.) Pfeiffer also showed that the half-integer multiflow problem is NP-complete if the supply and demand digraphs form a directed planar graph.

74.3. Okamura's theorem

Okamura [1983] gave the following extension of the Okamura-Seymour theorem. We follow the proof found in 1984 by G. Tardos (cf. Frank [1990e]). The first half of the proof below is similar to the proof of the Okamura-Seymour theorem (Theorem 74.1).

Theorem 74.4 (Okamura's theorem)**.** *Let $G = (V, E)$ be a planar graph and let F_1 and F_2 be two of its faces. Let R be a set of pairs of vertices of G such that each $r = st \in R$ satisfies $s, t \in \mathrm{bd}(F_1)$ or $s, t \in \mathrm{bd}(F_2)$. Let the Euler condition hold. Then the edge-disjoint paths problem has a solution if and only if the cut condition holds.*

Proof. Necessity of the cut condition being trivial, we show sufficiency. The cut condition implies that $|R| \leq |E|$ (assuming that each $r \in R$ consists of two distinct vertices), since

(74.39) $$2|R| = \sum_{v \in V} \deg_R(v) \leq \sum_{v \in V} \deg_E(v) = 2|E|.$$

So we can consider a counterexample with $2|E| - |R|$ minimal. Then

(74.40) G is 2-connected.

Indeed, if G is disconnected, we can deal with the components separately. Suppose next that G is connected and has a cut vertex v. We may assume that for no $r = st \in R$, the vertices s and t belong to different components of $G - v$, since otherwise we can replace r by sv and vt, without violating the Euler or cut condition. For any component K of $G - v$ consider the graph induced by $K \cup \{v\}$. Again, the Euler and cut conditions hold (with respect to those nets contained in $K \cup \{v\}$). So by the minimality of $2|E| - |R|$ we know that paths as required exist in $K \cup \{v\}$. As this is the case for each component of $G - v$, we have paths as required in G. This proves (74.40).

If some $r \in R$ is parallel to an edge of G we can delete this edge from G, and r from R, to obtain a smaller counterexample. Hence such r, e do not exist.

Call a subset X of V *tight* if $d_E(X) = d_R(X)$. Let C_1 and C_2 be the circuits forming the boundaries of F_1 and F_2 respectively. Then

(74.41) Each tight set X with $|\delta_E(X) \cap EC_1| = 2$ intersects VC_2.

For suppose that $X \cap VC_2 = \emptyset$. Choose such a set X with $|X|$ minimal. Let e be one of the two edges in $\delta_E(X) \cap EC_1$. Say $e = uw$ with $u \notin X$ and $w \in X$.

Since $d_R(X) = d_E(X) \geq 2$, we know $\delta_R(X) \neq \emptyset$. For each $r \in \delta_R(X)$, let s_r be the vertex in $r \cap X$, and t_r the vertex in $r \setminus X$. Choose $r \in \delta_R(X)$ such that t_r is as close as possible to u in the graph $C_1 - X$.

Since $\{u, w\} \neq \{s_r, t_r\}$, we can choose $v \in \{u, w\}$ with $v \notin \{s_r, t_r\}$. Let $R' := (R \setminus \{r\}) \cup \{s_r v, v t_r\}$. Trivially the Euler condition is maintained. We prove that also the cut condition is maintained, which is a contradiction as $2|E| - |R'| < 2|E| - |R|$ and as a solution for R' yields a solution for R.

To see that the cut condition is maintained, suppose to the contrary that there is a $Y \subseteq V$ satisfying

(74.42) $$d_E(Y) < d_{R'}(Y).$$

By Theorem 70.4, we can take Y such that $G[Y]$ and $G - Y$ are connected. By symmetry we can assume that $t_r \notin Y$. By the Euler condition, (74.42) implies $d_E(Y) \leq d_{R'}(Y) - 2$. So

(74.43) $$d_{R'}(Y) \geq d_E(Y) + 2 \geq d_R(Y) + 2 \geq d_{R'}(Y).$$

Hence we have equality throughout. So $\delta_{R'}(Y)$ contains both $s_r v$ and vt_r, that is, $s_r, t_r \notin Y$ and $v \in Y$. Moreover, $d_E(Y) = d_R(Y)$.

As Y and $V \setminus Y$ intersect VC_1 and as $G[Y]$ and $G - Y$ are connected, we know $|\delta_E(Y) \cap EC_1| = 2$. By the choice of r, there is no pair r' in R connecting $X \setminus Y$ and $Y \setminus X$ (otherwise, $t_{r'} \in Y \setminus X$ and hence $t_{r'}$ would be closer than t_r to u in $C_1 - X$). So (using Theorem 3.1)

(74.44) $$d_R(X \cap Y) + d_R(X \cup Y) = d_R(X) + d_R(Y).$$

Moreover,

(74.45) $$d_E(X \cap Y) + d_E(X \cup Y) \leq d_E(X) + d_E(Y).$$

As the cut condition holds for $X \cap Y$ and $X \cup Y$, we have equality in (74.45), and therefore $X \cap Y$ is tight. Since $s_r \in X \setminus Y$, we know $|X \cap Y| < |X|$. So by the minimality of X we have $X \cap Y = \emptyset$. So $w \notin Y$, hence $u = v \in Y$. Then edge $e = uw$ connects $X \setminus Y$ and $Y \setminus X$, contradicting equality in (74.45). This proves (74.41).

Now choose $r = st \in R$. By symmetry of F_1 and F_2, we may assume that $s, t \in VC_1$. Let P and Q be the two $s - t$ paths along C_1. Deleting the edges of P from G and r from R, must violate the cut condition (as the Euler condition is maintained, and as for the new data there is no solution, since with P it gives a solution for the original data). So $|\delta_{E \setminus EP}(K)| < |\delta_{R \setminus \{r\}}(K)|$ for some $K \subseteq V$, with $G[K]$ and $G - K$ connected (by Theorem 70.4 taking $c := \chi^{E \setminus EP}$ and $d := \chi^{R \setminus \{r\}}$). Since $G[K]$ and $G - K$ are connected, and using (74.41), $|\delta_E(K) \cap EC_i| = 2$ for $i = 1, 2$. Moreover, K is tight, $\delta_E(K)$ contains two edges of P, and K does not split r. So we may assume that $s, t \notin K$. Similarly, there is a tight subset L of V such that $|\delta_E(L) \cap EC_i| = 2$ for $i = 1, 2$, such that $\delta_E(L)$ contains two edges of Q, and such that $s, t \notin L$.

As each of K, $V \setminus K$, L, and $V \setminus L$ intersects VC_2, each $s - t$ path in G intersects $K \cup L$ (since K contains a path from VP to VC_2 and L contains a path from VQ to VC_2). Hence we can partition $V \setminus (K \cup L)$ into sets M and N, with $s \in M$, $t \in N$, and $E[M, N] = \emptyset$. (Here and below, $E[X, Y]$ and $R[X, Y]$ denote the set of pairs xy in E and R respectively with $x \in X$ and $y \in Y$.)

We can assume by symmetry that $R[M, K \cap L] = \emptyset$. For suppose that $R[M, K \cap L] \neq \emptyset$ and $R[N, K \cap L] \neq \emptyset$. Since $K \cap L$ does not intersect VC_1, it would follow that both M and N intersect VC_2. However, this implies $K \cap L = \emptyset$, and hence $R[M, K \cap L] = \emptyset$.

Then we have the contradiction

$$
\begin{aligned}
(74.46) \qquad & d_R(K) + d_R(L) = d_E(K) + d_E(L) \\
& = (d_E(K \cup M) + |E[M, K]| - |E[M, L \setminus K]| - |E[M, N]|) \\
& +(d_E(L \cup M) + |E[M, L]| - |E[M, K \setminus L]| - |E[M, N]|) \\
& \geq d_E(K \cup M) + d_E(L \cup M) \geq d_R(K \cup M) + d_R(L \cup M) \\
& = (d_R(K) - |R[M, K]| + |R[M, L \setminus K]| + |R[M, N]|) \\
& + (d_R(L) - |R[M, L]| + |R[M, K \setminus L]| + |R[M, N]|) \\
& > d_R(K) + d_R(L).
\end{aligned}
$$

This follows from a straightforward count of edges, and from the facts that $E[M, L \setminus K] \subseteq E[M, L]$, $E[M, K \setminus L] \subseteq E[M, K]$, $E[M, N] = \emptyset$, $R[M, K] = R[M, K \setminus L]$ (as $R[M, K \cap L] = \emptyset$), $R[M, L] = R[M, L \setminus K]$ (similarly), and $R[M, N] \neq \emptyset$ (as $st \in R[M, N]$). ∎

Notes. Suzuki, Nishizeki, and Saito [1985b,1989] gave an $O(kn + nt_1 \cdot \mathrm{SP}_+(n))$-time algorithm for finding the edge-disjoint paths in this case (similarly for the capacitated case), where $k := |R|$, t_1 is the number of vertices on the boundary of F_1, and $\mathrm{SP}_+(n)$ is any upper bound on the time needed to find a shortest path in a planar n-vertex graph with nonnegative edge lengths.

The example of Figure 70.2 shows that Okamura's theorem cannot be extended to more than two selected faces, and also is not maintained if we allow 'mixed pairs'; that is, nets that connect the two selected faces. Under certain conditions one can allow such pairs — see (74.55) and (76.50) below.

74.3a. Distances and cut packing

By Theorem 70.5, Okamura's theorem implies that for any planar graph $G = (V, E)$ and any choice of two faces F_1 and F_2, there is a fractional packing of cuts such that any two vertices s, t that are either both incident with F_1 or both incident with F_2, are separated by $\mathrm{dist}_G(s, t)$ of these cuts. In fact, there is a half-integer packing, as follows from the following result of Schrijver [1989a], generalizing Corollary 74.2a:

(74.47) Let $G = (V, E)$ be a bipartite planar graph and let F_1 and F_2 be two of its faces. Then there exist edge-disjoint cuts such that any two vertices s, t with $s, t \in \mathrm{bd}(F_1)$ or $s, t \in \mathrm{bd}(F_2)$ are separated by $\mathrm{dist}_G(s, t)$ of these cuts.

Karzanov [1990a] gave an alternative proof of this, yielding a strongly polynomial-time algorithm for finding the cuts, also for the weighted case (that is, for length function $l : E \to \mathbb{Z}_+$ with $l(C)$ even for each circuit C of G, finding an integer packing of cuts).

74.3b. The Klein bottle

In Schrijver [1989b] the following relation between Okamura's theorem and graphs embedded in the Klein bottle is given. It generalizes the relation between the Okamura-Seymour theorem and graphs embedded in the projective plane, as described in Section 74.1b.

We can represent the Klein bottle as obtained from the 2-sphere by adding two cross-caps. A closed curve C on the Klein bottle is called *orientation preserving* if after one turn of C the meaning of 'left' and 'right' is unchanged. Otherwise, it is called *orientation-reversing.*

Thus a closed curve is orientation-preserving if and only if it traverses the cross-caps an even number of times. It is orientation-reversing if and only if it traverses the cross-caps an odd number of times. So, if $G = (V, E)$ is a graph embedded in the Klein bottle, there is a subset R of E such that a circuit in G is orientation-reversing if and only if it traverses the edges in R an odd number of times.

Let $G = (V, E)$ be a graph embedded in the Klein bottle. Define

(74.48) $\mathcal{C} :=$ collection of orientation-reversing circuits in G;
$\mathcal{D} :=$ collection of edge sets intersecting each orientation-reversing circuit of G.

(Here we take circuits as edge sets.)

In Schrijver [1989b], the following is derived from (74.47):

(74.49) Let $G = (V, E)$ be a bipartite graph embedded in the Klein bottle. Then the minimum length of an orientation-reversing circuit in G is equal to the maximum number of disjoint sets in $\mathcal{D}$.

(In fact it suffices to require, instead of bipartiteness, that each face of G is surrounded by an even number of edges.)

(74.49) implies that the up hull of the incidence vectors of sets in $\mathcal{C}$ is determined by:

(74.50)
$$\begin{aligned} x_e &\geq 0 && \text{for } e \in E,\\ x(D) &\geq 1 && \text{for } D \in \mathcal{D}. \end{aligned}$$

This follows from the fact that for any $l : E \to \mathbb{Z}_+ \setminus \{0\}$, the minimum value of

(74.51)
$$\sum_{e \in E} l(e) x_e$$

over (74.50) is achieved by an integer vector x. To see this, we may assume that $l(e)$ is even for each $e \in E$. Now replace each edge e of G by a path of length $l(e)$. We obtain a bipartite graph G'. Let C' be a minimum-length orientation-reversing circuit in G'. By (74.49), there exist disjoint edge sets $D'_1, \ldots, D'_t$ in G' each intersecting all orientation-reversing circuits in G', such that t is equal to the number of edges in C'. Let $C, D_1, \ldots, D_t$ be the edge sets in G corresponding to $C', D'_1, \ldots, D'_t$. So $D_1, \ldots, D_t \in \mathcal{D}$. Then

(74.52)
$$\sum_{e \in E} l(e) \chi^C(e) = t = \sum_{i=1}^{t} 1 \text{ and } \sum_{i=1}^{t} \chi^{D_i} \leq l.$$

So $D_1, \ldots, D_t$ give a dual solution to minimizing (74.51) over (74.50) of value t, and hence $x := \chi^C$ is an optimum solution.

So the vertices of the polyhedron determined by (74.50) are incidence vectors of orientation-reversing circuits. By the theory of blocking polyhedra, this implies that the up hull of the incidence vectors of the sets in $\mathcal{D}$ is determined by:

(74.53) $x_e \geq 0$ for $e \in E$,
$x(C) \geq 1$ for $C \in \mathcal{C}$.

From this the following stronger property has been derived in Schrijver [1989b], generalizing Lins' theorem (Corollary 74.1b):

(74.54) Let $G = (V, E)$ be an Eulerian graph embedded in the Klein bottle. Then the maximum number of edge-disjoint orientation-reversing circuits is equal to the minimum number of edges intersecting all orientation-reversing circuits.

This result cannot be extended to compact surfaces with more than two cross-caps, as we can embed K_5 in such a surface in such a way that the orientation-reversing circuits are exactly the odd-size circuits of K_5. Then the maximum number of edge-disjoint orientation-reversing circuits is equal to 2, while at least 4 edges are needed to intersect all orientation-reversing circuits.

From (74.54) one can derive Okamura's theorem (Theorem 74.4) and also another disjoint paths theorem for planar graphs (Schrijver [1989b]):

(74.55) Let $G = (V, E)$ be a planar graph, and let $H = (V, R)$ be a graph, with $R = \{s_1t_1, \ldots, s_kt_k\}$, such that G has two bounded faces F_1 and F_2 with the property that $s_1, \ldots, s_k$ occur in clockwise order along $\mathrm{bd}(F_1)$ and $t_1, \ldots, t_k$ occur in clockwise order along $\mathrm{bd}(F_2)$. Let $G+H$ be Eulerian. Then there exist edge-disjoint paths P_r, where P_r is an r-path for $r \in R$, if and only if the cut condition holds.

(Here $G+H$ is the graph $(V, E \cup R)$, taking multiplicities of edges into account. An *r-path* is a path connecting the vertices in r.) To see this, we can extend the plane to a Klein bottle, by adding a cylinder between the boundaries of F_1 and F_2. (That is, we first make the plane to a sphere, next take out the interiors of the faces F_1 and F_2, and then add the cylinder, in such a way that we obtain a nonorientable surface.) By the condition on the orders of the s_i and t_i along the boundaries of F_1 and F_2, we can extend the embedding of G to an embedding of $G + H$ in the Klein bottle, by embedding the edges s_it_i over the cylinder. Then a circuit in $G + H$ is orientation-reversing if and only if it contains an odd number of edges in R. So it suffices to show that $G + H$ contains k orientation-reversing circuits.

By (74.54) one must show that each set D of edges of $G + H$ intersecting all orientation-reversing circuits has size at least k. We may assume that D is a minimal set of edges in $G+H$ intersecting all orientation-reversing circuits in $G+H$. This implies that for each circuit C of $G + H$, $|D \cap C|$ is odd if and only if C is orientation-reversing. (Indeed, for each $e \in D \cap C$ there is an orientation-reversing circuit C_e disjoint from $D \setminus \{e\}$ (by the minimality of D). As C_e intersects D we know $e \in C_e$. Hence the symmetric difference X of C and the C_e for $e \in D \cap C$ is disjoint from D. So X contains no orientation-reversing circuit. Therefore, X is the symmetric difference of an even number of orientation-reversing circuits. So C is orientation-reversing if and only if $|D \cap C|$ is odd.)

In particular, $|D \cap C|$ is even for each circuit C in G. So $D \cap E$ is a cut $\delta_E(X)$ in G. Then for each $i = 1, \ldots, k$:

(74.56) if X does not separate s_i and t_i, then $s_i t_i \in D$.

Indeed, if X does not separate s_i and t_i, then there is an $s_i - t_i$ path P in G containing an even number of edges in D. As $P \cup \{s_i t_i\}$ is an orientation-reversing circuit, it intersects D an odd number of times, and hence $s_i t_i \in D$.

(74.56) implies $|D \cap R| \geq |R \setminus \delta_R(X)|$. Hence

(74.57) $$|D| = |D \cap E| + |D \cap R| \geq |\delta_E(X)| + |R \setminus \delta_R(X)| \geq |R| = k,$$

since $|\delta_E(X)| \geq |\delta_R(X)|$ by the cut condition. So $|D| \geq k$ as required.

One can similarly derive Okamura's theorem. First one may assume, without loss of generality, that $R = \{s_1 t_1, \ldots, s_k t_k\}$ such that $s_1, \ldots, s_l, t_1, \ldots, t_l$ occur cyclically around $\mathrm{bd}(F_1)$ and $s_{l+1}, \ldots, s_k, t_{l+1}, \ldots, t_k$ occur cyclically around $\mathrm{bd}(F_2)$. This can be achieved with the construction described in Section 74.1b (cf. Figure 74.1).

Now we can obtain a Klein bottle by adding a cross-cap in the interior of F_1 and a cross-cap in the interior of F_2 (assuming that G is embedded in the 2-sphere). We can extend the embedding of G to an embedding of $G + H$, by adding edges $s_i t_i$ for $i = 1, \ldots, l$ over the first cross-cap, and adding edges $s_i t_i$ for $i = l+1, \ldots, k$ over the second cross-cap. Applying (74.54), we obtain Okamura's theorem.

74.3c. Commodities spanned by three or more faces

Karzanov [1994c,1994b] showed that Okamura's theorem and the dual cut packing result (74.47) can be extended in a certain way to planar graphs where the nets are on three or more faces. These results can be compared to those in Section 72.2a.

We repeat the definition of Γ-metric. Let Γ be a graph, and let V be a finite set. A metric μ on V is called a *Γ-metric* if there is a function $\phi : V \to V\Gamma$ with

(74.58) $$\mu(u, v) = \mathrm{dist}_\Gamma(\phi(u), \phi(v))$$

for all $u, v \in V$. (Here $\mathrm{dist}_\Gamma(x, y)$ denotes the distance of x and y in Γ.)

The *Γ-metric condition*, a necessary condition for the existence of a feasible multiflow in a supply graph $G = (V, E)$ with demand graph $H = (V, R)$, capacities $c : E \to \mathbb{R}_+$ and demands $d : R \to \mathbb{R}_+$, reads:

(74.59) $$\sum_{r=st \in R} d(r)\mu(s, t) \leq \sum_{e=uv \in E} c(e)\mu(u, v) \text{ for each } \Gamma\text{-metric } \mu \text{ on } V.$$

The $K_{2,3}$-metric condition generalizes the cut condition.

For the edge-disjoint paths problem, Karzanov [1994b] showed that for extending Okamura's theorem to three faces, adding the $K_{2,3}$-metric condition suffices:

(74.60) Let $G = (V, E)$ be a planar graph, let F_1, F_2, and F_3 be three of its faces, and let $H = (V, R)$ be a graph such that for each $r = st \in R$ there is an $i = 1, 2, 3$ with s and t on the boundary of F_i. Let $G + H$ be Eulerian. Then there exist edge-disjoint paths P_r for $r \in R$, where P_r connects the vertices in r, if and only if the $K_{2,3}$-metric condition holds.

In particular, if a fractional solution exists, then an integer solution exists.

Karzanov [1994b] derived (74.60) from a dual result on packing cuts and $K_{2,3}$-metrics, proved in Karzanov [1994c]:

(74.61) Let $G = (V, E)$ be a bipartite planar graph and let $\mathcal{F}$ be a set of three of its faces. Then there exist $K_{2,3}$-metrics $\mu_1, \ldots, \mu_k$ such that $\text{dist}_G(u, v) \geq \mu_1(u, v) + \cdots + \mu_k(u, v)$ for all $u, v \in V$, with equality if there is an $F \in \mathcal{F}$ with both u and v incident with F.

Sebő [1993a] showed that a related result on surfaces with three cross-caps also holds (in the same way as the results on the Klein bottle above relate to Okamura's theorem (Theorem 74.4)). Let S be the compact surface with three cross-caps. Let $G = (V, E)$ be a graph embedded in S, and consider the system:

(74.62)
$$\begin{array}{ll} x_e \geq 0 & \text{for each } e \in E, \\ x(C) \geq 1 & \text{for each orientation-reversing circuit } C. \end{array}$$

Sebő showed that the polyhedron determined by (74.62) has half-integer vertices only. Moreover, if Z denotes the set of minimal $\{0, \frac{1}{2}, 1\}$ solutions of (74.62), then the system

(74.63)
$$\begin{array}{ll} x_e \geq 0 & \text{for each } e \in E, \\ z^{\mathsf{T}} x \geq 1 & \text{for each 0,1 vector } z \in Z, \\ 2z^{\mathsf{T}} x \geq 2 & \text{for each } z \in Z \end{array}$$

(which determines the blocking polyhedron of (74.62)) is totally dual half-integral. More strongly, for each $c : E \to \mathbb{Z}_+$ with $c(C)$ even for each circuit C of G, the dual of minimizing $c^{\mathsf{T}} x$ over (74.63) has an integer optimum solution.

From this, Sebő derived a result related to (74.61), in the same was as (74.54) is related to Okamura's theorem (Theorem 74.4):

(74.64) Let $G = (V, E)$ be a bipartite planar graph and let F_1, F_2, F_3 be three of its faces, with F_1 and F_2 bounded. Let $s_1, \ldots, s_k$ occur clockwise along $\text{bd}(F_1)$ and let $t_1, \ldots, t_k$ occur clockwise along $\text{bd}(F_2)$. Then there exist $K_{2,3}$-metrics $\mu_1, \ldots, \mu_k$ such that $\text{dist}_G(u, v) \geq \mu_1(u, v) + \cdots + \mu_k(u, v)$ for all $u, v \in V$, with equality if there is an i with $u = s_i$, $v = t_i$, or if both u and v are incident with F_3.

For the extension of (74.61) to four or more faces, there is not a finite collection $\mathcal{G}$ of graphs such that in (74.60) and (74.61) one can consider Γ-metrics for Γ in $\mathcal{G}$. However, for four faces, Karzanov [1994c] proved:

(74.65) Let $G = (V, E)$ be a bipartite planar graph and let $\mathcal{F}$ be a set of four of its faces. Then there exists a collection of metrics $\mu_1, \ldots, \mu_k$ such that each μ_i is a Γ-metric for some bipartite planar graph Γ with four faces, and such that $\text{dist}_G(u, v) \geq \mu_1(u, v) + \cdots + \mu_k(u, v)$ for all $u, v \in V$, with equality if there is an $F \in \mathcal{F}$ with both u and v incident with F.

This implies, with the usual polarity argument, that if $G = (V, E)$ is a planar graph, $\mathcal{F}$ a set of four of its faces, $H = (V, R)$ a graph such that for each $r = st \in F$ there is an $F \in \mathcal{F}$ with s and t incident with F, $c : E \to \mathbb{R}_+$, and $d : R \to \mathbb{R}_+$, then there is a feasible multiflow if and only if the Γ-metric condition (74.59) holds for each planar bipartite graph Γ with four faces.

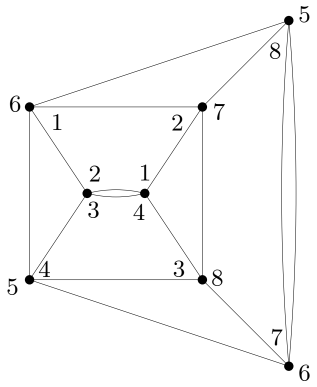


Figure 74.2

An example of a planar graph where each commodity is spanned by one of the four 4-sided faces and where there exists a half-integer, but no integer multiflow, while the Euler condition holds. The nets are indicated by pairs of indices at the vertices. All capacities and demands are 1. The half-integer multiflow is obtained by putting, for each index i, a flow of value $\frac{1}{2}$ along each of the two paths along the boundary of the (unique) face incident with both vertices i.

However, if $G + H$ is Eulerian, an integer solution (for $c = \mathbf{1}$, $d = \mathbf{1}$) need not exist, as is shown in Karzanov [1994b]. In fact, Karzanov gave an example where $G + H$ is Eulerian and where a half-integer solution exists, but no integer solution (Figure 74.2). Karzanov [1995] however showed that if c and d are integer and satisfy the Euler condition, then the existence of a fractional multiflow implies the existence of a half-integer multiflow. Hence, if c and d are integer (but not necessarily satisfy the Euler condition), then the existence of a fractional multiflow implies the existence of a quarter-integer multiflow.

Karzanov [1994c] showed that (74.65) cannot be extended to a set $\mathcal{F}$ of five faces by adding Γ-metrics for planar bipartite graphs Γ with five faces.

74.4. Further results and notes

74.4a. Another theorem of Okamura

Next to Theorem 74.4 ('Okamura's theorem'), Okamura [1983] gave another generalization of the Okamura-Seymour theorem:

Theorem 74.5. *Let $G = (V, E)$ be a planar graph. Let R be a set of nets such that there is a vertex q on the outer boundary of G with the property that each net is spanned by the outer boundary of G or it contains q. Let the Euler condition hold. Then the edge-disjoint paths problem has a solution if and only if the cut condition holds.*

Proof. Necessity being trivial, we show sufficiency. As in the proof of Theorem 74.4 we consider a counterexample with $2|E| - |R|$ minimal. It again implies that G is 2-connected, and that no $r \in R$ is parallel to an edge of G. Moreover, R contains at least one pair r with $q \notin r$, as otherwise the theorem follows easily from Menger's theorem.

Let C be the circuit formed by the outer boundary of G. Consider any pair $g = xy$ in R with $q \notin g$ (so $x, y \in VC$), such that the $x - y$ path P along C not containing q, is as short as possible. Deleting the edges in P from G, and net g from R, the cut condition is not maintained (as otherwise we have a smaller counterexample). As in the proof of Theorem 74.4 it implies that there exists a tight X with $x, y \notin X$ and such that X intersects C in a subpath of P. Choose X with $|X|$ minimal. Note that by the choice of g, X spans no pair in R.

If $\delta_R(X)$ contains no pair $r = st$ with both ends on C, it contains only pairs qv with $v \in X$. Hence we can contract X to one vertex and obtain a smaller counterexample (note that $|X| \geq 2$, since any net in $\delta_R(X)$ is equal to $qv \in R$ for some $v \in X$ with $v \notin VC$).

So we can assume that $\delta_R(X)$ contains a pair with both ends on C. Let e be one of the (two) edges in EC that belong to $\delta_E(X)$. We choose e such that there is a pair $r = st$ in $\delta_R(X)$ such that $s, t \in VC$ and such that the $s - t$ path along C containing e does not traverse q except possibly at its ends. Let $e = uw$ with $u \notin X$ and $w \in X$. For each $r \in \delta_R(X)$, let s_r be the vertex in $r \cap X$, and t_r the vertex in $r \setminus X$. Since $q \notin X$, we know that each such t_r is on C. Choose $r \in \delta_R(X)$ such that s_r belongs to VC, such that the $s_r - t_r$ path along C containing e does not traverse q except possibly at its ends, and such that t_r is as close as possible to u when following $C - X$. By the choice of e, such an r exists.

Since s_r and t_r are nonadjacent, we know that $\{s_r, t_r\} \neq \{u, w\}$. So we can choose $v \in \{u, w\}$ with $v \notin \{s_r, t_r\}$. Let $R' := (R \setminus \{r\}) \cup \{s_r v, v t_r\}$. Trivially the Euler condition is maintained. We show that also the cut condition is maintained, contradicting the minimality of the counterexample.

To see that the cut condition is maintained, suppose to the contrary that there is a $Y \subseteq V$ satisfying

$$(74.66) \qquad d_E(Y) < d_{R'}(Y).$$

By Theorem 70.4, we can assume that $G[Y]$ and $G - Y$ are connected. By symmetry we can assume that $t_r \notin Y$. By the Euler condition, (74.66) implies $d_E(Y) \leq d_{R'}(Y) - 2$. So

$$(74.67) \qquad d_{R'}(Y) \geq d_E(Y) + 2 \geq d_R(Y) + 2 \geq d_{R'}(Y).$$

Hence we have equality throughout. So $\delta_{R'}(Y)$ contains both $s_r v$ and $v t_r$, that is, $s_r, t_r \notin Y$ and $v \in Y$. Moreover, $d_E(Y) = d_R(Y)$.

By the choice of r, there is no pair in R connecting $X \setminus Y$ and $Y \setminus X$. So (using Theorem 3.1)

$$(74.68) \qquad d_R(X \cap Y) + d_R(X \cup Y) = d_R(X) + d_R(Y).$$

Moreover,

$$(74.69) \qquad d_E(X \cap Y) + d_E(X \cup Y) \leq d_E(X) + d_E(Y).$$

As the cut condition holds for $X \cap Y$ and $X \cup Y$, we have equality in (74.69), and therefore $d_E(X \cap Y) = d_R(X \cap Y)$. Since $s_r \in X \setminus Y$, we know $|X \cap Y| < |X|$. So by the minimality of X we have $X \cap Y = \emptyset$. So $w \notin Y$, hence $u = v \in Y$. Then edge $e = uw$ connects $X \setminus Y$ and $Y \setminus X$, contradicting equality in (74.69). ∎

Suzuki, Nishizeki, and Saito [1985a,1985b] gave an $O(t^2 n + n \cdot \mathrm{SP}_+(n))$-time algorithm for finding the edge-disjoint paths in this case (similarly for the capacitated case), where t is the number of vertices on the outer boundary, and where $\mathrm{SP}_+(n)$ is any upper bound on the time needed to find a shortest path in a planar n-vertex graph with nonnegative edge lengths.

With Theorem 70.5, Theorem 74.5 implies that for any planar graph $G = (V, E)$ and any vertex q on the outer boundary, there is a fractional cut packing such that any pair s, t of vertices, with s, t both on the outer boundary or $s = q$, is separated by $\mathrm{dist}_G(s, t)$ of these cuts. It seems to be open if the corresponding integer packing theorem for bipartite planar graphs holds.

74.4b. Some other planar cases where the cut condition is sufficient

It was announced by Gerards [1993] that if $G = (V, E)$ is a bipartite planar graph and $s, t \in V$, then there exist disjoint cuts such that for each $u, v \in V$ with u, v both on the outer boundary, or with $u = s$, $v = t$, the distance of u and v is equal to the number of cuts separating u and v. By Theorem 70.5, this implies that the cut condition implies the existence of a fractional multiflow, if each net is spanned by the outer boundary or is equal to some fixed pair $\{s, t\}$ of vertices.

Gerards [1993] also announced that if G is a graph embedded in the Möbius strip, and if $\{s_1, t_1\}, \ldots, \{s_k, t_k\}$ are nets such that the terminals are either in the order $s_1, \ldots, s_k, t_1, \ldots, t_k$ along the boundary, or in the order or $s_1, \ldots, s_k, t_k, \ldots, t_1$, then the cut condition and the Euler condition imply the existence of an integer multiflow.

74.4c. Vertex-disjoint paths in planar graphs

Let $G = (V, E)$ be a planar graph, embedded in the plane $\mathbb{R}^2$ and let $\{s_1, t_1\}, \ldots, \{s_k, t_k\}$ be disjoint pairs of vertices (the 'nets'). Robertson and Seymour [1986] observed that there is an easy greedy-type algorithm for the vertex-disjoint paths problem if all vertices $s_1, t_1, \ldots, s_k, t_k$ belong to the outer boundary of G. That is, there exists a polynomial-time algorithm for the following problem:

(74.70) given: a planar graph $G = (V, E)$ and disjoint pairs $\{s_1, t_1\}, \ldots, \{s_k, t_k\}$ of vertices on the outer boundary of G,
find: vertex-disjoint paths $P_1, \ldots, P_k$ in G, where P_i connects s_i and t_i ($i = 1, \ldots, k$).

We describe the simple intuitive idea of the method. (Pinter [1983] attributed this idea to C.P. Hsu (1982), and applied it to the vertex-disjoint paths problem in rectangular grids.)

We say that two disjoint pairs $\{s, t\}$ and $\{s', t'\}$ cross (around G) if there exist no disjoint curves in the unbounded face, connecting s and t, and connecting s' and t'. The following *noncrossing condition* is a necessary condition for (74.70) to have a solution:

(74.71) No two distinct nets $\{s_i, t_i\}, \{s_j, t_j\}$ cross.

The noncrossing condition implies that there exists an i such that at least one of the two $s_i - t_i$ paths along $\mathrm{bd}(F)$ contains no s_j or t_j for $j \neq i$: just choose i such that the shortest $s_i - t_i$ path along the outer boundary is shortest among all $i = 1, \dots, k$.

Without loss of generality, $i = k$. Let Q be a shortest $s_k - t_k$ path along the outer boundary. Let $G' := G - VQ$. Next solve the vertex-disjoint paths problem for input G', $\{s_1, t_1\}, \dots, \{s_{k-1}, t_{k-1}\}$. If this gives a solution $P_1, \dots, P_{k-1}$, then $P_1, \dots, P_{k-1}, Q$ forms a solution to the original problem (trivially).

If the reduced problem turns out to have no solution, then the original problem also has no solution. This follows from the fact that if $P_1, \dots, P_{k-1}, P_k$ would be a solution to the original problem, we may assume without loss of generality that $P_k = Q$, since we can 'push' P_k 'against' the outer boundary. Hence $P_1, \dots, P_{k-1}$ would form a solution to the reduced problem. This intuitive idea is the basis of a polynomial-time algorithm for problem (74.70):

Theorem 74.6. *The vertex-disjoint paths problem is polynomial-time solvable for planar graphs with all terminals on the outer boundary.*

Proof. See above. ∎

Linear-time implementations were given by Suzuki, Akama, and Nishizeki [1988c,1990] and Liao and Sarrafzadeh [1991].

The method implies moreover a characterization by means of a cut condition for the existence of a solution to (74.70). A *simple closed curve* C in $\mathbb{R}^2$ is by definition a one-to-one continuous function from the unit circle to $\mathbb{R}^2$. We will identify the function C with its image.

We say that C *separates* the pair $\{s, t\}$ if each curve connecting s and t intersects C. (In particular, if s or t is on C.) Now the following *cut condition* clearly is necessary for the existence of a solution to the vertex-disjoint paths problem in planar graphs:

(74.72) each simple closed curve in $\mathbb{R}^2$ intersects G at least as often as it separates pairs $\{s_1, t_1\}, \dots, \{s_k, t_k\}$.

Robertson and Seymour [1986] showed with the method above:

Theorem 74.7. *Let $G = (V, E)$ be a planar graph embedded in $\mathbb{R}^2$ and let $\{s_1, t_1\}, \dots, \{s_k, t_k\}$ be pairs of vertices on the outer boundary of G. Then there exist vertex-disjoint paths $P_1, \dots, P_k$ where P_i connects s_i and t_i ($i = 1, \dots, k$) if and only if the noncrossing condition* (74.71) *and the cut condition* (74.72) *hold.*

Proof. Necessity of the conditions is trivial. We show sufficiency by induction on k, the case $k = 0$ being trivial. Let $k \geq 1$ and let (74.71) and (74.72) be satisfied. Suppose that paths $P_1, \dots, P_k$ as required do not exist. Trivially, $\{s_1, t_1\}, \dots, \{s_k, t_k\}$ are disjoint (otherwise there would exist a simple closed curve C with $|C \cap G| = 1$ and intersecting two nets, thus violating the cut condition).

We may assume that G is connected, as we can decompose G into its components. (If some s_i and t_i would belong to different components, there trivially exists a closed curve C violating the cut condition.) We can also assume that there is no cut vertex v such that $G-v$ has a component K containing no terminal (otherwise we could delete K from G without violating the cut condition).

Now there exists an i and a simple $s_i - t_i$ path P_i such that P_i follows the outer boundary and traverses no other terminals than s_i and t_i. We can assume that $i = k$. Let $G' := G - VP_k$.

Then G' contains no vertex-disjoint $s_i - t_i$ paths ($i = 1, \ldots, k-1$), since otherwise G contains vertex-disjoint $s_i - t_i$ paths ($i = 1, \ldots, k$). Hence, by the induction hypothesis, there exists a simple closed curve C with $|C \cap G'|$ smaller than the number of pairs $\{s_1, t_1\}, \ldots, \{s_{k-1}, t_{k-1}\}$ separated by C.

We can assume that C traverses the unbounded face of G' exactly once and that it intersects G only in vertices of G. We choose C such that it has a minimum number of intersections with P_k. Then C intersects P_k at most once. If C does not intersect P_k, then $|C \cap G| = |C \cap G'|$, and C violates the cut condition also for G. If C intersects P_k, then $|C \cap G| = |C \cap G'| + 1$ and C separates s_k and t_k, and so again C violates the cut condition for G. ∎

It is easy to extend the algorithm and Theorem 74.7 to the directed case, and also to the following *vertex-disjoint trees problem*:

(74.73) given: a planar graph $G = (V, E)$ and sets $S_1, \ldots, S_k$ of vertices on the outer boundary of G,
find: vertex-disjoint subtrees $T_1, \ldots, T_k$ of G such that T_i covers S_i ($i = 1, \ldots, k$).

More generally, with similar techniques, Ding, Schrijver, and Seymour [1992] generalized Theorem 74.7 (and the polynomial-time algorithm) as follows.

Theorem 74.8. *Let $D = (V, A)$ be a directed planar graph, let B be a family of ordered pairs of vertices on the outer boundary of D (with $s \neq t$ if $(s,t) \in B$), for each $b \in B$ let $A_b \subseteq A$, and let H be a set of unordered pairs from B. Then there exist paths P_b for $b \in B$ such that:*

(74.74) (i) *for $b = (s,t) \in B$, P_b is a directed $s - t$ path in (V, A_b),*
(ii) *P_b and P_c are vertex-disjoint for each $\{b, c\} \in H$,*

if and only if the following two conditions hold: the 'noncrossing condition':

(74.75) *if $\{(r,s),(t,u)\} \in H$, then (r,s) and (t,u) are disjoint and do not cross,*

and the 'cut condition':

(74.76) *for each curve C starting and ending in the unbounded face and not intersecting any s, t with $(s,t) \in B$ and for each choice of $b_1, \ldots, b_n \in B$ satisfying:*
- *$\{b_j, b_{j+1}\} \in H$ for $j = 1, \ldots, n-1$,*
- *$f, x_1, \ldots, x_n, l, y_n, \ldots, y_1$ are all distinct and occur in this order clockwise around the outer boundary, where x_j and y_j are such that $b_j = (x_j, y_j)$ or $b_j = (y_j, x_j)$, and where f and l denote the first and last point of intersection of C with D,*

there exist distinct points $p_1, \ldots, p_n$ traversed by C in this order such that for each $j = 1, \ldots, n$:

- *p_i is on the image of D in $\mathbb{R}^2$, if $b_j = (x_j, y_j)$, then some arc in A_{b_j} is entering C at p_j from the left and some arc in A_{b_j} is leaving C at p_j from the right,*
- *if $b_j = (y_j, x_j)$, then some arc in A_{b_j} is entering C at p_j from the right and some arc in A_{b_j} is leaving C at p_j from the left.*

(The points p_i can be vertices of D or be on arcs of D.)

Theorem 74.8 implies an even more general characterization and algorithm for disjoint rooted subarborescences. Let $D = (V, A)$ be a planar digraph, let B be a collection of ordered pairs (r, S) where r is a vertex on the outer boundary of D, and S is a set of vertices on the outer boundary of D with $r \notin S$. For each $b \in B$, let $A_b \subseteq A$, and let H be a set of unordered pairs from B. Then Theorem 74.8 implies necessary and sufficient conditions for the existence of rooted subarborescences T_b in D (for $b \in B$), with the property that

(74.77) (i) for $b = (r, S) \in B$, T_b is rooted at r, covers S, and is contained in A_b,
(ii) T_b and T_c are vertex-disjoint for each $\{b, c\} \in H$.

The reduction to Theorem 74.8 is by replacing each pair (r, S) in B by the pairs (r, s) for $s \in S$, and reset H to all pairs $\{(r, s), (r', s')\}$ coming from pairs $\{(r, S), (r', S')\}$ in the original H.

Notes. Suzuki, Akama, and Nishizeki [1988c,1990] and Liao and Sarrafzadeh [1991] gave linear-time algorithms for problem (74.73). For a description, see also Wagner [1993].

Theorem 74.6 implies that the vertex-disjoint paths problem is polynomial-time solvable for outerplanar graphs. This was generalized to series-parallel graphs by Korach and Tal [1993].

Takahashi, Suzuki, and Nishizeki [1992] gave an $O(n \log n)$-time algorithm to find pairwise noncrossing paths of minimum total length, connecting prescribed terminals in a planar graph with all terminals on two specified face boundaries.

74.4d. Grid graphs

Grid graphs form a class of planar graphs that are of special interest for disjoint paths problem, as they arise in the design of VLSI-circuits, in particular in routing the wires on the layers of a chip.

Any finite subgraph of the 2-dimensional rectangular grid is called a *grid graph*. So its vertex set is a finite subset of $\mathbb{Z}^2$, and any two adjacent vertices have Euclidean distance 1. (It is not required conversely that any two vertices at Euclidean distance 1 are adjacent; so the subgraph need not be an *induced* subgraph.)

Kramer and van Leeuwen [1984] showed that both the vertex-disjoint and the edge-disjoint paths problems are NP-complete even when restricted to grid graphs. Pinter [1983] showed that the vertex-disjoint paths problem remains NP-complete for grid graphs in which all faces are bounded by a rectangle (including a square).

A *rectangular grid* is a grid graph whose outer boundary is a rectangle and whose bounded faces all are unit squares. The *channel routing problem* is the vertex-disjoint paths problem in a rectangular grid, where all nets connect a vertex on the upper horizontal border with one on the lower horizontal border. A criterion for the feasibility of the channel routing problem was given by Dolev, Karplus, Siegel, Strong, and Ullman [1981], while Rivest, Baratz, and Miller [1981] gave a heuristic algorithm approximating the minimal height of the rectangle, given the positions of the terminals (cf. Preparata and Lipski [1984] and Mehlhorn, Preparata, and Sarrafzadeh [1986]). A linear-time algorithm for channel-routing, allowing also multiterminal nets, was given by Greenberg and Maley [1992].

The feasibility criterion was extended by Pinter [1983] to *switchboxes*, which are rectangular grids in which the terminals can be anywhere along the outer boundary. For the vertex-disjoint paths problem in switchboxes, Pinter showed Theorem 74.7 and described the corresponding greedy-type algorithm. He attributes the idea to C.P. Hsu (1982).

Algorithms for the *edge*-disjoint paths problem in a switchbox were given by Frank [1982c] ($O(n \log n)$) and Mehlhorn and Preparata [1986] ($O(u \log u)$, where u is the circumference of the rectangle — note that this is sufficient to specify the graph). Frank also showed that solvability only depends on horizontal and vertical cuts.

A *generalized switchbox* is a grid graph with all bounded faces being unit squares. Nishizeki, Saito, and Suzuki [1985] gave an $O(n^2)$-time algorithm for routing in generalized switchboxes for which any two vertices on the outer boundary are connected by a path with at most one bend; all terminals are on the outer boundary. They also showed that in this case one may restrict the cuts to those that are either horizontal or vertical, if the global Euler condition holds. (A correction and generalization was given by Lai and Sprague [1987].)

Kaufmann and Mehlhorn [1986] described an $O(n \log^2 n + q^2)$-time algorithm for the edge-disjoint paths problem in a generalized switchbox, with all terminals on the outer boundary. Here q denotes the number of vertices v with $\deg_G(v)+\deg_H(v)$ odd. So if the Euler condition holds, the time bound is $O(n \log^2 n)$.

Kaufmann and Mehlhorn [1986] also showed that in a generalized switchbox satisfying the Euler condition and such that no vertex is end point of more than two curves, the cut condition holds whenever it holds for all 1-bend cuts. (A cut is called a 1-*bend cut* if it is the set of edges crossed by the union of some horizontal and some vertical halfline with one common end vertex.)

Kaufmann and Klär [1993] gave an $O(u \log^2 u)$-time algorithm for generalized switchboxes, whose outer boundary is simple and has no 'rectilinearly visible corners'. (Two corners p and q of the outer boundary are called *rectilinearly visible* if the (unique) rectangle of which p and q are opposite vertices, has a nonempty interior and intersects the outer boundary only in p and q.)

Wagner and Weihe [1993,1995] showed that for such problems, if $G + H$ is Eulerian, then there is even a linear-time algorithm, even for general planar graphs. (This improves earlier results of Becker and Mehlhorn [1986] and Kaufmann [1990].)

If G is a rectangle with one rectangular hole, and all nets join two vertices either on the outer rectangle or on the inner rectangle, and if the Euler condition holds, Suzuki, Ishiguro, and Nishizeki [1990] gave a linear-time algorithm. Related results are given in Frank, Nishizeki, Saito, Suzuki, and Tardos [1992].

Takahashi, Suzuki, and Nishizeki [1993] gave a polynomial-time algorithm for the minimum-length 'noncrossing' paths problem in certain grid graphs.

The problem of finding edge-disjoint trees connecting specified sets of vertices on the outer boundary of a rectangle is NP-complete (Sarrafzadeh [1987b]). More on channel routing can be found in Preparata and Sarrafzadeh [1985], Sarrafzadeh and Preparata [1985], Mehlhorn, Preparata, and Sarrafzadeh [1986], Sarrafzadeh [1987a], Formann, Wagner, and Wagner [1991,1993], Greenberg and Shih [1995, 1996], and Chan and Chin [1997,2000]. Surveys on disjoint paths problems in grid graphs are given by Kaufmann and Mehlhorn [1990] and in the book by Lengauer [1990].

74.4e. Further notes

The Lucchesi-Younger theorem (Theorem 55.2) implies the following. Let $D = (V, A)$ and $H = (V, R)$ be digraphs with D acyclic and $(V, A \cup R)$ planar. Then D has arc-disjoint paths P_r for $r \in R$, where P_r runs from s to t if $r = (s, t)$, if and only if for each $B \subseteq A$:

(74.78) $|B| \geq$ number of $r = (s, t) \in R$ such that B intersects each $s - t$ path in D.

Trivially, this condition is necessary. The derivation of sufficiency from the Lucchesi-Younger theorem is as follows. Consider the planar digraph $Q = (V, A \cup R^{-1})$. We need to show that if (74.78) holds for each $B \subseteq A$, then Q contains $|R|$ arc-disjoint directed circuits. Equivalently, the planar dual Q^* contains $|R|$ disjoint directed cuts. Applying the Lucchesi-Younger theorem to Q^* yields for Q that we should show that (74.78) implies that each set C of arcs of Q intersecting each directed circuit of Q has size at least $|R|$. Set $B := C \cap A$ and $R' := C^{-1} \cap R$. Then for each $r = (s, t) \in R \setminus R'$, each $s - t$ path in D intersects B. So by (74.78), $|B| \geq |R \setminus R'|$, and hence $|C| = |B| + |R'| \geq |R|$.

Similarly the polynomial-time solvability of the corresponding arc-disjoint paths problem follows (using Theorem 55.7).

Korte, Prömel, and Steger [1990] showed that the edge-disjoint trees problem is NP-complete, even if we ask for two disjoint trees in a planar graph, where the trees should cover two prescribed sets of vertices.

Surveys on linear-time methods for disjoint paths problems in planar graphs were given by Wagner [1993] and Ripphausen-Lipa, Wagner, and Weihe [1995]. For extensions to nets spanned by a fixed number of faces, see Section 76.7a.

Chapter 75

Cuts, odd circuits, and multiflows

Minimum-size cuts in a graph are well under control from an algorithmic point of view, as we saw in Parts I and V. Finding a *maximum*-size cut is however an NP-complete problem.
The complement of a maximum-size cut is a minimum-size odd circuit cover — a set of edges intersecting all odd circuits. By duality, this relates to maximum collections of edge-disjoint odd circuits. This in turn relates to multiflows.
Weakly bipartite graphs are those graphs where the polyhedral approach works. It makes that the maximum cut problem is polynomial-time solvable for these graphs.
Key result in this chapter is a theorem of Guenin characterizing weakly bipartite graphs, and its extension by Geelen and Guenin to evenly bipartite graphs. These results turn out to unify several multiflow and odd circuit packing theorems.

75.1. Weakly and strongly bipartite graphs

Let $G = (V, E)$ be an undirected graph. Call a subset B of E *bipartite* if (V, B) is bipartite; equivalently, if B does not contain the edge set of any odd circuit; equivalently, if B is contained in some cut C. So finding a maximum-size bipartite set of edges is equivalent to finding a maximum-size cut, and hence it is NP-complete (cf. Section 75.1a).

The *bipartite subgraph polytope* $P_{\text{bipartite subgraph}}(G)$ of G is the convex hull of the incidence vectors (in $\mathbb{R}^E$) of bipartite subsets B of E:

(75.1) $$P_{\text{bipartite subgraph}}(G) := \text{conv.hull}\{\chi^B \mid B \subseteq E \text{ bipartite}\}.$$

Any vector x in the bipartite subgraph polytope satisfies

(75.2)
$$\begin{array}{lll} \text{(i)} & 0 \leq x_e \leq 1 & \text{for each } e \in E, \\ \text{(ii)} & x(C) \leq |C| - 1 & \text{for each odd circuit } C. \end{array}$$

In general, these constraints are not enough to determine the bipartite subgraph polytope: for the complete graph K_5, the vector x with $x_e = \frac{2}{3}$ for each edge e satisfies (75.2), but does not belong to the bipartite subgraph polytope (since the largest bipartite subgraph has 6 edges, while $10 \cdot \frac{2}{3} > 6$).

Following Grötschel and Pulleyblank [1981], a graph G is called *weakly bipartite* if its bipartite subgraph polytope is determined by (75.2). An equivalent characterization is in terms of odd circuit covers. An *odd circuit cover* in an undirected graph $G = (V, E)$ is a set of edges intersecting all odd circuits. The *odd circuit cover polytope* is the convex hull of the incidence vectors of odd circuit covers. It is contained in the polytope determined by

(75.3) $\quad 0 \leq x_e \leq 1 \quad$ for each $e \in E$,
$\quad\quad\quad x(C) \geq 1 \quad$ for each odd circuit C.

Then a graph is weakly bipartite if and only the odd circuit polytope is determined by (75.3). This follows directly from the facts that a set of edges is an odd circuit cover if and only if its complement is bipartite, and that x satisfies (75.2) if and only if $\mathbf{1} - x$ satisfies (75.3).

The relevance of weakly bipartite graphs comes from the fact that a maximum-capacity cut in these graphs can be found in strongly polynomial time, with the ellipsoid method, since the separation problem over the polytopes (75.2) is polynomial-time solvable (cf. Section 5.11). Indeed, checking (75.2) is equivalent to checking (75.3). One can check the constraints in (75.3)(i) one by one, and so one may assume that $\mathbf{0} \leq x \leq \mathbf{1}$. Next, considering x as length function, one checks if there is an odd circuit of length < 1 (like in Theorem 68.1). If so, we find a violated constraint. If not, x satisfies (75.3).

Weakly bipartite graphs were characterized by Guenin [1998a,2001a], proving a conjecture of Seymour [1981a]. This characterization also holds for the more general structure of *signed graphs*, for which it is easier to prove as it allows a finer contraction operation — see Sections 75.2 and 75.5. For just undirected graphs the characterization can be formulated as follows.

Call a graph H an *odd minor* of a graph G if H arises from G by deleting edges and vertices and by contracting all edges in a cut. The class of weakly bipartite graphs is closed under taking odd minors. To see this, it is easily seen that this class is closed under deleting edges and vertices. To see that it is closed under contracting a cut, let $G = (V, E)$ be a weakly bipartite graph, let $U \subseteq V$ and $G' = G/\delta(U)$, and take $x \in \mathbb{R}^{E'}$, where $E' = E \setminus \delta(U)$ is the edge set of G'. Let x satisfy (75.3) with respect to G'. Define $x_e := 0$ for each $e \in \delta(U)$. Then the extended x satisfies (75.3) with respect to G. So the extended x belongs to the odd circuit cover polytope of G, implying that the original x belongs to the odd circuit cover polytope of G'.

Now Guenin's characterization reads for undirected graphs:

(75.4) $\quad$ an undirected graph G is weakly bipartite $\iff$ K_5 is not an odd minor of G.

Related is a characterization of those graphs for which (75.2) is totally dual integral. These graphs are called *strongly bipartite*[22]. A general hypergraph theorem of Seymour [1977b] implies a characterization of strongly bipartite graphs. They are precisely the graphs containing no odd K_4-subdivision — equivalently, the graphs not having K_4 as odd minor. Again, this is easier to handle in the context of signed graphs — see Section 75.4.

75.1a. NP-completeness of maximum cut

In this section we show (Karp [1972b]):

Theorem 75.1. *Finding the maximum size of a cut in an undirected graph is* NP-*complete.*

Proof. We reduce the problem of finding the minimum size of a vertex cover in a graph $G = (V, E)$ to the maximum-size cut problem. This is sufficient, since the first problem is NP-complete by Corollary 64.1a.

We can assume that G has no isolated vertices, since they will not occur in any minimum-size vertex cover. Extend G by a new vertex u and, for each $v \in V$, by $\deg_G(v) - 1$ parallel edges connecting v and u. Let G' be the extended graph. Then

(75.5) the minimum size of a vertex cover in G is equal to $2|E|$ minus the maximum size of a cut in G'.

To see this, we have for any $U \subseteq V$:

(75.6)
$$\begin{aligned} |\delta_{G'}(U)| &= |\delta_G(U)| + \sum_{v \in U} (\deg_G(v) - 1) \\ &= 2|\{e \in E \mid e \text{ intersects } U\}| - |U|. \end{aligned}$$

Hence, if U is a minimum-size vertex cover of G, then $|\delta_{G'}(U)| = 2|E| - |U|$, proving $\geq$ in (75.5).

To see the reverse inequality, choose a subset U of V that determines a maximum-size cut $\delta_{G'}(U)$ in G'. Then U is a vertex cover of G. Otherwise, $V \setminus U$ spans an edge e of G. Then extending U by one of the ends of e increases (75.6), a contradiction. So U is a vertex cover and $|U| = 2|E| - |\delta_{G'}(U)|$, proving $\leq$ in (75.5). ∎

75.1b. Planar graphs

Although we do not use these results in later sections, we first show that planar graphs are weakly bipartite, as it gives an interesting relation with T-joins (Barahona [1980]):

Theorem 75.2. *A planar graph is weakly bipartite.*

Proof. Consider the dual graph $G^* = (V^*, E^*)$. An odd circuit in G corresponds to an odd-size cut in G^*, that is, to a T-cut, where T is the set of vertices of G^* of

[22] A strongly bipartite graph need not be bipartite, as is shown by K_3.

odd degree. For G it means that an odd circuit cover in G corresponds to a set of edges of G^* containing a T-join. By Corollary 29.2b, the convex hull of these edge sets in G^* is determined by

(75.7) $$\begin{array}{ll} 0 \leq x(e^*) \leq 1 & \text{for } e^* \in E^*, \\ x(C) \geq 1 & \text{for each } T\text{-cut } C \text{ in } G^*. \end{array}$$

Hence the odd circuit cover polytope of G is determined by (75.3). ∎

With the help of the decomposition theorem of Wagner [1937a] (Theorem 3.3), this result can be extended to graphs without K_5 minor (Fonlupt, Mahjoub, and Uhry [1992]). We will however derive this from Guenin's more general characterization of weakly bipartite graphs.

75.2. Signed graphs

Guenin's characterization of weakly bipartite graph is valid, and easier to prove, in the more general context of signed graphs. In this section we collect some general terminology and facts on signed graphs.

A *signed graph* is a triple $G = (V, E, \Sigma)$, where (V, E) is an undirected graph and $\Sigma \subseteq E$. The graph (V, E) is called the *underlying graph* and Σ is called a *signing*.

Call a set of edges, or a path, or a circuit *odd* (*even*, respectively) if it contains an odd (even, respectively) number of edges in Σ. An *odd circuit cover* is a set of edges intersecting all odd circuits.

It is easy to show that, for any undirected graph (V, E),

(75.8) Two signings Σ and Σ' give the same collection of odd circuits $\iff$ $\Sigma \triangle \Sigma'$ is a cut of (V, E).

If $\Sigma \triangle \Sigma'$ is a cut, we call the two signed graphs, or the two signings, *equivalent*. The following is an important observation: for any signed graph $G = (V, E, \Sigma)$,

(75.9) the collection of inclusionwise minimal odd circuit covers of G is equal to the collection of inclusionwise minimal signings equivalent to Σ.

Indeed, any signing Σ' equivalent to Σ intersects each odd circuit in an odd number of edges, and hence is an odd circuit cover. Conversely, any inclusionwise minimal odd circuit cover B intersects each odd circuit C in an odd number of edges: by the minimality of B, for each $e \in B \cap C$ there exists an odd circuit C_e disjoint from $B \setminus \{e\}$. If $|B \cap C|$ is even, the symmetric difference of C and the C_e gives an odd cycle disjoint from B, a contradiction.

(75.9) has several consequences. The inclusionwise minimal sets among $\Sigma \triangle \delta(U)$ (for $U \subseteq V$) are precisely the inclusionwise minimal odd circuit covers. For any two inclusionwise minimal odd circuit covers B_1, B_2 there exists a subset U of V with

(75.10) $B_1 \triangle B_2 = \delta(U)$

(since $B_1 = \Sigma \triangle \delta(U_1)$ and $B_2 = \Sigma \triangle \delta(U_2)$ for some $U_1, U_2 \subseteq V$, hence $B_1 \triangle B_2 = \delta(U_1) \triangle \delta(U_2) = \delta(U_1 \triangle U_2)$).

(75.9) also implies that for each inclusionwise minimal odd circuit cover B of G, the set $B \triangle \Sigma$ is a cut. (We recall that, by definition, the empty set is also a cut.)

We can define the concepts of deletion, contraction, subgraph, and minor in a signed graph $G = (V, E, \Sigma)$. *Deleting* an edge e means replacing G by $G - e := (V, E \setminus \{e\}, \Sigma \setminus \{e\})$. Similarly, *deleting* a vertex v means deleting v in V and deleting in E and Σ all edges incident with v.

Contracting a (nonloop) edge e means: if $e \notin \Sigma$, replacing G by $G/e := (\widetilde{V}, \widetilde{E}, \Sigma)$, where $(\widetilde{V}, \widetilde{E})$ is obtained from (V, E) by contracting e; if $e \in \Sigma$, choose $v \in e$, replace Σ by $\Sigma \triangle \delta(v)$, and apply the previous operation. So the operation of contraction is not uniquely defined, but the outcome is unique up to equivalence of signings. This is sufficient for our purposes.

A *subgraph* of a signed graph is obtained by a series of deletions of vertices and edges. A *minor* is obtained by a series of deletions of vertices and edges and contractions of edges, and by replacing the signing by an equivalent signing.

For any complete graph K_n, let *odd-*K_n be the signed graph

(75.11) odd-$K_n := (VK_n, EK_n, EK_n)$.

A signed graph (V, E, Σ) is called an *odd* K_4*-subdivision* if (V, E) is a subdivision of K_4 such that each triangle has become an odd circuit (with respect to Σ). It is not difficult to show that:

(75.12) a signed graph contains an odd K_4-subdivision if and only if it has odd-K_4 as minor.

75.3. Weakly, evenly, and strongly bipartite signed graphs

In an obvious way, the notions of weakly and strongly bipartite graphs can be lifted to signed graphs. A signed graph $G = (V, E, \Sigma)$ is *weakly bipartite* if each vertex of the polyhedron (in $\mathbb{R}^E$) determined by:

(75.13)
(i) $0 \le x_e \le 1$ for each edge e,
(ii) $x(C) \ge 1$ for each odd circuit C,

is integer, that is, the incidence vector of an odd circuit cover.

System (75.13) gives rise to two stronger properties. First, a signed graph is called *strongly bipartite* if (75.13) is totally dual integral. Equivalently, for each function $w : E \to \mathbb{Z}_+$ the minimum of $w^{\mathsf{T}}x$ over (75.13) has integer primal and dual optimum solutions. Or: for each weight function $w : E \to \mathbb{Z}_+$, the minimum weight of an odd circuit cover is equal to the maximum size of a family of odd circuits such that each edge e is in at most $w(e)$ of them.

We also define an intermediate property (only seemingly intermediate, since it will turn out to be equivalent to weakly bipartite). A signed graph $G = (V, E, \Sigma)$ is called *evenly bipartite* if for each $w : E \to \mathbb{Z}_+$ with $w(\delta(v))$ even for each $v \in V$, the minimum of $w^\mathsf{T} x$ over (75.13) is attained by integer primal and dual optimum solutions. Equivalently, for each weight function $w : E \to \mathbb{Z}_+$ with $w(\delta(v))$ even for all $v \in V$, the minimum weight of an odd circuit cover is equal to the maximum size of a family of odd circuits such that each edge e is in at most $w(e)$ of them.

There are the following direct implications:

(75.14) strongly bipartite $\Longrightarrow$ evenly bipartite $\Longrightarrow$ weakly bipartite.

It is easy to check that the classes of weakly, evenly, and strongly bipartite signed graphs are closed under taking minors. So each class can be characterized by forbidden minors.

Now a theorem of Seymour [1977b] implies that a signed graph G is strongly bipartite if and only if it has no odd-K_4 minor (Corollary 75.3a below). Guenin [1998a,2001a] showed that G is weakly bipartite if and only if it has no odd-K_5 minor. This was sharpened by Geelen and Guenin [2001], who proved that G is evenly bipartite if and only if G has no odd-K_5 minor (Corollary 75.4a below). So weakly and evenly bipartite are equivalent.

75.4. Characterizing strongly bipartite signed graphs

A general hypergraph theorem of Seymour [1977b] (Theorem 80.1) implies a characterization of strongly bipartite signed graphs. This will be derived from the following equivalent result, which we prove with a method of Geelen and Guenin [2001]:

Theorem 75.3. *In a signed graph G without odd-K_4 minor, the maximum number of edge-disjoint odd circuits is equal to the minimum size of an odd circuit cover.*

Proof. For any signed graph $G = (V, E, \Sigma)$, let $\pi(G)$ denote the minimum size of an odd circuit cover and let $\mu(G)$ denote the maximum number of edge-disjoint odd circuits. We must show $\mu(G) = \pi(G)$ for any signed graph G without odd-K_4 minor.

Suppose that this is not true. Choose a counterexample $G = (V, E, \Sigma)$, with $\pi(G)$ minimum, $|V|$ minimum, and $|E|$ maximum, in this order of priority. Such a graph exists, since if there are more than $\pi(G)$ parallel edges connecting two vertices, we can contract them to obtain a counterexample with $|V|$ smaller.

Define $\pi := \pi(G)$. Fix an edge $e = xy$ not contained in every minimum-size odd circuit cover. By adding a parallel edge connecting x and y, we do not change $\pi(G)$ or $|V|$, but we increase $|E|$. Hence in the extended graph there exist π edge-disjoint odd circuits. This means that in the original graph G there exist odd circuits $C_1, \ldots, C_\pi$ with $e \in C_1 \cap C_2$ and with $C_1 \setminus \{e\}$, $C_2 \ldots, C_\pi$ disjoint. (Here we take circuits as edge sets.) We choose the C_i with $|C_1 \cup C_2|$ minimal.

For $i = 1, 2$, let P_i be the $x - y$ path $C_i \setminus \{e\}$, for $i = 1, 2$. (Also the paths are taken as edge sets.) Then

(75.15) $\qquad P_1 \cup P_2$ contains no odd circuit C.

Otherwise, replacing C_1 and C_2 by C and $C_1 \triangle C_2 \triangle C$ gives π edge-disjoint odd circuits, a contradiction.

Moreover, let $x = v_0, v_1, \ldots, v_k = y$ be the common vertices of P_1 and P_2, in the order on which they occur along P_1. Then

(75.16) $\qquad v_0, v_1, \ldots, v_k$ occur in this order also along P_2.

Indeed, orient P_1 and P_2 from x to y. Then we create no directed circuit, since otherwise there exist circuits $C'_1, C'_2 \subseteq C_1 \cup C_2$ with $C'_1 \cap C'_2 = \{e\}$ and $|C'_1 \cup C'_2| < |C_1 \cup C_2|$. Then C'_1 and C'_2 are odd (since otherwise $C_1 \triangle C'_1$ is odd, and hence contains an odd circuit, contradicting (75.15)). This contradicts the minimality of $|C_1 \cup C_2|$.

Now choose j with $0 \leq j \leq k$ such that

(75.17) $\qquad \pi(G - (P \cup \{e\})) \leq \pi - 2$

for each $v_j - y$ path P in $P_1 \cup P_2$ and such that j is as large as possible. Such a j exists, as (75.17) holds for each $x - y$ path P in $P_1 \cup P_2$ (otherwise, $G - (P \cup \{e\})$ contains $\pi - 1$ disjoint odd circuits; hence, with $P \cup \{e\}$ it gives π disjoint odd circuits in G as required).

Since $\pi(G) = \pi$ we know $\pi(G - \{e\}) \geq \pi - 1$, and hence $j < k$. By the maximality of j, there is a $v_{j+1} - y$ path R in $P_1 \cup P_2$ such that

(75.18) $\qquad \pi(G - (R \cup \{e\})) \geq \pi - 1.$

Let Q_1 and Q_2 be the two $v_j - v_{j+1}$ paths in $P_1 \cup P_2$. By (75.17) we know

(75.19) $\qquad \pi(G - (Q_i \cup R \cup \{e\})) \leq \pi - 2$

for $i = 1, 2$. Hence for each $i = 1, 2$ there exists an inclusionwise minimal odd circuit cover B_i with $|B_i \setminus (Q_i \cup R \cup \{e\})| \leq \pi - 2$. So B_i contains one edge of $C_3, \ldots, C_\pi$ each, and consists for the rest of edges in $Q_i \cup R \cup \{e\}$. As $C_1 \cup C_2$ contains an odd circuit disjoint from $Q_i \cup R$, we know $e \in B_i$.

Since B_1 and B_2 are minimal odd circuit covers, there exists a subset U of V with

(75.20) $\qquad B_1 \triangle B_2 = \delta(U)$

and U disjoint from e (as $e \notin B_1 \triangle B_2$). So U is disjoint from all $x - v_j$ paths and from the $v_{j+1} - y$ path R' in $P_1 \cup P_2$ edge-disjoint from R (since B_1 and B_2 are edge-disjoint from these paths).

As G has no odd-K_4 minor, there is no path contained in U that connects VQ_1 and VQ_2 and that consists only of edges out of B_1. (It creates with Q_1, Q_2, R', e and any $x - v_j$ path in $P_1 \cup P_2$ an odd K_4-subdivision, as B_1 can serve as a signing.) So U has a subset X such that $VQ_1 \cap U \subseteq X$ and $X \cap VQ_2 = \emptyset$ and such that each edge connecting X and $U \setminus X$ belongs to B_1. So $\delta(X) \subseteq B_1 \cup \delta(U) \subseteq B_1 \cup B_2$. Define

(75.21) $\qquad B := B_1 \triangle \delta(X).$

Then B is an odd circuit cover. We show that $|B \setminus (R \cup \{e\})| \leq \pi - 2$, contradicting (75.18).

Since $U \cap VQ_1 \subseteq X \subseteq U$, we know that $B_1 \cap Q_1 \subseteq \delta(U) \cap Q_1 \subseteq \delta(X)$, and hence $B \cap Q_1 = \emptyset$. Also $B \cap Q_2 = \emptyset$, as $\delta(X)$ contains no edge of Q_2, since X is disjoint from VQ_2.

As $\delta(X) \subseteq B_1 \cup B_2$, we know that $B \subseteq B_1 \cup B_2$. As $|B_i \cap C_h| = 1$ for each $h = 3, \ldots, \pi$, this implies that $|B \cap C_h| \leq 2$, and hence $|B \cap C_h| = 1$ (as it is odd). So we have $|B \setminus (R \cup \{e\})| \leq \pi - 2$. This contradicts (75.18). ∎

This theorem implies a characterization of strongly bipartite graphs:

Corollary 75.3a. *A signed graph G is strongly bipartite if and only if G has no odd-K_4 minor.*

Proof. Necessity follows from the fact that odd-K_4 is not strongly bipartite. To see sufficiency, let $G = (V, E, \Sigma)$ be a signed graph without odd-K_4 minor. Let $w : E \to \mathbb{Z}_+$. We must show that minimizing $w^{\mathsf{T}}x$ over (75.13) has an integer optimum dual solution.

Let G' arise from G by replacing (in E and in Σ) any edge e by $w(e)$ parallel edges. Then the minimum value of $w^{\mathsf{T}}x$ over integer vectors x satisfying (75.13) is equal to the minimum size of an odd circuit cover in G'. As G' has no odd-K_4 minor, by Theorem 75.3 this is equal to the maximum number of edge-disjoint odd circuits in G'. This gives an integer optimum dual solution to minimizing $w^{\mathsf{T}}x$ over (75.13). ∎

To interpret this characterization for (nonsigned) undirected graphs, and to get some subtleties straight, it is good to realize that for any undirected graph $G = (V, E)$ one has:

(75.22) $\qquad$ the signed graph (V, E, E) has odd-K_4 as minor $\iff$ the undirected graph (V, E) has K_4 as odd minor $\iff$ the undirected graph (V, E) contains an odd K_4-subdivision.

(Recall that an undirected graph H is an *odd minor* of an undirected graph G if H arises from G by deleting edges and vertices and contracting all edges in some cut. A subdivision of K_4 is called *odd* if each triangle of K_4 becomes an odd circuit.)

Hence:

Corollary 75.3b. *An undirected graph is strongly bipartite if and only if it contains no odd K_4-subdivision as subgraph.*

Proof. See above. ∎

For multiflows, Seymour's theorem implies (where we take $(V, E \setminus R)$ as supply graph and (V, R) as demand graph, and where $c|E \setminus R$ and $c|R$ are the capacity and demand function, respectively):

Corollary 75.3c. *Let $G = (V, E)$ be a graph and let $R \subseteq E$ be such that the signed graph (V, E, R) has no odd-K_4 minor. Then for each $c : E \to \mathbb{Z}_+$, the cut condition implies the existence of an integer multiflow.*

Proof. Let $c : E \to \mathbb{Z}_+$ satisfy the cut condition. So for each cut D we have $c(D \cap R) \leq c(D \setminus R)$. Hence for each cut D:

(75.23) $$c(D \triangle R) = c(D \setminus R) + c(R \setminus D) \geq c(D \cap R) + c(R \setminus D) = c(R).$$

So R minimizes $c(R)$ over all odd circuit covers. Therefore, as (V, E, R) has no odd-K_4 minor, by Corollary 75.3a, there exist odd circuits $C_1, \ldots, C_k$ such that each edge e is in at most $c(e)$ of the C_i and such that $k = c(R)$. Hence

(75.24) $$\sum_{i=1}^{k} |C_i \cap R| \leq c(R) = k.$$

This implies, since each $|C_i \cap R|$ is odd, that $|C_i \cap R| = 1$ for each i, and hence we have equality in (75.24). This gives the required multiflow. ∎

75.5. Characterizing weakly and evenly bipartite signed graphs

Guenin [1998a,2001a] showed that odd-K_5 is the only minor-minimal signed graph that is not weakly bipartite (unique up to resigning). It proves a special case of a hypergraph conjecture of Seymour [1977b] (cf. Section 78.3). We prove Guenin's theorem using shortenings of his proof found by Geelen and Guenin [2001] (yielding a similar characterization of evenly bipartite graphs) and Schrijver [2002a].

We use the following lemma on undirected graphs. (Recall that a K_4-subdivision is called *odd* if each triangle of K_4 has become a circuit with an odd number of edges.)

Lemma 75.4α. *Let $G = (V, E)$ be a graph, let u be a vertex of G, and let v_1, v_2, and v_3 be three of its neighbours. Let S_1, S_2, and S_3 be disjoint stable sets in G, with $v_i \in S_i$ for $i = 1, 2, 3$. Suppose that for all distinct $i, j \in \{1, 2, 3\}$, the subgraph induced by $S_i \cup S_j$ contains a $v_i - v_j$ path. Then G contains an odd K_4-subdivision containing the edges uv_1, uv_2, and uv_3.*

Proof. Consider a counterexample with $|V|+|E|$ minimal. So $V = S_1 \cup S_2 \cup S_3 \cup \{u\}$ and E consists of the edges uv_1, uv_2, and uv_3, and of the edges contained in the paths as described. Hence for distinct i, j, there is a unique path $P_{i,j}$ from v_i to v_j contained in $S_i \cup S_j$. Then

(75.25) for distinct i, j: $S_i \cup S_j = VP_{i,j}$.

For if (say) $v \in S_1 \setminus VP_{1,2}$, then v is only on $P_{1,3}$, and hence has degree 2. Then we can contract the two edges incident with v to obtain a smaller counterexample, a contradiction.

(75.25) implies $|S_1| = |S_2| = |S_3|$. If $|S_1| = 1$, $G = K_4$ and we are done. So we can assume that each $|S_i| \geq 2$. Hence each path $P_{i,j}$ has length at least 3. Let v_2' be the second vertex along $P_{1,2}$, v_3' the second vertex along $P_{2,3}$, and v_1' the second vertex along $P_{3,1}$. Contract the edges incident with u. The new vertex u' is adjacent to v_1', v_2', and v_3'. For $i = 1, 2, 3$, let $S_i' := S_i \setminus \{v_i\}$. So S_i' contains v_i' and is a stable set in the contracted graph G'. Moreover,

(75.26) for distinct i, j, $S_i' \cup S_j'$ contains a $v_i' - v_j'$ path.

To prove this, we can assume $i = 1$, $j = 2$. By (75.25), since $v_1' \in S_1$, we know that v_1' is on $P_{1,2}$. Since also v_2' is on $P_{1,2}$, $S_1 \cup S_2$ contains a $v_1' - v_2'$ path avoiding v_1 and v_2. This proves (75.26).

As G' is smaller than G, G' contains an odd K_4-subdivision containing $u'v_1'$, $u'v_2'$, and $u'v_3'$. By decontracting we obtain an odd K_4-subdivision in G as required. ∎

(The proof implies that the odd K_4-subdivision found in fact is a *bad* K_4-subdivision (cf. Section 68.4).)

This lemma is used in the characterization of Geelen and Guenin [2001] of evenly bipartite signed graphs. The following is the kernel of this characterization (a signed graph is called *Eulerian* if its underlying graph is Eulerian):

Theorem 75.4. *In an Eulerian signed graph without odd-K_5 minor, the maximum number of edge-disjoint odd circuits is equal to the minimum size of an odd circuit cover.*

Proof. For any signed graph $G = (V, E, \Sigma)$, let $\pi(G)$ denote the minimum size of an odd circuit cover and let $\mu(G)$ denote the maximum number of edge-disjoint odd circuits. It suffices to show $\mu(G) = \pi(G)$ for any Eulerian signed graph G without odd-K_5 minor.

Suppose that this is not true. Choose a counterexample $G = (V, E, \Sigma)$, with $\pi(G)$ minimum, $|V|$ minimum, and $|E|$ maximum, in this order of priority. Such a graph exists, since if there are more than $\pi(G)$ parallel edges connecting two vertices, we can contract them to obtain a counterexample with $|V|$ smaller.

Fix an edge $e = xy$ not contained in every minimum-size odd circuit cover. By adding two parallel edges connecting x and y, we do not change $\pi(G)$ or $|V|$, but we increase $|E|$. Hence in the extended graph there exist $\pi(G)$ edge-disjoint odd circuits. This means that in the original graph G

(75.27) there exist odd circuits $C_1, \ldots, C_{\pi(G)}$ with $e \in C_1 \cap C_2 \cap C_3$ and with $C_1 \setminus \{e\}$, $C_2 \setminus \{e\}$, C_3, $C_4, \ldots, C_{\pi(G)}$ disjoint

(describing circuits by edge sets). $G, C_1, \ldots, C_{\pi(G)}$ moreover satisfy:

(75.28) $\pi(G - C) \leq \pi(G) - 3$ for each odd circuit $C \subseteq C_1 \cup C_2 \cup C_3$ such that $((C_1 \cup C_2 \cup C_3) \setminus C) \cup \{e\}$ contains an odd circuit.

Otherwise, by the minimality of $\pi(G)$, $G - C$ contains disjoint odd circuits $C'_1, \ldots, C'_{\pi(G)-2}$. Then $E' := E \setminus (C \cup C'_1 \cup \cdots \cup C'_{\pi(G)-2})$ contains an odd circuit C'', since E is Eulerian and since G has a minimum-size odd circuit cover B of size $\pi(G)$; so, as B is an equivalent signing of G, $|E' \cap B|$ is odd. Hence $C, C'', C'_1, \ldots, C'_{\pi(G)-2}$ form $\pi(G)$ disjoint odd circuits in G, contradicting our assumption. This proves (75.28).

We show that for signed Eulerian graphs G, conditions (75.27) and (75.28) imply that G has an odd-K_5 minor, which finishes the proof.

We delete our earlier minimality assumptions, and now choose a counterexample to this with $|E|$ minimal and (secondly) $|C_1 \cup C_2 \cup C_3|$ minimal. Let P_i be the $x - y$ path $C_i \setminus \{e\}$ for $i = 1, 2, 3$ (describing paths by edge sets). Then:

Claim 1. P_1, P_2, P_3 are internally vertex-disjoint.

Proof of Claim 1. Suppose not. Define $F := P_1 \cup P_2 \cup P_3$. We first show:

(75.29) F contains no odd circuit.

To see this, first observe that any $P_i \cup P_j$ contains no odd circuit, since otherwise, for the third path P_k there exist $\pi(G) - 2$ disjoint odd circuits in $G - (P_k \cup \{e\})$, contradicting (75.28).

Hence there exists an inclusionwise minimal odd circuit cover B disjoint from $P_1 \cup P_2$. Then for each vertex v in VP_3 that is also in $VP_1 \cup VP_2$, the $x - v$ part of P_3 has an even number of edges in B (as it forms with part of P_1 or P_2 an even cycle). Hence between two contacts of P_3 with $VP_1 \cup VP_2$, P_3 has an even number of edges in B. This implies (75.29).

Orient the edges in $C_1 \cup C_2 \cup C_3$ by orienting each P_i from x to y, and by orienting edge e from y to x. Then

(75.30) F contains no directed circuit C, for otherwise $F \setminus C$ contains three edge-disjoint $x-y$ paths. They yield odd circuits C_1', C_2', C_3' avoiding C, with $e \in C_1' \cap C_2' \cap C_3'$ and with $C_1' \setminus \{e\}$, $C_2' \setminus \{e\}$, $C_3' \setminus \{e\}$ disjoint. This contradicts the minimality of $C_1 \cup C_2 \cup C_3$.

So F is acyclic, and hence there exists a total order $\leq$ on V with $s < t$ for each arc (s,t) in F. So all vertices v in $VP_1 \cup VP_2 \cup VP_3$ have $x \leq v \leq y$.

Then for each undirected $x-y$ path P in F:

(75.31) P is a directed path $\iff$ $(C_1 \cup C_2 \cup C_3) \setminus P$ contains an odd circuit.

To prove $\Rightarrow$, let P be a directed path. Then there exists a directed $x-y$ path edge-disjoint from Q. Hence $Q \cup \{e\}$ is an odd circuit disjoint from $(C_1 \cup C_2 \cup C_2) \setminus P$.

To prove $\Leftarrow$, let C be an odd circuit in $(C_1 \cup C_2 \cup C_3) \setminus P$. Then $C \setminus \{e\}$ is an $x-y$ path Q edge-disjoint from P. If P is not directed, there is a vertex v such that P traverses two arcs entering v. Now there exist precisely three arcs (s,t) with $s < v \leq t$. Hence P contains all three, and nothing is left for Q, a contradiction.

This proves (75.31). So any circuit C qualifies for (75.28) if and only if it is a directed circuit.

Let W be the set of vertices that are in at least two of the P_i. Since P_1, P_2, and P_3 are not internally vertex-disjoint by assumption, we know $|W| \geq 3$.

Call a directed path in F a *link* if it connects two distinct vertices in W, while each internal vertex is not in W. Then:

(75.32) there exist vertices u, v and a $u-v$ link Q such that $u \neq x$ and such that there is at least one directed $u-v$ path edge-disjoint from Q and such that each directed $x-u$ path is a link.

To see this, first observe that there is a directed $x-y$ path P traversing all vertices in W. Indeed, for all $s, t \in W$ with $s < t$, there is a directed $s-t$ path. This follows from the fact that at least two of the P_i leave s and at least two of the P_i enter t, and that hence at least one of the P_i leaves s and enters t.

Now to prove (75.32), let u be the smallest vertex in W with $u \neq x$. Then each directed $x-u$ path is a link. Let Q be a link leaving u which is not on P. Taking for v the end vertex of Q, we obtain (75.32).

Let X be the set of edges that are on directed $u-v$ paths $\neq Q$. We may assume that if C_i intersects X, then C_i traverses both u and v. So X consists of one or two $u-v$ paths. Then

(75.33) $(C_1 \cup C_2 \cup C_3) \setminus (Q \cup X)$ contains no arc leaving u or no arc entering v.

Otherwise, by (75.32), F has three arcs leaving u and three arcs entering v. So each C_i contains a $u-v$ path, which hence is in $Q \cup X$. This proves (75.33).

Consider $G' := G/Q$ and $C_i' := C_i \setminus (Q \cup X)$ for $i = 1, 2, 3$. Then (75.27) is maintained for G', C_1', C_2', C_3'. Hence, by the minimality of $C_1 \cup C_2 \cup C_3$, there is a directed circuit C' in G' with

(75.34) $$\pi(G' - C') \geq \pi(G') - 2.$$

Now $\pi(G') \geq \pi(G)$ (as this is true for any contraction of G). If C' is also a directed circuit in G, we have $\pi(G - C') \leq \pi(G) - 3$ by (75.28), and hence $G - C'$ has an odd circuit cover B of size $\leq \pi(G) - 3$. By (75.27), B does not intersect Q. Hence B is an odd circuit cover of $G - C'/Q = G' - C'$ of size $\leq \pi(G) - 3$, a contradiction.

So C' is not a directed circuit in G. Then $C' \cup Q$ forms a circuit in G, and, by (75.33), it is a directed circuit. Hence C' contains a link R entering u. As $u \in W$, there is another link, S say, entering u.

Consider $G'' := (G - R)/S$ and $C_i'' := C_i \setminus (R \cup S)$ for $i = 1, 2, 3$. Then (75.27) is maintained. Moreover, $\pi(G'') \geq \pi(G)$. For suppose that G'' has an odd circuit cover B of size $\pi(G'') \leq \pi(G) - 1$. By (75.27), $|B| = \pi(G) - 2$ (since it intersects each C_i in an odd number of edges), B does not intersect Q, and contains e. Hence $\pi((G - (R \cup \{e\}))/Q) \leq \pi(G) - 3$. This implies (since $R \cup \{e\} \subseteq C'$):

(75.35) $$\pi(G' - C') \leq \pi(G' - (R \cup \{e\})) = \pi(G - (R \cup \{e\})/Q) \leq \pi(G) - 3,$$

contradicting (75.34). This proves that $\pi(G'') \geq \pi(G)$.

Now, by the minimality of G, C_1, C_2, C_3, (75.28) is not maintained. So there is a directed circuit C'' in G'' with $\pi(G'' - C'') \geq \pi(G) - 2$. Then $C'' \cup S$ contains an odd circuit of G, hence also $C'' \cup R$ contains an odd circuit of G (since R and S are parallel links). So (by (75.28) for G) $\pi(G - (C'' \cup R)) \leq \pi(G) - 3$. Hence $G - (C'' \cup R)$ has an odd circuit cover B of size $\pi(G) - 3$, which by (75.27) is disjoint from $F \cup \{e\}$. Then B is an odd circuit cover of $G - (C'' \cup R)/S = G'' - C''$, and so $\pi(G'' - C'') \leq \pi(G) - 3$, contradicting our assumption. *End of Proof of Claim 1*

Set $\pi := \pi(G)$. Since by (75.28), for each $i = 1, 2, 3$, $\pi(G - C_i) \leq \pi - 3$, there is an inclusionwise minimal odd circuit cover B_i of G with $|B_i \setminus C_i| \leq \pi - 3$. By (75.27), we know that $B_i \cap P_j = \emptyset$ for $j \leq 3$ with $j \neq i$, and that $|B_i \cap C_j| = 1$ for $j \geq 4$. Since B_i intersects each of C_1, C_2, C_3, we have $e \in B_i$.

By (75.10), there exist $U_1, U_2, U_3 \subseteq V$ such that

(75.36) $$B_j \triangle B_k = \delta(U_i)$$

for distinct $i, j, k \in \{1, 2, 3\}$. We can assume that each U_i is disjoint from e, since $e \notin B_j \triangle B_k$ (as $e \in B_j \cap B_k$). Moreover, we can assume that $U_3 = U_1 \triangle U_2$ — otherwise, just reset $U_3 := U_1 \triangle U_2$. (This works, since $\delta(U_1 \triangle U_2) = \delta(U_1) \triangle \delta(U_2) = (B_2 \triangle B_3) \triangle (B_1 \triangle B_3) = B_1 \triangle B_2$.)

Define

(75.37) $$S_i := U_j \cap U_k$$

for distinct $i, j, k \in \{1, 2, 3\}$. So S_1, S_2, S_3 are disjoint and

(75.38) $U_i = S_j \cup S_k$

for distinct $i, j, k \in \{1, 2, 3\}$ (since $U_1 \triangle U_2 \triangle U_3 = \emptyset$). Define

(75.39) $S_0 := V \setminus (S_1 \cup S_2 \cup S_3)$.

Then

(75.40) each $f \in E \setminus (B_1 \cup B_2 \cup B_3)$ is spanned by S_0, S_1, S_2, or S_3.

Otherwise, f belongs to some $\delta(U_i)$, and hence to some B_j, by (75.36).

Moreover,

(75.41) $VP_i \subseteq S_0 \cup S_i$

for each $i \in \{1, 2, 3\}$, since $VP_i \cap \delta(U_i) = \emptyset$ by (75.36) and since $x, y \notin U_i$.

We in fact have for each $i \in \{1, 2, 3\}$:

(75.42) $C_i \subseteq B_i$.

For suppose that $f \in C_1 \setminus B_1$. Then G/f again satisfies (75.27) and (75.28), for $C'_1 := C_1 \setminus \{f\}$, $C'_2 := C_2$, $C'_3 := C_3$. Indeed, each odd circuit C of G/f contained in $C'_1 \cup C'_2 \cup C'_3$ is equal to one of the C'_i, and moreover

(75.43) $\pi((G/f) - C'_i) \leq |B_i \setminus C_i| \leq \pi(G) - 3 \leq \pi(G/f) - 3$.

This contradicts the minimality of $|E|$. So we have (75.42).

Similarly,

(75.44) each $f \in E \setminus (B_1 \cup B_2 \cup B_3)$ is spanned by $VP_1 \cup VP_2 \cup VP_3$.

Otherwise, we can contract f to obtain a smaller example satisfying (75.27) and (75.28) (by (75.43) for $C'_i := C_i$).

Now let E' be the set of edges in $B_1 \triangle B_2 \triangle B_3$ that are in $C_1 \cup C_2 \cup C_3$ or connect two distinct sets among S_1, S_2, S_3. So $C_1 \cup C_2 \cup C_3 \subseteq E'$. As $E' \subseteq B_1 \triangle B_2 \triangle B_3$ and as $B_1 \triangle B_2 \triangle B_3$ is a signing equivalent to Σ, it suffices to show that the undirected graph $G' = (V, E')$ has K_5 as odd minor.

By definition of E', for each $i \in \{1, 2, 3\}$:

(75.45) S_i is a stable set of G'.

Moreover, for all distinct $i, j \in \{1, 2, 3\}$,

(75.46) G' has a path contained in $S_i \cup S_j$ and connecting VP_i and VP_j.

To see this, we may assume $i = 1$, $j = 2$. Suppose that no such path exists. Then U_3 $(= S_1 \cup S_2)$ has a subset X such that $S_1 \cap VP_1 \subseteq X$ and $X \cap VP_2 = \emptyset$ and such that no edge of G' connects X and $U_3 \setminus X$. So

(75.47) $\delta_{E'}(X) \subseteq \delta(U_3)$.

Then

(75.48) $\delta_E(X) \subseteq B_1 \cup B_2 \cup B_3$.

Indeed, let $f \in \delta_E(X) \setminus (B_1 \cup B_2 \cup B_3)$. By (75.44), f is spanned by $VP_1 \cup VP_2 \cup VP_3$. Moreover, by (75.40), as f is incident with X, f is spanned by S_1 or S_2. So f is spanned by $S_1 \cap VP_1$ or by $S_2 \cap VP_2$, contradicting the fact that f leaves X. This proves (75.48).

Define

(75.49) $\qquad B := B_1 \triangle \delta_E(X).$

Then B is an odd circuit cover of G. So $|B| \geq \pi$. Since $S_1 \cap VP_1 \subseteq X$ and $P_1 \subseteq B_1$, we know that $P_1 \subseteq \delta(X)$, and so B is disjoint from P_1. For $i = 2, 3$, B is disjoint from P_i, as $\delta(X)$ contains no edge of P_i, since X is disjoint from VP_i. Hence, as $B \subseteq B_1 \cup B_2 \cup B_3$ by (75.48), we know that $|B \cap C_j| \geq 2$ for some $j = 4, \ldots, \pi$. Then $|B \cap C_j| \geq 3$. As $|B_1 \cap C_j| = 1$ and $|B_2 \cap C_j| = 1$, it follows that there exists an edge $f \in B \cap C_j$ with $f \notin B_1 \cup B_2$. So $f \in \delta_E(X)$, hence $f \in B_3$, therefore $f \in \delta(U_1) \cap \delta(U_2)$, and so $f \in E'$. Therefore, $f \in \delta_{E'}(X)$, and hence by (75.47), $f \in \delta(U_3)$, contradicting (75.36). This proves (75.46).

Consider the minor H of G' obtained by contracting, for each $i = 1, 2, 3$, $VC_i \setminus \{x, y\}$ to one vertex, z_i say. By Lemma 75.4α, $H - y$ has an odd K_4-subdivision containing the edges xz_1, xz_2, and xz_3. Since y is adjacent to x, z_1, z_2, and z_3, H has K_5 as odd minor. ∎

A consequence of this is a characterization of weakly and evenly bipartite graphs. (The equivalence of (i) and (iii) is Guenin's theorem (Guenin [1998a,2001a]), and the equivalence with (ii) was found by Geelen and Guenin [2001].)

Corollary 75.4a. *For any signed graph G the following are equivalent:*

(75.50) (i) *G is weakly bipartite;*
(ii) *G is evenly bipartite;*
(iii) *G has no odd-K_5 minor.*

Proof. The implications (ii)⇒(i)⇒(iii) follow from (75.14) and from the facts that weak bipartiteness is closed under taking minors and that odd-K_5 is not weakly bipartite.

The implication (iii)⇒(ii) follows from Theorem 75.4. Let $G = (V, E, \Sigma)$ be a signed graph without odd-K_5 minor and let $c : E \to \mathbb{Z}_+$ be such that $c(\delta(v))$ is even for each $v \in V$. We must show that the dual of minimizing $c^{\mathsf{T}}x$ over (75.13) has an integer optimum dual solution.

Let G' arise from G by replacing (in E and in Σ) any edge e by $c(e)$ parallel edges. So G' is Eulerian. Then the minimum value of $c^{\mathsf{T}}x$ over integer vectors x satisfying (75.13) is equal to the minimum size of an odd circuit cover in G'. As G' has no odd-K_5 minor, by Theorem 75.4 this is equal to the maximum number of edge-disjoint odd circuits in G'. This gives an integer optimum dual solution to minimizing $c^{\mathsf{T}}x$ over (75.13). ∎

For (nonsigned) undirected graphs, this characterization can be described in terms of odd minors as follows. (Recall that an undirected graph H is an *odd minor* of an undirected graph G if H arises from G by deleting edges and vertices and contracting all edges in some cut.) Then for any undirected graph $G = (V, E)$:

(75.51) the signed graph (V, E, E) has odd-K_5 as minor $\iff$ the undirected graph (V, E) has K_5 as odd minor.

(This is a simple exercise.) Hence:

Corollary 75.4b. *An undirected graph G is weakly bipartite if and only if K_5 is not an odd minor of G.*

Proof. See above. ∎

Notes. Special cases of the equivalence of (i) and (iii) in Corollary 75.4a were shown by Barahona [1980] (for planar graphs; cf. Theorem 75.2), Fonlupt, Mahjoub, and Uhry [1992] (for graphs without K_5 minor), Barahona [1983a] (for graphs G such that $G - u - v$ is bipartite for two of its vertices u, v), and Gerards [1992a] (for graphs G such that $G - v$ is planar with at most two odd faces, for some vertex v).

75.6. Applications to multiflows

Geelen and Guenin's theorems also have consequences for multiflows (where again we take $(V, E \setminus R)$ as supply graph and (V, R) as demand graph, and where $c|E \setminus R$ and $c|R$ are the capacity and demand function, respectively):

Corollary 75.4c. *Let $G = (V, E)$ be a graph and let $R \subseteq E$ be such that the signed graph (V, E, R) has no odd-K_5 minor. Then for each $c : E \to \mathbb{R}_+$, the cut condition implies the existence of a fractional multiflow. If moreover c is integer, there is a half-integer multiflow. If moreover the Euler condition holds, there is an integer multiflow.*

Proof. By Corollary 75.4a, (V, E, R) is weakly bipartite. Let c satisfy the cut condition. So for each cut D we have $c(D \cap R) \leq c(D \setminus R)$. Hence for each cut D:

(75.52) $$c(D \triangle R) = c(D \setminus R) + c(R \setminus D) \geq c(D \cap R) + c(R \setminus D) = c(R).$$

So R minimizes $c(R)$ over all odd circuit covers. Therefore, as (V, E, R) is weakly bipartite, there exist odd circuits $C_1, \ldots, C_k$ and $\lambda_1, \ldots, \lambda_k > 0$ with

(75.53) $$\sum_{i=1}^{k} \lambda_i \chi^{C_i} \leq c \text{ and } \sum_{i=1}^{k} \lambda_i = c(R).$$

Hence

(75.54) $$\sum_{i=1}^{k} \lambda_i |C_i \cap R| \leq c(R) = \sum_{i=1}^{k} \lambda_i.$$

This implies, since each $|C_i \cap R|$ is odd, that $|C_i \cap R| = 1$ for each i, and that we have equality in (75.54). This gives the required multiflow.

The integrality results follow from Theorem 75.4. ∎

This implies the following generalization of Theorem 74.3, due to Seymour [1981a] (who derived it from Theorem 74.3 by using Wagner's theorem on the decomposition of K_5-free graphs (Theorem 3.3)):

Corollary 75.4d. *A graph $G = (V, E)$ has no K_5 minor if and only if for each $R \subseteq E$ and each $c : E \to \mathbb{R}_+$, the cut condition implies the existence of a multiflow. Moreover, if G has no K_5 minor and c is integer, the cut condition implies the existence of a half-integer multiflow. If moreover the Euler condition holds, then it implies the existence of an integer multiflow.*

Proof. Directly from Corollary 75.4c. ∎

For planar graphs, these integrality results can be derived also from results on packing T-cuts (Theorem 29.2), using duality like in Theorem 75.2 (cf. Theorem 74.3).

75.7. The cut cone and the cut polytope

Let $G = (V, E)$ be an undirected graph. Recall that a subset C of E is called a *cut* if $C = \delta(U)$ for some $U \subseteq V$. The *cut polytope* $P_{\text{cut}}(G)$ of G is the convex hull of the incidence vectors (in $\mathbb{R}^E$) of cuts in G:

(75.55) $$P_{\text{cut}}(G) := \text{conv.hull}\{\chi^C \mid C \text{ cut in } G\}.$$

As $\emptyset$ is a cut, the cut polytope contains the origin.

Since a set of edges is bipartite if and only if it is contained in a cut, the bipartite subgraph polytope can be expressed in terms of the cut polytope:

(75.56) $$P_{\text{bipartite subgraph}}(G) = \{x \in \mathbb{R}_+^E \mid \exists y \geq x : y \in P_{\text{cut}}(G)\}.$$

Any vector x in the cut polytope of G satisfies

(75.57)
(i) $0 \leq x_e \leq 1$ for each $e \in E$,
(ii) $x(F) - x(C \setminus F) \leq |F| - 1$ for each circuit C and $F \subseteq C$ with $|F|$ odd.

A full characterization is known of those graphs for which (75.57) determines the cut polytope: they are the graphs without K_5 minor (Seymour [1981a], Barahona [1983b]). This can be deduced from the characterization of weakly bipartite graphs.

This characterization can be formulated equivalently in terms of the *cut cone* of a graph $G = (V, E)$, which is the convex cone generated by the incidence vectors of the cuts. Necessary conditions for its elements are:

$$(75.58) \qquad \begin{array}{ll} x_e \geq 0 & \text{for } e \in E, \\ x_f \leq x(C \setminus \{f\}) & \text{for each circuit } C \text{ and } f \in C. \end{array}$$

The graph K_5 shows that these conditions generally are not sufficient: fix distinct $u, v \in VK_5$; then $x := \mathbf{2} - \chi^{\delta(\{u,v\})}$ satisfies (75.58). However, x does not belong to the cut cone of K_5, since the incidence vector z of any cut satisfies $2z(\delta(\{u,v\})) \geq z(EK_5)$.

K_5 is the only minor-minimal example, as Seymour [1981a] showed:

Corollary 75.4e. *The cut cone is determined by* (75.58) *if and only if* G *has no* K_5 *minor.*

Proof. Necessity is shown by the example above, and by the closedness of the property under taking minors. If G has the property, and we contract an edge e, then any x satisfying (75.58) for G/e can be extended to a vector x satisfying (75.58) for G by defining $x_e := 0$. Then the extended x is in the cut cone of G, and hence the original x is in the cut cone of G/e.

If we delete e, extend x satisfying (75.58) for $G - e$ by defining x_e to be the distance in $G - e$ between the end vertices of e, taking x as length function. Again, the extended x is in the cut cone of G, and hence the original x is in the cut cone of $G - e$. This shows necessity.

To see sufficiency, let G have no K_5 minor. Let $c^{\mathsf{T}}x \geq 0$ be a valid inequality for the cut cone. Define $R := \{e \in E \mid c(e) < 0\}$. Taking $(V, E \setminus R)$ as supply graph and (V, R) as demand graph, the cut condition holds for capacity function $c|E \setminus R$ and demand function $-c|R$. By Corollary 75.4d, there exists a multiflow subject to $c|E \setminus R$ and of value $-c|R$. It means that c is a nonnegative combination of vectors $-\chi^f + \chi^{C\setminus\{f\}}$ where C is a circuit and $f \in C$, and of vectors χ^e where $e \in E \setminus R$. Hence the inequality $c^{\mathsf{T}}x \geq 0$ is a nonnegative linear combination of the inequalities (75.58). ∎

Using the symmetry of the cut polytope (as observed by Barahona and Grötschel [1986]), Corollary 75.4e has as a consequence (Barahona [1983b]):

Corollary 75.4f. *The cut polytope of a graph* $G = (V, E)$ *is determined by* (75.57) *if and only if* G *has no* K_5 *minor.*

Proof. By Corollary 75.4e, it suffices to show that the cut polytope is determined by (75.57) if and only if the cut cone is determined by (75.58).

First assume that the cut polytope is determined by (75.57). Since the origin belongs to the cut polytope, the cut cone is determined by the inequalities among (75.57) with right-hand side 0 — that is, by (75.58).

Conversely, assume that the cut cone is determined by (75.58). Then (75.58) determines the cut polytope in the neighbourhood of the origin.

Consider now any vertex χ^D of the cut polytope, where D is a cut of G. For each $x \in \mathbb{R}^E$, define $\tilde{x} \in \mathbb{R}^E$ by:

(75.59) $$\tilde{x}_e := \begin{cases} 1 - x_e & \text{if } e \in D, \\ x_e & \text{if } e \notin D. \end{cases}$$

The function $x \to \tilde{x}$ brings the cut polytope to itself (since $D' \triangle D$ is a cut for any cut D' and $\tilde{x} = \chi^{D' \triangle D}$ if $x = \chi^{D'}$), and χ^D to $\mathbf{0}$. Since the cut cone is determined by (75.58), it implies that in the neighbourhood of χ^D, the cut polytope is determined by the inequalities (75.58) applied to $\tilde{x}$:

(75.60) $$\begin{array}{lll} \text{(i)} & \tilde{x}_e \geq 0 & \text{for } e \in E, \\ \text{(ii)} & \tilde{x}_f \leq \tilde{x}(C \setminus \{f\}) & \text{for each circuit } C \text{ and } f \in C. \end{array}$$

Now inequality (75.60)(i) follows from (75.57)(i). To see inequality (75.60)(ii), we first consider the case $f \notin D$. Define $F := (C \cap D) \cup \{f\}$. Then, using (75.57)(ii), (75.60)(ii) follows from

(75.61) $$\begin{aligned} \tilde{x}_f = x_f = x(F) - x(C \cap D) &\leq x(C \setminus F) - x(C \cap D) + |F| - 1 \\ &= \tilde{x}(C \setminus F) + \tilde{x}(C \cap D) = \tilde{x}(C \setminus \{f\}). \end{aligned}$$

If $f \in D$, define $F := (C \cap D) \setminus \{f\}$. Then, again using (75.57)(ii), (75.60)(ii) follows from

(75.62) $$\begin{aligned} \tilde{x}_f = 1 - x_f = 1 + x(F) - x(C \cap D) &\leq x(C \setminus F) - x(C \cap D) + |F| \\ &= x(C \setminus D) + x_f - x(C \cap D) + |C \cap D| - 1 = \tilde{x}(C \setminus \{f\}). \end{aligned}$$ ∎

Notes. By Corollary 75.4f, the cut polytope of any planar graph is determined by (75.57). As cuts in planar graphs correspond to $\emptyset$-joins ($\equiv$ cycles) in the dual graph (Orlova and Dorfman [1972]), one may also derive this from Corollary 29.2e on the T-join polytope.

Hadlock [1975] showed in a similar way that a maximum-capacity cut in a planar graph can be found in strongly polynomial time. Using the decomposition of graphs without K_5 minors into planar graphs and copies of V_8 (Theorem 3.3), Barahona [1983b] derived from this a combinatorial strongly polynomial-time algorithm to find a maximum-capacity cut in graphs without K_5 minor.

Poljak [1992] showed that for each graph G, the polytope determined by (75.2) is the down hull of the polytope determined by (75.57).

Karzanov [1985b] showed that the separation problem for the cut cone is co-NP-complete.

Barahona and Mahjoub [1986] showed that the separation problem over (75.57) is polynomial-time solvable, hence any linear objective function can be optimized over (75.57) in strongly polynomial time (with the ellipsoid method).

Integer decomposition. What about integer decomposition in the cut cone? A theorem of Chvátal [1980] implies that it is NP-complete to decompose a given metric as a nonnegative integer sum of incidence vectors of cuts. Let $\mathcal{H}$ be the class of graphs such that each integer vector x in the cut cone with $x(C)$ even for each circuit C, is a nonnegative integer combination of incidence vectors of cuts. (Equivalently, the incidence vectors of the cuts form a Hilbert base.)

By (74.34), each planar graph belongs to $\mathcal{H}$. This was extended (using Wagner's theorem (Theorem 3.3)) by Fu and Goddyn [1999] who showed that each graph without K_5 minor belongs to $\mathcal{H}$.

Goddyn [1993] also conjectured that each graph not having the Petersen graph as minor, belongs to $\mathcal{H}$. However, Laurent [1996b] showed that K_6 does not belong to $\mathcal{H}$. She also showed that all proper subgraphs of K_6 belong to $\mathcal{H}$. (More on this can be found in Laburthe [1995] and in the survey by Goddyn [1993].)

Fu and Goddyn [1999] asked: is $\mathcal{H}$ closed under taking minors?

Metrics and hypermetrics. The following *metric inequalities* are valid for the vectors in the cut cone of the complete graph K_V on a vertex set V:

(75.63)
(i) $x_{uv} \geq 0$ for distinct $u, v \in V$,
(ii) $x_{uv} + x_{vw} \geq x_{uw}$ for distinct $u, v, w \in V$.

The cone determined by these inequalities is called the *metric cone*.

Tylkin [1960,1962] (= M.M. Deza) introduced a stronger set of valid inequalities, the *hypermetric inequalities*:

(75.64)
(i) $x_{uv} \geq 0$ for distinct $u, v \in V$,
(ii) $\displaystyle\sum_{\substack{u,v \in V\\ u \neq v}} c_u c_v x_{uv} \leq 0$ for each $c : V \to \mathbb{Z}$ with $c(V) = 1$.

These inequalities are valid for the vectors in the cut cone, since for each cut $\delta(U)$ one has (setting $x := \chi^{\delta(U)}$):

(75.65)
$$\begin{aligned}\sum_{\substack{u,v \in V\\ u \neq v}} c_u c_v x_{uv} &= 2\sum_{u\in U}\sum_{v \in V\setminus U} c_u c_v = 2c(U)c(V \setminus U)\\ &= \tfrac{1}{2}(c(U) + c(V \setminus U))^2 - \tfrac{1}{2}(c(U) - c(V \setminus U))^2\\ &= \tfrac{1}{2} - \tfrac{1}{2}(c(U) - c(V \setminus U))^2 \leq 0,\end{aligned}$$

since $|c(U) - c(V \setminus U)| \geq 1$, as $c(V) = 1$ and c is integer.

Hypermetric inequalities generalize the metric inequalities, since (75.63)(ii) is equivalent to taking $c := \chi^u + \chi^w - \chi^v$ in (75.64)(ii).

The cone determined by (75.64) is called the *hypermetric cone*. Deza, Grishukhin, and Laurent [1993] showed that this cone is polyhedral (despite that there are infinitely many inequalities in (75.64)(ii)).

Avis and Grishukhin [1993] showed that it is co-NP-complete to decide if a given vector is in the hypermetric cone. Relations with the geometry of numbers are given in Deza, Grishukhin, and Laurent [1995]. More on the metric cone can be found in Avis [1980b,1980c], Grishukhin [1992], Laurent and Poljak [1992,1995b], Lomonosov and Sebő [1993], and Laurent [1996a], and on metrics and hypermetrics in the book by Deza and Laurent [1997].

75.8. The maximum cut problem and semidefinite programming

The maximum-capacity cut problem has a natural semidefinite relaxation. Let V be a finite set and denote

(75.66) $\mathcal{M}_V :=$ the set of all symmetric positive semidefinite $V \times V$ matrices M with $M_{v,v} = 1$ for each $v \in V$.

Let $c : V \times V \to \mathbb{R}_+$ be a 'capacity' function, with $c(u,v) = c(v,u)$ for all $u, v \in V$. Consider c as a capacity function on the complete graph K_V on V. Let C be the $V \times V$ matrix with (u,v)th entry equal to $c(u,v)$. The maximum-capacity cut problem asks for the maximum of

$$(75.67) \qquad \sum_{u \in U} \sum_{v \in V \setminus U} c(u,v).$$

Theorem 75.1 implies that this is an NP-complete problem.

A relaxation is to maximize

$$(75.68) \qquad \tfrac{1}{4}\mathrm{Tr}C(J - M)$$

over $M \in \mathcal{M}_V$. If we restrict M to matrices of rank 1 (so $M = xx^{\mathsf{T}}$ for some $\{-1,+1\}$ vector x in $\mathbb{R}^V$), we have the maximum-capacity cut problem.

Goemans and Williamson [1994,1995b] showed the following surprising bound (surprising also since the proof is very simple). Define

$$(75.69) \qquad \alpha := \min_{0 < \phi \leq \pi} \frac{\phi}{1 - \cos\phi} \frac{2}{\pi} = 0.87856....$$

(The latter estimate results from a numerical computation.)

Theorem 75.5. *Let μ be the maximum capacity of a cut and let ν be the maximum value of* (75.68). *Then*

$$(75.70) \qquad \alpha\nu \leq \mu \leq \nu.$$

Proof. The inequality $\mu \leq \nu$ was shown above. To see the first inequality, let M maximize (75.68). As M is positive semidefinite, there exist vectors $x_v \in \mathbb{R}^n$ for $v \in V$ such that $x_u^{\mathsf{T}} x_v = M_{u,v}$ for all $u, v \in V$. (Here $n := |V|$.) So $\|x_v\| = 1$ for each $v \in V$.

For any hyperplane H in $\mathbb{R}^n$ with $\mathbf{0} \in H$, let D be the set of edges uv of K_V with u and v at different sides of H. Choosing H at random, the set D is a cut, with probability 1. Any edge uv of K_V is in D with probability

$$(75.71) \qquad \frac{\angle(x_u, x_v)}{\pi}.$$

($\angle(a,b)$ is the angle of a and b.) This follows from the fact that (75.71) is the probability that x_u and x_v are at different sides of H.

So the expected value of the capacity of D is equal to

$$(75.72) \qquad \sum_{uv \in EK_V} \frac{\angle(x_u, x_v)}{\pi} c(u,v).$$

Now if $\phi = \angle(x_u, x_v)$, then $x_u^{\mathsf{T}} x_v = \cos\phi$. Hence, by definition of α, (75.73) $$\frac{\angle(x_u, x_v)}{\pi} = \frac{\phi}{\pi} \geq \tfrac{1}{2}\alpha(1 - \cos\phi) = \tfrac{1}{2}\alpha(1 - x_u^\mathsf{T} x_v).$$

Hence (75.72) is at least

(75.74) $$\sum_{uv \in EK_V} \alpha \cdot \tfrac{1}{2} c(u, v)(1 - x_u^\mathsf{T} x_v) = \tfrac{1}{4}\alpha \mathrm{Tr} C(J - M) = \alpha\nu.$$

Concluding, there exists a cut of capacity at least $\alpha\nu$. So $\mu \geq \alpha\nu$. ∎

Since the separation problem over $\mathcal{M}_V$ is solvable in polynomial time, in a certain approximation model (cf. Grötschel, Lovász, and Schrijver [1988]), with the ellipsoid method, one can optimize any linear objective function over $\mathcal{M}_V$ in strongly polynomial time, or rather approximate the optimum. Hence the value of ν can be approximated in polynomial time. As Goemans and Williamson [1994,1995b] pointed out, this gives a randomized polynomial-time algorithm to find a cut of capacity at least $\alpha\nu \geq 0.87856\nu$: choosing a random hyperplane H as above gives a random cut of expected capacity as required. By derandomization, such a cut can in fact be found deterministically in polynomial-time (Mahajan and Ramesh [1995,1999]).

This approach also gives a *relaxation* ($\equiv$ superset) of the cut polytope. Indeed, let $G = (V, E)$ be an undirected graph. For any $M \in \mathcal{M}_V$, define $x_M : E \to \mathbb{R}_+$ by:

(75.75) $$x_M(e) := \tfrac{1}{2}(1 - M_{u,v})$$

for $e = uv \in E$. Then

(75.76) $$P_{\mathrm{cut}}(G) \subseteq K := \{x_M \mid M \in \mathcal{M}_V\},$$

since for each cut $\delta(U)$, the matrix M given by

(75.77) $$M := (\mathbf{1} - 2\chi^U)(\mathbf{1} - 2\chi^U)^\mathsf{T}$$

belongs to $\mathcal{M}_V$ and satisfies $x_M = \chi^{\delta(U)}$.

So K is a relaxation of the cut polytope. With the ellipsoid method, one can optimize over $\mathcal{M}_V$, and hence over K in polynomial time. What Goemans and Williamson's theorem tells is that for nonnegative $c : E \to \mathbb{R}_+$, maximizing $c^\mathsf{T} x$ over K has only a small relative error compared to maximizing over $P_{\mathrm{cut}}(G)$. In other words:

(75.78) $$K \subseteq \alpha^{-1} \cdot P_{\text{bipartite subgraph}}(G).$$

Here we use that $P_{\text{bipartite subgraph}}(G)$ is the down hull in $\mathbb{R}_+^E$ of $P_{\mathrm{cut}}(G)$.

Feige and Schechtman [2001,2002] showed that for each $\varepsilon > 0$ there is a graph for which the ratio of the semidefinite programming bound ν and the maximum cut-size is no better than $\alpha + \varepsilon$.

Notes. Before Goemans and Williamson found their theorem, only a factor of 2 was known to be achievable in polynomial time, by just taking $c(E)$ as upper bound. This gives a factor 2, since a random cut has expected capacity $\frac{1}{2}c(EK_V)$, as each edge has probability $\frac{1}{2}$ to be in the random cut (Johnson and Lafuente [1970] and Sahni and Gonzalez [1976]).

Håstad [1997,2001] showed that if NP≠P, then there is no polynomial-time algorithm that finds a cut of capacity more than $\frac{16}{17}$ of the maximum cut-capacity (cf. Trevisan, Sorkin, Sudan, and Williamson [1996,2000]).

Related work can be found in Bellare, Goldreich, and Sudan [1995,1998], Karloff [1996,1999], Zwick [1999b], Alon and Sudakov [2000], and Alon, Sudakov, and Zwick [2001,2002].

Earlier eigenvalue methods for the maximum cut problem include Poljak [1992] and Delorme and Poljak [1993a,1993b,1993c].

Surveys on semidefinite methods for the maximum cut problem (and more generally in combinatorial optimization) are given by Goemans [1997], Reed [2001a], and Laurent and Rendl [2002]. Alizadeh [1995] gives a survey of applying interior-point methods to semidefinite programming in combinatorial optimization. More on the semidefinite relaxation of the cut polytope can be found in Laurent and Poljak [1995a,1996a,1996b]. Other approximation algorithms for the maximum cut problem were given by Arora, Karger, and Karpinski [1995,1999], Fernandez de la Vega [1996], Frieze and Kannan [1996,1999], Fernandez de la Vega and Kenyon [1998,2001], Feige and Langberg [2001], and Halperin, Livnat, and Zwick [2002].

An extension of the semidefinite programming bound to 3-cuts was given by Goemans and Williamson [2001]. For extensions to directed graphs, see Feige and Goemans [1995], Matuura and Matsui [2001], and Lewin, Livnat, and Zwick [2002].

For a survey on approximation algorithms, see Shmoys [1995] and the book by Vazirani [2001].

75.9. Further results and notes

75.9a. Cuts and stable sets

The vertex cover polytope of a graph $G = (V, E)$ can be considered as a face of the cut polytope of the graph $\widetilde{G} = (\widetilde{V}, \widetilde{E})$ obtained from G by adding one new vertex u adjacent to all vertices of G. Since the stable set polytope can be expressed in terms of the vertex cover polytope (as S is a stable set if and only if $V \setminus S$ is a vertex cover), this gives a relation between cuts and stable sets.

To see the relation between $P_{\text{vertex cover}}(G)$ and $P_{\text{cut}}(\widetilde{G})$, first note that each $x \in P_{\text{cut}}(\widetilde{G})$ satisfies

(75.79) $\qquad x(T) \leq 2$ for each triangle $T \subseteq \widetilde{E}$.

Therefore, the set of vectors x in $P_{\text{cut}}(\widetilde{G})$ satisfying

(75.80) $\qquad x(T) = 2$ for each triangle $T = \{uv, uw, vw\}$ containing u

(so vw is an edge of G), forms a face F of $P_{\text{cut}}(\widetilde{G})$.

Now we project $\mathbb{R}^{\widetilde{E}}$ on $\mathbb{R}^{\widetilde{E}\setminus E}$ by deleting the coordinates indexed by E. Moreover, we identify any edge uv in $\widetilde{E} \setminus E$ with vertex v of G. This brings F one-to-one to the vertex cover polytope of G.

More precisely, define a projection $\pi : \mathbb{R}^{\widetilde{E}} \to \mathbb{R}^V$ by $\pi(x)_v := x_{uv}$ for $v \in V$ and $x \in \mathbb{R}^{\widetilde{E}}$. Then:

Theorem 75.6. *$\pi|F$ is a bijection between F and $P_{\text{vertex cover}}(G)$.*

Proof. First, $\pi|F$ is injective, since if $\pi(x) = \pi(y)$ for $x, y \in F$, then for each $v \in V$, $x_{uv} = y_{uv}$. Hence, by (75.80), for each $vw \in E$, $x_{vw} = 2 - x_{uv} - x_{uw} = 2 - y_{uv} - y_{uw} = y_{vw}$. So $x = y$.

To see that $\pi(F) \subseteq P_{\text{vertex cover}}(G)$, let C be a cut in $\widetilde{G}$ with $\chi^C \in F$ (that is, χ^C is a vertex of F). Then for each edge vw of G, precisely two of the edges uv, uw, vw belong to C. Hence at least one of uv, uw belongs to C. So $\pi(\chi^C)$ is the incidence vector of a vertex cover of G.

Conversely, to see $P_{\text{vertex cover}}(G) \subseteq \pi(F)$, let U be a vertex cover of G. So $U \subseteq V$. Let C be the cut in $\widetilde{G}$ determined by U. Then χ^C belongs to F, since for each edge vw of G we have that precisely two of uv, uw, vw belong to C. Moreover, $\pi(\chi^C) = \chi^U$, since for each $v \in V$: $v \in U \iff uv \in C$. ∎

The relation given in this theorem can be useful when we have a good description of the cut polytope for certain classes of graphs. The description then can be transferred to the vertex cover polytope, hence to the stable set polytope, for certain derived classes of graphs.

In particular, we can derive from Guenin's theorem the t-perfection of graphs without odd K_4-subdivision (a consequence of Theorem 68.3 (Gerards and Schrijver [1986])):

Theorem 75.7. *A graph G without odd K_4 subdivision as subgraph is t-perfect.*

Proof. Let $G = (V, E)$ be a graph without odd K_4-subdivision as subgraph. By (75.22), G has no K_4 as odd minor. Let $\widetilde{G} = (\widetilde{V}, \widetilde{E})$ be the graph obtained from G by adding a new vertex u, adjacent to all vertices in V.

Then $\widetilde{G}$ has no K_5 as odd minor. For suppose it has. Then by deleting the vertex from the K_5 to which u has been contracted (if any) we obtain a graph being K_4 or K_5. It implies that G has K_4 as odd minor, a contradiction.

Now let $y \in \mathbb{R}^V$ satisfy

$$(75.81)\qquad \begin{array}{ll} 0 \le y_v \le 1 & \text{for } v \in V, \\ y_u + y_v \le 1 & \text{for } uv \in E, \\ y(VC) \le \lfloor \frac{1}{2}|VC| \rfloor & \text{for each odd circuit } C. \end{array}$$

Define $x \in \mathbb{R}^{\widetilde{E}}$ by

$$(75.82)\qquad \begin{array}{ll} x(vw) := y_v + y_w & \text{for each edge } vw \text{ of } G, \\ x(uv) := 1 - y_v & \text{for each } v \in V. \end{array}$$

Then x satisfies (75.2) with respect to $\widetilde{G}$. Indeed, (75.2)(i) trivially holds. To see (ii), we can restrict ourselves to chordless odd circuits C. If C traverses u, it is a triangle containing u, and we have $x(EC) = 2 = |VC| - 1$. If C does not traverse u, then $x(EC) = 2y(VC) \le |VC| - 1$.

So by Corollary 75.4a, x is a convex combination of incidence vectors of bipartite subgraphs B:

$$(75.83)\qquad x = \sum_B \lambda_B \chi^B.$$

Since $x(C) = 2$ for each triangle C containing u, those B with $\lambda_B > 0$ intersect each such C in precisely two edges. Hence (since each circuit in $\widetilde{G}$ is a symmetric difference of triangles containing u) B intersects each circuit in an even number of edges. So B is a cut $\delta(U)$ of $\widetilde{G}$. We can assume that $u \notin U$. Then $V \setminus U$ is a stable set of G, and

$$y = \sum_U \lambda_{\delta(U)} \chi^{V \setminus U} \tag{75.84}$$

describes y as a convex combination of incidence vectors of stable sets. ∎

It should be noted that the face F of the cut polytope described above is at the same time a face of the (larger) bipartite subgraph polytope (while the cut polytope need not be a face of the bipartite subgraph polytope). Indeed, also the bipartite subgraph polytope satisfies (75.79). Moreover, any set B of edges having even intersection with the triangle $\{uv, uw, vw\}$ for each edge vw of G, has even intersection with each circuit of $\widetilde{G}$, as it is a binary sum of such triangles. So B is a cut.

Laurent, Poljak, and Rendl [1997] showed how the set $\mathrm{TH}(G)$ (defined in Section 67.4a) can be derived as an affine image from the convex body K in Section 75.8.

75.9b. Further notes

Chvátal, Cook, and Hartmann [1989] showed that the Chvátal rank of system (75.2) is at least $\frac{1}{4}|V| - 1$ for complete graphs $G = (V, E)$.

Conforti and Gerards [2000] described (by forbidden odd minors) another class of Eulerian graphs for which the maximum number of edge-disjoint odd circuits is equal to the minimum size of an odd circuit cover.

Barahona [1983b] showed that the maximum-size cut problem is NP-complete for *apex graphs*, that is graphs G having a vertex v with $G - v$ planar. (More strongly, Barahona proved NP-completeness if $G - v$ is cubic and planar.)

Grötschel and Nemhauser [1984] showed that for each fixed k there is a polynomial-time algorithm to solve the maximum-capacity cut problem for graphs without odd circuits of length $\geq k$.

Facets of the bipartite subgraph polytope were studied by Barahona, Grötschel, and Mahjoub [1985] (cf. Gerards [1985]), and facets of the cut polytope and cut cone by Barahona and Mahjoub [1986], De Simone [1989,1990], Deza and Laurent [1990, 1992a,1992b], Deza, Laurent, and Poljak [1992], and Laurent and Poljak [1996a]. Compositions in the bipartite subgraph polytope were given by Fonlupt, Mahjoub, and Uhry [1992].

Conforti and Rao [1987] showed that a minimum-weight odd circuit cover can be found in strongly polynomial time, if its weight is less than the minimum weight of a nonempty cut.

For more geometric background on the cut cone and the cut polytope, see the book by Deza and Laurent [1997]. Gerards [1990] gave a survey on signed graphs without odd K_4-subdivision. For more background on the relations between odd circuits and multicommodity flows, see Sebő [1990a] and Gerards [1993]. For surveys on maximum cut and the cut cone, see Deza, Grishukhin, and Laurent [1995] (hypermetrics) and Poljak and Tuza [1995]. For related work, see Conforti, Rao, and

Sassano [1990a,1990b], Jerrum and Sorkin [1993,1998], Feige and Goemans [1995], Frieze and Jerrum [1995,1997], Ageev and Sviridenko [1999], Ageev, Hassin, and Sviridenko [2001], Feige and Langberg [2001], Halperin and Zwick [2001a,2001b, 2002], Ye [2001], Han, Ye, and Zhang [2002], and Lewin, Livnat, and Zwick [2002].

Chapter 76

Homotopy and graphs on surfaces

As we saw in Chapter 74, disjoint paths and multiflow problems are generally hard even for planar graphs. In some special cases, these problems are polynomial-time solvable.
If we require the paths (or flows) to have certain homotopies, the range of polynomial-time solvable problems can be extended. By enumerating homotopies, it sometimes implies polynomial-time solvability for nonhomotopic versions of the problems.
This can be extended to general surfaces and yield polyhedral characterizations for circulations, flows, and paths of prescribed homotopies.

76.1. Graphs, curves, and their intersections: terminology and notation

Let S be a compact surface. A *closed curve* on S is a continuous function $C : S^1 \to S$, where S^1 is the unit circle in $\mathbb{C}$. It is *simple* if C is one-to-one.

Two closed curves C and C' are called *freely homotopic*, in notation $C \sim C'$, if there exists a continuous function bringing C to C'; that is, a continuous function $\Phi : S^1 \times [0, 1] \to S$ with $\Phi(z, 0) = C(z)$ and $\Phi(z, 1) = C'(z)$ for each $z \in S^1$.

For any pair of closed curves C, D on S, let $\mathrm{cr}(C, D)$ denote the number of intersections of C and D, counting multiplicities:

(76.1) $$\mathrm{cr}(C, D) := |\{(w, z) \in S^1 \times S^1 \mid C(w) = D(z)\}|.$$

Moreover, $\mathrm{mincr}(C, D)$ denotes the minimum of $\mathrm{cr}(C', D')$ where C' and D' range over closed curves freely homotopic to C and D, respectively:

(76.2) $$\mathrm{mincr}(C, D) := \min\{\mathrm{cr}(C', D') \mid C' \sim C, D' \sim D\}.$$

Similarly, $\mathrm{cr}(C)$ denotes the number of self-intersections of C:

(76.3) $$\mathrm{cr}(C) := \tfrac{1}{2}|\{(w, z) \in S^1 \times S^1 \mid C(w) = C(z), w \neq z\}|,$$

and $\mathrm{mincr}(C)$ denotes the minimum of $\mathrm{cr}(C')$ where C' ranges over all closed curves freely homotopic to C:

(76.4) $\qquad \operatorname{mincr}(C) := \min\{\operatorname{cr}(C') \mid C' \sim C\}.$

As is well-known, $\operatorname{mincr}(C, D)$ and $\operatorname{mincr}(C)$ are finite numbers.

Let $G = (V, E)$ be an undirected graph embedded in S. We identify G with its topological graph, and with its embedding in S.

For any closed curve D on S, $\operatorname{cr}(G, D)$ denotes the number of intersections of G and D (counting multiplicities):

(76.5) $\qquad \operatorname{cr}(G, D) := |\{z \in S^1 \mid D(z) \in G\}|.$

Moreover, $\operatorname{mincr}(G, D)$ denotes the minimum of $\operatorname{cr}(G, D')$ where D' ranges over all closed curves freely homotopic to D and not intersecting V:

(76.6) $\qquad \operatorname{mincr}(G, D) := \min\{\operatorname{cr}(G, D') \mid D'$ is a closed curve in $S \setminus V$ freely homotopic to $D\}$.

(It would seem more consistent with definition (76.2) if we would also allow to shift G over S so as to obtain G' and minimize $\operatorname{cr}(G', D')$, where G' is possibly not one-to-one mapped in S. However, Theorem 76.1 below implies that this would not change the minimum value.)

We say that a closed curve C is *in* a graph G if $C : S^1 \to G$.

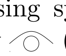
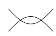
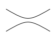
76.2. Making curves minimally crossing by Reidemeister moves

The proof of Theorem 76.1 below is based on the following result of de Graaf and Schrijver [1997b]. Let $C_1, \ldots, C_k$ be closed curves on S. Call $C_1, \ldots, C_k$ *minimally crossing* if

(76.7) $\qquad$ (i) $\operatorname{cr}(C_i) = \operatorname{mincr}(C_i)$ for each $i = 1, \ldots, k$;
(ii) $\operatorname{cr}(C_i, C_j) = \operatorname{mincr}(C_i, C_j)$ for all $i, j = 1, \ldots, k$ with $i \neq j$.

Call $C_1, \ldots, C_k$ *regular* if $C_1, \ldots, C_k$ have only a finite number of (self-)intersections, each being a crossing of only two curve parts. (That is, each point of S traversed twice by the $C_1, \ldots, C_k$ has a disk-neighbourhood on which the curve parts are topologically homeomorphic to two crossing straight lines.)

In de Graaf and Schrijver [1997b] the following was shown:

(76.8) $\qquad$ Any regular system of closed curves on a compact surface S can be transformed to a minimally crossing system by a series of *Reidemeister moves*: replacing  by  (*type 0*); replacing  by (*type I*); replacing  by  (*type II*); replacing  by  (*type III*).

The pictures in (76.8) represent the intersection of the union of $C_1, \ldots, C_k$ with a closed disk on S — no other curve parts than those shown intersect this disk.

It is important to note that in (76.8) we do not allow to apply the operations in the reverse direction — otherwise the result would follow quite straightforwardly with the techniques of simplicial approximation, and would not be powerful enough for our purposes. The Reidemeister moves given in (76.8) do not increase the number of intersections.

76.3. Decomposing the edges of an Eulerian graph on a surface

We first show a homotopic analogue of the theorems in previous chapters relating distances and cut packings. It will be used to derive that the cut condition is sufficient for the existence of a fractional packing of circuits of prescribed homotopies in a graph on a surface (analogous to the line of proof developed in Section 70.12).

A graph is called *Eulerian* if each vertex has even degree. (We do not assume connectedness of the graph.) Moreover, *decomposing* the edges into closed curves $C_1, \ldots, C_k$ means that $C_1, \ldots, C_k$ are closed curves in G such that each edge is traversed by exactly one C_i, and by that C_i exactly once.

We now give the theorem, due to de Graaf and Schrijver [1997a], which was proved for the projective plane by Lins [1981] (Corollary 74.1b above) and for compact orientable surfaces by Schrijver [1991a].

Theorem 76.1. *Let $G = (V, E)$ be an Eulerian graph embedded in a compact surface S. Then the edges of G can be decomposed into closed curves $C_1, \ldots, C_k$ such that*

$$\text{(76.9)} \qquad \operatorname{mincr}(G, D) = \sum_{i=1}^{k} \operatorname{mincr}(C_i, D)$$

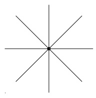
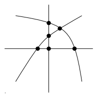
for each closed curve D on S.

Proof. First note that the inequality $\geq$ in (76.9) trivially holds, for *any* decomposition of the edges into closed curves $C_1, \ldots, C_k$: by definition of $\operatorname{mincr}(G, D)$, there exists a closed curve $D' \sim D$ in $S \setminus V$ with $\operatorname{mincr}(G, D) = \operatorname{cr}(G, D')$, and hence

$$\text{(76.10)} \qquad \operatorname{mincr}(G, D) = \operatorname{cr}(G, D') = \sum_{i=1}^{k} \operatorname{cr}(C_i, D') \geq \sum_{i=1}^{k} \operatorname{mincr}(C_i, D).$$

The content of the theorem is that there exists a decomposition attaining equality.

To prove this, we may assume that each vertex v of G has degree at most 4. If v would have a degree larger than 4, we can replace G in a neighbourhood of v like

 by  .

This modification does not change the value of $\mathrm{mincr}(G, D)$ for any D. Moreover, closed curves decomposing the edges of the modified graph satisfying (76.9), directly yield closed curves decomposing the edges of the original graph satisfying (76.9).

For any graph G embedded in S with each vertex having degree 2 or 4, we define the *straight decomposition* of G as the regular system of closed curves $C_1, \ldots, C_k$ such that $G = C_1 \cup \cdots \cup C_k$. So each vertex of G of degree 4 represents a (self-)crossing of $C_1, \ldots, C_k$.

Up to some trivial operations, such a decomposition is unique, and conversely, it uniquely describes G. Moreover, any Reidemeister move applied to $C_1, \ldots, C_k$ carries over a modification of G. So we can speak of Reidemeister moves applied to G. Then straightforwardly:

(76.11) if G' arises from G by one Reidemeister move of type III, then $\mathrm{mincr}(G', D) = \mathrm{mincr}(G, D)$ for each closed curve D.

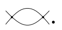
Call any graph $G = (V, E)$ that is a counterexample to the theorem such that each vertex has degree at most 4 and such that it has a minimum number of faces, a *minimal counterexample*. (A *face* is a connected component of $S \setminus G$.)

From (76.11) it directly follows that:

(76.12) if G' arises from a minimal counterexample G by one Reidemeister move of type III, then G' is a minimal counterexample again.

Moreover:

(76.13) if G is a minimal counterexample, then no Reidemeister move of type 0, I or II can be applied to G.

For suppose that a Reidemeister move of type II can be applied to G. Then G contains the following subconfiguration: . Replacing this by  would give a smaller counterexample (since the function $\mathrm{mincr}(G, D)$ does not change by this operation), contradicting the minimality of G. One similarly sees that no Reidemeister move of type I can be applied. No Reidemeister move of type 0 can be applied, as otherwise we can delete the circuit to obtain a smaller counterexample. This proves (76.13).

The proof now is finished by showing that the straight decomposition $C_1, \ldots, C_k$ of any minimal counterexample G satisfies (76.9) — contradicting the fact that G is a counterexample.

Choose a closed curve D. We may assume that $D, C_1, \ldots, C_k$ form a regular system. By (76.8) we can apply Reidemeister moves so as to obtain a minimally crossing system $D', C_1', \ldots, C_k'$. Let G' be the graph formed by $C_1', \ldots, C_k'$.

By (76.12) and (76.13) we did not apply Reidemeister moves of type 0, I or II to $C_1, \ldots, C_k$. Hence, by (76.11), $\text{mincr}(G', D) = \text{mincr}(G, D)$. So

$$\begin{aligned} \text{(76.14)} \qquad \text{mincr}(G, D) &= \text{mincr}(G', D) = \text{mincr}(G', D') \leq \text{cr}(G', D') \\ &= \sum_{i=1}^{k} \text{cr}(C'_i, D') = \sum_{i=1}^{k} \text{mincr}(C'_i, D') = \sum_{i=1}^{k} \text{mincr}(C_i, D). \end{aligned}$$

Since the converse inequality holds by (76.10), we have (76.9). ∎

The theorem can be sharpened to include compact surfaces with holes, just by replacing holes by handles.

76.4. A corollary on lengths of closed curves

Using surface duality we derive the following consequence of Theorem 76.1 (Schrijver [1991a], de Graaf and Schrijver [1997a]). If G is a graph embedded in a surface S and C is a closed curve in G, then $\text{minlength}_G(C)$ denotes the minimum length of any closed curve $C' \sim C$ in G. Here the *length* $\text{length}_G(C')$ of C' is the number of edges traversed by C', counting multiplicities. So

$$\text{(76.15)} \qquad \text{minlength}_G(C) = \min\{\text{length}_G(C') \mid C' \sim C, C' \text{ in } G\}.$$

Corollary 76.1a. *Let $G = (V, E)$ be a bipartite graph embedded in a compact surface S and let $C_1, \ldots, C_k$ be closed curves in G. Then there exist closed curves $D_1, \ldots, D_t$ in $S \setminus V$ such that each edge of G is crossed by exactly one D_j and by this D_j only once, and such that*

$$\text{(76.16)} \qquad \text{minlength}_G(C_i) = \sum_{j=1}^{t} \text{mincr}(C_i, D_j)$$

for each $i = 1, \ldots, k$.

Proof. Let

$$\text{(76.17)} \qquad d := \max\{\text{minlength}_G(C_i) \mid i = 1, \ldots, k\}.$$

We can extend G to a bipartite graph L embedded in S, such that each face of L is an open disk. By inserting d new vertices on each edge of L not occurring in G, we obtain a bipartite graph H satisfying $\text{minlength}_H(C_i) = \text{minlength}_G(C_i)$ for each $i = 1, \ldots, k$ (since the new edges cannot be used to obtain a closed curve shorter than $\text{minlength}_G(C_i)$).

Consider a surface dual graph H^* of H. Then for each $i = 1, \ldots, k$,

$$\text{(76.18)} \qquad \text{mincr}(H^*, C_i) = \text{minlength}_H(C_i) = \text{minlength}_G(C_i).$$

Since H is bipartite, H^* is Eulerian. Hence by Theorem 76.1, the edges of H^* can be decomposed into closed curves $D_1, \ldots, D_t$ such that

(76.19) $$\operatorname{mincr}(H^*, C) = \sum_{j=1}^{t} \operatorname{mincr}(D_j, C)$$

for each closed curve C. With (76.18), this gives (76.16). ∎

Notes. This proof also implies that we can replace $C_1, \ldots, C_k$ by the set of *all* closed curves on S if G is *cellularly embedded* (i.e., each face is an open disk) — in that case we need not extend G to L and H.

It is not difficult to see that this also holds for not-cellularly embedded bipartite graphs in the torus, since then there is essentially only one closed curve C in G to consider.

This is not true for the double torus (a surface with two handles), as is shown by the example of Figure 76.1 (from Schrijver [1991a]).

76.5. A homotopic circulation theorem

By linear programming duality (Farkas' lemma) we derive from Corollary 76.1a the following 'homotopic circulation theorem' — a fractional packing theorem for closed curves of given homotopies in a graph on a compact surface.

Let $G = (V, E)$ be a graph embedded in a compact surface S. For any closed curve C in G define the vector tr^C in $\mathbb{Z}_+^E$ by:

(76.20) $$\operatorname{tr}^C(e) := \text{number of times } C \text{ traverses } e,$$

for $e \in E$.

Let C_0 be a closed curve on S. Call a function $f : E \to \mathbb{R}$ a *circulation freely homotopic to* C_0 (of *value* 1) if f is a convex combination of functions tr^C, where the C are closed curves in G freely homotopic to C_0.

Corollary 76.1b (homotopic circulation theorem)**.** *Let $G = (V, E)$ be an undirected graph embedded in a compact surface S and let $C_1, \ldots, C_k$ be closed curves on S. Then there exist circulations $f_1, \ldots, f_k$ freely homotopic to $C_1, \ldots, C_k$ respectively, such that*

(76.21) $$f_1(e) + \cdots + f_k(e) \le 1$$

for each edge e, if and only if

(76.22) $$\operatorname{cr}(G, D) \ge \sum_{i=1}^{k} \operatorname{mincr}(C_i, D)$$

for each closed curve D in $S \setminus V$.

Proof. *Necessity.* First note that if f is a circulation freely homotopic to a closed curve C_0, then

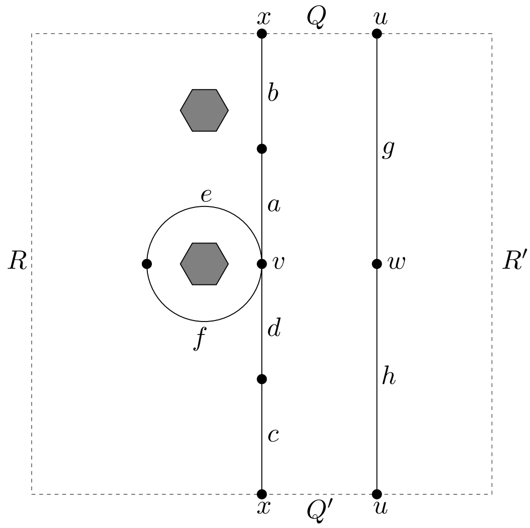


Figure 76.1

A not-cellularly embedded bipartite graph G in the double torus S for which Corollary 76.1a is not true if we replace $C_1, \ldots, C_k$ by all closed curves on S. The double torus is obtained from the square by identifying R and R' and identifying Q and Q' (so as to obtain the torus) and next deleting the interiors of the two hexagons and identifying their boundaries (so as to obtain the double torus).

For $i = 0, 1, 2, \ldots$ let C_i be the closed curve in G which, starting at v, follows e and f once, and next follows i times the closed curve a, b, c, d. Then $\text{minlength}_G(C_i) = 4i+2$. Suppose now that $D_1, \ldots, D_t$ are closed curves as described in Corollary 76.1a. Choose an arbitrary curve P from v to w. Then C_i is homotopic to the closed curve $\widetilde{C}_i$ obtained by, starting at v, first following e and f, next following P, then following i times the closed curve g, h, and finally following P back from w to v. Hence for each i (where B is the closed curve from w to w following g and h):

$$
\begin{aligned}
&4i + 2 = \text{minlength}_G(C_i) = \sum_{j=1}^{t} \text{mincr}(C_i, D_j) \\
&\leq \sum_{j=1}^{t} \text{cr}(\widetilde{C}_i, D_j) = \sum_{j=1}^{t} (\text{cr}(C_0, D_j) + 2 \cdot \text{cr}(P, D_j) + i \cdot \text{cr}(B, D_j)) \\
&= \sum_{j=1}^{t} (\text{cr}(C_0, D_j) + 2 \cdot \text{cr}(P, D_j)) + 2i.
\end{aligned}
$$

As the first term in the last sum is independent of i, this is a contradiction.

(76.23) $$\sum_{e\in E} f(e)\mathrm{cr}(e,D) \geq \mathrm{mincr}(C_0,D).$$

for each closed curve D in $S\setminus V$ (denoting by $\mathrm{cr}(e,D)$ the number of times D intersects edge e). This follows from the fact that (76.23) holds for $f := \mathrm{tr}^C$ for each C freely homotopic to C_0 as

(76.24) $$\sum_{e\in E} \mathrm{tr}^C(e)\mathrm{cr}(e,D) = \mathrm{cr}(C,D) \geq \mathrm{mincr}(C_0,D),$$

and hence also for any convex combination of such functions.

Suppose now that there exist circulations $f_1,\ldots,f_k$ as required. Let D be a closed curve in $S\setminus V$. Then, using (76.23):

(76.25) $$\begin{aligned}\mathrm{cr}(G,D) &= \sum_{e\in E}\mathrm{cr}(e,D) \geq \sum_{e\in E}\mathrm{cr}(e,D)\sum_{i=1}^{k} f_i(e)\\ &= \sum_{i=1}^{k}\sum_{e\in E} f_i(e)\mathrm{cr}(e,D) \geq \sum_{i=1}^{k}\mathrm{mincr}(C_i,D).\end{aligned}$$

Sufficiency. Suppose that (76.22) is satisfied for each closed curve D in $S\setminus V$. Let $I := \{1,\ldots,k\}$ and let K be the convex cone in $\mathbb{R}^I\times\mathbb{R}^E$ generated by the vectors[23]

(76.26) $$\begin{array}{ll}(\chi^i;\mathrm{tr}^C) & (i\in I; C \text{ closed curve in } G \text{ with } C\sim C_i),\\ (\mathbf{0}_I;\chi^e) & (e\in E).\end{array}$$

Here χ^i denotes the ith unit base vector in $\mathbb{R}^I$ and χ^e denotes the eth unit base vector in $\mathbb{R}^E$. Moreover, $\mathbf{0}_I$ denotes the all-zero vector in $\mathbb{R}^I$.

Although generally there are infinitely many vectors (76.26), K is finitely generated. This can be seen by observing that, for each $i\in I$, we can restrict the vectors $(\chi^i;\mathrm{tr}^C)$ in the first line of (76.26) to those that are minimal with respect to the usual partial order $\leq$ on $\mathbb{Z}_+^I\times\mathbb{Z}_+^E$ (with $(x;y)\leq(x';y') \iff x_i\leq x'_i$ for all $i\in I$ and $y_e\leq y'_e$ for all $e\in E$). They form an 'antichain' in $\mathbb{Z}_+^I\times\mathbb{Z}_+^E$ (i.e., a set of pairwise incomparable vectors). Since each antichain in $\mathbb{Z}_+^I\times\mathbb{Z}_+^E$ is finite, K is finitely generated.

We must show that the vector $(\mathbf{1}_I;\mathbf{1}_E)$ belongs to K. Here $\mathbf{1}_I$ and $\mathbf{1}_E$ denote the all-one vectors in $\mathbb{R}^I$ and $\mathbb{R}^E$, respectively. By Farkas' lemma, it suffices to show that each vector $(d;l)\in\mathbb{Q}^I\times\mathbb{Q}^E$ having nonnegative inner product with each of the vectors (76.26), also has nonnegative inner product with $(\mathbf{1}_I;\mathbf{1}_E)$. Thus let $(d;l)\in\mathbb{Q}^I\times\mathbb{Q}^E$ have nonnegative inner product with each vector among (76.26). This is equivalent to:

(76.27) $$\begin{array}{lll}\text{(i)} & d_i + \sum_{e\in E} l(e)\mathrm{tr}^C(e)\geq 0 & (i\in I; C \text{ closed curve in } G \text{ with } C\sim C_i),\\ \text{(ii)} & l(e)\geq 0 & (e\in E).\end{array}$$

[23] We write $(x;y)$ for $\begin{pmatrix}x\\y\end{pmatrix}$.

Suppose now that $(d;l)^{\mathsf{T}}(\mathbf{1}_I;\mathbf{1}_E) < 0$. By increasing l slightly, we may assume that $l(e) > 0$ for each $e \in E$. Next, by multiplying $(d;l)$ appropriately, we may assume that each entry in $(d;l)$ is an even integer.

Let G' be the graph arising from G by replacing each edge e of G by a path of length $l(e)$. That is, we insert $l(e) - 1$ new vertices on e. Then by (76.27)(i),

$$(76.28) \qquad -d_i \leq \text{minlength}_{G'}(C_i)$$

for each $i \in I$. Since G' is bipartite, by Corollary 76.1a there exist closed curves $D_1, \ldots, D_t$ intersecting no vertex of G' such that each edge of G' is intersected by exactly one D_j and only once by that D_j and such that

$$(76.29) \qquad \text{minlength}_{G'}(C_i) = \sum_{j=1}^{t} \text{mincr}(C_i, D_j)$$

for each $i \in I$. So

$$(76.30) \qquad l(e) = \sum_{j=1}^{t} \text{cr}(e, D_j)$$

for each edge e of G. Hence (76.22), (76.28) and (76.29) give

$$(76.31) \qquad \begin{aligned} \sum_{e\in E} l(e) &= \sum_{j=1}^{t}\sum_{e\in E} \text{cr}(e, D_j) = \sum_{j=1}^{t} \text{cr}(G, D_j) \\ &\geq \sum_{j=1}^{t}\sum_{i=1}^{k} \text{mincr}(C_i, D_j) = \sum_{i=1}^{k}\sum_{j=1}^{t} \text{mincr}(C_i, D_j) \\ &= \sum_{i=1}^{k} \text{minlength}_{G'}(C_i) \geq -\sum_{i=1}^{k} d_i. \end{aligned}$$

So $(d;l)^{\mathsf{T}}(\mathbf{1}_I;\mathbf{1}_E) \geq 0$. ∎

This corollary has an equivalent capacitated version. Let C_0 be a closed curve on S. Call a function $f : E \to \mathbb{R}$ a *circulation freely homotopic to* C_0 of *value* d if f is a nonnegative linear combination of functions tr^C, where the C are closed curves in G freely homotopic to C_0 and where the scalars add up to d.

Corollary 76.1c. *Let $G = (V,E)$ be an undirected graph embedded in a compact surface S and let $C_1, \ldots, C_k$ be closed curves on S. Let $c : E \to \mathbb{R}_+$ and $d_1, \ldots, d_k \in \mathbb{R}_+$. Then there exist circulations $f_1, \ldots, f_k$ freely homotopic to $C_1, \ldots, C_k$ respectively and of values $d_1, \ldots, d_k$ respectively, such that*

$$(76.32) \qquad \sum_{i=1}^{k} f_i(e) \leq c(e)$$

for each edge e if and only if

$$(76.33) \qquad \sum_{e\in E} c(e)\mathrm{cr}(e,D) \geq \sum_{i=1}^{k} d_i \cdot \mathrm{mincr}(C_i, D)$$

for each closed curve D in $S \setminus V$.

Proof. Using the argument on the finite generation of the convex cone K in the proof of Corollary 76.1b, we can assume that c and the d_i are rational, and hence integer. Replace each edge e of G by $c(e)$ parallel edges, and replace any C_i by d_i copies of C_i. Then the present corollary follows from Corollary 76.1b. ∎

Notes. Frank and Schrijver [1992] showed that if S is the torus and each C_i is a simple closed curve, then there exist half-integer circulations in Corollary 76.1b — that is, where the scalars of the tr^C are $\frac{1}{2}$ (similarly, in Corollary 76.1c if c and the d_i are integer). More generally, it is shown that there are integer circulations if the following Euler condition holds:

(76.34) for each closed curve D in $S \setminus V$, the number of crossings of D with G has the same parity as the number of crossings with $C_1, \ldots, C_k$.

This condition in particular implies that each vertex of G has even degree. This result can be formulated equivalently as:

(76.35) Let $G = (V, E)$ be a graph embedded in the torus S and let $C_1, \ldots, C_k$ be simple closed curves on S such that the Euler condition (76.34) holds. Then G has edge-disjoint closed walks $C'_1, \ldots, C'_k$ (each traversing no edge more than once) with C'_i freely homotopic to C_i for $i = 1, \ldots, k$, if and only if condition (76.22) holds.

The C'_i need not be simple; they may have self-intersections at vertices. (See Schrijver [1992] for a survey on disjoint circuits in graphs on the torus.)

Figures 76.2 and 76.3 show that we cannot delete in (76.35) the Euler condition or the condition that the C_i are simple. Moreover, Figure 76.4 shows that (76.35) does not extend to the double torus (a surface with two handles).

76.6. Homotopic paths in planar graphs with holes

As was shown in Schrijver [1991a], Corollary 76.1b gives a ‘homotopic flow-cut theorem’, stating that a homotopic cut condition implies the existence of a fractional solution for the planar edge-disjoint paths problem, if the paths have prescribed homotopies in the surface obtained from the plane by deleting the interiors of certain faces covering all terminals. (This answers a question of C.St.J.A. Nash-Williams.)

Before formulating the result, we introduce some notation and terminology. Fix some subset T of $\mathbb{R}^2$. A *curve* in T is a continuous function $D : [0, 1] \to T$. The points $D(0)$ and $D(1)$ are the *end points* of D.

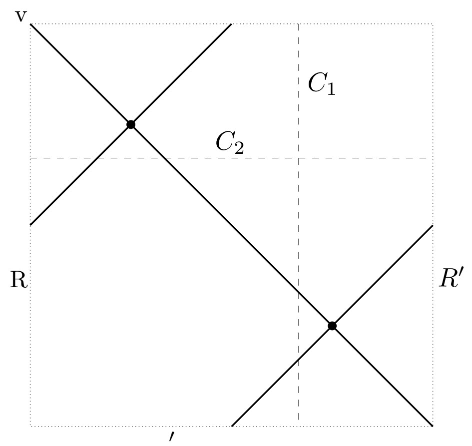


Figure 76.2
A graph G and curves C_1, C_2 on the torus satisfying the cut condition (76.22) (but not the Euler condition (76.34)), where G has no edge-disjoint circuits C_1' and C_2' with C_i' freely homotopic to C_i ($i = 1, 2$).

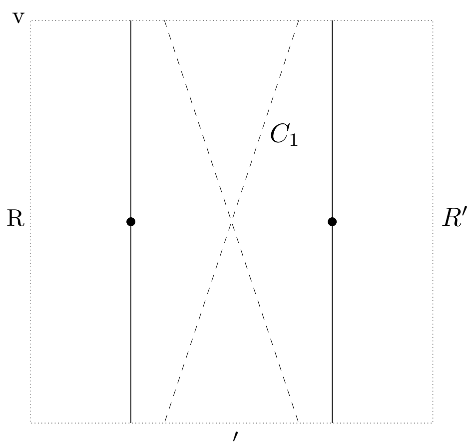


Figure 76.3
A graph G and a *nonsimple* curve C_1 on the torus satisfying the cut condition (76.22) and the Euler condition (76.34), where G has no closed curve C_1' freely homotopic to C_1 such that C_1' traverses any edge of G at most once.

Two curves D, D' are called *homotopic* (*in* T), denoted by $D \sim D'$, if there exists a continuous function $\Phi : [0,1] \times [0,1] \to T$ with $\Phi(x, 0) = D(x)$,

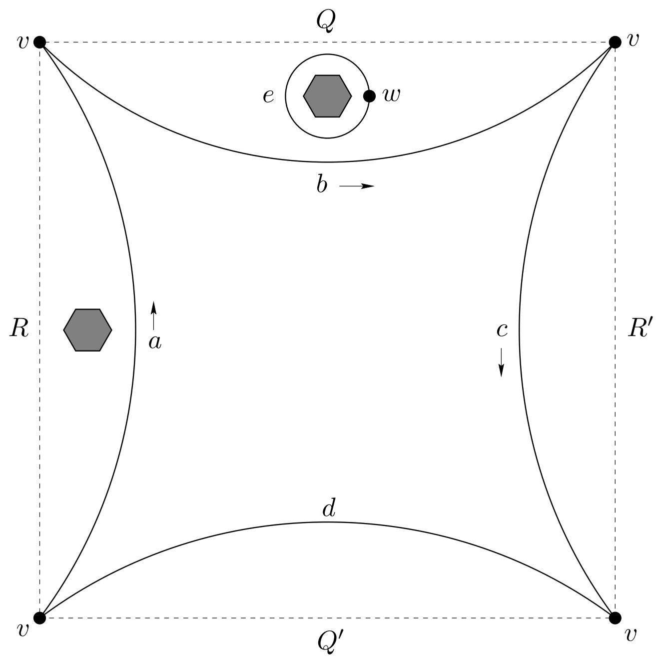


Figure 76.4
A graph G and curves C_1, C_2 on the double torus satisfying the cut condition (76.22) and Euler condition (76.34), but where no integer feasible circulations exist. The double torus is obtained from the square by identifying R and R' and identifying Q and Q' (so as to obtain the torus) and next deleting the interiors of the two hexagons and identifying their boundaries (so as to obtain the double torus). The graph G has two vertices, v and w, and four loops, a, b, c, d, at v, and one loop, e, at w. Curve C_1 follows the edges a and b, and curve C_2 follows the edges b and c — in the directions indicated. The cut condition (76.22) and the Euler condition (76.34) hold, but G has no edge-disjoint closed walks freely homotopic to C_1 and C_2 respectively. (The cut condition follows from the existence of a fractional solution.)

$\Phi(x,1) = D'(x)$, $\Phi(0,x) = D(0)$, $\Phi(1,x) = D(1)$ for each $x \in [0,1]$. (It follows that $D(0) = D'(0)$ and $D(1) = D'(1)$.)

If C and D are curves in T, then we denote:

(76.36) $$\mathrm{cr}(C,D) := |\{(x,y) \in [0,1] \times [0,1] \mid C(x) = D(y)\}|,$$ $$\mathrm{mincr}(C,D) := \min\{\mathrm{cr}(C',D') \mid C' \sim C, D' \sim D\}.$$

Let $G = (V,E)$ be a graph embedded in T. For any curve D in T and any $e \in E$, let

(76.37) $$\mathrm{cr}(e,D) := |\{x \in [0,1] \mid D(x) \in e\}|,$$

and

$$\text{cr}(G, D) = \sum_{e \in E} \text{cr}(e, D). \tag{76.38}$$

For any walk P in G, let tr^P be the vector in $\mathbb{Z}_+^E$ defined by

$$\text{tr}^P(e) := \text{number of times } P \text{ traverses } e, \tag{76.39}$$

for $e \in E$. For any curve C in T, a *flow homotopic to C* (*of value* 1) is a convex combination of functions tr^P where P is a walk in G being (as a curve) homotopic to C in T.

Corollary 76.1d. *Let $G = (V, E)$ be a planar graph embedded in $\mathbb{R}^2$. Let $F_1, \ldots, F_p$ be (the interiors of) some of the faces of G, including the unbounded face. Let $T := \mathbb{R}^2 \setminus (F_1 \cup \cdots \cup F_p)$. Let $C_1, \ldots, C_k$ be curves in T with ends points being vertices of G on the boundary of T. Then there exist flows $f_1, \ldots, f_k$ homotopic to $C_1, \ldots, C_k$ respectively, each of value* 1, *such that*

$$f_1(e) + \cdots + f_k(e) \leq 1 \tag{76.40}$$

for each edge e of G if and only if

$$\text{cr}(G, D) \geq \sum_{i=1}^{k} \text{mincr}(C_i, D) \tag{76.41}$$

for each curve D in $T \setminus V$ with end points on the boundary of T.

Proof. Necessity is shown similarly as in the proof of Corollary 76.1b. To see sufficiency, let the condition hold. We construct a compact orientable surface S. First embed $\mathbb{R}^2$ in the 2-dimensional sphere S^2. Next for each $i = 1, \ldots, k$ make a handle H_i between the faces among $F_1, \ldots, F_k$ incident with the end points of C_i. This yields S.

Let G' be the graph obtained from G by adding, for each $i = 1, \ldots, k$, an edge f_i between the end points of C_i, by routing f_i over H_i. This can be done in such a way that the new edges do not intersect each other, and do no intersect the edges of G. Each curve C_i now can be extended to a closed curve C'_i by adding f_i.

We apply Corollary 76.1b to G' and S. The circulations described in Corollary 76.1b give flows as required in the present corollary. So it suffices to check condition (76.22) for G' and S. That is, for any closed curve D in $S \setminus V$ we must show

$$\text{cr}(G', D) \geq \sum_{i=1}^{k} \text{mincr}'(C'_i, D). \tag{76.42}$$

Here mincr' denotes the function mincr with respect to S. To show (76.42), we distinguish three cases.

Case 1: D is contained in T. Let y be some point on D, let z be some point on the boundary of T with $z \notin V$, and let R be some curve in T connecting z and y, such that R does not intersect V, and intersects G only a finite number of times. For $n \in \mathbb{Z}_+$, let Q_n be the curve from z to z which follows R from z to y, then follows n times the closed curve D, and next returns from y to z over R. Let $r := \mathrm{cr}(G, R)$. Let D^n be the closed curve that follows n times D. Then for all $n \in \mathbb{Z}_+$,

$$\begin{aligned}(76.43)\qquad & n \cdot \sum_{i=1}^{k} \mathrm{mincr}'(C_i', D) = \sum_{i=1}^{k} \mathrm{mincr}'(C_i', D^n) \leq \sum_{i=1}^{k} \mathrm{mincr}(C_i, Q_n) \\ & \leq \mathrm{cr}(G, Q_n) = 2r + n \cdot \mathrm{cr}(G', D).\end{aligned}$$

Here the first equality is a general relation for curves on compact orientable surfaces (see Proposition 5 in Schrijver [1991a]). The first inequality holds as any curve homotopic to Q_n is equal to a *closed* curve *freely* homotopic to D^n. The second inequality follows from (76.41). The last equality follows from the definition of Q_n. Since (76.43) holds for each n, while r is fixed, we have (76.42).

Case 2: D does not intersect T. Then

$$(76.44)\qquad \mathrm{cr}(G', D) = \sum_{i=1}^{k} \mathrm{cr}(C_i', D) \geq \sum_{i=1}^{k} \mathrm{mincr}'(C_i', D).$$

Case 3: D intersects both T and $S \setminus T$. Set $H := S \setminus T$. Then we can split D into curves $D_1, D_2, \ldots, D_{2q}$, such that for odd i, D_i is contained in T and connects two points on the boundary of T, while for even i, D_i is contained in H, except for its end points. Then we have:

$$\begin{aligned}(76.45)\qquad \mathrm{cr}(G', D) &= \sum_{j=1}^{q} \mathrm{cr}(G, D_{2j-1}) + \sum_{j=1}^{q} \sum_{i=1}^{k} \mathrm{cr}(f_i, D_{2j}) \\ &\geq \sum_{j=1}^{q} \sum_{i=1}^{k} \mathrm{mincr}(C_i, D_{2j-1}) + \sum_{j=1}^{q} \sum_{i=1}^{k} \mathrm{cr}(f_i, D_{2j}) \\ &= \sum_{i=1}^{k} \sum_{j=1}^{q} (\mathrm{mincr}(C_i, D_{2j-1}) + \mathrm{cr}(f_i, D_{2j})) \geq \sum_{i=1}^{k} \mathrm{mincr}'(C_i', D).\end{aligned}$$

The first inequality follows from (76.41). The last inequality can be derived as follows. Fix $i = 1, \ldots, k$. Then there exist curves $\widetilde{C}_i \sim C_i$ and $\widetilde{D}_{2j-1} \sim D_{2j-1}$ ($j = 1, \ldots, q$) with $\mathrm{mincr}(C_i, D_{2j-1}) = \mathrm{cr}(\widetilde{C}_i, \widetilde{D}_{2j-1})$ for $j = 1, \ldots, q$. (This can be derived, for instance, from (76.8).) Hence $\widetilde{C}_i$ attains the minimum simultaneously for all D_{2j-1}. So

$$(76.46)\qquad \sum_{j=1}^{q} \mathrm{mincr}(C_i, D_{2j-1}) = \sum_{j=1}^{q} \mathrm{cr}(\widetilde{C}_i, \widetilde{D}_{2j-1}).$$

Hence, where $\widetilde{D}$ is the concatenation of $\widetilde{D}_1, D_2, \widetilde{D}_3, D_4, \ldots, \widetilde{D}_{2q-1}, D_{2q}$, and $\widetilde{C}_i'$ is the concatenation of $\widetilde{C}_i$ and f_i,

(76.47)
$$\begin{aligned}&\sum_{j=1}^{q}(\mathrm{mincr}(C_i, D_{2j-1}) + \mathrm{cr}(f_i, D_{2j}))\\ &= \sum_{j=1}^{q}(\mathrm{cr}(\widetilde{C}_i, \widetilde{D}_{2j-1}) + \mathrm{cr}(f_i, D_{2j})) = \mathrm{cr}(\widetilde{C}_i', \widetilde{D}) \geq \mathrm{mincr}'(C_i', D),\end{aligned}$$

proving the last inequality in (76.45). ∎

Notes. Related is the following *homotopic edge-disjoint paths problem*:

(76.48) given: a planar graph $G = (V, E)$, a subcollection $F_1, \ldots, F_p$ of the faces of G (including the unbounded face), curves $C_1, \ldots, C_k$ in $T := \mathbb{R}^2 \setminus (F_1 \cup \cdots \cup F_p)$, with end points in vertices of G on the boundary of T,
find: edge-disjoint walks $P_1, \ldots, P_k$, such that P_i traverses any edge at most once and is homotopic to C_i in T ($i = 1, \ldots, k$).

In this context, the faces $F_1, \ldots, F_p$ are called the *holes*.

Clearly, the *homotopic cut condition* (76.41) is a necessary condition for the feasibility of (76.48), while Figure 70.3 shows that it is generally not sufficient. By Corollary 76.1d, it is equivalent to the existence of a *fractional* solution of (76.48).

We can add the following *Euler condition* (or *local Euler condition*):

(76.49) for each vertex $v \in V$, the degree of v in G has the same parity as the number of times v is end point of the C_i (counting for 2 if C_i begins and ends at v).

By the Okamura-Seymour theorem, if $p = 1$ the homotopic cut and local Euler conditions are sufficient for the feasibility of (76.48). This was extended to $p = 2$ by van Hoesel and Schrijver [1990]:

(76.50) if $p = 2$ and the local Euler condition (76.49) holds, then the homotopic edge-disjoint paths problem (76.48) has a solution if and only if the homotopic cut condition (76.41) holds.

It implies that if $p = 2$, we can take the flows in Corollary 76.1d half-integer.

(76.50) cannot be extended to $p = 3$, as is shown by Figure 76.5. In fact, Kaufmann and Maley [1993] showed that it is NP-complete to solve the homotopic edge-disjoint paths problem even if the local Euler condition (76.49) holds and the graph is a grid graph (with as holes all faces enclosed by more than four edges).

We can consider a stronger parity condition, the *global Euler condition*:

(76.51) $\mathrm{cr}(G, D) \equiv \sum_{i=1}^{k} \mathrm{mincr}(C_i, D) \pmod 2$ for each curve D in $T \setminus V$ with end points on $\mathrm{bd}(T)$ and having no touchings with G.

Kaufmann and Mehlhorn [1992] showed that if G is a grid graph and the holes are those faces enclosed by more than four edges, and if the global Euler condition holds, then the homotopic edge-disjoint paths problem has a solution if and only if

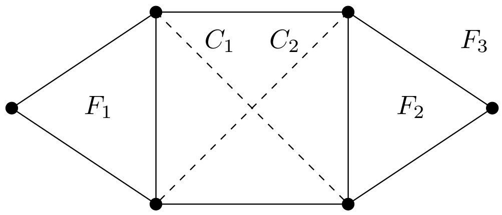


Figure 76.5
This graph G with curves C_1 and C_2 satisfies the cut condition (76.41) (as there is a fractional solution) and the (local) Euler condition, but G has no edge-disjoint walks $P_1 \sim C_1$ and $P_2 \sim C_2$ (in $\mathbb{R}^2 \setminus (F_1 \cup F_2 \cup F_3)$).

the homotopic cut condition holds. Kaufmann and Mehlhorn also gave an $O(n^2)$-time algorithm to find the paths. This was improved to a linear-time algorithm by Kaufmann [1987] and Kaufmann and Mehlhorn [1994].

Other types of grids, like the hexagonal and the octo-square grid, were considered by Kaufmann [1987]. A generalization to 'straight-line planar graphs' was given by Schrijver [1991d]. A *straight-line planar graph* is a planar graph G such that each edge is a straight line segment, where $F_1, \ldots, F_p$ are such that for each edge e of G and each vertex v on e, when extending the line segment forming e slightly at v we arrive either in another edge of G or in one of the faces F_i. In this case, if the global Euler condition holds, then the homotopic edge-disjoint paths problem has a solution if and only if the homotopic cut condition holds. Moreover, the problem is solvable in polynomial time in this case.

76.7. Vertex-disjoint paths and circuits of prescribed homotopies

76.7a. Vertex-disjoint circuits of prescribed homotopies

As for the vertex-disjoint analogue of the results studied above, the existence of vertex-disjoint circuits of prescribed homotopies in a graph on a compact surface can be fully characterized.

Let $G = (V, E)$ be a graph embedded in a compact surface S and let $C_1, \ldots, C_k$ be pairwise disjoint simple closed curves on S. We say that a closed curve D on a surface S is *doubly odd* (with respect to G, $C_1, \ldots, C_k$), if D is the concatenation of two closed curves D_1 and D_2, with common end point not on G, such that

(76.52) $$\mathrm{cr}(G, D_j) \not\equiv \sum_{i=1}^{k} \mathrm{mincr}(C_i, D_j) \pmod 2 \text{ for } j = 1, 2.$$

Then the following was shown in Schrijver [1991b] (conjectured by L. Lovász and P.D. Seymour):

Theorem 76.2. *There exist disjoint circuits $C'_1, \ldots, C'_k$ in G where C'_i is freely homotopic to C_i ($i = 1, \ldots, k$) if and only if for each closed curve D on S one has*

(76.53) $$\mathrm{cr}(G,D) \geq \sum_{i=1}^{k} \mathrm{mincr}(C_i,D),$$

with strict inequality if D is doubly odd.

We only show necessity of the condition. To that end, we can assume that $C_1,\ldots,C_k$ are disjoint circuits in G. Then the inequality follows from

(76.54) $$\mathrm{cr}(G,D) \geq \sum_{i=1}^{k} \mathrm{cr}(C_i,D) \geq \sum_{i=1}^{k} \mathrm{mincr}(C_i,D).$$

Moreover, if D is doubly odd, let D_1 and D_2 be as above. Then:

(76.55) $$\mathrm{cr}(G,D) = \mathrm{cr}(G,D_1) + \mathrm{cr}(G,D_2) > \sum_{i=1}^{k} (\mathrm{cr}'(C_i,D_1) + \mathrm{cr}'(C_i,D_2)) = \sum_{i=1}^{k} \mathrm{cr}'(C_i,D) \geq \sum_{i=1}^{k} \mathrm{mincr}(C_i,D).$$

Here $\mathrm{cr}'(C,D)$ counts the number of crossing (and not touchings) of C and D. The strict inequality holds as

(76.56) $$\mathrm{cr}(G,D_j) \not\equiv \sum_{i=1}^{k} \mathrm{mincr}(C_i,D_j) \equiv \sum_{i=1}^{k} \mathrm{cr}'(C_i,D_j) \pmod 2$$

for $j = 1,2$.

For the proof of sufficiency, based on solving a system of linear inequalities in integers, we refer to Schrijver [1991b]. The proof also implies a polynomial-time algorithm to find disjoint circuits as required in Theorem 76.2.

For the torus, the condition in Theorem 76.2 on the strictness of the inequality is superfluous, and the characterization can be formulated as:

(76.57) Let G be a graph embedded in the torus T, and let C be a simple closed curve on T. Then G contains k disjoint circuits each freely homotopic to C if and only if

$$\mathrm{cr}(G,D) \geq k \cdot \mathrm{mincr}(C,D)$$

for each closed curve D on T.

This was extended to directed graphs by Seymour [1991] (including polynomial-time solvability). A shorter proof of this, together with an extension to the Klein bottle, was given by Ding, Schrijver, and Seymour [1993]. A survey is given in Schrijver [1992].

76.7b. Vertex-disjoint homotopic paths in planar graphs with holes

In a similar way one can prove (or derive from Theorem 76.2 as in the proof of Corollary 76.1d for the fractional edge-disjoint case) results on vertex-disjoint homotopic paths in a planar graph with holes.

Consider the following *disjoint homotopic paths problem*:

(76.58) given: A planar graph $G = (V, E)$, faces $F_1, \ldots, F_p$ of G, including the unbounded face, disjoint curves $C_1, \ldots, C_k$ in $T := \mathbb{R}^2 \setminus (F_1 \cup \cdots \cup F_p)$, each with end points in vertices of G on the boundary of T,
find: disjoint paths $P_1, \ldots, P_k$ in G, where P_i is homotopic to C_i in T $(i = 1, \ldots, k)$.

Frank and Schrijver [1990] and Schrijver [1991c] gave polynomial-time algorithms for this problem, and gave the following characterization (the first paper gives an algorithm using the ellipsoid method, the second paper a combinatorial algorithm):

Theorem 76.3. *Problem* (76.58) *has a solution if and only if for each curve D in T with end points on* $\mathrm{bd}(T)$ *we have*

$$\text{(76.59)} \qquad \mathrm{cr}(G, D) \geq \sum_{i=1}^{k} \mathrm{mincr}(C_i, D),$$

and for each doubly odd closed curve D in T traversing no fixed point of any C_i we have

$$\text{(76.60)} \qquad \mathrm{cr}(G, D) > \sum_{i=1}^{k} \mathrm{mincr}(C_i, D).$$

Here a point p is called a *fixed point of* C if each curve homotopic to C traverses p. (In particular, the ends points of C are fixed points of C.)

Figure 76.6 (due to L. Lovász, cf. Robertson and Seymour [1986]) shows that condition (76.60) cannot be deleted in Theorem 76.3.

Theorem 76.3 was proved by Cole and Siegel [1984] for the special case where G is a grid graph (a subgraph of the rectangular grid), and the F_i are precisely the faces that are not surrounded by exactly four edges of the grid, and the boundary of each face F_i is a rectangle. In this case, condition (76.60) is superfluous. Cole and Siegel [1984] also gave a polynomial-time ($O(n \log n)$) algorithm for this case (answering a question of Pinter [1983]), using an oracle to test homotopy of curves. A polynomial-time algorithm for such graphs, not using a homotopy testing oracle, was given by Leiserson and Maley [1985]. Maley [1987] gave an $O(n^2 \log n)$-time algorithm (where the solution has the additional property that each of the paths found is shortest among all possible solutions), while Maley [1996] gave an $O(n \log n)$-time algorithm to test routability (not constructing the solution), under some mild conditions on the routing rules and the input layout.

Theorem 76.3 and the polynomial-time solvability of (76.58) was proved for $p \leq 2$ by Robertson and Seymour [1986], where again condition (76.60) is superfluous. (A linear-time algorithm for $p = 2$ was given by Ripphausen-Lipa, Wagner, and Weihe [1993a], if at least one of the curves C_i connects F_1 and F_2.) A short proof for the case $p = 2$ was given by Frank [1990c], which also extends to the directed case, implying a result of Seymour [1991].

The polynomial-time solvability of (76.58) implies the following for *nonhomotopic* disjoint paths (Schrijver [1991c]):

(76.61) for each fixed p, the vertex-disjoint paths problem is polynomial-time solvable for planar graphs if the terminals can be covered by the boundaries of at most p faces.

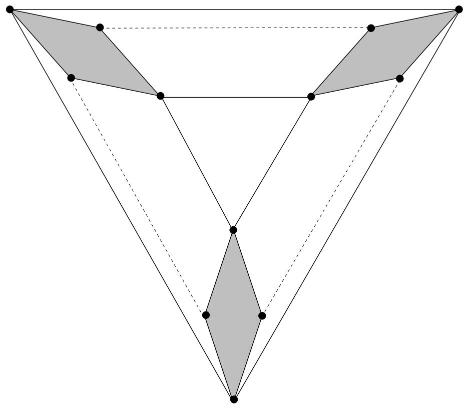


Figure 76.6

The three holes are indicated by grey regions, and the curves by dashed lines. We assume that the graph is embedded in the 2-sphere, such that there is no unbounded face.
The cut condition (76.59) holds, as there is a fractional solution, but no vertex-disjoint paths homotopic to the given curves exist.

(This was conjectured by Robertson and Seymour [1986], who proved it for $p \leq 2$ (see Section 74.4c for $p = 1$). For $p \leq 2$, Suzuki, Akama, and Nishizeki [1988a, 1988b,1988c,1990] gave an $O(n \log n)$-time algorithm, improved to linear-time by Ripphausen-Lipa, Wagner, and Weihe [1993a,1993b]. (The algorithm of Suzuki, Akama, Nishizeki is linear-time if each net is spanned by F_1 or by F_2.) For $p = 3$, a linear-time algorithm if each net is spanned by F_1, F_2, or F_3 was announced by H. Suzuki, T. Kumagai, and T. Nishizeki (1993; cf. Ripphausen-Lipa, Wagner, and Weihe [1995]).)

The idea of proof of (76.61) is that for each net r we choose a curve C_r connecting the points in r such that the C_r are disjoint, and next try to find paths as in (76.58); it can be proved that we need to consider only a polynomially bounded number of homotopy classes of curves C_r (for fixed p), which gives the required result.

In Schrijver [1993] this was extended, by similar methods, to the directed case:

(76.62) for each fixed p, the disjoint paths problem is polynomial-time solvable for directed planar graphs if the terminals can be covered by the boundaries of at most p faces.

This remains the case if we prescribe for each net (s_i, t_i) a subset A_i of the arc set that path P_i is allowed to use.

For a sketch of the method for (76.58), see Schrijver [1990b].

Robertson and Seymour [1995] proved that if the number of terminals is fixed, the vertex-disjoint paths problem in undirected graphs is $O(n^3)$-time solvable, also for nonplanar graphs. If moreover the graph is planar, Reed, Robertson, Schrijver, and Seymour [1993] gave a *linear*-time algorithm. This connects to the results described in Section 70.13a.

76.7c. Disjoint trees

The polynomial-time solvability of finding paths (76.58) can be generalized to disjoint trees (Schrijver [1991c]). The following problem is solvable in polynomial time:

(76.63) given: a planar graph G, faces $F_1, \ldots, F_p$ of G (including the unbounded face), curves $C_{1,1}, \ldots, C_{1,t_1}, \ldots, C_{k,1}, \ldots, C_{k,t_k}$ in the space $S := \mathbb{R}^2 \setminus (F_1 \cup \cdots \cup F_p)$, with end points in vertices of G on $\mathrm{bd}(S)$, such that for each $i = 1, \ldots, k$, $C_{i,1}, \ldots, C_{i,t_i}$ have the same starting vertex;
find: disjoint subtrees $T_1, \ldots, T_k$ of G such that for each $i = 1, \ldots, k$ and $j = 1, \ldots, t_i$, T_i contains a path homotopic to $C_{i,j}$ in S.

Again, by enumerating homotopy classes, it can be derived that, for each fixed p, the problem

(76.64) given: a graph $G = (V, E)$ and disjoint subsets $W_1, \ldots, W_k$ of V;
find: disjoint subtrees $T_1, \ldots, T_k$ of G such that T_i spans W_i for $i = 1, \ldots, k$,

is polynomial-time solvable if G is planar and $W_1, \ldots, W_k$ can be covered by the boundaries of at most p faces of G. (For $p \leq 2$, Suzuki, Akama, and Nishizeki [1988a, 1988b,1988c,1990] gave an $O(n \log n)$-time algorithm, improved to linear-time by Ripphausen-Lipa, Wagner, and Weihe [1993a,1993b].)

Robertson and Seymour [1995] showed that for each fixed p, (76.64) is $O(n^3)$-time solvable for any graph if $|W_1 \cup \cdots \cup W_k| \leq p$. If moreover the graph is planar, Reed, Robertson, Schrijver, and Seymour [1993] gave a *linear*-time algorithm.

For minimum-length homotopic routing in grid graphs, see Ho, Suzuki, and Sarrafzadeh [1993]. Surveys of homotopic routing methods are given by Schrijver [1990b,1994b], and of applications of polyhedral combinatorics to multiflows on surfaces by Schrijver [1990a]. ‘Gridless’ homotopic routing (that is, routing in the plane (not in a graph), observing mutual distances between curves) was studied by Tompa [1981] and Maley [1988].