Part VI

Cliques, Stable Sets, and Colouring

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We now arrive at a class of problems that are in general NP-complete: finding a maximum-size clique or stable set or a minimum vertex-colouring in an undirected graph. These problems relate to each other: a stable set in a graph is a clique in the complementary graph, a colouring is a partitioning of the vertex set into stable sets, and the maximum size of a clique is a lower bound for the minimum number of colours.

Graph colouring was motivated originally by the four-colour conjecture formulated in the 1850s, stating that each planar map can be coloured with at most four colours — since 1977 a theorem of Appel and Haken. Later, colouring turned out to have several other applications, like in school scheduling, timetabling, and warehouse planning and in bungalow, terminal, platform, and frequency assignment. Finding optimum cliques of stable sets again can be used in frequency assignment, and in set packing problems, which show up for instance in crew scheduling.

While these problems are in general NP-complete, some are polynomial-time solvable for special classes of graphs: perfect graphs, t-perfect graphs, claw-free graphs. They form the body of this part.

Perfect graphs carry one of the deepest theorems in graph theory, the strong perfect graph theorem — recently proved by Chudnovsky, Robertson, Seymour, and Thomas. The proof is highly complicated, and we cannot give it in this book.

We refer to Part III for stable sets in and colouring of *line graphs* — equivalently, matchings and edge-colouring.

Chapters:

Chapter 64

Cliques, stable sets, and colouring

This chapter studies cliques, stable sets, and colouring for general graphs: complexity, polyhedra, fractional solutions, weighted versions.
In studying later chapters of this part, one can do largely without the results of the present chapter. Only some elementary definitions and terminology will be needed. It suffices to use this chapter just for reference.
In this chapter, all graphs can be assumed to be simple.

64.1. Terminology and notation

Let $G = (V, E)$ be an undirected graph. A *clique* is a set of vertices any two of which are adjacent. The maximum size of a clique in G is the *clique number* of G, and is denoted by $\omega(G)$.

A *stable set* is a set of vertices any two of which are nonadjacent. The maximum size of a stable set in G is called the *stable set number* of G, and is denoted by $\alpha(G)$.

A *vertex cover* is a set of vertices intersecting all edges. The minimum size of a vertex cover in G is called the *vertex cover number* of G, and is denoted by $\tau(G)$.

A *(vertex-)colouring* of G is a partition of V into stable sets $S_1, \ldots, S_k$. The sets $S_1, \ldots, S_k$ are called the *colours* of the colouring. The minimum number of colours in a vertex-colouring of G is called the *(vertex-)colouring number* of G, denoted by $\chi(G)$. A graph G is called *k-(vertex-)colourable* if $\chi(G) \leq k$, and *k-chromatic* if $\chi(G) = k$. A *minimum (vertex-)colouring* is a colouring with $\chi(G)$ colours. A *k-(vertex-)colouring* is a colouring with k colours.

A *clique cover* of G is a partition of V into cliques. The minimum number of cliques in a clique cover of G is called the *clique cover number* of G, and is denoted by $\overline{\chi}(G)$. A *minimum clique cover* is a clique cover with $\overline{\chi}(G)$ cliques.

The following relations between these parameters are immediate:

(64.1) $\alpha(G) = \omega(\overline{G})$, $\overline{\chi}(G) = \chi(\overline{G})$, $\omega(G) \leq \chi(G)$, $\alpha(G) \leq \overline{\chi}(G)$, $\tau(G) = |V| - \alpha(G)$.

64.2. NP-completeness

It is NP-complete to find a maximum-size stable set in a graph. To be more precise, the *stable set problem*: given a graph G and a natural number k, decide if $\alpha(G) \geq k$, is NP-complete (according to Karp [1972b] this is implicit in the work of Cook [1971] and was also known to R. Reiter):

Theorem 64.1. *Determining the stable set number is* NP*-complete.*

Proof. We reduce the satisfiability problem to the stable set problem. Let $C_1 \wedge \cdots \wedge C_k$ be a Boolean expression, where each C_i is of the form $y_1 \vee \cdots \vee y_m$, with $y_1, \ldots, y_m \in \{x_1, \neg x_1, \ldots, x_n, \neg x_n\}$. Call $x_1, \neg x_1, \ldots, x_n, \neg x_n$ the *literals*. Consider the graph $G = (V, E)$ with $V := \{(\sigma, i) \mid \sigma$ is a literal in $C_i\}$ and $E := \{\{(\sigma, i), (\tau, j)\} \mid i = j$ or $\sigma = \neg\tau\}$. Then the expression is satisfiable if and only if G has a stable set of size k. ∎

It can be shown that the stable set problem remains NP-complete if the graphs are restricted to 3-regular planar graphs (Garey, Johnson, and Stockmeyer [1976]) or to triangle-free graphs (Poljak [1974]).

Since a subset U of VG is a vertex cover if and only if $VG \setminus U$ is a stable set, we also have:

Corollary 64.1a. *Determining the vertex cover number is* NP*-complete.*

Proof. By Theorem 64.1, since the vertex cover number of a graph G is equal to $|VG|$ minus the stable set number. ∎

A subset C of VG is a clique in a graph G if and only if C is a stable set in the complementary graph $\overline{G}$. So finding a maximum-size clique in G is equivalent to finding a maximum-size stable set in $\overline{G}$, and $\omega(G) = \alpha(\overline{G})$. Hence, as determining $\alpha(G)$ is NP-complete, also determining $\omega(G)$ is NP-complete.

Also, it is NP-complete to decide if a graph is k-colourable (Karp [1972b]):

Theorem 64.2. *Determining the vertex-colouring number is* NP*-complete.*

Proof. We show that the stable set problem can be reduced to the vertex-colouring problem. Let $G = (V, E)$ be an undirected graph and let $k \in \mathbb{Z}_+$. We want to decide if $\alpha(G) \geq k$. To this end, let V' be a copy of V and let C be a set of size k, where V, V', and C are disjoint. Make a graph H with vertex set $V \cup V' \cup C$ as follows. A pair of vertices in V is adjacent in H if and only if it is adjacent in G. The sets V' and C are cliques in H. Each vertex in V is adjacent to each vertex in $V' \cup C$, except to its copy in V'. No vertex in V' is adjacent to any vertex in C.

This defines the graph H. Then $\alpha(G) \geq k$ if and only if $\chi(H) \leq |V| + 1$. ∎

Well-known is the *four-colour conjecture* (or *4CC*), stating that $\chi(G) \leq 4$ for each loopless planar graph G. This conjecture was proved by Appel and Haken [1977] and Appel, Haken, and Koch [1977], and is now called the *four-colour theorem.* (A shorter proof was given by Robertson, Sanders, Seymour, and Thomas [1997], leading to an $O(n^2)$-time 4-colouring algorithm for planar graphs (Robertson, Sanders, Seymour, and Thomas [1996]).)

However, it is NP-complete to decide if a planar graph is 3-colourable, even if the graph has maximum degree 4 (Garey, Johnson, and Stockmeyer [1976]). Moreover, determining the colouring number of a graph G with $\alpha(G) \leq 4$ is NP-complete (cf. Garey and Johnson [1979]). Holyer [1981] showed that deciding if a 3-regular graph is 3-edge-colourable is NP-complete (see Section 28.3). Note that one can decide in polynomial time if a graph G is 2-colourable, since bipartiteness can be checked in polynomial time.

These NP-completeness results imply that if NP≠co-NP, then one may not expect a min-max relation characterizing the stable set number $\alpha(G)$, the vertex cover number $\tau(G)$, the clique number $\omega(G)$, or the colouring number $\chi(G)$ of a graph G.

64.3. Bounds on the colouring number

A lower bound on the colouring number is given by the clique number:

(64.2) $\omega(G) \leq \chi(G)$.

This is easy, since in any clique all vertices should have different colours.

There are several graphs which have strict inequality in (64.2). We mention the odd circuits C_k, with k odd and ≥ 5: then $\omega(C_k) = 2$ and $\chi(C_k) = 3$. Moreover, for the complement $\overline{C}_k$ of any such graph we have: $\omega(\overline{C}_k) = \lfloor k/2 \rfloor$ and $\chi(\overline{C}_k) = \lceil k/2 \rceil$.

It was a conjecture of Berge [1963a] that these graphs are crucial. In May 2002, M. Chudnovsky, N. Robertson, P.D. Seymour, and R. Thomas announced that they have found a proof of this conjecture.

Strong perfect graph theorem: Each graph G with $\omega(G) < \chi(G)$ has C_k or $\overline{C}_k$ as induced subgraph for some odd $k \geq 5$.

It is convenient to define a *hole* of a graph G to be an induced subgraph of G isomorphic to C_k for some $k \geq 4$. Moreover, an *antihole* is an induced subgraph isomorphic to $\overline{C}_k$ for some $k \geq 4$. A hole or antihole is *odd* if it has an odd number of vertices. Then the strong perfect graph theorem can be formulated as: each graph G with $\omega(G) < \chi(G)$ has an odd hole or odd antihole.

For more on this we refer to Chapter 65.

64.3a. Brooks' upper bound on the colouring number

There is a trivial upper bound on the colouring number:

(64.3) $\chi(G) \leq \Delta(G) + 1,$

where $\Delta(G)$ denotes the maximum degree of G. This bound follows by colouring the vertices 'greedily' one by one: at any stage, at least one colour (out of $\Delta(G)+1$ colours) is not used by the neighbours.

Brooks [1941] sharpened this inequality as follows. We follow the proof given by Lovász [1975d].

Theorem 64.3 (Brooks' theorem)**.** *For any connected graph G one has $\chi(G) \leq \Delta(G)$, except if G is a complete graph or an odd circuit.*

Proof. We can assume that G is 2-connected, since otherwise we can apply induction. Moreover, we can assume that $\Delta(G) \geq 3$. Let $k := \Delta(G)$.

I. First assume that G has nonadjacent vertices u and w with $G - u - w$ disconnected. Let V_1 and V_2 be proper subsets of V such that $V_1 \cup V_2 = V$, $V_1 \cap V_2 = \{u, w\}$, and no edge connects $V_1 \setminus \{u, w\}$ and $V_2 \setminus \{u, w\}$. Let $G_1 := G[V_1]$ and $G_2 := G[V_2]$.

For $i = 1, 2$, we know by induction that $\chi(G_i) \leq k$, since G_i is not complete (as u and w are nonadjacent), and since $\Delta(G_i) \leq k$ and $k \geq 3$. By symmetry of G_1 and G_2, we can assume that each k-colouring of G_1 gives u and w the same colour (otherwise G_1 and G_2 have k-colourings that coincide on u and w, yielding a k-colouring of G). This implies that both u and w have degree at least $k-1$ in G_1. Hence they have degree at most 1 in G_2. Therefore, as $k \geq 3$, G_2 has a k-colouring giving u and w the same colour. So G is k-colourable.

II. Now choose a vertex v of maximum degree. As G is not a complete graph, v has two nonadjacent neighbours, say u and w. By part I, we can assume that $G - u - w$ is connected. Hence it has a spanning tree T. Orient T so as to obtain a rooted tree, rooted at v. Hence we can order the vertices of G as $v_1, \ldots, v_n$ such that $v_1 = v$, $v_{n-1} = u$, $v_n = w$, and such that each v_i with $i > 1$ is adjacent to some v_j with $j < i$. Give u and w colour 1. Next successively for $i = n-2, n-1, \ldots, 1$, we can give a colour from $1, \ldots, k$ to v_i different from the colours given to the neighbours v_j of v_i with $j > i$. Such a colour exists, since if $i > 1$, there are less than k neighbours v_j of v_i with $j > i$; and if $i = 1$, there are k such neighbours, but neighbours u and w have the same colour. ∎

(A related proof was given by Ponstein [1969], and a strengthening of Brooks' theorem by Reed [1999a]. For another proof of Brooks' theorem, see Melnikov and Vizing [1969].)

64.3b. Hadwiger's conjecture

Another upper bound on the colouring number is conjectured by Hadwiger [1943]. Since there exist graphs with $\omega(G) < \chi(G)$, it is not true that if $\chi(G) \geq k$, then G contains the complete graph K_k on k vertices as a subgraph. However, Hadwiger conjectured the following, where a graph H is called a *minor* of a graph G if H arises from some subgraph of G by contracting some (possibly none) edges.

Hadwiger's conjecture: If $\chi(G) \geq k$, then G contains K_k as a minor.

In other words, for each k, the graph K_k is the only graph G with the property that G is not $(k-1)$-colourable and each proper minor of G is $(k-1)$-colourable.

Hadwiger's conjecture is trivial for $k = 1, 2, 3$, and was shown by Hadwiger [1943] for $k = 4$ (also by Dirac [1952]):

Theorem 64.4. *If G has no K_4 minor, then $\chi(G) \leq 3$.*

Proof. One may assume that $G = (V, E)$ is not a forest or a circuit. Then G has a circuit C not covering all vertices of G. Choose $v \in V \setminus VC$. If G is 3-connected, there are three paths from v to VC, disjoint except for v. This creates a K_4 minor, a contradiction.

So G is not 3-connected, that is, G has a vertex-cut of size less than 3. Then $\chi(G) \leq 3$ follows by induction: if G is disconnected or has a 1-vertex-cut, this is trivial, and if G is 2-connected and has a 2-vertex-cut $\{u, w\}$, we can apply induction to the graphs $G - K$ after adding an edge uw, for each component K of $G - u - w$. ∎

(For another proof, see Woodall [1992].)

As planar graphs contain no K_5 minor, Hadwiger's conjecture for $k = 5$ implies the four-colour theorem. In fact, Wagner [1937a] showed that his decomposition theorem (Theorem 3.3) implies that Hadwiger's conjecture for $k = 5$ is equivalent to the four-colour conjecture. (Young [1971] gave a 'quick' proof of this equivalence.) The four-colour conjecture was proved by Appel and Haken [1977] and Appel, Haken, and Koch [1977]. (Robertson, Sanders, Seymour, and Thomas [1997] gave a shorter proof.)

Robertson, Seymour, and Thomas [1993] showed that Hadwiger's conjecture is true also for $k = 6$, by reducing it again to the four-colour theorem. For $k \geq 7$, Hadwiger's conjecture is unsettled.

Halin [1964] has proved that if G has no K_k minor, then $\chi(G) \leq 2^{k-2}$ (Wagner [1964] gave a short proof). Further progress on Hadwiger's conjecture was made by Wagner [1960], Mader [1968], Jakobsen [1971], Duchet and Meyniel [1982], Kostochka [1982], Fernandez de la Vega [1983], Thomason [1984], and Reed and Seymour [1998].

Hajós' conjecture. G. Hajós[1] conjectured (more strongly than Hadwiger) that any k-chromatic graph contains a subdivision of K_k as subgraph. For $k \leq 4$, Hajós' conjecture is equivalent to Hadwiger's conjecture.

Hajós' conjecture was refuted by Catlin [1979] for $k = 8$. He showed that the line graph $L(G)$ of the graph G obtained from the 5-circuit C_5 by replacing each

[1] According to Toft [1996], Hajós considered the conjecture already in the 1940s in connection with the four-colour conjecture, but he never published it. (The paper Hajós [1961] commonly referred to, does not give Hajós' conjecture.) An early written record of the conjecture is in the review of Tutte [1961b], in the January 1961 issue of *Mathematical Reviews*, of the book *Färbungsprobleme auf Flächen und Graphen* (Colouring Problems on Surfaces and Graphs) by Ringel [1959]. This book itself however does not mention the conjecture.

edge by three parallel edges, has colouring number 8 (as $L(G)$ has 15 vertices and stable set number 2), but contains no subdivision of K_8.

Catlin in fact gave a counterexample to Hajós' conjecture for each $k \geq 7$. Erdős and Fajtlowicz [1981] showed that almost all graphs are counterexamples to Hajós' conjecture.

Related is the following result of Hajós [1961]: any graph G with $\chi(G) \geq k$ can be obtained from the complete graph K_k by a series of the following operations on graphs (each preserving $\chi \geq k$):

(64.4)
(i) add vertices or edges;
(ii) identify two nonadjacent vertices;
(iii) take two disjoint graphs G_1 and G_2, choose edges $e_1 = u_1v_1$ of G_1 and $e_2 = u_2v_2$ of G_2, identify u_1 and u_2, delete e_1 and e_2, and add edge v_1v_2.

64.4. The stable set, clique, and vertex cover polytope

The *stable set polytope* $P_{\text{stable set}}(G)$ of a graph $G = (V, E)$ is the convex hull of the incidence vectors of the stable sets in G. Since finding a maximum-size stable set is NP-complete, one may not expect a polynomial-time checkable system of linear inequalities describing the stable set polytope (Corollary 5.16a). More precisely, if NP≠co-NP, then there do not exist inequalities satisfied by the stable set polytope such that their validity can be certified in polynomial time and such that the inequality $\mathbf{1}^{\mathsf{T}}x \leq \alpha(G)$ is a nonnegative linear combination of them.

The *clique polytope* $P_{\text{clique}}(G)$ of a graph $G = (V, E)$ is the convex hull of the incidence vectors of cliques. Trivially

(64.5) $$P_{\text{clique}}(G) = P_{\text{stable set}}(\overline{G}).$$

Hence, similar observations hold for the clique polytope.

Another related polytope is the *vertex cover polytope* $P_{\text{vertex cover}}(G)$ of G, being the convex hull of the incidence vectors of vertex covers in G. Since a subset U of V is a vertex cover if and only if $V \setminus U$ is a stable set, we have

(64.6) $$x \in P_{\text{vertex cover}}(G) \iff \mathbf{1} - x \in P_{\text{stable set}}(G).$$

This shows that problems on the two types of polytopes can be reduced to each other.

64.4a. Facets and adjacency on the stable set polytope

Padberg [1973] (for facets induced by odd circuits) and Nemhauser and Trotter [1974] observed that

(64.7) each facet of the stable set polytope of an induced subgraph $G[U]$ of G, is the restriction to U of some unique facet of $P_{\text{stable set}}(G)$.

More precisely, for each facet F of $P_{\text{stable set}}(G[U])$ there is a unique facet F' of $P_{\text{stable set}}(G)$ with the property that $F = \{x \in \mathbb{R}^U \mid x' \in F'\}$, where $x'_v = x_v$ if $v \in U$ and $x'_v = 0$ if $v \in V \setminus U$.

To prove (64.7), it suffices to prove it for $U = V \setminus \{v\}$ for some $v \in V$. Let F be a facet of $P_{\text{stable set}}(G - v)$. We can consider F as a face of codimension 2 of $P_{\text{stable set}}(G)$ (by extending F with a 0 at coordinate v). Define $H := \{x \in \mathbb{R}^V \mid x_v = 0\}$. As F is on the facet $F'' := P_{\text{stable set}}(G) \cap H$ of $P_{\text{stable set}}(G)$, there is a unique facet F' of $P_{\text{stable set}}(G)$ with $F = F'' \cap F'$. This implies $F = F' \cap H$, since

(64.8) $$F = F' \cap F'' = F' \cap P_{\text{stable set}}(G) \cap H = F' \cap H.$$

Suppose now that $P_{\text{stable set}}(G)$ has another facet F''' with $F = F''' \cap H$. Then $F \subseteq F''' \cap F'' \subseteq F''' \cap H = F$, and hence $F = F'' \cap F'''$, contradicting the unicity of F'. This proves (64.7).

Padberg [1973] also showed the following:

Theorem 64.5. *Let $G = (V, E)$ be a graph and let $a \in \mathbb{Z}_+^V$. Then the inequality*

(64.9) $$a^{\mathsf{T}} x \leq 1$$

is valid for the stable set polytope of G if and only if a is the incidence vector of a clique C. Moreover, (64.9) *determines a facet if and only if C is an inclusionwise maximal clique.*

Proof. Trivially, inequality (64.9) is valid if $a = \chi^C$ for some clique C. Conversely, if (64.9) is valid, then a is a $0,1$ vector, and hence the incidence vector of a subset C of V. Then C is a clique, since otherwise C contains a stable set S of size 2, implying that (64.9) is not valid for $x := \chi^S$.

In proving the second statement, we can assume that $a = \chi^C$ for some clique C. Suppose that (64.9) determines a facet, and that C is not an inclusionwise maximal clique. Then there is a clique C' properly containing C. Hence for each $x \in P_{\text{stable set}}(G)$, if $x(C) = 1$, then $x(C') = 1$. This implies that the inequality $x(C) \leq 1$ is not facet-inducing, a contradiction.

Finally suppose that C is an inclusionwise maximal clique. To see that (64.9) determines a facet, let $a^{\mathsf{T}} x = \beta$ be satisfied by all x in the stable set polytope with $x(C) = 1$. So $a(S) = \beta$ for each stable set S with $|S \cap C| = 1$. Then $a_v = \beta$ for each $v \in C$, as $S := \{v\}$ is stable. Also, $a_u = 0$ for each $u \in V \setminus C$, since by the maximality of C, there is a vertex $v \in C$ that is not adjacent to u. So $S := \{u, v\}$ is stable, and hence $a_u + a_v = \beta$. So $a_u = 0$. Concluding, $a^{\mathsf{T}} x = \beta$ is some multiple of $x(C) = 1$, and hence $x(C) \leq 1$ determines a facet. ∎

Graphs for which the nonnegativity and clique inequalities determine all facets, are precisely the *perfect* graphs — see Chapter 65.

Trivially, the vertices of the stable set polytope are precisely the incidence vectors of the stable sets. Chvátal [1975a] characterized adjacency:

Theorem 64.6. *The incidence vectors of two different stable sets R, S are adjacent vertices of the stable set polytope if and only if $R \triangle S$ induces a connected subgraph of G.*

Proof. To see necessity, if $G[R \triangle S]$ is not connected, then (as it is bipartite) it has two colour classes U and W with $\{U, W\} \neq \{R \setminus S, S \setminus R\}$. Let $U' := U \cup (R \cap S)$ and

$W' := W \cup (R \cap S)$. Then U' and W' are stable sets and $\frac{1}{2}(\chi^R + \chi^S) = \frac{1}{2}(\chi^{U'} + \chi^{W'})$, contradicting the adjacency of χ^R and χ^S.

To see sufficiency, if χ^R and χ^S are not adjacent, then there exist stable sets U and W, and $\lambda, \mu \in (0,1)$ such that $\lambda\chi^R + (1-\lambda)\chi^S = \mu\chi^U + (1-\mu)\chi^W$ and $\{U, W\} \neq \{R, S\}$. So $U \cap W = R \cap S$. Hence $U \setminus W, W \setminus U$ forms a bipartition of $G[R \triangle S]$ different from the bipartition $R \setminus S, S \setminus R$. This contradicts the connectedness of $G[R \triangle S]$. ∎

64.5. Fractional stable sets

The incidence vectors of stable sets in an undirected graph $G = (V, E)$ are precisely the integer vectors $x \in \mathbb{R}^V$ satisfying

$$\text{(64.10)} \qquad \begin{array}{rll} \text{(i)} & 0 \leq x_v \leq 1 & \text{for } v \in V, \\ \text{(ii)} & x_u + x_v \leq 1 & \text{for } \{u, v\} \in E. \end{array}$$

(The inequalities (64.10)(ii) are called the *edge inequalities*.) Any (not necessarily integer) solution x of (64.10) is called a *fractional stable set*. By definition, its *size* is equal to $x(V)$.

The maximum size of a fractional stable set is called the *fractional stable set number* and is denoted by $\alpha^*(G)$. By linear programming duality, $\alpha^*(G)$ is equal to the *fractional edge cover number* $\rho^*(G)$ (assuming that G has no isolated vertices), which is the minimum value of $y(E)$ over all $y \in \mathbb{R}^E$ satisfying

$$\text{(64.11)} \qquad \begin{array}{rll} \text{(i)} & 0 \leq y_e \leq 1 & \text{for } e \in E, \\ \text{(ii)} & y(\delta(v)) \geq 1 & \text{for } v \in V. \end{array}$$

Any solution y of (64.11) is called a *fractional edge cover*.

This was also discussed in Section 30.11, where it was shown that each vertex of the polytope determined by (64.11) (the fractional edge cover polytope) is half-integer. A similar result holds for the *fractional stable set polytope*, which is the polytope determined by (64.10) (the result is implicit in Balinski [1965]):

Theorem 64.7. *Each vertex of the fractional stable set polytope P is half-integer.*

Proof. Let x be a vertex of P. Let $U := \{v \in V \mid 0 < x_v < \frac{1}{2}\}$ and let $W := \{v \in V \mid \frac{1}{2} < x_v < 1\}$. Then there is an $\varepsilon > 0$ such that both $x + \varepsilon(\chi^U - \chi^W)$ and $x - \varepsilon(\chi^U - \chi^W)$ belong to P. As x is a vertex, it follows that $\chi^U - \chi^W = 0$. So $U = W = \emptyset$. ∎

(This proof was provided to Nemhauser and Trotter [1974] by a referee of their paper.)

The theorem also follows from the observation of Balinski [1965] that each nonsingular submatrix of the incidence matrix of a graph has a half-integer inverse.

Theorem 64.7 implies that $\alpha^*(G) = \frac{1}{2}\alpha_2(G)$, where $\alpha_2(G)$ is the maximum size of a 2-*stable set*, which is an integer vector $x \in \mathbb{R}^V$ satisfying

(64.12) (i) $x_v \geq 0$ for $v \in V$,
(ii) $x_u + x_v \leq 2$ for $\{u, v\} \in E$

(cf. Section 30.9).

Moreover, it implies a characterization of the 2-*stable set polyhedron*, which is the convex hull of the 2-stable sets:

Corollary 64.7a. *The* 2-*stable set polyhedron is determined by* (64.12).

Proof. Directly from Theorem 64.7. ∎

With the following construction, the problem of finding a maximum-weight fractional stable set (and similarly, a maximum-weight 2-stable set), can be reduced to the problem of finding a maximum-weight stable set in a bipartite graph. The latter problem is strongly polynomial-time solvable, by Theorem 21.10.

Let $G = (V, E)$ be a graph. Let V' be a copy of V. For any $v \in V$, let v' denote the copy of V in V'. Define $\widetilde{V} := V \cup V'$. Let $\widetilde{E}$ be the set of pairs $u'v$ and uv', over all edges uv of G. Then $\widetilde{G} := (\widetilde{V}, \widetilde{E})$ is a bipartite graph.

For any weight function $w : V \to \mathbb{R}_+$, define $\tilde{w} : \widetilde{V} \to \mathbb{R}_+$ by $\tilde{w}(v) := \tilde{w}(v') := w(v)$ for $v \in V$. Then any stable set S in $\widetilde{G}$ maximizing $\tilde{w}(S)$ gives a 2-stable set x in G maximizing $w^{\mathsf{T}}x$, by defining $x_v := |S \cap \{v, v'\}|$. Indeed, for any 2-stable set x' in G we can define a stable set S' in $\widetilde{G}$ by

(64.13) $$S' := \{v \in V \mid x'_v \geq 1\} \cup \{v \in V' \mid x'_v \geq 2\}.$$

Then $w^{\mathsf{T}}x' = \tilde{w}(S') \leq \tilde{w}(S) = w^{\mathsf{T}}x$. (Here we assume without loss of generality that G has no isolated vertices.)

64.5a. Further on the fractional stable set polytope

Nemhauser and Trotter [1974] characterized the vertices of the fractional stable set polytope:

Theorem 64.8. *A vector $x \in \mathbb{R}^V$ is a vertex of the fractional stable set polytope P of G if and only if $x = \chi^{U_2} + \frac{1}{2}\chi^{U_1}$, where U_2 is a stable set of G, where U_1 is disjoint from $U_2 \cup N(U_2)$, and where each component of $G[U_1]$ is nonbipartite.*

Proof. *Necessity.* Let x be a vertex of P, and define $U_2 := \{v \in V \mid x_v = 1\}$ and $U_1 := \{v \in V \mid x_v = \frac{1}{2}\}$. Then U_2 is a stable set and no vertex in U_1 is adjacent to any vertex in U_2. So U_1 is disjoint from $U_2 \cup N(U_2)$.

If some component of $G[U_1]$ would be bipartite, say with colour classes S and T, then $x \pm \varepsilon(\chi^S - \chi^T)$ would belong to P for some $\varepsilon \neq 0$. This contradicts the fact that x is a vertex of P.

Sufficiency. Suppose that x satisfies the condition, and that x is not a vertex of P. Then there is a nonzero vector y such that both $x + y$ and $x - y$ belong to P. Necessarily, $y_v = 0$ if $v \notin U_1$. Moreover, for each edge uw of $G[U_1]$ one has $y_u + y_w = 0$, since $x_u + x_w = 1$. As each component of $G[U_1]$ contains an odd circuit, this implies $y_v = 0$ for each $v \in U_1$. So $y = \mathbf{0}$, a contradiction. ∎

A useful condition was given by Nemhauser and Trotter [1975]:

Theorem 64.9. *Let $G = (V, E)$ be a graph, let $w : V \to \mathbb{R}$ be a weight function, and let $S \subseteq V$ be a stable set. If S is a maximum-weight stable set in the subgraph of G induced by $S \cup N(S)$, then S is contained in some maximum-weight stable set of G.*

Proof. Let T be a maximum-weight stable set of G. Define $U := (S \cup T) \setminus N(S)$. Trivially, U is stable. Also, $w(N(S) \cap T) \leq w(S \setminus T)$, since $w((S \cup N(S)) \cap T) \leq w(S)$, as S has maximum weight in $G[S \cup N(S)]$. Hence

(64.14) $$w(U) = w(T) + w(S \setminus T) - w(N(S) \cap T) \geq w(T),$$

implying that U is a maximum-weight stable set in G. ∎

This implies (Nemhauser and Trotter [1975]):

Corollary 64.9a. *Let $G = (V, E)$ be a graph, let $w : V \to \mathbb{R}$ be a weight function, and let x be a maximum-weight fractional stable set. Then $S := \{v \mid x_v = 1\}$ is contained in a maximum-weight stable set.*

Proof. This follows from Theorem 64.9, since S is a maximum-weight stable set in $G[S \cup N(S)]$. For if T would be a stable set in $G[S \cup N(S)]$ with $w(T) > w(S)$, then $x + \varepsilon(\chi^T - \chi^S)$ would belong to the fractional stable set polytope for some $\varepsilon > 0$, while it has weight larger than x, a contradiction. ∎

Picard and Queyranne [1977] showed that, for any graph $G = (V, E)$ and any weight function $w : V \to \mathbb{R}$, there is a unique minimal subset of vertices that has fractional values in some optimum fractional stable set (solving a problem posed by Nemhauser and Trotter [1975]):

Theorem 64.10. *Let $G = (V, E)$ be a graph, let $w : V \to \mathbb{R}$ be a weight function, and let x and y be maximum-weight fractional stable sets. Then there is a maximum-weight fractional stable set z such that, for each vertex v, z_v is integer if x_v or y_v integer.*

Proof. We can assume that x and y are half-integer (as we can assume that x and y are vertices of the fractional stable set polytope). For $i = 0, 1, 2$, let $X_i := \{v \mid x_v = i/2\}$ and $Y_i := \{v \mid y_v = i/2\}$. Then

(64.15) $$w(Y_0 \cap X_2) \leq w(X_0 \cap Y_2),$$

since

(64.16) $$y + \tfrac{1}{2}(\chi^{Y_0 \cap X_2} - \chi^{X_0 \cap Y_2})$$

is a fractional stable set. Otherwise, since X_2 is stable, there is an edge uv with $y_u + y_v = 1$, $u \in Y_0 \cap X_2$, and $v \notin X_0 \cap Y_2$. So $y_u = 0$, and hence $y_v = 1$. Also, $x_u = 1$, and hence $x_v = 0$. So $v \in X_0 \cap Y_2$, a contradiction. This shows (64.15).

Moreover,

(64.17) $$w(X_0 \setminus Y_0) \leq w(X_2 \setminus Y_2),$$

since

(64.18) $$x + \tfrac{1}{2}(\chi^{X_0 \setminus Y_0} - \chi^{X_2 \setminus Y_2})$$

is a fractional stable set. Otherwise there is an edge uv with $x_u + x_v = 1$, $u \in X_0 \setminus Y_0$, and $v \notin X_2 \setminus Y_2$. So $x_u = 0$, and hence $x_v = 1$. Also, $y_u > 0$, and hence $y_v < 1$. So $v \in X_2 \setminus Y_2$, a contradiction. This shows (64.17).

(64.15) and (64.17) imply that

(64.19) $$\begin{aligned} w(Y_1 \cap X_2) &= w(X_2 \setminus Y_2) - w(X_2 \cap Y_0) \geq w(X_0 \setminus Y_0) - w(Y_2 \cap X_0) \\ &= w(Y_1 \cap X_0). \end{aligned}$$

Hence

(64.20) $$z := y + \tfrac{1}{2}(\chi^{Y_1 \cap X_2} - \chi^{Y_1 \cap X_0})$$

has weight at least that of y. Moreover, z is a fractional stable set. Otherwise, as X_2 is stable, there is an edge uv with $y_u + y_v = 1$, $u \in Y_1 \cap X_2$ and $v \notin Y_1 \cap X_0$. So $y_u = y_v = \frac{1}{2}$, $x_u = 1$, hence $x_v = 0$. So $v \in Y_1 \cap X_0$, a contradiction. Hence z is a fractional stable set as required. ∎

Nemhauser and Trotter [1975] and Picard and Queyranne [1977] gave a polynomial-time algorithms to find a half-integer maximum-weight fractional stable set attaining the minimum number of fractional values. (This can be derived from the uniqueness of the minimal set of fractional vertices: just try $x_v = 0$ and $x_v = 1$ for each $v \in V$, and see if the fractional stable set number drops.)

Pulleyblank [1979a] and Bourjolly and Pulleyblank [1989] characterized the minimal set of fractional values. Related results were given by Grimmett [1986].

64.6. Fractional vertex covers

Similar results hold for *fractional vertex covers*, which are vectors $x \in \mathbb{R}^V$ satisfying

(64.21) (i) $0 \leq x_v \leq 1$ for $v \in V$,
(ii) $x_u + x_v \geq 1$ for $\{u, v\} \in E$.

Trivially, a vector x is a fractional vertex cover if and only if $\mathbf{1} - x$ is a fractional stable set.

The minimum size of a fractional vertex cover is called the *fractional vertex cover number*, and is denoted by $\tau^*(G)$. So

(64.22) $\tau^*(G) + \alpha^*(G) = |V|$.

Again, by linear programming duality, $\tau^*(G)$ is equal to the *fractional matching number* $\nu^*(G)$, which is the maximum value of $y(E)$ over all $y \in \mathbb{R}^E$ satisfying

(64.23) (i) $0 \leq y_e \leq 1$ for $e \in E$,
(ii) $y(\delta(v)) \leq 1$ for $v \in V$.

Any solution y of (64.23) is called a *fractional matching*. This was also discussed in Section 30.3, where it was shown that each vertex of the polytope determined by (64.23) (the fractional matching polytope) is half-integer. A similar result holds for the *fractional vertex cover polytope*, which is the polytope determined by (64.21):

Theorem 64.11. *Each vertex of the fractional vertex cover polytope P is half-integer.*

Proof. Directly from Theorem 64.7, since x belongs to the fractional vertex cover polytope if and only if $\mathbf{1}-x$ belongs to the fractional stable set polytope. ∎

Theorem 64.11 implies that $\tau^*(G) = \frac{1}{2}\tau_2(G)$, where $\tau_2(G)$ is the minimum size of a 2-*vertex cover*, which is an integer vector $x \in \mathbb{R}^V$ satisfying

(64.24) (i) $x_v \geq 0$ for $v \in V$,
(ii) $x_u + x_v \geq 2$ for $\{u, v\} \in E$

(cf. Section 30.10).

It also implies a characterization of the 2-*vertex cover polyhedron*, which is the convex hull of the 2-vertex covers:

Corollary 64.11a. *The 2-vertex cover polyhedron is determined by* (64.24).

Proof. Directly from Theorem 64.11. ∎

By the results on fractional stable sets and 2-stable sets given in Section 64.5, and using the reductions described above, a minimum-weight fractional vertex cover and a minimum-weight 2-vertex cover can be found in strongly polynomial time.

Notes. Corollary 64.9a and Theorem 64.10 have direct analogues for vertex covers: given a graph $G = (V, E)$ and a weight function $w : V \to \mathbb{R}$,

(64.25) for each minimum-weight fractional vertex cover x there is a minimum-weight vertex cover contained in $\{v \mid x_v \neq 0\}$,

and

(64.26) for any two minimum-weight fractional vertex covers x and y there is a minimum-weight fractional vertex cover z such that for each $v \in V$: $x_v \in \mathbb{Z}$ or $y_v \in \mathbb{Z} \Rightarrow z_v \in \mathbb{Z}$.

These statements can be derived from Corollary 64.9a and Theorem 64.10 by again observing that a vector x is a (fractional) stable set if and only if $\mathbf{1} - x$ is a (fractional) vertex cover. Similarly, Theorem 64.8 implies a characterization of the vertices of the fractional vertex cover polytope.

64.6a. A bound of Lorentzen

The fractional stable set and vertex cover numbers give upper and lower bounds on the stable set and vertex cover number, respectively. These bounds are computable in polynomial time. A better polynomial-time computable bound was given by Lorentzen [1966]:

Theorem 64.12. *For each graph $G = (V, E)$:*

(64.27) $2\nu^*(G) - \nu(G) \leq \tau(G)$.

Proof. Since $\nu^*(G) = 2\nu_2(G)$ (cf. Section 30.2), there is a half-integer fractional matching $x : E \to \mathbb{R}_+$ with $x(E) = \nu^*(G)$, such that the support of x is the disjoint union of a matching and a number t of odd circuits. We can assume that each edge of G belongs to the support of x (as deleting edges increases neither $\nu(G)$ nor $\tau(G)$). Also we can assume that G has no isolated vertices. Then $\nu(G) = \frac{1}{2}(|V|-t)$, $\tau(G) = \frac{1}{2}(|V|+t)$, and $\nu^*(G) = \frac{1}{2}|V|$. ∎

Bound (64.27) is generally a better lower bound on $\tau(G)$ than $\tau^*(G)$ (for example, for $G = K_3$). It implies an upper bound for $\alpha(G)$, generally better than $\alpha(G) \leq \rho^*(G)$:

Corollary 64.12a. *For each graph $G = (V, E)$ without isolated vertices:*

(64.28) $\alpha(G) \leq 2\rho^*(G) - \rho(G)$.

Proof. Using Theorem 30.9, we have $\alpha(G) = |V| - \tau(G) \leq |V| - 2\nu^*(G) + \nu(G) = 2(|V| - \nu^*(G)) - (|V| - \nu(G)) = 2\rho^*(G) - \rho(G)$. ∎

64.7. The clique inequalities

A set of constraints stronger than the edge inequalities (64.10)(ii) is obtained by the 'clique inequalities'. Let $P(G)$ be the polytope in $\mathbb{R}^V$ determined by

(64.29) (i) $x_v \geq 0$ for each $v \in V$,
(ii) $x(C) \leq 1$ for each clique C.

The inequalities (64.29)(ii) are called the *clique inequalities.*

Since the integer solutions of (64.29) are exactly the incidence vectors of stable sets, the stable set polytope of G is equal to the integer hull of $P(G)$ (the convex hull of the integer vectors in $P(G)$).

We call any vector x satisfying (64.29) a *strong fractional stable set.* We denote

(64.30) $\alpha^{**}(G)$:= *strong fractional stable set number* := the maximum size of a strong fractional stable set.

Since each strong fractional stable set is a fractional stable set, we know

(64.31) $\alpha(G) \leq \alpha^{**}(G) \leq \alpha^{*}(G)$.

So $\alpha^{**}(G)$ gives a better upper bound on $\alpha(G)$ than $\alpha^{*}(G)$ gives — however, $\alpha^{**}(G)$ is generally more difficult to compute.

Note that $P(G)$ is the antiblocking polyhedron of the clique polytope of G:

(64.32) $P(G) = A(P_{\text{clique}}(G))$.

(For background on antiblocking polyhedra, see Section 5.9.)

64.8. Fractional and weighted colouring numbers

For any graph $G = (V, E)$, the *fractional colouring number* $\chi^{*}(G)$ is the minimum value of $\lambda_1 + \cdots + \lambda_k$ with $\lambda_1, \ldots, \lambda_k \in \mathbb{R}_+$ such that there exist stable sets $S_1, \ldots, S_k$ with

(64.33) $\lambda_1 \chi^{S_1} + \cdots + \lambda_k \chi^{S_k} = \mathbf{1}$.

So if the λ_i are required to be integer, we have the colouring number.

By linear programming duality, the fractional colouring number is equal to the maximum of $\mathbf{1}^{\mathsf{T}}x$ over the polytope $\overline{P}(G)$ in $\mathbb{R}^V_+$ determined by

(64.34)
$$\begin{array}{ll} x_v \geq 0 & \text{for each } v \in V, \\ x(S) \leq 1 & \text{for each stable set } S. \end{array}$$

(So $\overline{P}(G) = P(\overline{G})$ and $\overline{P}(G) = A(P_{\text{stable set}}(G))$.) Hence we have:

(64.35) $\chi^{*}(\overline{G}) = \alpha^{**}(G)$.

We denote

(64.36) $\overline{\chi}^{*}(G) := \chi^{*}(\overline{G})$,

which is called the *fractional clique cover number* of G.

No polynomial-time algorithm is known to calculate $\chi^{*}(G)$. Note that the separation problem for $\overline{P}(G)$ is NP-complete, since the optimization problem over $P_{\text{stable set}}(G)$ is NP-complete.

Given a graph $G = (V, E)$ and a weight function $w : V \to \mathbb{Z}_+$, the *weighted colouring number* $\chi_w(G)$ is the minimum value of $\lambda_1 + \cdots + \lambda_k$ with $\lambda_1, \ldots, \lambda_k \in \mathbb{Z}_+$ such that there exist stable sets $S_1, \ldots, S_k$ with

(64.37) $\lambda_1 \chi^{S_1} + \cdots + \lambda_k \chi^{S_k} = w$.

So if $w = \mathbf{1}$, then $\chi_w(G)$ is equal to the colouring number $\chi(G)$ of G. Hence determining $\chi_w(G)$ is NP-complete.

For $w : V \to \mathbb{Z}_+$, let graph G^w arise from G by replacing each vertex by a clique C_v of size $w(v)$, two vertices in different cliques C_u,C_v being adjacent if and only if u and v are adjacent. Then

(64.38) $$\chi_w(G) = \chi(G^w).$$

We denote

(64.39) $$\overline{\chi}_w(G) := \chi_w(\overline{G}),$$

called the *weighted clique cover number* of G.

The fractional version is the *fractional weighted colouring number* $\chi^*_w(G)$, defined as the minimum value of $\lambda_1 + \cdots + \lambda_k$ with $\lambda_1, \ldots, \lambda_k \in \mathbb{R}_+$ such that there exist stable sets $S_1, \ldots, S_k$ with

(64.40) $$\lambda_1\chi^{S_1} + \cdots + \lambda_k\chi^{S_k} = w.$$

This value is equal to the maximum value of $w^{\mathsf{T}}x$ over the antiblocking polytope $A(P_{\text{stable set}}(G))$ pf $P_{\text{stable set}}(G)$. Since the optimization problem over $P_{\text{stable set}}(G)$ is NP-complete, determining $\chi^*_w(G)$ is NP-hard.

We denote

(64.41) $$\overline{\chi}^*_w(G) := \chi^*_w(\overline{G}),$$

called the *fractional weighted clique cover number* of G.

The complexity results above can be specialized to classes of graphs. By the results of Grötschel, Lovász, and Schrijver [1981,1984c]:

(64.42) For any collection $\mathcal{G}$ of graphs: there is a polynomial-time algorithm to find the fractional weighted colouring number for any graph in $\mathcal{G}$ and any weight function if and only there is a polynomial-time algorithm to find a maximum-weight stable set in any graph in $\mathcal{G}$ and for any weight function.

Since the problem of determining $\alpha(G)$ is NP-complete even if we restrict ourselves to planar cubic graphs, determining $\chi^*_w(G)$ for such graphs is NP-hard. As noticed in Grötschel, Lovász, and Schrijver [1981], determining $\chi^*_w(G)$ and $\chi(G)$ seem incomparable with respect to complexity. For cubic graphs G, $\chi(G)$ can be easily found in polynomial time (using Brooks' theorem (Theorem 64.3)), while determining $\chi^*_w(G)$ is NP-hard. In contrast to this, for the line graph G of a cubic graph H, $\chi(G)$ is NP-complete to compute by Holyer's theorem that 3-edge colourability is NP-complete (see Section 28.3), whereas $\chi^*_w(G)$ can be computed in polynomial time, since the separation problem over $A(P_{\text{stable set}}(G))$ is polynomial-time solvable, as the optimization problem over $P_{\text{stable set}}(G)$ is polynomial-time solvable (as it amounts to finding a maximum-weight matching in H).

64.8a. The ratio of $\chi(G)$ and $\chi^*(G)$

For later purposes we prove the following upper bound for the colouring number in terms of the fractional colouring number, obtained by applying a greedy-type algorithm (Johnson [1974a], Lovász [1975c]):

Theorem 64.13. *For any graph $G = (V, E)$:*

(64.43) $$\chi(G) \leq (1 + \ln \alpha(G))\chi^*(G).$$

Proof. Set $k := \alpha(G)$. Iteratively choose a maximum-size stable set S in G and reset G to $G - S$. We stop if VG is empty.

The stable sets found form a colouring $\mathcal{C}$ of the (original) vertex set V. So $\chi(G) \leq |\mathcal{C}|$.

For each $v \in V$, define

(64.44) $$x_v := \frac{1}{|S|},$$

where S is the set in $\mathcal{C}$ containing v. Then $x(V) = |\mathcal{C}|$, and hence

(64.45) $$\chi(G) \leq x(V).$$

Consider any stable set S' of G. Let S' consist of vertices $v_1, \ldots, v_k$, in the order by which they are covered by stable sets S in the algorithm. Then for each $i = 1, \ldots, k$, we have

(64.46) $$x_{v_i} \leq \frac{1}{k - i + 1}.$$

Indeed, let v_i be covered by $S \in \mathcal{C}$. When we selected S, the vertices $v_i, v_{i+1}, \ldots, v_k$ were uncovered yet. As we chose S, we know $|S| \geq |\{v_i, v_{i+1}, \ldots, v_k\}| = k - i + 1$, implying (64.46).

(64.46) implies

(64.47) $$x(S') \leq \sum_{i=1}^{k} \frac{1}{k - i + 1} = \sum_{i=1}^{k} \frac{1}{i} \leq 1 + \ln k \leq 1 + \ln \alpha(G).$$

So $(1 + \ln \alpha(G))^{-1} \cdot x$ satisfies (64.34), and hence

(64.48) $$(1 + \ln \alpha(G))^{-1} \cdot x(V) \leq \chi^*(G).$$

Together with (64.45), this implies (64.43). ∎

This theorem will be used in proving Theorem 67.17.

64.8b. The Chvátal rank

In Section 36.7a we defined the polyhedron P' for any rational polyhedron P and the notion of the Chvátal rank of a polyhedron P.

Let $P(G)$ denote the polytope of strong fractional stable sets, that is, the polytope determined by (64.29) (the nonnegativity and clique constraints). For any polyhedron P, let P_{I} denote the *integer hull* of P, that is, the convex hull of the integer vectors in P.

Chvátal [1973a] showed that there is no fixed t such that $P(G)^{(t)} = P(G)_{\mathrm{I}}$ for each graph G, even if we restrict G to graphs with $\alpha(G) = 2$. Chvátal, Cook, and Hartmann [1989] showed that t can be at least $\frac{1}{3}\log n$ for such graphs (where n is the number of vertices).

We will see in Corollary 65.2e that the class of graphs G with $P(G)_{\mathrm{I}} = P(G)$ is exactly the class of perfect graphs. By Edmonds' matching polytope theorem (Corollary 25.1a) if G is the line graph of some graph H, then $P(G)' = P(G)_{\mathrm{I}}$, which is the matching polytope of H.

The smallest t for which $P(G)^{(t)} = P(G)_{\mathrm{I}}$ might be an indication of the computational complexity of the stable set number $\alpha(G)$. For each fixed t, the stable set problem for graphs with $P(G)^{(t)} = P(G)_{\mathrm{I}}$ belongs to NP$\cap$co-NP. Chvátal [1973a] raised the question whether it belongs to P. (A negative indication is the result of Eisenbrand [1999] that given a polytope P by linear inequalities and given x, deciding if x belongs to P' is co-NP-complete.)

Another (weaker, but easier to compute) relaxation is: $Q(G)$ is the polytope of fractional stable sets; that is, the polytope in $\mathbb{R}^V$ determined by

(64.49) (i) $x_v \geq 0$ for each $v \in V$,
(ii) $x_v + x_w \leq 1$ for each $vw \in E$.

Again $Q(G)_{\mathrm{I}} = P_{\text{stable set}}(G)$. Since $Q(G) \supseteq P(G)$, there is no fixed t with $Q(G)^{(t)} = Q(G)_{\mathrm{I}}$ for each graph G. Chvátal [1973a] noticed that for $G = K_n$ the smallest t with $Q(G)^{(t)} = P_{\text{stable set}}(G)$ is about $\log n$.

It is not difficult to see that $Q(G)'$ is the polytope determined by (64.49) together with

(64.50) (iii) $x(VC) \leq \lfloor \frac{1}{2}|VC| \rfloor$ for each odd circuit C.

Graphs G with $Q(G)' = P_{\text{stable set}}(G)$ are called *t-perfect*. More on t-perfect graphs can be found in Chapter 68.

Chvátal [1975b] conjectures that there is no polynomial $p(n)$ such that for each graph G with n vertices we can obtain the inequality $x(V) \leq \alpha(G)$ from system (64.49) by adding at most $p(n)$ cutting planes. (That is, a list of at most $p(n)$ inequalities $a_i^{\mathsf{T}} x \leq \lfloor \beta_i \rfloor$ such that, for each i, a_i is an integer vector and the inequality $a_i^{\mathsf{T}} x \leq \beta_i$ is a nonnegative combination of inequalities from (64.49) and inequalities occurring earlier in the list.)

Chvátal, Cook, and Hartmann [1989] showed that the Chvátal rank of the following relaxation of the stable set polytope:

(64.51) $x_v \geq 0$ for $v \in V$,
$x(U) \leq \alpha(G[U])$ for $U \subseteq V$,

is $\Omega((n/\log n)^{\frac{1}{2}})$, where G is a graph with n vertices. This relaxation is stronger than the polytope determined by just the nonnegativity and clique constraints.

64.9. Further results and notes

64.9a. Graphs with polynomial-time stable set algorithm

In the remaining chapters of this part we will see that a maximum-weight stable set can be found in strongly polynomial time in perfect graphs and their complements, in t-perfect graphs, and in claw-free graphs. In perfect graphs and their complements, also a minimum vertex-colouring can be found in polynomial time. In this section we list some other classes of graphs where a maximum-size stable set or a minimum vertex-colouring can be found in polynomial time.

A graph is a *circular-arc graph* if it is the intersection graph of a set of intervals on a circle. Gavril [1974a] gave polynomial-time algorithms for finding a maximum-size clique, a maximum-size stable set, and a minimum clique cover in these graphs. Karapetyan [1980] showed that $\chi(G) \leq \frac{3}{2}\omega(G)$ for any circular-arc graph G (proving a conjecture of Tucker [1975]). More on circular-arc graphs can be found in Klee [1969], Tucker [1971,1974,1975,1978,1980], Trotter and Moore [1976], Garey, Johnson, Miller, and Papadimitriou [1980], Golumbic [1980], Orlin, Bonuccelli, and Bovet [1981], Gupta, Lee, and Leung [1982], Skrien [1982], Leung [1984], Hsu [1985, 1995], Teng and Tucker [1985], Apostolico and Hambrusch [1987], Golumbic and Hammer [1988], Masuda and Nakajima [1988], Spinrad [1988], Shih and Hsu [1989a, 1989b], Bertossi and Moretti [1990], Hell, Bang-Jensen, and Huang [1990], Hsu and Tsai [1991], Deng, Hell, and Huang [1992,1996], Eschen and Spinrad [1993], Hsu and Spinrad [1995], Bhattacharya, Hell, and Huang [1996], Bhattacharya and Kaller [1997], Hell and Huang [1997], Feder, Hell, and Huang [1999], and McConnell [2001]. See also Section 65.6d.

A graph is a *circle graph* if its vertex set is a set of chords of the circle, two chords being adjacent if they intersect or cross. For these graphs, Gavril [1973] gave polynomial-time algorithms for finding a maximum-size clique and a maximum-size stable set. Bouchet [1985,1987b,1994], Naji [1985], and Gabor, Supowit, and Hsu [1989] showed that circle graphs can be recognized in polynomial time; this was improved to quadratic time by Spinrad [1994]. (Related results can be found in Fournier [1978], Garey, Johnson, Miller, and Papadimitriou [1980], Golumbic [1980], Rotem and Urrutia [1981], de Fraysseix [1984], Hsu [1985], Naji [1985], Dagan, Golumbic, and Pinter [1988], Gabor, Supowit, and Hsu [1989], Masuda, Nakajima, Kashiwabara, and Fujisawa [1990], Felsner, Müller, and Wernisch [1994], Ma and Spinrad [1994], Spinrad [1994], and Elmallah and Stewart [1998]. See also Section 65.6d.)

The weighted stable set problem was shown to be polynomial-time solvable for graphs without $K_5 - e$ minor by Barahona and Mahjoub [1994b]. (The graph $K_5 - e$ is obtained from K_5 by deleting one edge.) Descriptions of the corresponding polytopes are given by Barahona and Mahjoub [1994b,1994c].

Hsu, Ikura, and Nemhauser [1981] gave, for each fixed k, a polynomial-time algorithm for the weighted stable set problem for graphs without odd circuits of length larger than $2k + 1$. A 'nice class for the vertex packing problem' (obtained from bipartite graphs and claw-free graphs by repeated substitution) was given by Bertolazzi, De Simone, and Galluccio [1997]. Another nice class was given by De Simone [1993].

In Section 60.3d (Corollary 60.5b) we gave a proof of Győri's theorem (Győri [1984]), stating that the following class of graphs G satisfies $\alpha(G) = \overline{\chi}(G)$. Let A be a $\{0, 1\}$ matrix such that the 1's in each row form a contiguous interval. Then G has vertex set $\{(i, j) \mid a_{i,j} = 1\}$, where two pairs (i, j) and (i', j') are adjacent if and only if $a_{i,j'} = a_{i',j} = 1$. The method of Frank and Jordán [1995b] also yields a polynomial-time algorithm to find a maximum-size stable set and a minimum clique cover. Frank [1999a] gave an alternative algorithmic proof. This class of graphs is not closed under taking induced subgraphs, and they need not be perfect.

Hammer, Mahadev, and de Werra [1985], Balas, Chvátal, and Nešetřil [1987], Balas and Yu [1989], De Simone and Sassano [1993], Hertz and de Werra [1993], Hertz [1995,1997], Brandstädt and Hammer [1999], Mosca [1999], and Lozin [2000] gave further classes of graphs for which the maximum-size or maximum-weight clique problem is polynomial-time solvable.

64.9b. Colourings and orientations

Let $D = (V, A)$ be an orientation of an undirected graph $G = (V, E)$. The following was shown by Gallai [1968a] and Roy [1967] (referring to conjectures by P. Erdős and C. Berge, respectively):

(64.52) $\chi(G) \leq \lambda(D)$,

where

(64.53) $\lambda(D) :=$ the maximum number of vertices on a directed path in D.

To see this, consider an inclusionwise maximal subset A' of A with the property that $D' = (V, A')$ is acyclic. For any $v \in V$, let $h(v)$ be the number of vertices in a longest directed path in D' ending at v. If $h(u) = h(v)$ for distinct vertices u and v, then u and v are nonadjacent, since otherwise we can add the arc joining u and v to A'. So h defines a colouring of V, with no more colours than the number of vertices in a longest directed path in D'.

This proves (64.52). Note that (64.52) implies that each tournament ($\equiv$ orientation of a complete graph) has a Hamiltonian path (a theorem of Rédei [1934] (Corollary 14.14a)).

Roy [1967] also observed that each undirected graph $G = (V, E)$ has an acyclic orientation in which the number of vertices in a longest directed path is equal to the colouring number of G. (This follows by colouring the vertices with colours $1, \ldots, \chi(G)$, and orienting any edge from u to v if the colour of u is smaller than that of v, which gives a digraph D with $\lambda(D) \leq \chi(G)$.)

This result is equivalent to the fact that for any undirected graph $G = (V, E)$:

(64.54) $\chi(G) = \min_{D} \lambda(D)$,

where D ranges over all acyclic orientations of G.

These results are essentially based on the (easy) fact that the minimum number of antichains needed to cover a partially ordered set is equal to the size of a maximum chain (Theorem 14.1).

Minty [1967] showed that for each graph $G = (V, E)$:

(64.55) $\chi(G) \leq k \iff G$ has an orientation such that each undirected circuit has at least $\frac{1}{k}|VC|$ forward arcs.

Necessity follows by colouring the vertices with colours $1, \ldots, k$, and orienting any edge from u to v if colour$(u) <$ colour(v). To see sufficiency, let D be an orientation as described. Give each arc a length $k - 1$, and add an arc in the reverse direction of length -1. Then each directed circuit in the extended digraph has nonnegative length. Hence there is a 'potential' $p : V \to \mathbb{Z}$ with $1 \leq p(v) - p(u) \leq k - 1$ for each arc (u, v) of D. Reducing p mod k gives a k-colouring as required.

Note that each orientation as in (64.55) is acyclic, and that any orientation D with $\lambda(D) \leq k$ is as in (64.55). The equivalence (64.55) gives a vertex-free description of the colouring number, and implies that $\chi(G)$ only depends on the cycle matroid of G.

Deming [1979a] showed that dual statements can be derived from Dilworth's decomposition theorem (Theorem 14.2), where 'chain' and 'antichain' are interchanged.

First one has, as a dual to (64.52), that for any orientation $D = (V, A)$ of an undirected graph $G = (V, E)$:

(64.56) $\alpha(G) \geq \xi(D)$,

where

(64.57) $\xi(D) :=$ the minimum number of directed paths in D needed to cover V.

To see (64.56), again consider an inclusionwise maximal subset A' of A with $D' = (V, A')$ acyclic. By Dilworth's decomposition theorem, V has a subset U of size $\xi(D)$ such that no two vertices in U are connected by a directed path in D. Then U is a stable set in G, since if two distinct $u, v \in U$ are adjacent in G, say $(u, v) \in A$, then $(u, v) \notin A'$, and hence $A' \cup \{(u, v)\}$ is not acyclic. But then A' contains a directed path from v to u, a contradiction.

This shows (64.56). Deming [1979a] showed also a dual form of (64.54):

(64.58) $\alpha(G) = \max_D \xi(D)$,

where D ranges over all acyclic orientations of G. Indeed, $\geq$ in (64.58) follows from (64.56). To see $\leq$, let U be a maximum-size stable set in G. Let D be any acyclic orientation of G in which each vertex in U is a source. Then $\xi(D) \geq |U| = \alpha(G)$.

64.9c. Algebraic methods

Lovász [1994] gave the following relations between stable sets, cliques, and colourings, using Hilbert's Nullstellensatz (extending Li and Li [1981] and unpublished work of D.J. Kleitman and L. Lovász). For any graph $G = (V, E)$, define the polynomial p_G in the variables x_v ($v \in V$) by:

(64.59) $$p_G := \prod_{uv \in E} (x_u - x_v)$$

(fixing some orientation of the edges). Then $\alpha(G) \leq k$ if and only if there are graphs $H_1, \ldots, H_t$ on V satisfying

(64.60) $p_G = p_{H_1} + \cdots + p_{H_t}$,

with $\overline{\chi}(H_i) \leq k$ for $i = 1, \ldots, t$. The number t can be exponentially large — hence (64.60) gives no good characterization for the stable set number. Similarly, $\chi(G) \geq k$ if and only if there are graphs satisfying (64.60) with $\omega(H_i) \geq k$ for $i = 1, \ldots, t$.

Let $G = (V, E)$ be a (simple) graph, with adjacency matrix A_G. Motzkin and Straus [1965] showed that the maximum value of $x^{\mathsf{T}} A_G x$ over $x : V \to \mathbb{R}_+$ satisfying $x(V) = 1$, is equal to $1 - \omega(G)^{-1}$.

The proof of this is easy: for any two nonadjacent vertices u, v with $x_u > 0$ and $x_v > 0$, we can reset $x_u := x_u + \varepsilon$, $x_v := x_v - \varepsilon$ for some $\varepsilon \neq 0$ without decreasing $x^{\mathsf{T}} A_G x$. Hence the maximum value is attained by a vector x whose support is a clique C. As x takes the maximum value, we should have $x_v = 1/|C|$ for each $v \in C$. Then $x^{\mathsf{T}} A_G x$ is maximized if C is a maximum-size clique.

Motzkin and Straus' theorem implies the result of Korn [1968] that the minimum value of $x^{\mathsf{T}}(I + A_G)x$ over $x : V \to \mathbb{R}_+$ with $x(V) = 1$, is equal to $\alpha(G)^{-1}$. Indeed,

(64.61) $$\min_x x^{\mathsf{T}}(I + A_G)x = \min_x x^{\mathsf{T}}(J - A_{\overline{G}})x = 1 - \max_x x^{\mathsf{T}} A_{\overline{G}} x = \omega(\overline{G})^{-1} = \alpha(G)^{-1}.$$

More on this can be found in Gibbons, Hearn, Pardalos, and Ramana [1997].

Lovász [1982,1994] gave surveys of algebraic, topological, and other methods for the stable set and the vertex colouring problem.

64.9d. Approximation algorithms

Lund and Yannakakis [1993,1994] showed that unless NP=P, there do not exist a constant c and a polynomial-time algorithm that finds a vertex-colouring of any graph G using at most $c\chi(G)$ colours. (This was proved for $c < 2$ by Garey and Johnson [1976].)

More generally, Lund and Yannakakis [1993,1994] showed that there exists an $\varepsilon > 0$ such that, unless NP=P, there is no polynomial-time algorithm to find the colouring number of a graph up to a factor of n^{ε} (where n is the number of vertices).

A similar result for maximum-size stable sets was proved by Arora, Lund, Motwani, Sudan, and Szegedy [1992,1998]. Håstad [1996,1999] showed that, if NP$\neq$P, then there is no $\varepsilon > 0$ and a polynomial-time algorithm that finds a clique that is maximum-size up to a factor $n^{1/2-\varepsilon}$. Under a slightly stronger complexity assumption (NP$\neq$ZPP), Håstad proved a factor of $n^{1-\varepsilon}$.

For background, see Johnson [1992] and Papadimitriou [1994]. Related results can be found in Hochbaum [1983a], Wigderson [1983], Berger and Rompel [1990], Feige, Goldwasser, Lovász, Safra, and Szegedy [1991,1996], Berman and Schnitger [1992], Boppana and Halldórsson [1992], Bellare, Goldwasser, Lund, and Russell [1993], Khanna, Linial, and Safra [1993,2000], Bellare and Sudan [1994], Feige and Kilian [1994,1996,1998a,1998b,2000], Karger, Motwani, and Sudan [1994,1998], Bellare, Goldreich, and Sudan [1995,1998], Feige [1995,1997], Fürer [1995], Håstad [1996,1999], Alon and Kahale [1998], Arora and Safra [1998], Engebretsen and Holmerin [2000], Srinivasan [2000], and Khot [2001].

In contrast, there is an easy algorithm to obtain a vertex cover in a graph $G = (V, E)$ of size at most $2\tau(G)$ (F. Gavril 1974 (cf. Garey and Johnson [1979])): choose any inclusionwise maximal matching M (greedily); then the set of vertices covered by M is a vertex cover of size $2|M|$. Since $\tau(G) \geq |M|$, this is a vertex cover as described.

No polynomial-time algorithm yielding a factor better than 2 is known. Håstad [1997,2001] showed that, if NP$\neq$P, no factor better than $\frac{7}{6}$ is achievable in polynomial time.

See also Section 67.4f below.

64.9e. Further notes

Yannakakis [1988,1991] showed that the stable set polytope of the line graph $L(K_n)$ of a complete graph K_n cannot be represented as the projection of a polytope in higher dimensions that is symmetric under the automorphism group of $L(K_n)$. Cao and Nemhauser [1998] characterized line graphs as those graphs whose stable set polytope is determined by the inequalities corresponding to the matching polytope constraints.

Euler, Jünger, and Reinelt [1987] extended results of Padberg [1973] on facets of the stable set polytope, to more general 'independence' polytopes.

More on the stable set polytope can be found in Fulkerson [1971a], Chvátal [1973a,1975a,1985a], Padberg [1973,1974b,1977,1979,1980,1984], Nemhauser and Trotter [1974], Trotter [1975], Wolsey [1976], Balas and Zemel [1977], Naddef and Pulleyblank [1981a], Sekiguchi [1983], Ikura and Nemhauser [1985], Grötschel, Lovász, and Schrijver [1986], Lovász and Schrijver [1989,1991], Cheng and Cunningham [1995,1997], Cánovas, Landete, and Marín [2000], Lipták and Lovász [2000, 2001], and Cheng and de Vries [2002a,2002b].

The convex hull of the incidence vectors of the stable sets of size at most a given k is studied by Janssen and Kilakos [1999]. Generalizations of the stable set polytope to more general $0, \pm 1$ programming and satisfiability problems were studied by Johnson and Padberg [1982], Hooker [1996], and Sewell [1996].

Methods for and computational results on the stable set problem (or the equivalent clique, vertex cover, and set packing problems) are given by Balas and Samuelsson [1977], Chvátal [1977], Houck and Vemuganti [1977], Tarjan and Trojanowski [1977], Geoffroy and Sumner [1978], Gerhards and Lindenberg [1979], Hansen [1980b], Bar-Yehuda and Even [1981,1982,1985], Billionnet [1981], Chiba, Nishizeki, and Saito [1982] (planar graphs), Hochbaum [1982,1983a], Loukakis and Tsouros [1982], Baker [1983,1994], Clarkson [1983], Monien and Speckenmeyer [1983,1985], Balas and Yu [1986], Jian [1986], Robson [1986], Shindo and Tomita [1988], Hurkens and Schrijver [1989], Carraghan and Pardalos [1990], Nemhauser and Sigismondi [1992], Balas and Xue [1991,1996], Boppana and Halldórsson [1992], Pardalos and Rodgers [1992], Paschos [1992], Khuller, Vishkin, and Young [1993,1994], Berman and Fürer [1994], Mannino and Sassano [1994], Halldórsson [1995], Balas, Ceria, Cornuéjols, and Pataki [1996], Bourjolly, Laporte, and Mercure [1997], Halldórsson and Radhakrishnan [1997], Alon and Kahale [1998], Arkin and Hassin [1998], Feige and Kilian [1998a], Kleinberg and Goemans [1998], Chandra and Halldórsson [1999, 2001], Nagamochi and Ibaraki [1999b], Bar-Yehuda [2000], Halperin [2000,2002], Krivelevich and Vu [2000], Chen, Kanj, and Jia [2001], Krivelevich, Nathaniel, and Sudakov [2001a,2001b], and Guha, Hassin, Khuller, and Or [2002].

Methods for graph colouring are proposed and investigated by Christofides [1971], Brown [1972], Matula, Marble, and Isaacson [1972], Corneil and Graham [1973], Johnson [1974b], Wang [1974], Lawler [1976a], McDiarmid [1979], Matula and Beck [1983], Sysło, Deo, and Kowalik [1983], Wigderson [1983], Edwards [1986], Berger and Rompel [1990], Hertz [1991], Halldórsson [1993], Blum [1994], Demange, Grisoni, and Paschos [1994], Karger, Motwani, and Sudan [1994,1998], Schiermeyer [1994], Beigel and Eppstein [1995], Blum and Karger [1997], Krivelevich and Vu [2000], Eppstein [2001], Halperin, Nathaniel, and Zwick [2001], Molloy and Reed [2001], Stacho [2001], Alon and Krivelevich [2002], and Charikar [2002].

For computational results on clique, stable set, and colouring problems, consult also Johnson and Trick [1996].

A survey of graph colouring algorithms was given by Matula, Marble, and Isaacson [1972]. Chiba, Nishizeki, and Saito [1981], Thomassen [1994], and Robertson, Sanders, Seymour, and Thomas [1996] gave linear-time 5-colouring algorithms for planar graphs. The worst-case behaviour of graph colouring algorithms was investigated by Johnson [1974b].

Mycielski [1955] showed that triangle-free graphs can have arbitrarily large colouring number. King and Nemhauser [1974] and Gyárfás [1987] and Fouquet, Giakoumakis, Maire, and Thuillier [1995] studied classes of graphs for which the colouring number can be bounded by a function of the clique number.

Gyárfás [1987] conjectures that there exists a function $g : \mathbb{Z}_+ \to \mathbb{Z}_+$ such that $\chi(G) \leq g(\omega(G))$ for each graph G without odd holes. Equivalently, for $\omega \in \mathbb{Z}_+$, let $g(\omega)$ be the maximum colouring number of a graph without odd holes and cliques of size $> \omega$. Then Gyárfás' conjectures that g is finite. It is easy to see that $g(2) = 2$. N. Robertson, P.D. Seymour, and R. Thomas announced that they proved $g(3) = 4$ (this was conjectured by G. Ding).

Upper bounds for the stable set number of a graph in terms of the degrees were presented by Hansen [1979,1980b]. Relations between the colouring number and the fractional colouring number are investigated by Kilakos and Marcotte [1997]. Reed [1998] discussed bounding the chromatic number of a graph by a convex combination of its clique number and its maximum degree plus 1. Gerke and McDiarmid [2001a,2001b] investigated the ratio of the weighted colouring and the weighted clique number.

A theorem of Turán [1941] implies that any simple graph G with n vertices and m edges satisfies:

$$\text{(64.62)} \qquad \alpha(G) \geq \frac{n^2}{n + 2m}.$$

Bondy [1978] showed that $m \geq 2\tau(G) - 1$ if G is connected. A study of the relations between several parameters derived from stability and colouring was given by Hell and Roberts [1982].

A survey on the stable set problem is given by Padberg [1979], on approximation methods for the stable set problem by Halldórsson [1998], and on colourings by Jensen and Toft [1995] and Toft [1995]. Colouring is also discussed in most graph theory books mentioned in Chapter 1.

Chapter 65

Perfect graphs: general theory

In this and the next two chapters, we consider the 'perfect' graphs, introduced by C. Berge in the 1960s. They turn out to unify several results in combinatorial optimization, in particular, min-max relations and polyhedral characterizations.
Berge proposed two conjectures, the *weak* and the *strong* perfect graph conjecture. The second implies the first.
The weak perfect graph conjecture says that perfection is maintained under taking the complementary graph. This was proved by Lovász [1972c]: the perfect graph theorem.
The strong perfect graph conjecture characterizes perfect graphs by excluding odd holes and odd antiholes. A proof of this was announced in May 2002 by Chudnovsky, Robertson, Seymour, and Thomas, resulting in the strong perfect graph theorem. The announced proof is highly complicated, and we cannot give it here.
Many of the results described in this and the next chapter follow directly as a consequence of the strong perfect graph theorem (while some of them are used in the proof). Where possible and appropriate, we give direct proofs of these consequences.
In this chapter, we give general theory, in Chapter 66 we discuss classes of perfect graphs, and in Chapter 67 we show the polynomial-time solvability of the maximum-weight clique and minimum colouring problems for perfect graphs.

65.1. Introduction to perfect graphs

As we saw before, the clique number $\omega(G)$ and the colouring number $\chi(G)$ of a graph $G = (V, E)$ are related by the inequality

(65.1) $\omega(G) \leq \chi(G)$.

Strict inequality can occur, for instance, for any odd circuit of length at least five, and its complement.

Having equality in (65.1) does not say that much about the internal structure of a graph: any graph $G = (V, E)$ can be extended to a graph $G' = (V', E')$ satisfying $\omega(G') = \chi(G')$, simply by adding to G a clique of size $\chi(G)$, disjoint from V.

However, the condition becomes much more powerful if we require that equality in (65.1) holds for each induced subgraph of G. The idea for this was formulated by Berge [1963a]. He defined a graph $G = (V, E)$ te be *perfect* if $\omega(G') = \chi(G')$ holds for each induced subgraph G' of G.

Various classes of graphs could be shown to be perfect, like the bipartite graphs (trivially) and the line graphs of bipartite graphs (by Kőnig's edge-colouring theorem).

Berge [1960a,1963a] observed the important phenomenon that for several of these classes, also the complementary graphs are perfect. Berge therefore conjectured that the complement of any perfect graph is perfect again — the *weak perfect graph conjecture*. This conjecture was proved by Lovász [1972c], by proving an equivalent form of the conjecture given by Fulkerson [1972a] (the replication lemma — see Corollary 65.2c below).

As mentioned, obvious examples of imperfect graphs are the odd circuits of length at least five, and their complements. Berge [1963a] and P.C. Gilmore (cf. Berge [1966]) made the conjecture that this characterizes perfect graphs, which is the strong perfect graph conjecture. A proof was announced in May 2002 by Chudnovsky, Robertson, Seymour, and Thomas.

To simplify formulation, it is convenient to introduce the notions of 'hole' and 'antihole'. A *hole* in a graph G is an induced subgraph of G isomorphic to a circuit with at least four vertices. An *antihole* in G is an induced subgraph of G isomorphic to the complement of a circuit with at least four vertices. A hole or antihole is *odd* if it has an odd number of vertices.

Theorem 65.1 (Strong perfect graph theorem)**.** *A graph G is perfect if and only if G contains no odd hole and no odd antihole.*

A graph containing no odd hole or odd antihole is called a *Berge graph*[2]. So the strong perfect graph theorem says that Berge graphs are precisely the perfect graphs.

An alternative formulation is in terms of minimally imperfect graphs. A *minimally imperfect* (or *critically imperfect*) graph is an imperfect graph such that each proper induced subgraph is perfect. Then the strong perfect graph theorem says that the only minimally imperfect graphs are the odd circuits of length at least five, and their complements.

It is (as yet) unknown if perfection of a graph can be tested in polynomial time. (Lovász [1986] 'would guess' that such an algorithm exists.) The clique number of a perfect graph can be determined in polynomial time, with the help of the ellipsoid method — see Chapter 67. However, no combinatorial polynomial-time algorithm is known.

We will next discuss perfect graph theory in greater detail (although we cannot give a proof of the strong perfect graph theorem). Let us make a useful observation:

[2] This term was introduced by Chvátal and Sbihi [1987].

(65.2) any minimally imperfect graph $G = (V, E)$ has no stable set S with $\omega(G - S) < \omega(G)$.

Otherwise, $\omega(G) \geq \omega(G - S) + 1 = \chi(G - S) + 1 \geq \chi(G)$, since we can use S as colour.

Similarly, for any class $\mathcal{G}$ of graphs closed under taking induced subgraphs:

(65.3) each graph $G \in \mathcal{G}$ is perfect $\iff$ each graph $G \in \mathcal{G}$ with $VG \neq \emptyset$ has a stable set S with $\omega(G - S) < \omega(G)$.

Here necessity follows from the fact that we can take for S any of the colours in a minimum colouring of G. Sufficiency follows by induction on $|VG|$: $\chi(G) \leq \chi(G - S) + 1 = \omega(G - S) + 1 \leq \omega(G)$.

65.2. The perfect graph theorem

Lovász [1972a] proved the weak perfect graph conjecture in the following stronger form (suggested by A. Hajnal), which we show with the elegant linear-algebraic proof found by Gasparian [1996].

Theorem 65.2. *A graph G is perfect if and only if $\omega(G')\alpha(G') \geq |VG'|$ for each induced subgraph G' of G.*

Proof. Necessity is easy, since if G is perfect, then $\omega(G') = \chi(G')$ for each induced subgraph G' of G, and since $\chi(G')\alpha(G') \geq |VG'|$ for any graph G'.

To see sufficiency, it suffices to show that each minimally imperfect graph G satisfies $|VG| \geq \alpha(G)\omega(G) + 1$. We can assume that $VG = \{1, \ldots, n\}$. Define $\omega := \omega(G)$ and $\alpha := \alpha(G)$.

We first construct

(65.4) stable sets $S_0, \ldots, S_{\alpha\omega}$ such that each vertex is covered by exactly α of the S_i.

Let S_0 be a stable set in G of size α. By the minimality of G, we know that for each $v \in S_0$, the graph $G - v$ is perfect, and that hence $\chi(G - v) = \omega(G - v) \leq \omega$. Therefore, $V \setminus \{v\}$ can be partitioned into ω stable sets. Doing this for each $v \in S_0$, we obtain stable sets as in (65.4).

Now for each $i = 0, \ldots, \alpha\omega$, there exists a clique C_i of size ω with $C_i \cap S_i = \emptyset$ (by (65.2)). Then, for distinct i, j with $0 \leq i, j \leq \alpha\omega$, we have $|C_i \cap S_j| = 1$. This follows from the fact that C_i has size ω and intersects each S_j in at most one vertex, and hence, by (65.4), it intersects $\alpha\omega$ of the S_j. As $C_i \cap S_i = \emptyset$, we have that $|C_i \cap S_j| = 1$ if $i \neq j$.

Now consider the $(\alpha\omega+1) \times n$ incidence matrices M and N of $S_0, \ldots, S_{\alpha\omega}$ and $C_0, \ldots, C_{\alpha\omega}$ respectively. So M and N are $\{0, 1\}$ matrices, with $M_{i,j} = 1 \iff j \in S_i$, and $N_{i,j} = 1 \iff j \in C_i$, for $i = 0, \ldots, \alpha\omega$ and $j = 1, \ldots, n$. By the above, $MN^{\mathsf{T}} = J - I$, where J is the $(\alpha\omega+1) \times (\alpha\omega+1)$ all-1 matrix, and I the $(\alpha\omega+1)\times(\alpha\omega+1)$ identity matrix. As $J-I$ has rank $\alpha\omega+1$, we have $n \geq \alpha\omega+1$. ∎

Theorem 65.2 implies (Lovász [1972c]):

Corollary 65.2a (perfect graph theorem)**.** *The complement of a perfect graph is perfect again.*

Proof. Directly from Theorem 65.2, as the condition given in it is invariant under taking the complementary graph. ∎

As was observed by Cameron [1982], Theorem 65.2 implies that the question 'Given a graph, is it perfect?' belongs to co-NP. Indeed, to certify imperfection of a graph, it is sufficient, and possible, to specify:

(65.5) (i) an induced subgraph $G=(V,E)$,
(ii) integers $\alpha,\omega \geq 2$ with $|V| = \alpha\omega+1$, and
(iii) for each $v \in V$, an ω-colouring of $G-v$ and an α-colouring of $\overline{G}-v$.

This is possible, since, by Theorem 65.2, a minimally imperfect subgraph G has these properties for $\omega := \omega(G)$ and $\alpha := \alpha(G)$. It is also sufficient to certify imperfection, since (65.5)(iii) implies that $\omega(G) \leq \omega$ and $\alpha(G) \leq \alpha$, and hence by (65.5)(ii), that G is not perfect.

Theorem 65.2 implies:

Corollary 65.2b. *Each minimally imperfect graph G satisfies*

(65.6) $$|VG| = \alpha(G)\omega(G)+1.$$

Proof. We have $|VG| \leq \alpha(G)\omega(G)+1$, since for any vertex v of G, the graph $G-v$ is perfect, and hence

(65.7) $$|VG|-1 = |V(G-v)| \leq \alpha(G-v)\omega(G-v) \leq \alpha(G)\omega(G).$$

Conversely, $|VG| \geq \alpha(G)\omega(G)+1$, since if $|VG| \leq \alpha(G)\omega(G)$, then $|VG'| \leq \alpha(G')\omega(G')$ for each induced subgraph G' of G (by the minimal imperfection of G). This implies with Theorem 65.2 that G is perfect, a contradiction. ∎

65.3. Replication

Let $G=(V,E)$ be a graph and let $v \in V$. Extend G with some new vertex, v' say, which is adjacent to v and to all vertices adjacent to v in G. In this way we obtain a new graph H, which we say is obtained from G by *duplicating* v. Repeated duplicating a vertex is called *replicating*. *Replicating* a vertex v *by a factor* k means duplicating v $k-1$ times if $k \geq 1$, and deleting v if $k=0$.

Corollary 65.2c (replication lemma)**.** *Let H arise from G by duplicating vertex v. Then if G is perfect, also H is perfect.*

Proof. By the perfect graph theorem, it suffices to show that $\overline{H}$ is perfect, and hence (as we can apply induction) that $\omega(\overline{H}) = \chi(\overline{H})$.

By the perfect graph theorem, if G is perfect, then $\overline{G}$ is perfect. Hence $\omega(\overline{H}) = \omega(\overline{G}) = \chi(\overline{G}) = \chi(\overline{H})$. ∎

Repeated application of Corollary 65.2c implies the following (the weighted colouring number is defined in Section 64.8):

Corollary 65.2d. *Let G be a perfect graph and let $w : V \to \mathbb{Z}_+$ be a 'weight' function. Then the maximum weight of a clique is equal to the weighted colouring number $\chi_w(G)$ of G.*

Proof. Let G^w be the graph arising from G by replicating any vertex v by a factor $w(v)$. By Corollary 65.2c, G^w is perfect, and so $\omega(G^w) = \chi(G^w)$. Since $\omega(G^w)$ is equal to the maximum weight of a clique in G and since $\chi(G^w) = \chi_w(G)$, the corollary follows. ∎

65.4. Perfect graphs and polyhedra

The *clique polytope* of a graph $G = (V, E)$ is the convex hull of the incidence vectors of the cliques. Clearly, any vector x in the clique polytope satisfies:

(65.8) (i) $x_v \geq 0$ for each $v \in V$,
(ii) $x(S) \leq 1$ for each stable set S.

Fulkerson [1972a] and Chvátal [1975a] showed that Corollary 65.2d implies a polyhedral characterization of perfect graphs:

Corollary 65.2e. *A graph G is perfect if and only if its clique polytope is determined by* (65.8)*.*

Proof. *Necessity.* Let G be perfect. To prove that the clique polytope is determined by (65.8), it suffices to show that for each weight function $w : V \to \mathbb{Z}_+$, the maximum weight t of a clique in G is not less than the maximum of $w^{\mathsf{T}}x$ over (65.8). By Corollary 65.2d, there exist stable sets $S_1, \ldots, S_t$ with

(65.9) $$w = \chi^{S_1} + \cdots + \chi^{S_t}.$$

Hence for each x satisfying (65.8) we have

(65.10) $$w^{\mathsf{T}}x = x(S_1) + \cdots + x(S_t) \leq t.$$

Sufficiency. Let the clique polytope of G be determined by (65.8). Suppose that G is not perfect. Choose a minimal set U with $\omega(G[U]) < \chi(G[U])$. Let

$w := \chi^U$. The function $w^\mathsf{T} x$ is maximized over $P_{\text{clique}}(G)$ by the incidence vector of each maximum-size clique of $G[U]$. Moreover, by linear programming duality, there exists a stable set S with $x(S) = 1$ for each optimum solution x. So S intersects each maximum-size clique of $G[U]$, and hence

(65.11) $$\omega(G[U \setminus S]) \leq \omega(G[U]) - 1 < \chi(G[U]) - 1 \leq \chi(G[U \setminus S]),$$

contradicting the minimality of U. ∎

Corollary 65.2e is equivalent to: G is perfect if and only if $P_{\text{clique}}(G) = A(P_{\text{stable set}}(G))$. (Here $A(P)$ is the antiblocking polyhedron of P.) Hence it implies the perfect graph theorem (using the theory of antiblocking polyhedra (cf. Section 5.9)):

(65.12) $$\begin{aligned} &G \text{ is perfect} \iff P_{\text{clique}}(G) = A(P_{\text{stable set}}(G)) \\ &\iff P_{\text{stable set}}(G) = A(P_{\text{clique}}(G)) \\ &\iff P_{\text{clique}}(\overline{G}) = A(P_{\text{stable set}}(\overline{G})) \iff \overline{G} \text{ is perfect.} \end{aligned}$$

Corollary 65.2d also implies that perfect graphs can be characterized by total dual integrality:

Corollary 65.2f. *A graph G is perfect if and only if system* (65.8) *is totally dual integral.*

Proof. Directly from Corollaries 65.2d and 65.2e. ∎

So for any graph G we have that (65.8) determines an integer polytope if *and only if* it is totally dual integral.

65.4a. Lovász's proof of the replication lemma

The proof of Lovász [1972c] of the weak perfect graph theorem is based on proving the 'replication lemma' (Corollary 65.2c above), as follows.

By (65.2), it suffices to find a stable set S in H intersecting all maximum-size cliques of H, since any induced subgraph of H is an induced subgraph of G or arises from it by replication.

Consider an $\omega(G)$-colouring of G, and let S be the colour containing v. Then S intersects each maximum-size clique C of H. Indeed, if $v' \notin C$, then C is a maximum-size clique of G, and so it intersects S. If $v' \in C$, then also $v \in C$ (as $C \cup \{v\}$ is a clique), and so C intersects S.

This proves the replication lemma, which by repeated application gives Corollary 65.2d. Since the proof of Corollary 65.2e given above only uses Corollary 65.2d, this shows (with (65.12)) that the replication lemma implies the perfect graph theorem. This is Fulkerson's proof of the equivalence of the replication lemma and the weak perfect graph conjecture ($\equiv$ perfect graph theorem).

65.5. Decomposition of Berge graphs

The proof of the strong perfect graph conjecture is based on a decomposition theorem of Berge graphs, stating that each Berge graph can be decomposed into 'basic' graphs: bipartite graphs and their complements, and line graphs of bipartite graphs and their complements. We formulate the decomposition rules.

Let $G = (V, E)$ be a graph. A 2-*join* of G is a partition of V into sets V_1 and V_2 such that for $i = 1, 2$, $|V_i| \geq 3$ and V_i contains disjoint nonempty subsets A_i, B_i with the property that for all $v_1 \in V_1$ and $v_2 \in V_2$:

(65.13) v_1 and v_2 are adjacent $\iff$ $v_1 \in A_1$, $v_2 \in A_2$, or $v_1 \in B_1$, $v_2 \in B_2$.

A *skew partition* of G is a partition V_1, V_2 of V such that $G[V_1]$ and $\overline{G}[V_2]$ are disconnected. An *homogeneous pair* of G is a pair A, B of disjoint subsets of V such that $3 \leq |A| + |B| \leq |V| - 2$ and such that for all $x, y \in A \cup B$ and $z \in V \setminus (A \cup B)$, if $xz \in E$ and $yz \notin E$, then x and y belong to distinct sets A, B.

M. Chudnovsky, N. Robertson, P.D. Seymour, and R. Thomas announced in May 2002 that they proved the following[3]:

Theorem 65.3. *Let G be a Berge graph. Then G or $\overline{G}$ is bipartite or the line graph of a bipartite graph, or has a* 2*-join, a skew partition, or a homogeneous pair.*

Unfortunately, we cannot give the (highly complicated) proof of this theorem. M. Chudnovsky, N. Robertson, P.D. Seymour, and R. Thomas also showed that any minimum-size imperfect Berge graph has no skew partition[4]. Since such a graph has no 2-join (Cornuéjols and Cunningham [1985] and Kapoor [1994] — see Corollary 65.7a below) and no homogeneous pair (Chvátal and Sbihi [1987]), and since bipartite graphs and their line graphs are perfect (Kőnig [1916] — see Section 66.1), this implies:

Theorem 65.4 (strong perfect graph theorem)**.** *A graph is perfect if and only if it is a Berge graph.*

65.5a. 0- and 1-joins

A 0-*join* of a graph $G = (V, E)$ is a partition of V into nonempty sets V_1 and V_2 such that no edge connects V_1 and V_2. Let $G_1 := G[V_1]$ and $G_2 = G[V_2]$. Then G is called the 0-*join* of G_1 and G_2. Trivially:

[3] This was conjectured by M. Conforti, G. Cornuéjols, N. Robertson, P.D. Seymour, R. Thomas, and K. Vušković (cf. Cornuéjols [2002]). It builds on work of Roussel and Rubio [2001], and it was stimulated by interaction with concurrent work of Conforti, Cornuéjols, Vušković, and Zambelli [2002] and Conforti, Cornuéjols, and Zambelli [2002b].

[4] conjectured by Chvátal [1985c].

Theorem 65.5. *G is perfect $\iff$ G_1 and G_2 are perfect.*

Proof. This follows from the facts that $\omega(G) = \max\{\omega(G_1), \omega(G_2)\}$ and $\chi(G) = \max\{\chi(G_1), \chi(G_2)\}$, and that induced subgraphs of G arise by the same construction from induced subgraphs of G_1 and G_2. ∎

Hence

(65.14) no minimally imperfect graph has a 0-join.

A 1-*join* (or *join*) of a graph $G = (V, E)$ is a partition of V into subsets V_1 and V_2 such that $|V_1| \geq 2$, $|V_2| \geq 2$, and such that there exist nonempty $A_1 \subseteq V_1$ and $A_2 \subseteq V_2$ with the property that for all $v_1 \in V_1$ and $v_2 \in V_2$:

(65.15) v_1 and v_2 are adjacent $\iff$ $v_1 \in A_1$ and $v_2 \in A_2$.

Choose $v_1 \in A_1$ and $v_2 \in A_2$, and define $G_1 := G[V_1 \cup \{v_2\}]$ and $G_2 := G[V_2 \cup \{v_1\}]$. Then G is called the 1-*join* of G_1 and G_2.

Bixby [1984] proved (generalizing a result of Lovász [1972c]):

Theorem 65.6. *G is perfect $\iff$ G_1 and G_2 are perfect.*

Proof. Necessity follows from the fact that G_1 and G_2 are induced subgraphs of G. To prove sufficiency, it suffices to show $\omega(G) = \chi(G)$, since each induced subgraph of G arises by the same construction, or by a 0-join from induced subgraphs of G_1 and G_2. Let $\omega := \omega(G)$ and $a_i := \omega(G[A_i])$ for $i = 1, 2$. It suffices to show that for each $i = 1, 2$,

(65.16) $G[V_i]$ has an ω-colouring such that A_i uses a_i colours only,

since then we can assume that we use different colours for A_1 and A_2 (as $a_1 + a_2 \leq \omega$), yielding an ω-colouring of G.

To prove (65.16), we may assume that $i = 1$. Let G'_1 be the graph obtained from G_1 by replicating v_2 by a factor $\omega - a_1$. So $\omega(G'_1) = \omega$. By the replication lemma, G'_1 is perfect. Hence $\omega(G'_1) = \chi(G'_1)$. As the clique of vertices obtained from v_2 has size $\omega - a_1$, we use only a_1 colours for A_1, as required. ∎

An alternative proof follows from Cunningham [1982b]. Cunningham [1982a] described an $O(n^3)$-time algorithm to find a 1-join (if any).

Theorem 65.6 implies:

(65.17) no minimally imperfect graph has a 1-join,

since it has no 0-join, and hence G_1 and G_2 as above are proper induced subgraphs of G, implying that they are perfect. Therefore, by Theorem 65.6, G is perfect, a contradiction.

65.5b. The 2-join

We next show that a minimally imperfect graph has no 2-join, except if it is an odd circuit. This was shown by Cornuéjols and Cunningham [1985] (for a special case) and Kapoor [1994].

The proof uses the following 'special replication lemma' (Cornuéjols and Cunningham [1985]). Let $e = uv$ be an edge of a graph G. Let G' be the graph obtained from replicating v and deleting edge uv', where v' is the new vertex.

Lemma 65.7α (special replication lemma)**.** *If G is perfect and uv is not contained in a triangle of G, then G' is again perfect.*

Proof. It suffices to show that G' has a stable set S' such that $\omega(G' - S') < \omega(G')$. If $\omega(G') > \omega(G)$, we can take $S' = \{v'\}$. So we may assume $\omega(G') = \omega(G)$. Let S be the colour of an $\omega(G)$-colouring of G with $v \in S$. Let $S' := (S \setminus \{v\}) \cup \{v'\}$. Then S' is a stable set in G'. If $\omega(G' - S') < \omega(G')$ we are done. So assume that $\omega(G' - S') = \omega(G')$. Let C be a clique in $G' - S'$ of size $\omega(G)$. Since $\omega(G - S) < \omega(G)$ and since $G - S = G' - S' - v$, we know $v \in C$. Since $\omega(G') = \omega(G)$, we know $u \in C$. Hence, $C = \{u, v\}$ (since uv is not in a triangle). So $\omega(G') = 2$, and hence vv' is not contained in a triangle of G'. But then v' has degree 1 in G', implying $\chi(G') = \chi(G) = \omega(G) = \omega(G')$. ∎

Next we consider a *special* 2*-join*, namely where the sets A_i and B_i in the definition of 2-join are connected by a path in $G[V_i]$ (for $i = 1, 2$). For $i = 1, 2$, let P_i be a shortest $A_i - B_i$ path in $G[V_i]$. Define $G_1 := G[V_1 \cup VP_2]$ and $G_2 := G[V_2 \cup VP_1]$.

Theorem 65.7. *G is perfect $\iff$ G_1 and G_2 are perfect.*

Proof. Necessity follows from the fact that G_1 and G_2 are induced subgraphs of G. To prove sufficiency, it is enough to prove $\omega(G) = \chi(G)$, since each induced subgraph of G arises by the same construction, or by 1- or 0-joins, from induced subgraphs of G_1 and G_2. Define $\omega := \omega(G)$, and

(65.18) $\quad a_i := \omega(G[A_i])$ and $b_i := \omega(G[B_i])$ for $i = 1, 2$.

Note that perfection of G_1 implies that $|EP_1| \equiv |EP_2| \pmod 2$, since $VP_1 \cup VP_2$ induces a hole in G_1.

For any colouring ϕ of a graph and any set X of vertices, let $\phi(X)$ denote the set of colours used by X. We show that, for each $i = 1, 2$, $G[V_i]$ has an ω-colouring $\phi : V \to \{1, \ldots, \omega\}$ such that

(65.19)
(i) $\phi(A_i) = \{1, \ldots, a_i\}$;
(ii) if $|EP_i|$ is even, then $\phi(B_i) = \{1, \ldots, b_i\}$;
(iii) if $|EP_i|$ is odd, then $\phi(B_i) = \{\omega - b_i + 1, \ldots, \omega\}$.

This yields an ω-colouring of G, by replacing the colour, i say, of any vertex in V_2 by $\omega - i + 1$. (The correctness follows from $a_1 + a_2 \le \omega$ and $b_1 + b_2 \le \omega$.)

To prove the existence of a colouring satisfying (65.19), we may assume $i = 1$. Let $v_0, v_1, \ldots, v_k$ be the vertices (in order) of the $A_2 - B_2$ path P_2.

First assume that $k > 1$ or $a_1 + b_1 \ge \omega$. Let G_1' be the graph arising from G_1 by replicating v_j by a factor

(65.20)
$$\begin{array}{ll} \omega - a_1 & \text{if } j < k-1 \text{ and } j \text{ is even,} \\ a_1 & \text{if } j < k-1 \text{ and } j \text{ is odd,} \\ \min\{\omega - a_1, b_1\} & \text{if } j = k-1 \text{ and } j \text{ is even,} \\ \min\{a_1, b_1\} & \text{if } j = k-1 \text{ and } j \text{ is odd,} \\ \omega - b_1 & \text{if } j = k. \end{array}$$

Then $\omega(G_1') = \omega$, and any ω-colouring of G_1' yields a colouring satisfying (65.19). Indeed, if k is even, then (65.20) implies that the set of colours used by the copies of v_0 and the set of colours used by the copies of v_k are comparable[5]. If k is odd, then (65.20) implies that the set of colours used by the copies of v_0 and the set of colours *not* used by the copies of v_k are comparable.

Next assume that $k = 1$ and $a_1 + b_1 < \omega$. Extend G_1 by a new vertex v', adjacent to all vertices in B_1 and to v_1. By the special replication lemma (Lemma 65.7α), the new graph G_1'' is again perfect. Let G_1' be the graph arising from G_1'' by replicating v_0 by a factor $\omega - a_1$, v_1 by a factor a_1, and v' by a factor $\omega - a_1 - b_1$. Again, $\omega(G_1') = \omega$, and any ω-colouring of G_1' yields a colouring satisfying (65.19). ∎

This implies:

Corollary 65.7a. *Any minimally imperfect graph having a* 2*-join is an odd circuit.*

Proof. Let G be a minimally imperfect graph, and let V_i, A_i, B_i (for $i = 1, 2$) be as in the definition of 2-join. If for some i, the graph $G[V_i]$ has no $A_i - B_i$ path, then G has a 0- or 1-join, contradicting (65.14) or (65.17). So we can assume that, for $i = 1, 2$, $G[V_i]$ has an $A_i - B_i$ path. Let P_i be a shortest such path.

By Theorem 65.7 and by symmetry, we may assume that $G[V_1 \cup VP_2]$ is not perfect. Hence, by the minimal imperfection of G, $G = G[V_1 \cup VP_2]$.

We first show $\omega(G) = 2$. Choose an internal vertex u on P_2. (This exists, since $|V_2| \geq 3$.) Choose $v \in V \setminus \{u\}$. By the minimal imperfection of G, we know $\chi(\overline{G} - v) = \alpha(G - v)$. Therefore, $VG \setminus \{v\}$ can be partitioned into $\alpha(G - v)$ cliques. Since $|VG| = \alpha(G)\omega(G) + 1$ (by (65.6)), each of these cliques has size $\omega(G)$. In particular, u is in a clique of size $\omega(G)$. Hence, since u is an internal vertex of P_2, $\omega(G) = 2$.

As $\omega(G) = 2$, $\chi(G - v) \leq 2$ for each $v \in VG$; that is, $G - v$ is bipartite for each $v \in VG$. So each odd circuit is Hamiltonian. As G is not bipartite, G has an odd circuit. This circuit has no chords, as otherwise there exists a shorter odd circuit. ∎

Cornuéjols and Cunningham [1985] gave an $O(n^2m^2)$-time algorithm to find a 2-join in a given graph (if any).

65.6. Pre-proof work on the strong perfect graph conjecture

In this section we survey research done on the strong perfect graph conjecture before it was proved in general. Many of the results follow as a consequence of the strong perfect graph theorem. Since the proof of this theorem is very complicated, we will include proofs not based on the strong perfect graph theorem.

[5] Sets X and Y are called *comparable* if $X \subseteq Y$ or $Y \subseteq X$.

65.6a. Partitionable graphs

The strong perfect graph theorem implies that each minimally imperfect graph is a circuit or its complement, and hence is highly symmetric. Before the strong perfect graph theorem was proved, several regularity properties of minimally imperfect graphs were shown, initiated by the work of Padberg [1974a].

A graph $G = (V, E)$ is called *partitionable* if $|V| = \alpha(G)\omega(G)+1$ and $\chi(G-v) = \omega(G)$ and $\overline{\chi}(G - v) = \alpha(G)$ for each $v \in V$. By Corollary 65.2b, each minimally imperfect graph is partitionable. As each partitionable graph is imperfect, the strong perfect graph theorem is equivalent to: each partitionable graph has an odd hole or odd antihole.

Partitionable graphs are characterized as follows[6]. Our proof of necessity is based on Gasparian [1996] (and is similar to the proof of Theorem 65.2).

Theorem 65.8. *A graph G is partitionable if and only if $|VG| = \alpha(G)\omega(G) + 1$ and each vertex is contained in exactly $\alpha(G)$ stable sets of size $\alpha(G)$ and in exactly $\omega(G)$ cliques of size $\omega(G)$.*

Proof. Define $n := |VG|$, $\alpha := \alpha(G)$, and $\omega := \omega(G)$.

I. To see necessity, let G be partitionable. Then the proof method of Theorem 65.2 applies: We again construct

(65.21) stable sets $S_0, \ldots, S_{\alpha\omega}$ such that each vertex is covered by exactly α of the S_i.

Indeed, let S_0 be a stable set in G of size α. For each vertex v, as G is partitionable, we know $\chi(G - v) = \omega$. Therefore, $VG \setminus \{v\}$ can be partitioned into ω stable sets. Doing this for each $v \in S_0$, we obtain stable sets as in (65.21).

Next, for each $i = 0, \ldots, \alpha\omega$, there exists a clique C_i of size ω with $C_i \cap S_i = \emptyset$. To see this, choose $v \in S_i$. As G is partitionable, $\chi(\overline{G}-v) = \alpha$, and hence $VG \setminus \{v\}$ can be partitioned into α cliques. Since $n = \alpha\omega + 1$, each clique has size ω. Since $|S_i \setminus \{v\}| \leq \alpha - 1$, at least one of these cliques is disjoint from S_i.

Then, for distinct i, j with $0 \leq i, j \leq \alpha\omega$, we have $|C_i \cap S_j| = 1$. This follows from the fact that C_i has size ω and intersects each S_j in at most one vertex, and hence, by (65.21), C_i intersects $\alpha\omega$ of the S_j. As $C_i \cap S_i = \emptyset$, we have that $|C_i \cap S_j| = 1$ if $i \neq j$.

Now consider the $(\alpha\omega + 1) \times n$ incidence matrices M and N of $S_0, \ldots, S_{\alpha\omega}$ and $C_0, \ldots, C_{\alpha\omega}$ respectively. So M and N are $\{0, 1\}$ matrices, with $M_{i,j} = 1 \iff j \in S_i$, and $N_{i,j} = 1 \iff j \in C_i$, for $i = 0, \ldots, \alpha\omega$ and $j = 1, \ldots, n$. By the above, $MN^{\mathsf{T}} = J - I$, where J is the $(\alpha\omega + 1) \times (\alpha\omega + 1)$ all-1 matrix, and I the $(\alpha\omega + 1) \times (\alpha\omega + 1)$ identity matrix. So M and N are nonsingular.

It then suffices (by symmetry) to show that each maximum-size clique C occurs among $C_0, \ldots, C_n$. Now $(M\chi^C)_i$ is 1 if $|C \cap S_i| = 1$, and is 0 otherwise. As $|C| = \omega$ and as each $v \in V$ belongs to exactly α of the S_i, C intersects precisely $\alpha\omega$ of the S_i. That is, there is exactly one, say S_j, disjoint from C. Hence $M\chi^C = M\chi^{C_j}$, and therefore $C = C_j$, as M is nonsingular.

[6] Necessity of the condition for minimally imperfect graphs was shown by Padberg [1974a], and for partitionable graphs in general by Bland, Huang, and Trotter [1979]. As to sufficiency, Cameron [1982] referred to private communication with A. Lubiw in 1981, and Whitesides [1982] called it 'well known'.

II. To see sufficiency, let G satisfy the condition. As each vertex of G is in exactly α stable sets of size α, there are exactly n maximum-size stable sets. Similarly, there are exactly n maximum-size cliques.

Let M and N be the incidence matrix of the maximum-size stable sets and maximum-size cliques, respectively. We can order the rows such that $MN^{\mathsf{T}} = J - I$, where J is the all-one $n \times n$-matrix, and I the identity matrix of order n. To see this, each maximum-size stable set S intersects precisely $\alpha\omega$ maximum-size cliques, since $|S| = \alpha$ and each vertex $v \in S$ is in precisely ω maximum-size cliques. Hence there is a unique maximum-size clique C disjoint from S. Similarly, for each maximum-size clique C there is a unique maximum-size stable set S disjoint from C.

So $MN^{\mathsf{T}} = J - I$, implying

(65.22) $$\begin{aligned} M(J-I)N^{\mathsf{T}} &= MJN^{\mathsf{T}} - MN^{\mathsf{T}} = \alpha JN^{\mathsf{T}} - (J-I) = \alpha\omega J - J + I \\ &= nJ - 2J + I = (J-I)(J-I) = MN^{\mathsf{T}}MN^{\mathsf{T}}. \end{aligned}$$

Since M and N^{T} are nonsingular, this implies $N^{\mathsf{T}}M = J - I$.

Now choose $v \in V$. As $N^{\mathsf{T}}M = J - I$, for each $u \in V \setminus \{v\}$ there exists a unique pair of a maximum-size clique C_u and a maximum-size stable set S_u with $u \in C_u$, $v \in S_u$, and $C_u \cap S_u = \emptyset$. Then for each $w \in C_u$ we have $C_w = C_u$, since $w \in C_u$ and $v \in S_u$. So the C_u partition $V \setminus \{v\}$, and hence $\chi(\overline{G} - v) = \alpha$. Then also $\chi(G - v) = \omega$ by symmetry. ∎

A partitionable graph G with $\alpha(G) = \alpha$ and $\omega(G) = \omega$, is also called an (α, ω)-*graph*.

The proof of Theorem 65.8 also implies the following further properties of partitionable graphs (properties (i)-(iii) were proved for minimally imperfect graphs by Padberg [1974a] and for partitionable graphs by Bland, Huang, and Trotter [1979]; property (iv) was shown by Whitesides [1982]):

Theorem 65.9. *Let G be a partitionable graph with n vertices. Then:*

(65.23)
(i) *G contains exactly n maximum-size cliques and exactly n maximum-size stable sets;*
(ii) *the matrix N formed by the incidence vectors of the maximum-size cliques is nonsingular, and the matrix M formed by the incidence vectors of the maximum-size stable sets is nonsingular;*
(iii) *each maximum-size clique intersects all but one maximum-size stable sets, and each maximum-size stable set intersects all but one maximum-size cliques;*
(iv) *for any two distinct vertices u, v of G there is a unique pair of a maximum-size clique C and a maximum-size stable set S with $u \in C$, $v \in S$, and $C \cap S = \emptyset$.*

Proof. See the proof of Theorem 65.8. ∎

Notes. One may show that $|\det M| = \alpha(G)$ and $|\det N| = \omega(G)$ for any partitionable graph. Indeed, since $M\mathbf{1} = \alpha(G) \cdot \mathbf{1}$, we have that $M^{-1}\mathbf{1} = \alpha(G)^{-1} \cdot \mathbf{1}$. Hence $\alpha(G)$ divides $\det M$ (as $(\det M) \cdot M^{-1}$ is an integer matrix). Similarly, $\omega(G)$ divides $\det N$. Now $|\det M \cdot \det N| = |\det(MN^{\mathsf{T}})| = |\det(J - I)| = |VG| - 1 = \alpha(G)\omega(G)$. So $|\det M| = \alpha(G)$ and $|\det N| = \omega(G)$.

Shepherd [1994b] showed that a graph G is partitionable if and only if for some $p, q \geq 2$ with $|VG| = pq + 1$: (i) G has a family of $|VG|$ stable sets of size p such that each vertex is in precisely p of them, and (ii) G has no stable set S of size p that intersects each clique of size q. A polynomial-time recognition algorithm of partitionable graphs was given by Shepherd [2001].

65.6b. More characterizations of perfect graphs

It is not difficult to show that for any partitionable graph G one has:

(65.24) $$\chi^*(G) = \omega(G) + \frac{1}{\alpha(G)}.$$

Indeed, let $n := |VG|$, $\alpha := \alpha(G)$, $\omega := \omega(G)$, $\chi^* := \chi^*(G)$. To see $\geq$, observe that the vector $\alpha^{-1} \cdot \mathbf{1}$ satisfies all stable set inequalities (64.34), and hence $\chi^* \geq n\alpha^{-1} = \omega + \alpha^{-1}$. To see $\leq$, give each stable set of size α a value α^{-1}. This gives a fractional colouring of size $\omega + \alpha^{-1}$. So $\chi^* \leq \omega + \alpha^{-1}$, proving (65.24).

Hence perfect graphs can be characterized by:

Theorem 65.10. *A graph G is perfect $\iff$ $\chi^*(G')$ is an integer for each induced subgraph G' of G.*

Proof. See above. ∎

Berge [1973a] gave the following further characterization of perfect graphs. For any graph $G = (V, E)$, let $\chi_2(G)$ denote the *bicolouring number* of G, being the minimum number of stable sets $S_1, \ldots, S_t$ such that each vertex is in two of the S_i. Alternatively, it is the minimum number of colours such that we can assign to each vertex a pair of colours in such a way that any two adjacent vertices get two disjoint pairs of colours.

Theorem 65.11. *A graph G is perfect if and only if $\chi_2(G') = 2\chi(G')$ for each induced subgraph G' of G.*

Proof. To see necessity, we have $2\omega(G) \leq \chi_2(G) \leq 2\chi(G)$ for each graph G. Hence if G is perfect, then $\omega(G) = \chi(G)$, and hence $\chi_2(G) = 2\chi(G)$. As perfection is closed under taking induced subgraphs, necessity of the condition follows.

To see sufficiency, let G be a minimally imperfect graph. Consider two nonadjacent vertices u and v. Then $\chi_2(G) \leq \chi(G - u) + \chi(G - v) + 1$ (as we can take $\{u, v\}$ as a colour). Since, by the condition, $\chi_2(G) = 2\chi(G)$, we can assume, by symmetry, that $\chi(G) \leq \chi(G - u)$. Hence $\chi(G) \leq \chi(G - u) \leq \omega(G - u) \leq \omega(G)$, contradicting the fact that G is minimally imperfect. ∎

(This proof does not use the perfect graph theorem.)

65.6c. The stable set polytope of minimally imperfect graphs

The following theorem of Padberg [1976] is a direct consequence of the strong perfect graph conjecture, but we give a direct proof (we adapt the proof of Seymour [1990b]):

Theorem 65.12. *Let $G = (V, E)$ be a minimally imperfect graph. Then the polytope determined by*

(65.25) (i) $x_v \geq 0$ for $v \in V$,
(ii) $x(C) \leq 1$ for each clique C,

has precisely one noninteger vertex, namely $\omega^{-1} \cdot \mathbf{1}$, where $\omega := \omega(G)$.

Proof. Let $G = (V, E)$ be a minimally imperfect graph, and let x^* be a noninteger vertex of the polytope determined by (65.25). Set $n := |V|$.

First we have that

(65.26) $x^*_v > 0$ for each vertex v.

For suppose that $x^*_v = 0$. Then $x^*|V \setminus \{v\}$ is a noninteger vertex of the polytope (65.25) for $G - v$, contradicting the perfection of $G - v$ (by Corollary 65.2e). This proves (65.26).

Let $\mathcal{C}$ be a collection of cliques C such that $x^*(C) = 1$ for each $C \in \mathcal{C}$ and such that $\{\chi^C \mid C \in \mathcal{C}\}$ is a set of n linearly independent vectors. For $v \in V$, let $\mathcal{C}_v$ denote the collection of $C \in \mathcal{C}$ with $v \in C$. Then:

(65.27) $|\mathcal{C}_v| \leq \omega$ for each $v \in V$.

To see this, consider any $v \in V$ and any $C \in \mathcal{C} \setminus \mathcal{C}_v$. Since $G - v$ is perfect, the vector $x^*|V \setminus \{v\}$ is a convex combination $\sum_S \lambda_S \chi^S$ of stable sets S in $G - v$. For each $u \in C$, choose a stable set S_u with $u \in S_u$ and $\lambda_{S_u} > 0$. Then $|C' \cap S_u| = 1$ for each $C' \in \mathcal{C} \setminus \mathcal{C}_v$ (since $(x^*|V \setminus \{v\})(C') = 1$). So the incidence vectors χ^{S_u} for $u \in C$ are linearly independent. This implies that the vectors $\chi^{S_u} - x^*$ for $u \in C$ have rank at least $|C| - 1$. As each of these vectors is orthogonal to $\chi^{C'}$ for each $C' \in \mathcal{C} \setminus \mathcal{C}_v$, we have

(65.28) $|\mathcal{C} \setminus \mathcal{C}_v| \leq (n-1) - (|C| - 1) = n - |C|$.

Let U be the set of vertices not covered by all cliques in $\mathcal{C}$. Then:

(65.29)
$$n = \sum_{C \in \mathcal{C}} 1 = \sum_{C \in \mathcal{C}} \sum_{v \in V \setminus C} \frac{1}{n - |C|} = \sum_{v \in U} \sum_{C \in \mathcal{C} \setminus \mathcal{C}_v} \frac{1}{n - |C|}$$
$$\leq \sum_{v \in U} \sum_{C \in \mathcal{C} \setminus \mathcal{C}_v} \frac{1}{|\mathcal{C} \setminus \mathcal{C}_v|} = \sum_{v \in U} 1 = |U| \leq n.$$

So we have equality throughout; that is, $U = V$ and $|\mathcal{C} \setminus \mathcal{C}_v| = |V| - |C|$ for each $v \in V$ and each $C \in \mathcal{C} \setminus \mathcal{C}_v$. This gives equality in (65.28). So $|\mathcal{C}_v| = |C| \leq \omega$, proving (65.27).

Let $\mathcal{C}'$ denote the collection of maximum-size cliques in G. By Theorem 65.9, each $v \in V$ is in precisely ω sets in $\mathcal{C}'$. Hence

(65.30)
$$n = \sum_{C \in \mathcal{C}} 1 = \sum_{C \in \mathcal{C}} x^*(C) = \sum_{v \in V} |\mathcal{C}_v| x^*_v \leq \omega \sum_{v \in V} x^*_v = \sum_{C \in \mathcal{C}'} x^*(C)$$
$$\leq \sum_{C \in \mathcal{C}'} 1 = n.$$

Hence we have equality throughout. Therefore, x^* satisfies $x^*(C) = 1$ for each maximum-size clique C. Hence $x^* = \omega^{-1} \cdot \mathbf{1}$. ∎

By the antiblocking relation, Theorem 65.12 implies that the stable set polytope of a minimally imperfect graph has precisely one facet not given by the clique and nonnegative constraints:

Corollary 65.12a. *Let $G = (V, E)$ be a minimally imperfect graph. Then the stable set polytope of G is determined by:*

(65.31)
$$\begin{array}{ll} x_v \geq 0 & \text{for } v \in V, \\ x(C) \leq 1 & \text{for each clique } C, \\ x(V) \leq \alpha(G). & \end{array}$$

Proof. Directly from Theorem 65.12 applied to the antiblocking polytope of the polytope determined by (65.25) (for $\overline{G}$). ∎

Shepherd [1990,1994b] calls a graph *near-perfect* if its stable set polytope is determined by (65.31), and he showed that a graph G is minimally imperfect if and only if both G and its complement $\overline{G}$ are near-perfect.

65.6d. Graph classes

Before a proof of the strong perfect graph theorem in general was announced in 2002, it had been proved for several classes of graphs. Next to the classes to be discussed in Chapter 66, it was shown for (among other):

- *claw-free graphs*, that is, graphs not having $K_{1,3}$ (a *claw*) as induced subgraph (Parthasarathy and Ravindra [1976] (cf. Tucker [1979] and Maffray and Reed [1999], and Giles and Trotter [1981] for a simpler proof)).
  It follows that the line graph $L(G)$ of a graph G is perfect if and only if G contains no odd circuit with at least five vertices as (not necessarily induced) subgraph. So these graphs have edge-colouring number $\chi'(G)$ equal to the maximum degree $\Delta(G)$ (if $\Delta(G) \geq 3$); moreover, the matching number $\nu(G)$ is equal to the minimum number of stars and triangles needed to cover the edges of G; this extends Kőnig's edge-colouring and matching theorems (cf. Trotter [1977] and de Werra [1978]).
  Sbihi [1978,1980] and Minty [1980] showed that the weighted stable set problem is solvable in strongly polynomial time for claw-free graphs (see Chapter 69). A combinatorial polynomial-time algorithm for the colouring problem for claw-free perfect graphs was given by Hsu [1981], and for the weighted clique and clique cover problems by Hsu and Nemhauser [1981,1982,1984].
  Chvátal and Sbihi [1988] gave a polynomial-time algorithm to recognize claw-free perfect graphs, based on decomposition (cf. Maffray and Reed [1999]). Koch [1979] gave a polynomial-time algorithm which for any claw-free graph either finds a maximum-size stable set and a minimum-size clique cover of equal cardinalities, or else finds an odd hole or odd antihole.
  Perfection of line graphs was also studied by Cao and Nemhauser [1998]. The validity of the strong perfect graph conjecture for claw-free graphs was extended to 'pan-free' graphs by Olariu [1989b].
- *K_4-free graphs* — graphs not having K_4 as subgraph (Tucker [1977b], cf. Tucker [1979,1987a]).

- *diamond-free graphs* (Tucker [1987b][7]) — graphs not having $K_4 - e$ (a *diamond*) as induced subgraph (where $K_4 - e$ is the graph obtained from K_4 by deleting an edge) (Conforti [1989] gave an alternative proof). Tucker [1987b] gave an $O(n^2m)$-time algorithm to colour such graphs optimally. Fonlupt and Zemirline [1987] and Conforti and Rao [1993] gave polynomial-time perfection tests for diamond-free graphs. Related results can be found in Conforti and Rao [1989, 1992a,1992b] and Fonlupt and Zemirline [1992,1993].
- *paw-free graphs* — graphs not having a *paw* (a K_4 with two incident edges deleted) as induced subgraph. This follows from the perfection of Meyniel graphs (Theorem 66.6). It also follows from a characterization of Olariu [1988e] of paw-free graphs.
- *square-free graphs* (Conforti, Cornuéjols, and Vušković [2002]) — graphs not having a C_4 (a *square*) as induced subgraph.
- *bull-free graphs* (Chvátal and Sbihi [1987]) — graphs not having a *bull* as induced subgraph, where a bull is the (self-complementary) graph on five vertices a, b, c, d, e and edges ab, ac, bc, bd, ce (see Figure 65.1). Reed and Sbihi [1995] gave a polynomial-time perfection test for bull-free graphs. More on bull-free graphs can be found in de Figueiredo [1995], de Figueiredo, Maffray, and Porto [1997,2001], and Hayward [2001].
- *chair-free graphs* (Sassano [1997]) — graphs not having a *chair* as induced subgraph, where a chair is the graph on five vertices a, b, c, d, e and edges ab, bc, cd, be (see Figure 65.1).
- *dart-free graphs* (Sun [1991]) — graphs not having a *dart* as induced subgraph (a dart is a graph with vertices a, b, c, d, e and edges ab, ac, ad, ae, bc, cd (see Figure 65.1)); Chvátal, Fonlupt, Sun, and Zemirline [2000,2002] gave a polynomial-time recognition algorithm for dart-free perfect graphs.
- graphs having neither P_5 nor K_5 as induced subgraph (Maffray and Preissmann [1994], Barré and Fouquet [1999]).
- *circular-arc graphs* (Tucker [1975]) — these are the intersection graphs of families of intervals on a circle (cf. Section 64.9a).
- *circle graphs* (Buckingham and Golumbic [1984]) — these are the intersection graphs of families of chords of a circle (cf. Section 64.9a).
- planar graphs (Tucker [1973b]). Tucker [1984b] showed that this can be derived (without appealing to the four-colour theorem) from the validity of the strong perfect graph conjecture for K_4-free graphs: a K_4 subgraph in a planar graph $G \neq K_4$ contains a triangle that is a vertex-cut of G; hence one can apply induction to find a 4-colouring of G.
  Tucker and Wilson [1984] gave an $O(n^2)$ algorithm for finding a minimum colouring of a planar perfect graph, Hsu [1987b] gave an $O(n^3)$-time perfection test for planar graphs, and Hsu [1988] described strongly polynomial-time algorithms for the maximum-weight stable set, the weighted colouring, and the weighted clique cover problems for planar perfect graphs.
- graphs embeddable in the torus or in the Klein bottle (Grinstead [1980,1981]).
- *checked graphs* (Gurvich and Temkin [1992]) — graphs whose vertex set is a subset of $\mathbb{R}^2$, two vertices being adjacent if and only the line segment connecting them is horizontal or vertical.

[7] An partial proof was given by Parthasarathy and Ravindra [1979], cf. Tucker [1987b].

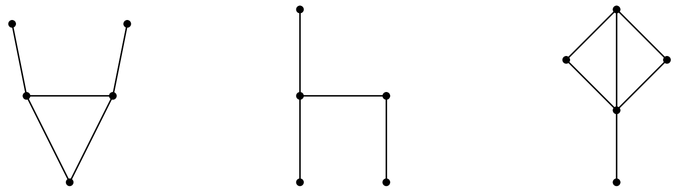

Figure 65.1

The perfect graph theorem implies that the strong perfect graph conjecture is true also for the classes of graphs complementary to those listed above.

Since C_k and $\overline{C}_k$ (for odd $k \geq 5$) are claw-free, the result of Parthasarathy and Ravindra [1976] implies that, to show the strong perfect graph theorem, it suffices to show that each minimally imperfect graph is claw-free.

Other classes of graphs for which the strong perfect graph conjecture holds were found by Rao and Ravindra [1977], Olariu [1988d], Jamison and Olariu [1989b], Carducci [1992], Galeana-Sánchez [1993], Lê [1993b], De Simone and Galluccio [1994], Maire [1994b], Maffray and Preissmann [1995], Xue [1995,1996], Ait Haddadene and Gravier [1996], Maffray, Porto, and Preissmann [1996], Aït Haddadène and Maffray [1997], Kroon, Sen, Deng, and Roy [1997], Babaĭtsev [1998], Hoàng and Le [2000b, 2001], and Gerber and Hertz [2001].

65.6e. The P_4-structure of a graph and a semi-strong perfect graph theorem

V. Chvátal noticed that the collection of 4-sets inducing the 4-vertex path P_4 as a subgraph, provides a useful tool in studying perfection. (Note that $\overline{P_4}$ is isomorphic to P_4.) It led Chvátal [1984a] to conjecture the following 'semi-strong perfect graph theorem', which was proved by Reed [1987] (announced in Reed [1985]).

Call two graphs G and H, with $VG = VH$, *P_4-equivalent* if for each $U \subseteq V$ one has: U induces a P_4-subgraph of G if and only if U induces a P_4-subgraph of H. Then Reed's theorem states that

(65.32) if G and H are P_4-equivalent and G is perfect, then H is perfect.

This theorem implies the perfect graph theorem, since G and $\overline{G}$ are P_4-equivalent. On the other hand, the theorem is implied by the strong perfect graph theorem, since any graph P_4-equivalent to an odd circuit of length at least 5 is equal to that circuit or to its complement.

Other relations between the P_4-structure and perfection were proved by Chvátal and Hoang [1985] and Chvátal, Lenhart, and Sbihi [1990]. Let $G = (V, E)$ be a graph and let V be partitioned into classes V_0 and V_1, with both $G[V_0]$ and $G[V_1]$ perfect. For each word $x = x_1x_2x_3x_4$ of length 4 over the alphabet $\{0, 1\}$, let $\mathcal{Q}_x$ denote the set of chordless paths in G with vertices v_1, v_2, v_3, v_4 (in order) with $v_i \in V_{x_i}$ for $i = 1, 2, 3, 4$. Then G is perfect if:

(65.33) (i) $\mathcal{Q}_{1000} = \mathcal{Q}_{0100} = \mathcal{Q}_{0111} = \mathcal{Q}_{1011} = \emptyset$, or

(ii) $\mathcal{Q}_{0000} = \mathcal{Q}_{0110} = \mathcal{Q}_{1001} = \mathcal{Q}_{1111} = \emptyset$, or
(iii) $\mathcal{Q}_{1000} = \mathcal{Q}_{0100} = \mathcal{Q}_{0110} = \mathcal{Q}_{1001} = \mathcal{Q}_{1111} = \emptyset$, or
(iv) $\mathcal{Q}_{1000} = \mathcal{Q}_{0101} = \mathcal{Q}_{0110} = \mathcal{Q}_{1001} = \mathcal{Q}_{1111} = \emptyset$, or
(v) $\mathcal{Q}_{1000} = \mathcal{Q}_{0101} = \mathcal{Q}_{0110} = \mathcal{Q}_{1001} = \mathcal{Q}_{0111} = \emptyset$, or
(vi) $\mathcal{Q}_{1000} = \mathcal{Q}_{1001} = \mathcal{Q}_{1011} = \emptyset$.

Sufficiency of (i) was shown by Chvátal and Hoang [1985], and of the other cases by Chvátal, Lenhart, and Sbihi [1990] ((ii) also by Gurvich [1993a,1993b]), who also showed that (65.33) essentially covers all cases where perfection of G follows from perfection of its constituents and the 'colouring' of the P_4-subgraphs. In fact, (65.33) and its symmetrical cases (interchanging V_0 and V_1 and/or replacing G by $\overline{G}$) describe exactly the cases excluded by G or $\overline{G}$ being an odd chordless circuit of length ≥ 5 or its complement.

A theorem of Seinsche [1974] states that each graph without an induced P_4 subgraph is perfect. (This follows from the perfection of Meyniel graphs (Theorem 66.6).[8])

Hence, case (65.33)(ii) implies the result of Hoang [1985] that any graph is perfect if there is a set U of vertices having an odd intersection with each chordless path with 4 vertices. More generally, it implies perfection of any graph if there is a set U of vertices such that each induced P_4 subgraph has exactly one of its two middle vertices in U or has exactly one of its ends in U.

Related (and more general than the results of Chvátal and Hoang quoted above) is the following theorem of Chvátal [1987a]. Let $G = (V, E)$ be a graph and let V be partitioned into two classes X and Y such that there are no $x \in X$, $y \in Y$, and $U \subseteq V$ such that both $U \cup \{x\}$ and $U \cup \{y\}$ induce a P_4 subgraph. Then G is perfect if and only if $G[X]$ and $G[Y]$ are perfect.

More work on the P_4-structure related to perfection is reported in Jung [1978], Jamison and Olariu [1989c,1992a,1992b,1995a,1995b], Hayward and Lenhart [1990], Hoàng [1990,1995,1999], Olariu [1991], Ding [1994], Rusu [1995a,1999b], Giakoumakis [1996], Hoàng, Hougardy, and Maffray [1996], Hougardy [1996b,1997,1999, 2001], Babel and Olariu [1997,1998,1999], Giakoumakis, Roussel, and Thuillier [1997], Giakoumakis and Vanherpe [1997], Hougardy, Le, and Wagler [1997], Babel [1998a,1998b], Babel, Brandstädt, and Le [1999], Brandstädt and Le [1999,2000], Roussel, Rusu, and Thuillier [1999], Brandstädt, Le, and Olariu [2000], Hoàng and Le [2000a,2001], Barré [2001], and Hayward, Hougardy, and Reed [2002].

65.6f. Further notes on the strong perfect graph conjecture

Markosyan and Karapetyan [1984] showed that the strong perfect graph conjecture is equivalent to: each minimally imperfect graph G is regular of degree $2\omega(G) - 2$.

For $k, n \in \mathbb{Z}_+$, let $C_{n,k}$ be the graph obtained from the circuit C_n (with n vertices) by adding all edges connecting two vertices at distance less than k. If $n \equiv 1 \pmod{k+1}$ and $n \geq 2k + 3$, then $C_{n,k}$ is partitionable. Chvátal [1976] showed that the strong perfect graph conjecture is equivalent to: each minimally imperfect graph G has $C_{|VG|,\omega(G)}$ as *spanning* subgraph (not necessarily induced).

[8] Arditti and de Werra [1976] claimed that Seinsche's result also follows from the 'fact' that any graph without induced P_4 subgraph is the comparability graph of a branching, therewith overlooking C_4.

Bland, Huang, and Trotter [1979] and Chvátal, Graham, Perold, and Whitesides [1979] gave examples of partitionable graphs G not containing $C_{|VG|,\omega(G)}$ as a spanning subgraph. Sebő [1996a] and Bacsó, Boros, Gurvich, Maffray, and Preissmann [1998] showed that these constructions give no counterexamples to the strong perfect graph conjecture. Related results are given by Chvátal [1984b]. A computer search for partitionable graphs was reported by Lam, Swiercz, Thiel, and Regener [1980].

Call a pair of vertices u, v in a graph an *even pair* if each induced $u - v$ path has even length. Meyniel [1987] showed that a minimally imperfect graph has no even pair. (This was extended to partitionable graphs by Bertschi and Reed [1987].) Hougardy [1996a] proved that the strong perfect graph conjecture is equivalent to: each properly induced subgraph of a minimally imperfect graph has an even pair or is a clique. Bienstock [1991] showed that it is NP-complete to test if a graph has no even pair. More on even pairs can be found in Hoàng and Maffray [1989], Bertschi [1990], Hougardy [1995], Rusu [1995c], Everett, de Figueiredo, Linhares-Sales, Maffray, Porto, and Reed [1997] (survey), Linhares Sales, Maffray, and Reed [1997], Linhares Sales and Maffray [1998], de Figueiredo, Gimbel, Mello, and Szwarcfiter [1999], Rusu [2000], and Everett, de Figueiredo, Linhares Sales, Maffray, Porto, and Reed [2001] (survey).

Prömel and Steger [1992] showed that 'almost all Berge graphs are perfect': the ratio of the number of n-vertex perfect graphs and the number of n-vertex Berge graphs, tends to 1 if $n \to \infty$.

The role of uniquely colourable perfect graphs for the strong perfect graph conjecture was investigated by Tucker [1983b]. Bacsó [1997] studied the conjecture that a uniquely colourable perfect graph G is either a clique or contains two maximum-size cliques intersecting each other in $\omega(G) - 1$ vertices. This is implied by the strong perfect graph theorem. Related work was given by Sakuma [2000].

Corneil [1986] investigated families of graphs 'complete' for the strong perfect graph conjecture (that is, proving the conjecture for these graphs suffices to prove the conjecture in general).

Equivalent versions of the strong perfect graph conjecture were given by Olaru [1972,1973b], Ravindra [1975], Markosyan [1981], Markosyan and Gasparyan [1987], Olariu [1990b], Huang [1991], Markosian, Gasparian, and Markosian [1992], Galeana-Sánchez [1993], De Simone and Galluccio [1994], Lonc and Zaremba [1995], and Padberg [2001].

Giles, Trotter, and Tucker [1984], Hsu [1984], Fonlupt and Sebő [1990], Croitoru and Radu [1992b], Sebő [1992], Panda and Mohanty [1997], and Rusu [1997] gave further techniques for proving the strong perfect graph conjecture.

Several other properties of minimally imperfect and partitionable graphs were derived by Olaru [1969,1972,1973a,1973b,1977,1980,1993,1998], Sachs [1970], Padberg [1974a,1974b,1975,1976], Parthasarathy and Ravindra [1976], Tucker [1977b, 1983a], Olaru and Suciu [1979], Markosyan [1981,1985], Sridharan and George [1982], Whitesides [1982], Buckingham and Golumbic [1983], Chvátal [1984c,1985c], Grinstead [1984], Olaru and Sachs [1984], Chvátal and Sbihi [1987], Meyniel [1987], Olariu [1988b,1988c,1990a,1991], Meyniel and Olariu [1989], Preissmann [1990], Sebő [1992,1996a,1996b], Cornuéjols and Reed [1993], Hougardy [1993], Maffray [1993], Olariu and Stewart [1993], Hayward [1995], Hoàng [1996c], Perz and Zaremba [1996], Fouquet, Maire, Rusu, and Thuillier [1997], Gasparyan [1998], Barré and Fouquet [1999,2001], Croitoru [1999], de Figueiredo, Klein, Kohayakawa, and Reed [2000], Barré [2001], Roussel and Rubio [2001], and Conforti, Cornuéjols, Gasparyan, and Vušković [2002]. Surveys were given by Preissmann and Sebő [2001] and Cornuéjols [2002].

65.7. Further results and notes

65.7a. Perz and Rolewicz's proof of the perfect graph theorem

An interesting proof of the perfect graph theorem was given by Perz and Rolewicz [1990]. It does not use the replication lemma, and is based on linear algebra, in a manner different from the proof of Gasparian given in Section 65.2, namely on the value of determinants.

In fact, Perz and Rolewicz [1990] show (in a different but equivalent terminology) that a graph $G = (V, E)$ is perfect if and only if $P_{\text{stable set}}(G)$ and $P_{\text{clique}}(G)$ form an antiblocking pair of polytopes. They prove sufficiency in a way similar to the proof of Fulkerson given for sufficiency in Corollary 65.2e above.

They proved necessity as follows. Choose a counterexample with $|V|$ minimal. So G is perfect, and $P_{\text{stable set}}(G)$ and $P_{\text{clique}}(G)$ do not form an antiblocking pair. Hence there exist $x \in A(P_{\text{stable set}}(G))$ and $y \in A(P_{\text{clique}}(G))$ with $x^{\mathsf{T}}y > 1$. Choose such x, y with $x^{\mathsf{T}}y$ maximal. Let $\nu := x^{\mathsf{T}}y$.

We first show

(65.34) $$\nu \leq \frac{n}{n-1},$$

where $n := |V|$. Indeed, for each $u \in V$, deleting the uth component of x and y, we obtain vectors in $A(P_{\text{stable set}}(G - u))$ and $A(P_{\text{clique}}(G - u))$, respectively. By the minimality of G, we have $\sum_{v \neq u} x_v y_v \leq 1$. Hence

(65.35) $$\nu = \sum_{v} x_v y_v = \frac{1}{n-1} \sum_{u} \Big(\sum_{v \neq u} x_v y_v \Big) \leq \frac{n}{n-1}.$$

This proves (65.34). By the minimality of G, we also have $x_v > 0$ and $y_v > 0$ for each v.

Now $\nu^{-1} \cdot x \in P_{\text{clique}}(G)$, since otherwise there is a $z \in A(P_{\text{clique}}(G))$ with $\nu^{-1}x^{\mathsf{T}}z > 1$, contradicting the maximality of $x^{\mathsf{T}}y$. So there exist cliques $C_1, \ldots, C_n$ and $\lambda_1, \ldots, \lambda_n > 0$ such that

(65.36) $$x = \sum_{i=1}^{n} \lambda_i \chi^{C_i} \text{ and } \sum_{i=1}^{n} \lambda_i = \nu.$$

Similarly, there exist stable sets $S_1, \ldots, S_n$ and $\mu_1, \ldots, \mu_n > 0$ such that

(65.37) $$y = \sum_{j=1}^{n} \mu_j \chi^{S_j} \text{ and } \sum_{j=1}^{n} \mu_j = \nu.$$

Then $y(C_i) = 1$ for $i = 1, \ldots, n$, since $y(C_i) \leq 1$ (as $y \in A(P_{\text{clique}}(G))$), and

(65.38) $$\nu = x^{\mathsf{T}}y = \sum_{i=1}^{n} \lambda_i y(C_i) \leq \sum_{i=1}^{n} \lambda_i = \nu.$$

Similarly, $x(S_j) = 1$ for $j = 1, \ldots, n$.

Consider, for some $i = 1, \ldots, n$,

$$(65.39) \qquad 1 = y(C_i) = \sum_{j=1}^{n} \mu_j |C_i \cap S_j| \leq \sum_{j=1}^{n} \mu_j = \nu.$$

So the inequality is strict, and hence there is at least one j with $C_i \cap S_j = \emptyset$. Then

$$(65.40) \qquad 1 = x(S_j) = \sum_{i'} \lambda_{i'} |C_{i'} \cap S_j| = \sum_{i' \neq i} \lambda_{i'} |C_{i'} \cap S_j| \leq \sum_{i' \neq i} \lambda_{i'} = \nu - \lambda_i.$$

Hence with (65.34),

$$(65.41) \qquad n \leq \sum_i (\nu - \lambda_i) = n\nu - \nu \leq n,$$

implying equality in (65.40) for each i. So if $C_i \cap S_j = \emptyset$, then $C_{i'} \cap S_j \neq \emptyset$ for each $i' \neq i$. Hence for each j, there is exactly one i with $C_i \cap S_j = \emptyset$, and conversely. We can assume that $C_i \cap S_j = \emptyset$ if and only if $i = j$.

Let M and N be the incidence matrices of $S_1, \ldots, S_n$ and of $C_1, \ldots, C_n$ respectively. So $MN^{\mathsf{T}} = J - I$. Hence $|\det M \det N| = |\det(J - I)| = n - 1$. Since $y(S_i) = 1$ for each i, we have $My = \mathbf{1}$. So $y' := |\det M| \cdot y$ is a positive integer vector. Similarly, $x' := |\det N| \cdot x$ is a positive integer vector. Then

$$(65.42) \qquad (x')^{\mathsf{T}} y' = |\det M \det N| x^{\mathsf{T}} y = |\det(J - I)| \nu = n.$$

The kernel of the argument now is that this implies that x' and y' are the all-one vectors, and therefore x and y each are scalar multiples of the all-one vector.

As $x \in A(P_{\text{stable set}}(G))$, $x(S) \leq 1$ for any stable set S, and hence $x = \alpha'^{-1} \cdot \mathbf{1}$ for some $\alpha' \geq \alpha(G)$. Similarly, $y = \omega'^{-1} \cdot \mathbf{1}$ for some $\omega' \geq \omega(G)$. As G is perfect, $\alpha'\omega' \geq \alpha(G)\omega(G) \geq n$. Hence $\nu = x^{\mathsf{T}} y = (\alpha'\omega')^{-1} n \leq 1$, a contradiction.

65.7b. Kernel solvability

The following generalization of the Gale-Shapley theorem on stable matchings was conjectured by Berge and Duchet [1986,1988a][9] and proved by Boros and Gurvich [1996], using a technique from game theory due to Scarf [1967]. With the strong perfect graph theorem it characterizes perfect graph by being kernel solvable.

Call a graph $G = (V, E)$ *kernel solvable* if the following holds: if for each clique C of G we have a total order $<_C$ of C, then there exists a stable set S such that for each $v \in V$ there is an $s \in S$ and a clique C such that $v, s \in C$ and $v \leq_C s$. Berge and Duchet conjectured that kernel solvable graphs are precisely the perfect graphs. With Theorem 65.14 below, this conjecture is implied by the strong perfect graph theorem.

Kernel solvability can be formulated equivalently in terms of kernels of digraphs. A *kernel* of a directed graph $D = (V, A)$ is a subset S of V such that S spans no arc of D and such that for each $v \in V \setminus S$ there is a $u \in S$ with $(v, u) \in A$.

For any graph $G = (V, E)$, a directed graph $D = (V, A)$ is called a *superorientation* of G if $E = \{\{u, v\} \mid (u, v) \in A\}$. (So $\{u, v\}$ is an edge of $G \iff$ at least

[9] Berge and Duchet [1986] refer to 'Séminaire du Lundi, MSH, Paris, Janvier 1983' (Monday Seminar, MSH, Paris, January 1983). See Jensen and Toft [1995] p. 140 for further references to the history of this conjecture.

one of (u, v) and (v, u) belongs to A.) Then a graph $G = (V, E)$ is kernel solvable if and only if any superorientation D of G has a kernel if each clique C of G induces a subgraph of D with a kernel.

Kernel solvability is closed under taking induced subgraphs: if $U \subseteq V$, and each clique C of $G[U]$ has a total order $<_C$, we can choose for each clique C of G a total order which coincides with $<_{C\cap U}$ on $C \cap U$ and which has $C \cap U$ as upper ideal.

Since neither C_k nor $\overline{C}_k$ is kernel solvable for odd $k \geq 5$, the strong perfect graph theorem implies that each kernel solvable graph is perfect.

Boros and Gurvich [1996] proved that a graph G is perfect if and only if each graph H arising from G by replicating vertices is kernel solvable. It implies that the strong perfect graph theorem is equivalent to: each Berge graph is kernel solvable (since the class of Berge graphs is closed under replicating vertices).

To show that each perfect graph is kernel solvable, we follow the proof method of Aharoni and Holzman [1998]. We first prove the following results of Scarf [1967].

Let M and N be disjoint finite nonempty sets, and for each $i \in M$ let $<_i$ be a total order of N. For any U, write $y <_i U$ if $y <_i u$ for each $u \in U$. Define $K := M \cup N$.

Call a subset L of K *light* if for each $j \in N$ there is an $i \in M \setminus L$ with $j \leq_i L \setminus M$. So any subset of a light set is light again. Let $m := |M|$ and define

(65.43) $\qquad \mathcal{S} := \{M\} \cup \{L \mid L \text{ light}, |L| = m\}.$

Note that M is not light.

Now Scarf first proved:

Lemma 65.13α. *Any light set L with $|L| = m - 1$ is contained in precisely two sets in $\mathcal{S}$.*

Proof. Extend each $<_i$ to a total order on K, with $i <_i j <_i i'$ for all $j \in N$ and all $i' \in M \setminus \{i\}$. Then

(65.44) $\qquad$ any subset L of K is light if and only if for each $k \in K$ there is an $i \in M$ with $k \leq_i L$.

To see necessity in (65.44), let $L \subseteq K$ be light and let $k \in K$. If $k \in M$, then $k \leq_k L$. If $k \in N$, then there is an $i \in M \setminus L$ with $k \leq_i L \setminus M$. As $i \notin L$, we have also $k \leq_i L \cap M$ (since $k \leq_i M \setminus \{i\}$). So $k \leq_i L$.

To see sufficiency in (65.44), suppose $\forall k \in K \exists i \in M : k \leq_i L$. Let $j \in N$. Then $\exists i \in M : j \leq_i L$. Then $i \notin L$ (as otherwise $j \leq_i i$). Moreover, $j \leq_i L \setminus M$. This proves (65.44).

For any $i \in M$ and any nonempty $U \subseteq K$, let $\min_i U$ and $\max_i U$ denote the minimal and maximal element of U with respect to $<_i$.

First assume that $L \subseteq M$, say $L = M \setminus \{i\}$. Then L is contained in M, which belongs to $\mathcal{S}$. Moreover, $z := \max_i N$ is the unique element with $L \cup \{z\}$ light.[10] This proves the lemma.

So henceforth we can assume that $L \not\subseteq M$. Define $\pi : M \to L$ by $\pi(i) := \min_i L$. Then π is onto, since, as L is light, for each $r \in L$ there is an $i \in M$ with $r \leq_i L$. So $r = \min_i L = \pi(i)$.

[10] For let $x \in N$. Then $L \cup \{x\}$ is light $\iff \forall j \in N \exists i \in M \setminus L : j \leq_i (L \cup \{x\}) \setminus M \iff \forall j \in N : j \leq_i x \iff x = \max_i N$.

Hence, as $|L| = |M| - 1$, there exist distinct $i_1, i_2 \in M$ with $\pi(i_1) = \pi(i_2)$, while all other values of π are mutually distinct and different from $\pi(i_1)$.

For $h = 1, 2$, define

(65.45) $$R_h := \{k \in K \mid k \not\leq_i L \text{ for all } i \neq i_h\}.$$

Then $R_h \neq \emptyset$, since $i_h \in R_h$, as there is an $r \in L \setminus M$ (as $L \not\subseteq M$), hence if $i \neq i_h$, then $i_h \not\leq_i r$. Also, $R_h \cap L = \emptyset$, since if $r \in L$, there is an $i \neq i_h$ with $\pi(i) = r$, so $\min_i L = r$, implying $r \leq_i L$, and hence $r \notin R_h$. Moreover, $R_1 \cap R_2 = \emptyset$, since otherwise there is a $k \in K$ with $k \not\leq_i L$ for all $i \in M$, contradicting the fact that L is light.

Define for $h = 1, 2$:

(65.46) $$r_h := \max_{i_h} R_h.$$

We first show that $L \cup \{r_1\}$ and $L \cup \{r_2\}$ are light. Suppose that (say) $L \cup \{r_1\}$ is not light. So there is a $k \in K$ with $k \not\leq_i L \cup \{r_1\}$ for each $i \in M$. Since $r_1 \not\leq_i L$ for each $i \neq i_1$ (by definition of R_1), it follows that $k \not\leq_i L$ for each $i \neq i_1$. Hence $k \in R_1$. However, $r_1 <_{i_1} L$, since $r_1 \in R_1$ and $r_1 \leq_i L$ for some $i \in M$. So $k \leq_{i_1} r_1 <_{i_1} L$, and therefore $k \leq_{i_1} L \cup \{r_1\}$, a contradiction. So $L \cup \{r_1\}$ and $L \cup \{r_2\}$ are light.

Finally we show that for any $s \in K \setminus L$, if $L \cup \{s\}$ is light, then $s = r_1$ or $s = r_2$. So let $L \cup \{s\}$ be light. Then the function $\pi' : M \to L \cup \{s\}$ defined by $\pi'(i) := \min_i(L \cup \{s\})$ is onto (as $L \cup \{s\}$ is light), implying that it is one-to-one (as $|M| = |L \cup \{s\}|$). Hence π' coincides with π on all but one element of M. Necessarily the exceptional element belongs to $\{i_1, i_2\}$. Say $\pi'(i) = \pi(i)$ for each $i \neq i_1$, while $\pi'(i_1) = s$. So $\min_i L = \pi(i) = \pi'(i) <_i s$ for each $i \neq i_1$; that is, $s \in R_1$. Suppose $s \neq r_1$. So $s <_{i_1} r_1$. Then $r_1 \not\leq_i L \cup \{s\}$ for each $i \in M$, contradicting the fact that $L \cup \{s\}$ is light. ∎

From this, Scarf derived:

Theorem 65.13 (Scarf's lemma). *Let A be a nonnegative $m \times n$ matrix and let $b \in \mathbb{R}^m_+$ be such that the polytope $P := \{x \in \mathbb{R}^n_+ \mid Ax \leq b\}$ is nonempty and bounded. For each $i = 1, \ldots, m$, let $<_i$ be a total order on $\{1, \ldots, n\}$. Then P has a vertex x such that*

(65.47) *for each $j \in \{1, \ldots, n\}$ there is an $i \in \{1, \ldots, m\}$ such that $a_i^{\mathsf{T}} x = b_i$ and such that $x_k = 0$ for each $k <_i j$.*

Proof. We can assume, by slightly perturbing b, that for each vertex x of P there are precisely n constraints among $x \geq \mathbf{0}$, $Ax \leq b$ satisfied with equality. Add n to each index i of $<_i$. (So $x <_{n+i} y$ in the new notation $\iff$ $x <_i y$ in the old notation.) Let $N := \{1, \ldots, n\}$, $M := \{n+1, \ldots, n+m\}$, and $K := N \cup M$. For each face f of P define

(65.48) $K_f := \{k \in K \mid$ the kth constraint in $x \geq \mathbf{0}$, $Ax \leq b$ is not tight at some point in $f\}$.

So $|K_v| = m$ for any vertex v and $|K_e| = m + 1$ for any edge e. Call an edge e of P *good* if $1 \in K_e$ and the set $K_e \setminus \{1\}$ belongs to $\mathcal{S}$ (cf. (65.43)).

Now $\mathbf{0}$ is incident with precisely one good edge. Hence there is a vertex $v \neq \mathbf{0}$ incident with an odd number of good edges. We show that $K_v \in \mathcal{S}$, and hence K_v is light (since $K_v \neq M$, as $v \neq \mathbf{0}$), implying that v satisfies (65.47).

Let e be a good edge incident with v. Then $K_e = K_v \cup \{k\}$ for some k. As e is good, we know that $1 \in K_e$ and $K_e \setminus \{1\} \in \mathcal{S}$.

If $1 \notin K_v$, then $k = 1$ and hence $K_v \in \mathcal{S}$. So we can assume that $1 \in K_v$. Applying Lemma 65.13α to the light set $K_v \setminus \{1\}$, there is precisely one $j \notin K_v \setminus \{1\}$ with $j \neq k$ and $K_v \setminus \{1\} \cup \{j\} \in \mathcal{S}$. If $j \neq 1$, then v is incident with precisely two good edges, a contradiction. So $j = 1$, and hence $K_v \in \mathcal{S}$. ∎

This implies the theorem of Boros and Gurvich [1996]:

Corollary 65.13a. *A perfect graph is kernel solvable.*

Proof. Let $G = (V, E)$ be a perfect graph, and for each clique C, let $<_C$ be a total order on C. We must prove that

(65.49) there exist a stable set S such that for each $v \in V$ there is a clique C and an element $s \in C \cap S$ with $v \in C$ and $v \leq_C s$.

Extend each $<_C$ to a total order on V with $w <_C v$ for each $w \in C$, $v \in V \setminus C$. Then by Theorem 65.13, the polytope in $\mathbb{R}^V$ determined by $x \geq \mathbf{0}$, $x(C) \leq 1$ (C clique), has a vertex x such that for each $v \in V$ there is a clique C with $x(C) = 1$ and such that $x_u = 0$ for each $u <_C v$. By Corollary 65.2e, x is the incidence vector of some stable set S. So for each $v \in V$ there is a clique C with $|C \cap S| = 1$ and with $u \notin S$ if $u <_C v$. Therefore, for the vertex s in $C \cap S$ we have $v \leq_C s$, and hence $v \in C$. This shows (65.49). ∎

It was conjectured by Berge and Duchet that conversely, each kernel solvable graph is perfect. This follows from the strong perfect graph theorem, since kernel solvability is closed under taking induced subgraphs and since odd circuits of length at least five and their complements are not kernel solvable.

It implies the following theorem found by Boros and Gurvich [1996], for which we give a direct proof. A graph H is called a *blow-up* of a graph G, if H arises from G by replicating vertices (replacing vertices by cliques).

Theorem 65.14. *A graph G is perfect if and only if each blow-up of G is kernel solvable.*

Proof. Since each blow-up of a perfect graph is perfect again (by the replication lemma (Corollary 65.2c)), necessity follows from Corollary 65.13a.

Sufficiency is shown by proving that each graph $G = (V, E)$ with $|V| \geq \alpha(G)\omega(G) + 1$ has a blow-up that is not kernel solvable. (This is sufficient by Theorem 65.2.)

Let $\mathcal{C}$ be the collection of cliques in G, and for each vertex v, let $\mathcal{C}_v$ be the collection of cliques containing v. Let $n := |V|$ and define

(65.50) $Y := \{y : \mathcal{C} \to \mathbb{Z}_+ \mid y(\mathcal{C}) \leq n|\mathcal{C}|\}$.

For each $y \in Y$, we choose a vertex v_y of G with

(65.51) $y(\mathcal{C}_{v_y}) \leq \omega(G)|\mathcal{C}|$.

This is possible since

(65.52) $$\sum_{v \in V} y(\mathcal{C}_v) = \sum_{C \in \mathcal{C}} |C| y_C \leq \omega(G) \sum_{C \in \mathcal{C}} y_C \leq \omega(G) n |\mathcal{C}|.$$

Let H be the graph with vertex set Y, two distinct vertices $y, z \in Y$ being adjacent if $v_y = v_z$ or v_y and v_z are adjacent in G. So H is a blow-up of G. We show that H is not kernel solvable.

For each clique $K \subseteq Y$ of H, the set $C := \{v_y \mid y \in K\}$ is a clique of G. Then choose a total order $<_K$ on K such that for all $y, z \in K$:

(65.53) if $y_C < z_C$, then $y <_K z$.

Assume that H is kernel solvable. Then H has a stable set $Z \subseteq Y$ such that for each $y \in Y$ there is a $z \in Z$ and a clique K of H with $y, z \in K$ and $y \leq_K z$. As Z is stable in H, the v_z for $z \in Z$ are distinct and form a stable set S in G. So for each clique C of G there is at most one $z \in Z$ with $v_z \in C$. Define $y : \mathcal{C} \to \mathbb{Z}_+$ by:

(65.54) $$y_C := \begin{cases} z_C + 1 & \text{if } v_z \in C, \text{ for } z \in Z, \\ 0 & \text{if } C \cap S = \emptyset. \end{cases}$$

Then y belongs to Y, since

(65.55) $$y(\mathcal{C}) = \sum_{z \in Z} \sum_{C \in \mathcal{C}_{v_z}} (z_C + 1) = \sum_{z \in Z} (z(\mathcal{C}_{v_z}) + |\mathcal{C}_{v_z}|) \leq |Z| \omega(G) |\mathcal{C}| + |\mathcal{C}|$$
$$\leq (\alpha(G)\omega(G) + 1)|\mathcal{C}| \leq n|\mathcal{C}|.$$

(The first inequality follows from (65.51).) Hence there exist a $z \in Z$ and a clique K of H with $y, z \in K$ and $y \leq_K z$. So for $C := \{v_x \mid x \in K\}$ we have, by (65.53), $y_C \leq z_C$. Since $v_z \in C$ (as $z \in K$), this contradicts (65.54). ∎

Before the strong perfect graph conjecture was settled, and hence the conjecture of Berge and Duchet, partial and related results on the latter conjecture were obtained by Blidia [1986], Maffray [1986,1992], Duchet [1987], Berge and Duchet [1988b,1990], Champetier [1989], Blidia and Engel [1992], Blidia, Duchet, and Maffray [1993,1994], Chilakamarri and Hamburger [1993], and Galeana-Sánchez [1995, 1996,1997].

65.7c. The amalgam

A composition generalizing the 1-join, the *amalgam*, was shown to preserve perfection by Burlet and Fonlupt [1984]. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be perfect graphs such that $K := V_1 \cap V_2$ is a clique in both graphs. For $i = 1, 2$, let $v_i \in V_i \setminus K$ be such that each vertex in K is adjacent to v_i and to each neighbour of v_i. Let H be the graph on $(V_1 \setminus \{v_1\}) \cup (V_2 \setminus \{v_2\})$ obtained from the union of $G_1 - v_1$ and $G_2 - v_2$ by adding all edges between $N(v_1) \setminus K$ and $N(v_2) \setminus K$.

Theorem 65.15. *If G_1 and G_2 are perfect, then H is perfect.*

Proof. It suffices to show that $\omega(H) = \chi(H)$, since each induced subgraph of H arises by the same construction.

For $i = 1, 2$, let $p_i := \omega(G_i[N(v_i)])$ and let G_i' be the graph obtained from G_i by replicating v_i by a factor $\omega(H) - p_i$. So $\omega(G_i') = \omega(H)$. By the replication lemma, G_i' is perfect. Hence $\omega(H) = \chi(G_i')$.

Consider colourings of G_1' and G_2' with colours $1, \ldots, \omega(H)$. So $N(v_i)$ uses precisely p_i colours. As $p_1 + p_2 - |K| \leq \omega(H)$, we have $(\omega(H) - p_1) + (\omega(H) - p_2) \geq \omega(H) - |K|$. Hence we can assume that in G_1 and G_2 the colourings of K are the same, and that all colours are used by the replication vertices of v_1 and v_2 and by K. Then $N(v_1) \setminus K$ and $N(v_2) \setminus K$ have no colours in common. Hence we obtain an $\omega(H)$-colouring of H. ∎

Cornuéjols and Cunningham [1985] gave an $O(n^2m)$-time algorithm to decide if a graph is the amalgam of smaller graphs.

Perfection is trivially closed under 'clique sums', that is, identifying two cliques in two graphs. Whitesides [1981] gave an $O(nm)$ algorithm to find a clique cut in a graph, that is, a vertex-cut that is a clique. Tarjan [1985] gave an $O(nm)$-time algorithm to find for any graph a decomposition by clique cuts.

Fonlupt and Uhry [1982] gave conditions under which identification of two vertices in a graph maintains perfection. Ravindra and Parthasarathy [1977], Ravindra [1978], Mândrescu [1991], and Kwaśnik and Szelecka [1997] investigated the behaviour of perfection under taking (various) products of graphs.

More on the (de)composition of perfect graphs can be found in Hsu [1986,1987a], Conforti and Rao [1992a,1992b], Corneil and Fonlupt [1993], Burlet and Fonlupt [1994], and Conforti, Cornuéjols, Kapoor, and Vušković [1995].

65.7d. Diperfect graphs

Berge [1982a] introduced a directed variant of perfect graphs. In fact, there are two symmetric variants, as no complementary phenomenon holds in the directed case.

A *stable set* or *clique* in a directed graph is a stable set of clique in the underlying undirected graph. A directed graph $D = (V, A)$ is called *α-diperfect* if for every induced subgraph $D' = (V', A')$ of D and for each maximum-size stable set S in D' there is a partition of V' into directed paths each intersecting S in exactly one vertex.

Then:

(65.56) if the underlying undirected graph G of D is perfect, then D is α-diperfect.

Indeed, if G is perfect, there is a maximum-size stable set S and a partition of V into cliques each intersecting S. Each clique C gives a tournament on C in D, and hence, by Rédei's theorem (Corollary 14.14a), it contains a directed path spanning C.

Another class of α-diperfect digraphs is formed by the *symmetric* digraphs: directed graphs $D = (V, A)$ such that if $(u, v) \in A$, then $(v, u) \in A$:

(65.57) each symmetric digraph is α-diperfect.

To see this, let S be a maximum-size stable set in D, and let D' arise from D by deleting all arcs entering S. By the Gallai-Milgram theorem (Theorem 14.14), V can be partitioned into $|S|$ directed paths in D'. These paths are as required.

Berge offered the following conjecture characterizing α-diperfect digraphs:

(65.58) (?) A directed graph $D = (V, A)$ is α-diperfect if and only if D has no induced subgraph C whose underlying undirected graph is a chordless odd circuit of length ≥ 5, say with vertices $v_1, \ldots, v_{2k+1}$ (in order) such that each of $v_1, v_2, v_3, v_4, v_6, v_8, \ldots, v_{2k}$ is a source or a sink. (?)

The odd circuit described is not α-diperfect, since $\{v_1, v_4, v_6, v_8, \ldots, v_{2k}\}$ is a maximum-size stable set, but there are no directed paths as required.

A 'dual' concept is that of a χ-*diperfect* graph, which is a digraph $D = (V, A)$ such that for each induced subgraph $D' = (V', A')$ of D and for each minimum vertex-colouring (in the underlying undirected graph of D') there exists a directed path intersecting each colour exactly once.

Again one has:

(65.59) if the underlying undirected graph G of D is perfect, then D is χ-diperfect.

Indeed, any maximum-size clique C intersects each colour in each minimum vertex-colouring, and, again by Rédei's theorem (Corollary 14.14a), there is a path spanning C.

Also:

(65.60) any symmetric digraph is χ-diperfect.

To see this, let $S_1, \ldots, S_k$ be an optimum vertex-colouring. Let D' be the graph obtained from D by deleting all arcs from S_j to S_i for all $j > i$. By the theorem of Gallai and Roy (see (64.52)), D' has a directed path of length k. Necessarily, it intersects each S_i exactly once.

One may show that the odd undirected circuit described in (65.58) is not χ-diperfect. So conjecture (65.58) would imply that each χ-diperfect digraph is α-diperfect.

In fact, any odd undirected circuit that contains three consecutive vertices v_1, v_2, v_3 that are sources or sinks, is not χ-diperfect (since there is an optimum 3-vertex-colouring where $\{v_2\}$ is one of the colours — hence v_2 should belong to a directed path with 3 vertices). In particular, the undirected circuit with vertices $v_1, \ldots, v_7$ and arcs

(65.61) $(v_1, v_2), (v_3, v_2), (v_3, v_4), (v_4, v_5), (v_5, v_6), (v_6, v_7), (v_1, v_7)$

is α-diperfect but not χ-diperfect (cf. Figure 65.2).

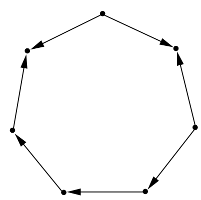

Figure 65.2

65.7e. Further notes

Cameron, Edmonds, and Lovász [1986] showed that if the edges of a complete graph are coloured with three colours such that no triangle gets three different colours, and two of these colours form perfect graphs, then so does the third. (This generalizes the perfect graph theorem.) A generalization and a related characterization of perfection in terms of decomposition was given by Cameron and Edmonds [1997].

Markosyan and Karapetyan [1976] characterize perfection with the help of *critical edges* (edges e with $\alpha(G-e) > \alpha(G)$) and *essential edges* (edges e with $\chi(G-e) > \chi(G)$). More on such edges can be found in Markosyan [1975], Karapetyan [1976], Sebő [1996a], and Markossian, Gasparian, Karapetian, and Markosian [1998]. Edge-minimal perfect graphs are studied by Wagler [1999], colouring perfect 'degenerate' graphs by Aït Haddadène and Maffray [1997], 'Gallai graphs' and 'anti-Gallai graphs' by Le [1993a,1996b], and 'edge perfect graphs' by Müller [1996].

An alternative polyhedral characterization of perfection of graphs was given by Zaremba and Perz [1982]. Related is the work of Zaremba [1991] and Hujter [1999].

Chandrasekaran and Tamir [1984] and Cook, Fonlupt, and Schrijver [1986] showed that, for any perfect graph $G = (V, E)$ and any weight $w : V \to \mathbb{Z}_+$, the weighted colouring number is attained by a weighted colouring using at most $|V|$ different stable sets.

Von Rimscha [1983] showed that if $G = (V, E)$ and $H = (V, F)$ are graphs with $G - v$ isomorphic to $H - v$ for each $v \in V$, then G is perfect if and only if H is perfect.

Bienstock [1991] showed that it is NP-complete to decide if a given graph has an odd hole containing a prescribed vertex. More on the complexity of finding odd holes can be found in Reed [1990]. A survey on forbidding holes and antiholes was given by Hayward and Reed [2001].

Le [1996a] showed that if a graph G is imperfect and has no odd hole, then the intersection graph of the edge sets of chordless circuits in G has an odd hole. Akiyama and Chvátal [1990] characterized for which graphs $G = (V, E)$ the intersection graph of the triples spanning at least two edges, is perfect. Olaru and Mândrescu [1992] considered perfection of products of graphs, and de Werra and Hertz [1999] perfection of sums of graphs. Hertz [1998] characterized the graphs for which all graphs obtained by 'switching' are perfect.

Variants of the notion of perfect graph were studied by Kőrner [1973], Duchet [1980], Galeana-Sánchez [1982,1986,1988], Duchet and Meyniel [1983], Galeana-Sánchez and Neumann-Lara [1986,1991a,1991b,1994,1996,1998], Lehel and Tuza [1986], Conforti, Corneil, and Mahjoub [1987], Cameron [1989], Brown, Corneil, and Mahjoub [1990], Markosyan and Gasparyan [1990], Reed [1990], Scheinerman and Trenk [1990,1993], Berge [1992b,1992a,1995], Kőrner, Simonyi, and Tuza [1992], Lehel [1994], Trenk [1995], Cai and Corneil [1996], Markossian, Gasparian, and Reed [1996], Tamura [1997,2000], Gutin and Zverovich [1998], De Simone and Kőrner [1999], Huang and Guo [1999], Fachini and Kőrner [2000], and de Figueiredo and Vušković [2000].

Introductions to and surveys of perfect graphs are given by Berge [1973b,1975, 1986], Golumbic [1980], Lovász [1983b], Berge and Chvátal [1984] (a collection of papers on perfect graphs), Chvátal [1985b,1987b], Jensen and Toft [1995], Toft [1995], Ravindra [1997], Brandstädt, Le, and Spinrad [1999], and Ramírez Alfonsín and Reed [2001] (a collection of survey papers on perfect graphs). The latter reference includes a bibliography on perfect graphs by Chvátal [2001]. Applications of perfect graphs to graph entropy were surveyed by Simonyi [2001] (cf. Simonyi [1995]). Algorithmic aspects are discussed in Golumbic [1984].

We refer for historical remarks on perfect graphs to Section 67.4g.

Chapter 66

Classes of perfect graphs

In this chapter we consider classes of perfect graphs. The phenomenon observed by Berge that clique number and colouring number are equal for bipartite graphs, their line graphs, comparability graphs, and chordal graphs, and for their complements, formed the motivation for him to raise the conjecture that the complement of a perfect graph is perfect again ($\equiv$ perfect graph theorem).
The perfection of the graphs considered in this chapter follows directly from the strong perfect graph theorem. However, since its proof is highly complicated, we will give direct proofs of the perfection of several of these graphs.

66.1. Bipartite graphs and their line graphs

The perfect graph theorem can be used to prove several min-max relations on bipartite graphs: Kőnig's matching theorem, the Kőnig-Rado edge cover theorem, and Kőnig's edge colouring theorem.

We start from the trivial observation that:

Theorem 66.1. $\omega(G) = \chi(G)$ *for each bipartite graph* G.

Proof. Trivial. ∎

Since the class of bipartite graphs is closed under taking induced subgraphs, this gives:

Corollary 66.1a. *Each bipartite graph is perfect.*

Proof. See above. ∎

Hence, by the perfect graph theorem, also the complements of bipartite graphs are perfect. This amounts to the Kőnig-Rado edge cover theorem (Theorem 19.4):

Corollary 66.1b (Kőnig-Rado edge cover theorem)**.** *For any bipartite graph* G*,* $\alpha(G) = \overline{\chi}(G)$*. Equivalently, the stable set number of any bipartite graph (without isolated vertices) is equal to its edge cover number.*

Proof. Directly from the perfect graph theorem, since by Theorem 66.1, any bipartite graph is perfect. Note that if G is a bipartite graph, then its cliques have size at most 2; hence $\overline{\chi}(G)$ is equal to the edge cover number of G if G has no isolated vertices. ∎

We saw in Section 16.2 that by Gallai's theorem (Theorem 19.1), the Kőnig-Rado edge cover theorem implies Kőnig's matching theorem (Theorem 16.2), saying that the matching number of a bipartite graph G is equal to its vertex cover number. That is, the stable set number of the line graph $L(G)$ of G is equal to the minimum number of cliques of $L(G)$ that cover all vertices of $L(G)$; in notation:

(66.1) $\alpha(L(G)) = \overline{\chi}(L(G)).$

As this is true for any induced subgraph of $L(G)$ we know that the complement $\overline{L(G)}$ of the line graph $L(G)$ of any bipartite graph G is perfect.

Hence with the perfect graph theorem we know:

Corollary 66.1c. *The line graph of any bipartite graph is perfect.*

Proof. See above. ∎

This amounts to Kőnig's edge-colouring theorem (Theorem 20.1):

Corollary 66.1d (Kőnig's edge-colouring theorem)**.** *If G is the line graph of a bipartite graph, then $\omega(G) = \chi(G)$. Equivalently, the edge-colouring number of any bipartite graph is equal to its maximum degree.*

Proof. Again directly from Kőnig's matching theorem and the perfect graph theorem. ∎

Complexity. In Part II on bipartite matching and covering, we saw that the optimization problems corresponding to the perfect graph parameters are solvable in polynomial time, and their weighted versions are solvable in strongly polynomial time, mainly by utilizing network flow techniques. We review the results.

The maximum-weight clique and the minimum colouring problem for bipartite graphs are trivially solvable in strongly polynomial time. Also the weighted colouring problem for bipartite graphs is easily solvable in strongly polynomial time.

A maximum-size stable set and a minimum clique cover in a bipartite graph can be found in polynomial time (cf. Corollary 19.3a). Note that in bipartite graphs, the minimum clique cover problem amounts to the minimum-size edge cover problem. Also the weighted versions are solvable in strongly polynomial time by max-flow techniques (cf. Corollary 21.25a). In bipartite graphs, the minimum weighted clique cover problem amounts to the minimum-size b-edge cover problem.

A bipartite graph is easily recognized, by checking if there is no odd circuit.

In *line graphs of bipartite graphs*, finding a maximum-weight clique is trivial (by checking all stars of the graph). In Sections 20.1 and 20.2 we saw that a minimum weighted colouring can be found in strongly polynomial time.

Finding a maximum clique and a minimum colouring in the complement of the line graph of a bipartite graph G amounts to finding a maximum-size matching and a minimum-size vertex cover in G, which can be found in polynomial time (cf. Theorem 16.3 and Corollary 16.6a). Their weighted versions can be found in strongly polynomial time with the methods for the assignment and the minimum-cost flow problems (cf. Theorems 17.4 and 17.6).

Van Rooij and Wilf [1965] showed that line graphs of bipartite graphs can be recognized in polynomial time, and that the corresponding bipartite graph can be reconstructed in polynomial time.

66.2. Comparability graphs

Also Dilworth's decomposition theorem (Theorem 14.2) can be derived from the perfect graph theorem. Let $(V, \leq)$ be a partially ordered set. Let $G = (V, E)$ be the graph with:

(66.2) $uv \in E$ if and only if $u < v$ or $v < u$.

Any graph G obtained in this way is called a *comparability graph*.

In Theorem 14.1 we saw the following easy 'dual' form of Dilworth's decomposition theorem:

Theorem 66.2. *In any partially ordered set $(V, \leq)$, the maximum size of a chain is equal to the minimum number of antichains needed to cover V.*

Proof. For any $v \in V$ define the *height* of v as the maximum size of a chain in V with maximum element v. Let k be the maximum height of the elements of V. For $i = 1, \ldots, k$, let A_i be the set of elements of height i. Then $A_1, \ldots, A_k$ are antichains covering V, and moreover, there is a chain of size k, since there is an element of height k. ∎

Equivalently, we have $\omega(G) = \chi(G)$ for any comparability graph. As the class of comparability graphs is closed under taking induced subgraphs we have:

Corollary 66.2a. *Each comparability graph is perfect.*

Proof. Directly from Theorem 66.2. ∎

Hence, by the perfect graph theorem, also the complement of a comparability graph is perfect. This implies:

Corollary 66.2b (Dilworth's decomposition theorem)**.** *In any partially ordered set $(V, \leq)$, the maximum size of an antichain is equal to the minimum number of chains needed to cover V.*

Proof. Directly from Corollary 66.2a. ∎

Complexity. The optimization problems corresponding to the perfect graph parameters for comparability graphs can be solved in strongly polynomial time by path and flow techniques, as we saw in Chapter 14. With a greedy method, one can find a maximum-weight clique in a comparability graph $G = (V, E)$ with weight function $w : V \to \mathbb{Q}_+$ (if the underlying partial order $\leq$ is given): if all weights are 0, the problem is trivial; if there exist vertices of positive weights, find the set S of minimal elements of positive weight, let $\alpha := \min_{v \in S} w(v)$, reset $w(v) := w(v) - \alpha$ for $v \in S$, and find recursively a maximum-weight clique C for the new weights. Then we can assume that $C \cap S \neq \emptyset$. Hence C is also a maximum-weight clique for the original weight function.

This method also solves the weighted colouring problem in strongly polynomial time. An $O(n^2)$ algorithm for the weighted colouring problem for comparability graphs was given by Hoàng [1994]. The weighted stable set and clique cover problems can be solved in strongly polynomial time with flow techniques (see Chapter 14).

Trivially, recognizing comparability graphs belongs to NP (by giving the underlying partial order), and membership of co-NP follows from the characterizations of Ghouila-Houri [1962a,1964] and Gilmore and Hoffman [1964]. A method of Gallai [1967] implies that the problem in fact is polynomial-time solvable (cf. Pnueli, Lempel, and Even [1971], Golumbic [1977], Spinrad [1985], Muller and Spinrad [1989], and McConnell and Spinrad [1994,1997,1999] (the latter paper gives a linear-time recognition algorithm)).

Golumbic, Rotem, and Urrutia [1983] and Lovász [1983b] characterized complements of comparability graphs as those graphs that are the intersection graph of a family of continuous functions $f : (0, 1) \to \mathbb{R}$. (Here f and g *intersect* if $f(x) = g(x)$ for some $x \in (0, 1)$.)

Permutation graphs. A *permutation graph* is a graph on $\{1, \ldots, n\}$ for which there exists a permutation π of $\{1, \ldots, n\}$ such that $i, j \in \{1, \ldots, n\}$ are adjacent if and only if $(i-j)(\pi(i)-\pi(j)) > 0$. A graph G is (isomorphic to) a permutation graph if and only if both G and $\overline{G}$ are comparability graphs (Dushnik and Miller [1941] (also Even, Pnueli, and Lempel [1972])). McConnell and Spinrad [1997] showed that permutation graphs can be recognized in linear time (improving McConnell and Spinrad [1994]). Another characterization was given by Baker, Fishburn, and Roberts [1972].

The books by Even [1973] and Golumbic [1980] devote chapters to comparability graphs and to permutation graphs.

66.3. Chordal graphs

We next consider a further class of perfect graphs, the 'chordal graphs' (or 'rigid circuit graphs' or 'triangulated graphs'). A graph G is called *chordal* if each circuit in G of length at least 4 has a chord. (A *chord* is an edge connecting two vertices of the circuit that are nonadjacent in the circuit.) Equivalently, a graph is chordal if it has no hole.

For any set U of vertices, let $N(U)$ denote the set of vertices not in U that are adjacent to at least one vertex in U. Call a vertex v *simplicial* if $N(\{v\})$ is a clique in G.

Dirac [1961] showed the following basic property of chordal graphs:

Theorem 66.3. *Each chordal graph G contains a simplicial vertex.*

Proof. We may assume that G has at least two nonadjacent vertices a, b. Let U be a maximal nonempty subset of V with $G[U]$ connected and with $U \cup N(U) \neq V$. Such a subset U exists as $U := \{a\}$ induces a connected subgraph of G and as $\{a\} \cup N(\{a\}) \neq V$.

Let $W := V \setminus (U \cup N(U))$. Then each vertex v in $N(U)$ is adjacent to each vertex in W, since otherwise we could increase U by v. Moreover, $N(U)$ is a clique, for suppose that $u, w \in N(U)$ are nonadjacent. Choose $v \in W$. Let P be a shortest path in $U \cup N(U)$ connecting u and w. Then $P \cup \{u, v, w\}$ would form a chordless circuit of length at least 4, a contradiction.

Now inductively we know that $G[W]$ contains a vertex v that is simplicial in $G[W]$. Since $N(U)$ is a clique and since each vertex in W is adjacent to each vertex in $N(U)$, v is also simplicial in G. ∎

(The proof of Theorem 66.3 implies that, in a chordal graph, each vertex v that is nonadjacent to at least one vertex $w \neq v$, is nonadjacent to at least one simplicial vertex $w \neq v$. Hence each noncomplete chordal graph contains at least two nonadjacent simplicial vertices.)

As was observed by Fulkerson [1972a], Theorem 66.3 implies a result of Berge [1963a] (the result was announced (with partial proof) in Berge [1960a]):

Theorem 66.4. *Any chordal graph G satisfies $\omega(G) = \chi(G)$.*

Proof. By Theorem 66.3, G has a simplicial vertex v. By induction we have $\omega(G - v) = \chi(G - v)$. In particular, $G - v$ has an $\omega(G)$-vertex-colouring. As $|N(v)| \leq \omega(G) - 1$ (since $\{v\} \cup N(v)$ is a clique), we can extend this to an $\omega(G)$-vertex-colouring of G. ∎

As the class of chordal graphs is closed under taking induced subgraphs, this implies:

Corollary 66.4a. *Each chordal graph is perfect.*

Proof. Directly from Theorem 66.4. ∎

With the perfect graph theorem, this implies the following result of Hajnal and Surányi [1958] (which also can be derived directly from Theorem 66.3):

Corollary 66.4b. *For any chordal graph G, $\alpha(G) = \overline{\chi}(G)$.*

Proof. Directly from Corollary 66.4a and the perfect graph theorem (Corollary 65.2a). ∎

Complexity. Dirac's theorem (Theorem 66.3) can be used to obtain strongly polynomial-time algorithms for the basic optimization problems for chordal graphs. The proof of Theorem 66.4 yields such an algorithm to find an optimum colouring and clique, also for the weighted versions. Similarly, the strong polynomial-time solvability of the weighted stable set and clique cover problems can be derived (Gavril [1972], Frank [1976b]). $O(n^2)$ algorithms for minimum weighted colouring for chordal graphs were given by Balas and Xue [1991] and Hoàng [1994].

Dirac's theorem also directly gives a polynomial-time recognition algorithm for chordal graphs: iteratively find and delete a simplicial vertex until the graph is empty. Linear-time algorithms were given by Lueker [1974], Rose and Tarjan [1975], Rose, Tarjan, and Lueker [1976], and Tarjan and Yannakakis [1984]. (Gavril [1974b] gave another polynomial-time algorithm.)

Dirac's theorem also implies the following other characterizations of chordal graphs (Dirac [1961] (stated explicitly by Fulkerson and Gross [1965] and Rose [1970])):

(66.3) A graph $G = (V, E)$ is chordal $\iff$ each induced subgraph has a simplicial vertex $\iff$ G has an acyclic orientation $D = (V, A)$ such that if (u, v), $(u, w) \in A$, then $\{v, w\} \in E$.

Dirac [1961] moreover showed that a graph is chordal if and only if each inclusionwise minimal vertex-cut is a clique.

Interval graphs. An *interval graph* is the intersection graph G of a family $\mathcal{C}$ of nonempty intervals on the real line[11]. Trivially, such a graph is the complement of a comparability graph: define $I < J \iff i < j$ for all $i \in I$, $j \in J$. This gives a partial order, and the corresponding comparability graph is equal to $\overline{G}$.

Perfection of the complements of interval graphs was observed by T. Gallai (see Hajnal and Surányi [1958]) — that is, the maximum number of disjoint intervals in $\mathcal{C}$ is equal to the minimum number of points intersecting all intervals in $\mathcal{C}$. This is not hard to prove, and can be proved similarly to the easy dual of Dilworth's decomposition theorem (Theorem 14.1). In fact, a graph is an interval graph if and only if it is chordal and its complement is a comparability graph.

The clique, stable set, colouring, and clique cover problem and their weighted versions can be solved in strongly polynomial time with the methods for comparability graphs described above. If the intervals are given in the order of their maximal elements, and we consecutively assign to each interval the smallest available colour (numbering the colours $1, 2, \ldots$), we obtain an optimum colouring. (Kierstead [1988] showed that if we get the intervals in an *arbitrary* order and we assign to any given interval the smallest possible colour ('on-line'), then we need at most $40\chi(G)$ colours.)

In fact, for any clique C in G there is a point x such that all intervals in C contain x (by Helly's theorem: a family of pairwise intersecting intervals has a nonempty intersection). So finding a maximum-weight clique is trivial. A maximum-size stable

[11] The *intersection graph* of a family $\mathcal{C}$ is the graph with vertex set $\mathcal{C}$, two sets in $\mathcal{C}$ being adjacent if and only if they intersect.

set can be found by a greedy method: first find an interval $I \in \mathcal{C}$ with $\sup I$ minimal. Next find recursively a maximum-size stable set S among the intervals in $\mathcal{C}$ disjoint from I. Then $S \cup \{I\}$ is a maximum-size stable set in G.

In reply to questions of Hajós [1957] and Benzer [1959], interval graphs have been characterized by Lekkerkerker and Boland [1962] (cf. Halin [1982]), Gilmore and Hoffman [1964], and Fulkerson and Gross [1965]. The latter paper gives a polynomial-time recognition algorithm. A linear-time recognition algorithm was given by Booth and Lueker [1975,1976]. This was simplified by Korte and Möhring [1987], Corneil, Olariu, and Stewart [1998], Hsu and Ma [1999], and Hsu [2002].

More on interval graphs can be found in the books by Golumbic [1980], Skrien [1982], Fishburn [1985], and Brandstädt, Le, and Spinrad [1999], and in the survey article by Golumbic [1985].

Split graphs. A *split graph* is a graph $G = (V, E)$ where V can be partitioned into a clique C and a stable set S. Trivially, a split graph is perfect, since C is contained in a maximum-size clique; hence we can assume that C is a maximum-size clique; so for each $s \in S$ there is a $c \in C$ nonadjacent to s; this yields a $|C|$-vertex-colouring of G.

A graph G is a split graph if and only if both G and $\overline{G}$ are chordal graphs (Foldes and Hammer [1977], Hammer and Simeone [1981]). The book by Golumbic [1980] devotes a chapter to split graphs.

Trivially perfect graphs. Golumbic [1978] calls a graph *trivially perfect* if for each induced subgraph, the stability number is equal to the number of inclusionwise maximal cliques. Trivially, each trivially perfect graph is perfect. Choudom, Parthasarathy, and Ravindra [1975] and Golumbic [1978] showed that a graph is trivially perfect if and only if it has no induced subgraph equal to the path P_4 or the circuit C_4 (each with 4 vertices). This implies (by a theorem of Wolk [1962] (proof simplified in Wolk [1965])) that a graph is trivially perfect if and only if it is the comparability graph coming from a branching. Another characterization of trivially perfect graphs was given by Alexe and Olaru [1997].

Threshold graphs. A *threshold graph* is a graph on vertex set V given by a function $w : V \to \mathbb{R}$, such that two distinct vertices u, v are adjacent if and only if $w(u) + w(v) > 0$. Chvátal and Hammer [1977] showed that a graph G is a threshold graph if and only if neither G nor $\overline{G}$ has an induced subgraph equal to the path P_4 or the circuit C_4 (each with 4 vertices) — that is, both G and $\overline{G}$ are trivially perfect.

Each threshold graph is a split graph (trivially) and a permutation graph (order the vertices as $v_1, \ldots, v_n$ such that $w(v_1) \leq w(v_2) \leq \cdots \leq w(v_n)$, and let π be the permutation given by ordering $|w(v_1)|, |w(v_2)|, \ldots, |w(v_n)|$). However, the path P_4 with 4 vertices is both a split graph and a permutation graph, but no threshold graph.

The book by Mahadev and Peled [1995] focuses on threshold graphs, and the book by Golumbic [1980] devotes a chapter to threshold graphs. A related class of graphs was described by Wang [1995,1996].

'Strongly chordal' graphs have been studied by Farber [1983,1984] and Kaplan and Shamir [1994], and an analogue of chordal graphs for bipartite graphs by Golumbic and Goss [1978].

66.3a. Chordal graphs as intersection graphs of subtrees of a tree

Chordal graphs can be characterized as intersection graphs of subtrees of a tree, as was shown by L. Surányi (see Gyárfás and Lehel [1970]) and also by Walter [1972, 1978], Buneman [1974], and Gavril [1974c].

Let $\mathcal{S}$ be a collection of nonempty subtrees of a tree T. The *intersection graph* of $\mathcal{S}$ is the graph with vertex set $\mathcal{S}$, where two vertices S, S' are adjacent if and only if S and S' have at least one vertex in common.

The class of graphs obtained in this way coincides with the class of chordal graphs. To see this, we first show the following elementary lemma:

Lemma 66.5α. *Let $\mathcal{S}$ be a collection of pairwise intersecting subtrees of a tree T. Then there is a vertex of T contained in all subtrees in $\mathcal{S}$.*

Proof. By induction on $|VT|$. If $|VT| = 1$ the lemma is trivial, so assume $|VT| \geq 2$. Let t be an end vertex of T. If there exists a subtree in $\mathcal{S}$ consisting only of t, the lemma is trivial. Hence we may assume that each subtree in $\mathcal{S}$ containing t also contains the neighbour of t. So deleting t from T and from all subtrees in $\mathcal{S}$ gives the lemma by induction. ∎

Then we have the subtree characterization of chordal graphs:

Theorem 66.5. *A graph is chordal if and only if it is isomorphic to the intersection graph of a collection of subtrees of some tree.*

Proof. *Necessity.* Let $G = (V, E)$ be chordal. By Theorem 66.3, G contains a simplicial vertex v. By induction, the graph $G - v$ is the intersection graph of a collection $\mathcal{S}$ of subtrees of some tree T. Let $\mathcal{S}'$ be the subcollection of $\mathcal{S}$ corresponding to the set N of neighbours of v in G. As N is a clique, $\mathcal{S}'$ consists of pairwise intersecting subtrees. Hence, by Lemma 66.5α, these subtrees have a vertex t of T in common. Now we extend T and all subtrees in $\mathcal{S}'$ with a new vertex s and a new edge st. Moreover, we introduce a new subtree $\{s\}$ representing v. In this way we obtain a subtree representation for G.

Sufficiency. Let G be the intersection graph of some collection $\mathcal{S}$ of subtrees of some tree T. By (66.3) it suffices to show that G has a simplicial vertex. Let s be an end vertex of T. If $\mathcal{S}$ contains a subtree only consisting of s, it gives a simplicial vertex in G. If $\mathcal{S}$ contains no such subtree, then each subtree in $\mathcal{S}$ containing s also contains the neighbour t (say) of s. So deleting s from T and from all subtrees in $\mathcal{S}$, does not modify the graph G. Hence we are done by induction. ∎

This theorem enables us to interpret the perfection of chordal graphs in terms of trees:

Corollary 66.5a. *Let $\mathcal{S}$ be a collection of nonempty subtrees of a tree T. Then the maximum number of pairwise vertex-disjoint trees in $\mathcal{S}$ is equal to the minimum number of vertices of T intersecting each tree in $\mathcal{S}$.*

Proof. Directly from Corollary 66.4b and Theorem 66.5, using Lemma 66.5α. ∎

(This result was also stated by Cockayne, Hedetniemi, and Slater [1979].)
Similarly we have:

Corollary 66.5b. *Let $\mathcal{S}$ be a collection of subtrees of a tree T. Let k be the maximum number of times that any vertex of T is covered by trees in $\mathcal{S}$. Then $\mathcal{S}$ can be partitioned into subcollections $\mathcal{S}_1, \ldots, \mathcal{S}_k$ such that each $\mathcal{S}_i$ consists of pairwise vertex-disjoint trees.*

Proof. Directly from Theorems 66.4 and 66.5, again using Lemma 66.5α. ∎

Variations of the problem of packing and covering a tree by subtrees were studied by Bárány, Edmonds, and Wolsey [1986]. More characterizations of chordal graphs were offered by Benzaken, Crama, Duchet, Hammer, and Maffray [1990]. More on chordal graphs can be found in the book of Golumbic [1980] and in Skrien [1982], Leung [1984], Seymour and Weaver [1984] (a generalization of chordal graphs), Lubiw [1987], Wallis and Wu [1995], and Nakamura and Tamura [2000] (a generalization to bidirected graphs).

66.4. Meyniel graphs

Markosyan and Karapetyan [1976] and Meyniel [1976] showed the perfection of graphs in which each odd circuit of length at least five has at least two chords (*Meyniel graphs*). This was conjectured by Olaru [1969,1972].

It implies the perfection of *Gallai graphs* — graphs in which each odd circuit of length at least five has two *noncrossing* chords (Gallai [1962], cf. Surányi [1968] for a shorter proof [12]), *parity graphs* — graphs in which each odd circuit of length at least five has two crossing chords (Olaru [1969,1972, 1977], cf. Sachs [1970]), and graphs that have no path P_4 as induced subgraph (Seinsche [1974]).

We follow the proof given by Lovász [1983b] (which is a simplification of Meyniel's original proof).

Theorem 66.6. *Each Meyniel graph is perfect.*

Proof. I. We first show that in a Meyniel graph $G = (V, E)$:

(66.4) for each odd circuit C and each vertex v on C, C has a chord disjoint from v or each vertex of $C - v$ is adjacent to v.

[12] Gallai [1962] published a proof that $\alpha(G) = \overline{\chi}(G)$ for graphs in which each odd circuit of length at least 5 has two noncrossing chords. Berge [1997] wrote that Gallai informed him in a letter that he knew that also $\omega(G) = \chi(G)$ holds for these graphs.

Let C have no chord disjoint from v. Then the subgraph of G induced by VC is outerplanar, with C as boundary. As each odd circuit of size at least five has a chord we know that each odd bounded face is a triangle. (A face is *odd* (*even*) if its is incident with an odd (even) number of edges.)

Moreover, as C is odd, there is at least one odd bounded face. So if v is not adjacent to all vertices of $C - v$, there is an even bounded face, neighbouring an odd bounded face. But then the union of these two faces forms an odd circuit with only one chord, contradicting the condition.

II. We now prove the theorem. It suffices to show that $\chi(G) = \omega(G)$ for any Meyniel graph $G = (V, E)$, as the class of Meyniel graphs is closed under taking induced subgraphs. We may assume that $V = \{1, \ldots, n\}$. Let $k := \chi(G)$.

For each colouring $\phi : V \to \{1, \ldots, k\}$, let the (ordered) clique $K_\phi = (v_1, \ldots, v_t)$ be obtained recursively as follows. If $v_1, \ldots, v_i$ have been determined (for $i \geq 0$), then v_{i+1} is the largest vertex of colour $i+1$ that is adjacent to each of $v_1, \ldots, v_i$. If no such vertex exists, we stop, setting $t := i$.

Let ϕ be a k-colouring with $K_\phi = (v_1, \ldots, v_t)$ lexicographically minimal. If $t = k$ we are done, so assume $t < k$. Consider the subgraph of G induced by the vertices coloured t and $t + 1$, and let H be its component containing v_t. Let ψ be the colouring obtained from ϕ by interchanging colours t and $t+1$ in H. We show that K_ψ is lexicographically less than K_ϕ, contradicting our assumption.

Trivially, $v_1, \ldots, v_{t-1}$ belong to K_ψ (since we did not change any of the colours $1, \ldots, t-1$). If no other vertex is in K_ψ we are done, so we can assume that K_ψ contains a vertex w with $\psi(w) = t$.

Then $w \neq v_t$, since $\psi(v_t) = t + 1$. If $w < v_t$ we are done, so we can assume that $w > v_t$. If $\phi(w) = t$, this contradicts the choice of $v_t \in K_\phi$. So $\phi(w) = t + 1$, and H contains a shortest path P from v_t to w. Necessarily, this path is odd, and has no chords.

Let u be the second vertex on P. So $\phi(u) = t + 1$. Since v_t is the last vertex in K_ϕ we know that there is an $i \in \{1, \ldots, t-1\}$ with v_i not adjacent to u. Let C be the circuit made by P, $v_i v_t$, and $v_i w$. As P has no chords, by (66.4) v_i is adjacent to u, a contradiction. ∎

Ravindra [1982] showed that each Meyniel graph is strongly perfect (see Section 66.5a below). This was extended by Hoàng [1987b], who showed that Meyniel graphs are precisely those graphs with the property that for each induced subgraph H and each vertex v of H, there exists a stable set in H containing v and intersecting all inclusionwise maximal cliques of H. (This was conjectured by Meyniel.)

Complexity. Burlet and Fonlupt [1984] showed that the class of Meyniel graphs is closed under amalgamation (see Section 65.7c) and that each Meyniel graph arises by amalgamation from chordal graphs and bipartite graphs added with one vertex connected to all vertices of the bipartite graph. They showed that it yields a polynomial-time recognition algorithm (speeded up by Roussel and Rusu [1999b]).

Hoàng [1987b] gave an $O(n^8)$-time algorithm to find a minimum colouring and a maximum clique. An $O(n^3)$ algorithm was given by Hertz [1990a].

(Conforti, Cornuéjols, Kapoor, and Vušković [1999] consider an extension by decomposing *cap-free* graphs (where a *cap* is a circuit with exactly one chord, connecting two vertices at distance two in the circuit) — a (not necessarily perfect) generalization of Meyniel graphs.)

Gallai graphs. As mentioned, these are graphs in which each odd circuit of length ≥ 5 has two noncrossing chords. Polynomial-time recognition algorithms were given by Burlet and Fonlupt [1984], Whitesides [1984], and Cicerone and Di Stefano [1999b] (linear-time). The latter paper also gives a linear-time maximum-weight clique algorithm. A linear-time colouring algorithm was found by Roussel and Rusu [1999a].

Parity graphs. As mentioned, these are graphs in which each odd circuit of length ≥ 5 has two crossing chords. Parity graphs can be characterized alternatively as those graphs such that for each pair u, v of vertices, all chordless $u - v$ paths have the same parity.

Combinatorial strongly polynomial-time algorithms to solve the weighted clique, stable set, colouring, and clique cover problems in parity graphs were given by Burlet and Uhry [1982], who also gave a polynomial-time recognition algorithm (by decomposition of the graph into smaller parity graphs).

The parity graphs include the *line-perfect graphs*, which are graphs whose line graph is perfect. They were characterized by Trotter [1977] — see the claw-free graphs in Section 65.6d. More on parity graphs can be found in Adhar and Peng [1990], Bandelt and Mulder [1991], Przytycka and Corneil [1991], Rusu [1995b], Jansen [1998], and Cicerone and Di Stefano [1999a].

66.5. Further results and notes

66.5a. Strongly perfect graphs

Following Berge and Duchet [1984], a graph $G = (V, E)$ is *strongly perfect* if each induced subgraph H has a stable set intersecting all inclusionwise maximal cliques of H. Each strongly perfect graph is perfect (by (65.2)). Berge and Duchet showed that comparability graphs, chordal graphs, and complements of chordal graphs are strongly perfect. Ravindra [1982] showed that Meyniel graphs are strongly perfect, and Chvátal [1984d] that perfectly orderable graphs are strongly perfect.

Berge and Duchet also showed that the recognition problem for strongly perfect graphs belongs to co-NP. No combinatorial polynomial-time algorithms are known for the optimization problems for strongly perfect graphs.

Olaru [1996] showed that the graphs that are both minimally strongly imperfect and imperfect are precisely the odd circuits of length at least five and their complements. Hence to prove the strong perfect graph theorem it suffices to show that each minimally imperfect graph is also minimally strongly imperfect.

More on strongly perfect graphs can be found in Ravindra [1981,1999], Berge [1983], Basavayya and Ravindra [1985,1987], Preissmann [1985], Preissmann and de Werra [1985], Olaru and Mîndrescu [1986a,1986b], Olaru [1987,1993], Ravindra and Basavayya [1988,1992,1994,1995], Olariu [1989a], Włoch [1995], Blidia, Duchet, and Maffray [1996], Szelecka and Włoch [1996], and Alexe and Olaru [1997].

66.5b. Perfectly orderable graphs

A graph $G = (V, E)$ is a *perfectly orderable graph* if it has an acyclic orientation $D = (V, A)$ such that if four vertices v_1, v_2, v_3, v_4 induce a chordless path in G with edges v_1v_2, v_2v_3, v_3v_4, then $(v_1, v_2) \in A$ or $(v_4, v_3) \in A$. Chvátal [1984d] showed that perfectly orderable graphs are perfect — in fact, strongly perfect:

Theorem 66.7. *Each perfectly orderable graph is strongly perfect.*

Proof. We can assume that $V = \{1, \ldots, n\}$ and that if $(i, j) \in A$, then $i < j$. Let S be the stable set with $\sum(2^{-i} \mid i \in S)$ maximal. Then each $v \notin S$ has a neighbour $u \in S$ with $u < v$, since otherwise $(S \setminus N(v)) \cup \{v\}$ is better than S.

We show that each inclusionwise maximal clique K intersects S. Suppose $K \cap S = \emptyset$. For $s \in S$, let K_s be the set of neighbours $v \in K$ with $s < v$. Choose $s \in S$ with $\sum(2^i \mid i \in K_s)$ maximal. As K is a maximal clique, s is nonadjacent to some $v \in K$. Let $u \in S$ be a neighbour of v with $u < v$. So $v \in K_u \setminus K_s$. By the choice of s, there is a vertex $i \in K_s \setminus K_u$ with $i > v$. So $u < v < i$, and hence u and i are nonadjacent (otherwise $i \in K_u$). As u and s are nonadjacent (since $u, s \in S$) and v and i are adjacent (since $v, i \in K$), u, v, i, s induce a P_4 subgraph with $(u, v), (v, i), (s, i) \in A$, a contradiction. ∎

(Another proof, and a generalization, of this was given by Duchet and Olariu [1991].)

Note that the set S in this proof can be found by a greedy method. So we can find an optimum colouring in polynomial time. Given an orientation as above, also a maximum-size clique can be found in a greedy way — see Chvátal [1984d]. Hoàng [1994] gave $O(nm)$-time algorithms, also for the weighted versions. Middendorf and Pfeiffer [1990a] showed that it is NP-complete to decide if a graph is perfectly orderable.

Comparability graphs, chordal graphs, and complements of chordal graphs are perfectly orderable.

More on perfectly orderable graphs can be found in Cochand and de Werra [1986], Preissmann, de Werra, and Mahadev [1986], Chvátal, Hoàng, Mahadev, and de Werra [1987], Lehel [1987], Hertz [1988,1990b], Hoàng and Khouzam [1988], Olariu [1988a,1993], Bielak [1989], Hoàng and Mahadev [1989], Hoàng and Reed [1989a,1989b], Jamison and Olariu [1989a], Chvátal [1990,1993], Hoàng, Maffray, and Preissmann [1991], Croitoru and Radu [1992a], Hoàng, Maffray, Olariu, and Preissmann [1992], Gavril, Toledano Laredo, and de Werra [1994], Arikati and Peled [1996], Giakoumakis [1996], Hoàng [1996a,1996b,2001], Rusu [1996], Hayward [1997a], Hoàng, Maffray, and Noy [1999], and Hoàng and Tu [2000].

More classes of graphs based on orienting or colouring edges are given by Hoàng [1987a].

66.5c. Unimodular graphs

A graph $G = (V, E)$ is *unimodular* if the following matrix M is totally unimodular: the columns are indexed by V and the rows are the incidence vectors of all inclusionwise maximal cliques of G. Any induced subgraph of a unimodular graph is unimodular again, since for each $v \in V$ and for each maximal clique C of $G - v$, either C or $C \cup \{v\}$ is a maximal clique of G.

Unimodular graphs include bipartite graphs, line graphs of bipartite graphs, and interval graphs.

Perfection of unimodular graphs and their complements was shown by Berge [1963a]. Perfection of the complements of unimodular graphs follows from the Hoffman-Kruskal theorem (Hoffman and Kruskal [1956]), since

(66.5) $$\alpha(G) = \max\{\mathbf{1}^{\mathsf{T}} x \mid x \geq \mathbf{0}, Mx \leq \mathbf{1}\} = \min\{y^{\mathsf{T}}\mathbf{1} \mid y \geq \mathbf{0}, y^{\mathsf{T}} M \geq \mathbf{1}\} = \chi(\overline{G}),$$

as the LP-optima are attained by integer vectors x and y.

The perfection of a unimodular graph $G = (V, E)$ can also be derived from the Hoffman-Kruskal theorem, with an idea which Berge [1963a] attributed to M.H. McAndrew. It suffices to find a stable set that intersects all maximum-size cliques. Let M' be the submatrix of M corresponding to the maximum-size cliques. The system $\mathbf{0} \leq x \leq \mathbf{1}$, $Mx \leq \mathbf{1}$, $M'x \geq \mathbf{1}$ has a solution (namely $x = \omega(G)^{-1}\mathbf{1}$). Hence, as M is total unimodular, it has an integer solution x. This is the incidence vector of a stable set as required.

By a result of Heller [1957] (cf. Theorem 21.4 in Schrijver [1986b]), a unimodular graph has at most $|V|(|V|+1)$ inclusionwise maximal cliques. As W.H. Cunningham (cf. Grötschel, Lovász, and Schrijver [1988]) observed, this gives a polynomial-time method to enumerate all maximal cliques: Choose $v \in V$. Enumerate the maximal cliques $C_1, \ldots, C_t$ of $G - v$ (recursively). Then the maximal cliques of G are among the cliques C_i $(i = 1, \ldots, t)$, and $(C_i \cap N(v)) \cup \{v\}$ $(i = 1, \ldots, t)$. We can select the maximal cliques among these cliques in polynomial time. Since $t \leq |V|(|V| + 1)$, this gives a polynomial-time method.

This directly gives a strongly polynomial-time method to find a maximum-weight clique. It also implies that the weighted versions of the stable set, colouring, and clique cover problems can be solved in strongly polynomial time, by solving an explicit linear programming problem (using Tardos [1986]). The colouring problem can be solved recursively by first finding (with LP-techniques) a $0, 1$ vector x satisfying $x(C) \leq 1$ for each maximal clique C and $x(C) = 1$ for each maximum-size clique C, and next colouring $G - S$ recursively (where $x = \chi^S$). The weighted version can be solved similarly.

Since by a theorem of Seymour [1980a], totally unimodular matrices can be recognized in polynomial time, this also yields a polynomial-time method to recognize a unimodular matrix.

Ghouila-Houri [1962b] showed that a graph $G = (V, E)$ is unimodular if and only if each nonempty subset U of V contains two disjoint sets U_1 and U_2 such that $U_1 \cup U_2 \neq \emptyset$ and such that each maximal clique C of G with $|C \cap U|$ even, satisfies $|C \cap U_1| = |C \cap U_2|$.

66.5d. Further classes of perfect graphs

Weakly chordal graphs. A graph $G = (V, E)$ is called *weakly chordal* (or *weakly triangulated*) if neither G nor its complement contains a chordless circuit of length at least 5. Hayward [1985] showed that weakly chordal graphs are perfect. Polynomial-time algorithms for the optimization problems related to weakly chordal graphs were given by Hayward, Hoàng, and Maffray [1989], Spinrad and Sritharan [1995], and Hayward, Spinrad, and Sritharan [2000], and polynomial-time recognition algorithms by Spinrad and Sritharan [1995] and Hayward, Spinrad, and Sritharan [2000]. The class of weakly chordal graphs contains both the chordal graphs and their complements.

More on weakly chordal graphs is given in Hoàng, Maffray, Olariu, and Preissmann [1992], Hayward [1996,1997a,1997b], and McMorris, Wang, and Zhang [1998]. Weakly chordal comparability graphs were studied by Eschen, Hayward, Spinrad, and Sritharan [1999].

Quasi-parity graphs. A graph $G = (V, E)$ is a *quasi-parity graph* if each induced subgraph H that is not a clique has two vertices that are not connected by a chordless path of odd length. Meyniel [1987] showed that these graphs are perfect, and that they include the Meyniel graphs and the perfectly orderable graphs. (A short proof of this last is given by Hertz and de Werra [1988].)

Berge [1986] showed that the class of quasi-parity graphs can be enlarged to those graphs in which for each induced subgraph H with at least two vertices, there exist two vertices such that in H or $\overline{H}$ there is no chordless odd-length path connecting them.

Edmonds-Giles graphs. Let $D = (V, A)$ be a directed graph and let $\mathcal{C}$ be a crossing collection of subsets of V with $\delta^{\text{out}}(U) = \emptyset$ for each $U \in \mathcal{C}$. Make an undirected graph G with vertex set A, two arcs a, a' being adjacent if and only if there is a $U \in \mathcal{C}$ such that both a and a' enter U. In Schrijver [1983a] such a graph is called an *Edmonds-Giles graph*. Each such graph is perfect, as can be seen as follows.

A special case of the Edmonds-Giles theorem (Theorem 60.1) is that the system (in $x \in \mathbb{R}^A$)

$$\text{(66.6)} \qquad \begin{array}{lll} \text{(i)} & 0 \leq x(a) \leq 1 & \text{for } a \in A, \\ \text{(ii)} & x(\delta^{\text{in}}(U)) \leq 1 & \text{for } U \in \mathcal{C}, \end{array}$$

is totally dual integral. Hence it determines an integer polytope. Now the integer vectors x satisfying (66.6) are exactly the incidence vectors of the stable sets of G. Each inequality (66.6)(ii) is a clique inequality. The stable set polytope of G therefore is determined by the clique inequalities, and hence G is perfect (Corollary 65.2e). It in particular implies that each clique of G is contained in $\delta^{\text{in}}(U)$ for some $U \in \mathcal{C}$.

A special case of Edmonds-Giles graphs was given by Kahn [1984], where $D = (V, A)$ is a directed graph and $\mathcal{C}$ is the collection of nonempty proper subsets U of V with $\delta^{\text{out}}(U) = \emptyset$ and $|\delta^{\text{in}}(U)|$ minimal. With the perfect graph theorem this implies that the arcs of a digraph can be coloured in such a way that each minimum-size directed cut contains each colour exactly once.

p-comparability graphs. Cameron and Edmonds [1992] showed perfection of the following graphs. Let $D = (V, A)$ be a directed graph and let $U \subseteq V$ be such that each directed circuit of D has precisely one vertex in U. Let G be the undirected graph on $V \setminus U$ with any two $u, v \in V \setminus U$ adjacent if and only if there is a directed circuit containing u and v. Cameron and Edmonds [1992] call such graphs *p-comparability graphs*. Any comparability graph is a p-comparability graph, but not conversely.

Each such graph G is perfect. The proof is by reduction to minimum-cost flow, using the facts that each clique of G is contained in some directed circuit of D and that by Theorem 65.11 it suffices to show that $\overline{\chi}^*(G) = \overline{\chi}(G)$. (The class of p-comparability graphs is closed under taking induced subgraphs, since adding all arcs (u, v) for which there is a directed $u - v$ path avoiding U, maintains the above property of D.)

Now a minimum fractional clique cover of G corresponds to a minimum fractional covering of $V \setminus U$ by directed circuits. By the integer flow theorem, this last is attained by an integer covering of $V \setminus U$ by directed circuits. Hence, the minimum fractional clique cover in G is attained by an integer clique cover. This amounts to $\overline{\chi}^*(G) = \overline{\chi}(G)$.

Polyominoes. A *polyomino* is a union of unit squares in the plane. (A unit square is a square with integer coordinates and area 1.)

Given a polyomino P, make a graph G with vertices all unit squares contained in P, two of them being adjacent if and only if P contains a rectangle (with horizontal and vertical sides) containing both squares. Győri [1984] showed that if P is horizontally convex, then $\alpha(G) = \overline{\chi}(G)$ (see Section 60.3d). (P is *horizontally convex* if each horizontal line has a convex intersection with P.) This extends a result of Chaiken, Kleitman, Saks, and Shearer [1981], who proved $\alpha(G) = \overline{\chi}(G)$ if P is orthogonally convex. (P is *orthogonally convex* if each horizontal or vertical line has a convex intersection with P.) The latter paper also mentions that E. Szemerédi gave an example that one cannot delete orthogonal convexity, and it gives an example of F.R.K. Chung (1979) showing that one cannot relax it to simple connectivity.

Saks [1982] showed that if P is orthogonally convex, then the subgraph of G induced by the boundary squares is perfect. (A *boundary square* of P is a unit square having a neighbouring square not in P.) (This was proved for the subset of corner squares by Chaiken, Kleitman, Saks, and Shearer [1981]. (A *corner square* of P is a unit square having at least two neighbouring squares not in P.))

Shearer [1982] showed that also the following graph G arising from a *simply connected* polyomino P is perfect: the vertices of G are the rectangles contained in P, where two of them are adjacent if and only if they have a unit square in common.

Motwani, Raghunathan, and Saran [1989] showed that the visibility graph of a horizontally convex polyomino is perfect; in fact, a permutation graph. More on this and related problems can be found in Berge, Chen, Chvátal, and Seow [1981], Győri [1985], Motwani, Raghunathan, and Saran [1988,1990], and Maire [1994a].

66.5e. Further notes

Hayward [1990] showed that graphs containing neither C_5 nor P_6 nor $\overline{P_6}$ as induced subgraphs, are perfect. Other classes of perfect graphs were studied by Ravindra

[1976], Payan [1983], Golumbic, Monma, and Trotter [1984], Hammer and Mahadev [1985], Monma, Reed, and Trotter [1988], Hertz [1989a,1989b,1989c], Hoàng and Maffray [1989,1992], Bertschi [1990], Lubiw [1991b], Sun [1991], Croitoru and Radu [1993], Gurvich, Temkin, Udalov, and Shapovalov [1993], Thomas [1993], Maire [1994b,1996], Rusu [1995b,1999c,1999a], Cheah and Corneil [1996], Gyárfás, Kratsch, Lehel, and Maffray [1996] Giakoumakis [1997], Giakoumakis and Rusu [1997], and Maffray and Preissmann [1999]. Le [2000] gave conjectures on the perfection of certain classes of graphs. A survey of several classes of perfect graphs and their recognition and interrelations, is given in the book by Brandstädt, Le, and Spinrad [1999]. The book of Simon [1992] studies efficient algorithms for classes some of perfect graphs.

Conforti, Cornuéjols, Kapoor, and Vušković [1997] investigated 'universally signable' graphs, a generalization of chordal graphs.

Hammer and Maffray [1993] introduced 'preperfect' graphs, and showed that each preperfect graph is perfect, and that preperfect graphs include the Gallai and the parity graphs (cf. Section 66.4).

Corneil and Stewart [1990] studied the complexity of finding minimum-size dominating sets in several classes of perfect graphs. (A *dominating set* is a set U of vertices with $U \cup N(U) = V$.)

Berge and Las Vergnas [1970] showed that a graph G is perfect if for each odd circuit C and each maximal clique K, the intersection of C and K does not consist of two vertices that form an edge of C.

Vertex cuts in perfect and minimally imperfect graphs were surveyed by Rusu [2001]. A characterization of perfect total graphs was given by Rao and Ravindra [1977].

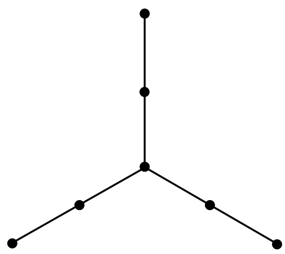

Figure 66.1

Lovász [1983b] calls a graph *k-perfect* if for each induced subgraph $G = (V, E)$ one has:

(66.7) $$\omega_k(G) = \min_{U \subseteq V} (k\chi(G - U) + |U|)$$

where $\omega_k(G)$ is the maximum size of a union of k cliques. By the results of Greene and Kleitman (Corollaries 14.8a and 14.10a), comparability graphs and their complements are k-perfect for each k. Also, complements of line graphs of bipartite graphs are k-perfect, by Corollary 21.4b. On the other hand, the line graph of the bipartite graph in Figure 66.1 is not 2-perfect (Greene [1976]). Related results were given by Berge [1989b,1992a,1992b] and Cameron [1989].

A.J. Hoffman and E.L. Johnson (cf. Golumbic [1980]) proposed the following sharpening of perfection. Let $G = (V, E)$ be a graph and let $w : V \to \mathbb{Z}_+$. A *k-interval colouring* is an assignment to each vertex v of an open subinterval of $[0, k]$ of length $w(v)$ such that adjacent vertices obtain disjoint intervals. Let $\chi_{\text{int}}(G, w)$ denote the minimum value of k for which G has a k-interval colouring. If $w(v) = 1$ for each vertex v, then $\chi_{\text{int}}(G, w) = \chi(G)$. Call G *superperfect* if $\chi_{\text{int}}(G, w)$ is equal to the maximum of $w(K)$ over all cliques K in G. As Hoffman observed, each comparability graph is superperfect (this can be derived from Dilworth's decomposition theorem), but none of the other known interesting classes of perfect graphs have this property.

A survey on subclasses of 'classical' perfect graphs (comparability graphs and chordal graphs) was given by Duchet [1984]. More examples and applications of perfect graphs were given by Shannon [1956], Berge [1967], and Tucker [1973a].

Chapter 67

Perfect graphs: polynomial-time solvability

In this chapter we show that a maximum-weight stable set and a minimum weighted clique cover in a perfect graph can be found in strongly polynomial time. This was shown by Grötschel, Lovász, and Schrijver [1981,1988] with the help of the ellipsoid method and of the function $\vartheta(G)$, introduced by Lovász [1979d] as upper bound on the Shannon capacity of a graph G. No combinatorial polynomial-time algorithms for these problems are known.

We should stress that the naive approach of applying the ellipsoid method to the stable set polytope of a perfect graph using the clique inequalities does not work: it reduces the problem of finding a maximum-weight stable set to deciding for any $x \in \mathbb{R}_+^V$ if there is a clique C satisfying $x(C) > 1$. This is equivalent to finding a maximum-weight clique, which is equivalent to finding a maximum-weight stable set in the complementary graph, which is perfect again. So this would give nothing but a reduction to itself.

In this chapter, all graphs can be assumed to be simple.

67.1. Optimum clique and colouring in perfect graphs algorithmically

Lovász [1979d] introduced the following real number $\vartheta(G)$ for any graph $G = (V, E)$. Let $\mathcal{M}_G$ be the collection of symmetric $V \times V$ matrices satisfying $M_{u,v} = 0$ for any two distinct adjacent vertices u and v and satisfying $\mathrm{Tr}M = 1$. Here $\mathrm{Tr}M$ is the trace of M (sum of diagonal elements). Define

(67.1) $$\vartheta(G) := \max\{\mathbf{1}^{\mathsf{T}} M \mathbf{1} \mid M \in \mathcal{M}_G \text{ positive semidefinite}\}.$$

Here $\mathbf{1}$ denotes the all-one vector in $\mathbb{R}^V$.

$\vartheta(G)$ has two important properties: it can be calculated (at least, approximated) in polynomial time, and it gives an, often close, upper bound on the stable set number $\alpha(G)$ (Lovász [1979d]).

First we show (where $\overline{\chi}^*(G)$ denotes the fractional clique cover number — cf. Section 64.8):

Theorem 67.1. *For any graph $G = (V, E)$:*

(67.2) $\alpha(G) \leq \vartheta(G) \leq \overline{\chi}^*(G)$.

Proof. To see $\alpha(G) \leq \vartheta(G)$, let S be a maximum-size stable set and let M be the matrix given by:

(67.3) $$M := \frac{1}{|S|}\chi^S(\chi^S)^{\mathsf{T}}.$$

Here χ^S is the incidence vector of S in $\mathbb{R}^V$. Then M belongs to $\mathcal{M}_G$ and is positive semidefinite. Hence $\alpha(G) = |S| = \mathbf{1}^{\mathsf{T}}M\mathbf{1} \leq \vartheta(G)$.

To see $\vartheta(G) \leq \overline{\chi}^*(G)$, let M attain the maximum in (67.1). Consider cliques $C_1, \ldots, C_k$ and $\lambda_1, \ldots, \lambda_k \geq 0$ with

(67.4) $$\sum_{i=1}^{k} \lambda_i \chi^{C_i} = \mathbf{1} \text{ and } \sum_{i=1}^{k} \lambda_i = \overline{\chi}^*(G).$$

Then, setting $\gamma := \overline{\chi}^*(G)$:

(67.5) $$\begin{aligned} 0 &\leq \sum_{i=1}^{k} \lambda_i(\gamma \cdot \chi^{C_i} - \mathbf{1})^{\mathsf{T}} M(\gamma \cdot \chi^{C_i} - \mathbf{1}) \\ &= \gamma^2 \sum_{i=1}^{k} \lambda_i (\chi^{C_i})^{\mathsf{T}} M \chi^{C_i} - 2\gamma \sum_{i=1}^{k} \lambda_i (\chi^{C_1})^{\mathsf{T}} M\mathbf{1} + \gamma \mathbf{1}^{\mathsf{T}} M\mathbf{1} \\ &= \gamma^2 \mathrm{Tr} M - 2\gamma \mathbf{1}^{\mathsf{T}} M\mathbf{1} + \gamma\mathbf{1}^{\mathsf{T}} M\mathbf{1} = \gamma^2 - \gamma\vartheta(G), \end{aligned}$$

since $\mathrm{Tr}M = 1$, $\mathbf{1}^{\mathsf{T}}M\mathbf{1} = \vartheta(G)$, and $M_{u,v} = 0$ if $u \neq v$ and $u, v \in C_i$ for some i.

(67.5) implies that $\vartheta(G) \leq \gamma = \overline{\chi}^*(G)$. ∎

Moreover, $\vartheta(G)$ can be approximated in polynomial time (Grötschel, Lovász, and Schrijver [1981]):

Theorem 67.2. *There is an algorithm that for any given graph $G = (V, E)$ and any $\varepsilon > 0$, returns a rational closer than ε to $\vartheta(G)$, in time bounded by a polynomial in $|V|$ and* $\log(1/\varepsilon)$.

Proof. This is a consequence of Corollary (4.3.12) in Grötschel, Lovász, and Schrijver [1988], stating that we can solve a convex optimization problem approximatively in polynomial time, if we know a ball contained in the feasible region and a ball containing the feasible region, and if we can test membership of the feasible region in polynomial time. These conditions are satisfied, if we restrict ourselves to the affine space $\mathcal{M}_G$. The convex body of all positive semidefinite matrices in $\mathcal{M}_G$ contains the ball with center $(1/|V|) \cdot I$ and radius $1/|V|^2$, and is contained in the ball with center the all-zero matrix and radius $|V|^2$. Membership can be tested in polynomial time, since we can test positive semidefiniteness in polynomial time. ∎

The two theorems above imply:

Corollary 67.2a. *For any graph G satisfying $\alpha(G) = \overline{\chi}(G)$, the stable set number can be found in polynomial time.*

Proof. Theorem 67.1 implies $\alpha(G) = \vartheta(G) = \overline{\chi}(G)$, and by Theorem 67.2 we can find a number closer than $\frac{1}{2}$ to $\vartheta(G)$ in time polynomial in $|V|$. Rounding to the closest integer yields $\alpha(G)$. ∎

To obtain an explicit maximum-size stable set, we need perfection of the graph:

Corollary 67.2b. *A maximum-size stable set in a perfect graph can be found in polynomial time.*

Proof. Let $G = (V, E)$ be a perfect graph. Iteratively, for each $v \in V$, replace G by $G - v$ if $\alpha(G - v) = \alpha(G)$. By the perfection of G, we can calculate these values in polynomial time, by Corollary 67.2a.

We end up with a graph that forms a maximum-size stable set in the original graph. ∎

As perfection is closed under taking complements, also a maximum-size clique in a perfect graph can be found in polynomial time.

The method described in the proof of Corollary 67.2b applies to all graphs G for which $\alpha(H) = \vartheta(H)$ holds for each induced subgraph H of G; but, as we shall see in Corollary 67.14a, these are precisely the perfect graphs.

From Corollary 67.2b one can derive that a minimum colouring of a perfect graph can also be found in polynomial time (we follow the method given in Grötschel, Lovász, and Schrijver [1988]):

Corollary 67.2c. *A minimum colouring in a perfect graph can be found in polynomial time.*

Proof. Let $G = (V, E)$ be a perfect graph. It suffices to find a stable set S intersecting each maximum-size clique in G; applying recursion to $G - S$ does the rest.

Starting with $t = 0$, we iteratively extend a list of maximum-size cliques $K_1, \ldots, K_t$ as follows. First, find a stable set S intersecting each of $K_1, \ldots, K_t$. This can be done by considering

(67.6) $$c := \chi^{K_1} + \cdots + \chi^{K_t},$$

and finding a stable set S maximizing $c(S)$. This can be found by replacing each vertex v by $c(v)$ nonadjacent vertices (adjacent to the new vertices that replace vertices adjacent to v), and finding a maximum-size stable set in the new graph. This gives a stable set S in the original graph maximizing $c(S)$.

Necessarily, $c(S) = t$, since G has a stable set intersecting each maximum-size clique (as G is perfect). So S intersects each K_i.

If $\omega(G-S) < \omega(G)$, then S intersects each maximum-size clique in G, and we are done. If $\omega(G-S) = \omega(G)$, we find a maximum-size clique K_{t+1} in $G-S$, add it to our list, and iterate.

The number of iterations is bounded by $|V|$, since in each iteration the dimension of the space L_t of vectors $x \in \mathbb{R}^V$ with $x(K_i) = 1$ for each i drops, as for the S found we have $\chi^S \in L_t$ and $\chi^S \notin L_{t+1}$. ∎

67.2. Weighted clique and colouring algorithmically

In a straightforward way, the results of the previous section can be extended to the weighted case. Let $G = (V, E)$ be a graph and let $w : V \to \mathbb{Z}_+$ be a weight function. Let G_w be the graph obtained from G by replacing each vertex v by a stable set S_v of size $w(v)$, where vertices in distinct S_u and S_v are adjacent if and only if u and v are adjacent in G. So the maximum *weight* of a stable set in G is equal to the maximum *size* of a stable set in G_w. Define:

(67.7) $$\alpha_w(G) := \alpha(G_w),\ \vartheta_w(G) := \vartheta(G_w),\ \overline{\chi}_w(G) := \overline{\chi}(G_w),$$
$$\overline{\chi}^*_w(G) := \overline{\chi}^*(G_w).$$

So $\alpha_w(G)$ is equal to the maximum weight of a stable set in G. The definitions of $\overline{\chi}_w(G)$ and $\overline{\chi}^*_w(G)$ agree with those in Section 64.8.

Theorem 67.1 gives the following inequalities:

Theorem 67.3. *For any graph $G = (V, E)$ and $w : V \to \mathbb{R}_+$:*

(67.8) $$\alpha_w(G) \leq \vartheta_w(G) \leq \overline{\chi}^*_w(G).$$

Proof. Directly from Theorem 67.1 and (67.7). ∎

In order to calculate $\vartheta_w(G)$, we need not construct G_w and calculate $\vartheta(G_w)$. This would not be a polynomial-time method. We can calculate $\vartheta_w(G)$ more concisely as follows.

Define $\sqrt{w} : V \to \mathbb{R}_+$ by:

(67.9) $$\sqrt{w}(v) := \sqrt{w(v)}$$

for $v \in V$. Then:

Theorem 67.4. *For any graph G and $w : VG \to \mathbb{Z}_+$:*

(67.10) $$\vartheta_w(G) = \max\{\sqrt{w}^{\mathsf{T}} M \sqrt{w} \mid M \in \mathcal{M}_G \text{ positive semidefinite}\}.$$

Proof. We may assume that $w > \mathbf{0}$. Let D be the $VG_w \times VG$ matrix defined by

(67.11) $$D_{u,v} := \begin{cases} w(v)^{-\frac{1}{2}} & \text{if } u \in S_v, \\ 0 & \text{if } u \notin S_v, \end{cases}$$

for $u \in VG_w$ and $v \in VG$.

First let M attain the maximum in (67.10). Then $M' := DMD^{\mathsf{T}}$ is positive semidefinite, and, moreover, belongs to $\mathcal{M}_{G_w}$. Indeed, for adjacent vertices u, u' of G_w, say $u \in S_v$ and $u' \in S_{v'}$, with v and v' adjacent vertices of G, we have $M_{v,v'} = 0$, and hence

(67.12) $$M'_{u,u'} = (DMD^{\mathsf{T}})_{u,u'} = \sum_{t,t' \in VG} D_{u,t} M_{t,t'} D_{u',t'} = w(v)^{-\frac{1}{2}} w(v')^{-\frac{1}{2}} M_{v,v'} = 0.$$

Also (setting $v_u := v$ if $u \in S_v$):

(67.13) $$\mathrm{Tr} M' = \mathrm{Tr}(DMD^{\mathsf{T}}) = \sum_{u \in VG_w} \sum_{v,v' \in VG} D_{u,v} D_{u,v'} M_{v,v'} = \sum_{u \in VG_w} w(v_u)^{-1} M_{v_u,v_u} = \sum_{v \in VG} w(v)^{-1} w(v) M_{v,v} = \mathrm{Tr} M = 1.$$

So $M' \in \mathcal{M}_{G_w}$. Hence

(67.14) $$\vartheta_w(G) = \vartheta(G_w) \geq \mathbf{1}^{\mathsf{T}} M' \mathbf{1} = \mathbf{1}^{\mathsf{T}} (DMD^{\mathsf{T}}) \mathbf{1} = \sqrt{w}^{\mathsf{T}} M \sqrt{w}.$$

This shows $\geq$ in (67.10).

To see the reverse inequality, let M' be a positive semidefinite matrix in $\mathcal{M}_{G_w}$ with $\mathbf{1}^{\mathsf{T}} M' \mathbf{1} = \vartheta(G_w)$. Then $M := D^{\mathsf{T}} M' D$ is positive semidefinite, and, moreover, belongs to $\mathcal{M}_G$. Indeed, for adjacent $v, v' \in VG$ we have

(67.15) $$M_{v,v'} = (D^{\mathsf{T}} MD)_{v,v'} = \sum_{u,u' \in VG_w} D_{u,v} D_{u',v'} M'_{u,u'} = \sum_{u \in S_v} \sum_{u' \in S_{v'}} w(v)^{-\frac{1}{2}} w(v')^{-\frac{1}{2}} M'_{u,u'} = 0.$$

Also:

(67.16) $$\mathrm{Tr} M = \sum_{v \in VG} \sum_{u,u' \in VG_w} D_{u,v} D_{u',v} M'_{u,u'} = \sum_{v \in VG} \sum_{u \in S_v} \sum_{u' \in S_v} w(v)^{-1} M'_{u,u'} \leq \sum_{v \in VG} \sum_{u \in S_v} M'_{u,u} = \mathrm{Tr} M' = 1.$$

The inequality holds as for any positive semidefinite matrix A one has: $\mathbf{1}^{\mathsf{T}} A \mathbf{1} \leq \mathbf{1}^{\mathsf{T}} \mathbf{1} \cdot \mathrm{Tr} A$, since the largest eigenvalue of A is at most $\mathrm{Tr} A$. This is applied to the $S_v \times S_v$ submatrix of M, for each $v \in V$.

Hence the matrix $\widetilde{M} := (\mathrm{Tr} M)^{-1} \cdot M$ belongs to $\mathcal{M}_G$, and so the maximum in (67.10) is at least $\sqrt{w}^{\mathsf{T}} \widetilde{M} \sqrt{w}$, and hence at least

(67.17) $$\sqrt{w}^{\mathsf{T}} M \sqrt{w} = \sqrt{w}^{\mathsf{T}} D^{\mathsf{T}} M' D \sqrt{w} = \mathbf{1}^{\mathsf{T}} M' \mathbf{1} = \vartheta_w(G).$$ ∎

This implies that $\vartheta_w(G)$ can be approximated in polynomial time:

Theorem 67.5. *There is an algorithm that for any given graph $G = (V, E)$, any $w : V \to \mathbb{Z}_+$, and any $\varepsilon > 0$, returns a rational closer than ε to $\vartheta_w(G)$, in time bounded by a polynomial in $|V|$, $\log \|w\|_\infty$, and $\log(1/\varepsilon)$.*

Proof. Similar to the proof of Theorem 67.2. ∎

The two theorems above imply:

Corollary 67.5a. *For any graph G and weight function $w : V \to \mathbb{Z}_+$ satisfying $\alpha_w(G) = \overline{\chi}_w(G)$, the weighted stable set number can be found in polynomial time.*

Proof. Theorem 67.3 implies $\alpha_w(G) = \vartheta_w(G) = \overline{\chi}_w(G)$, and by Theorem 67.5 we can find a number closer than $\frac{1}{2}$ to $\vartheta_w(G)$ in time polynomial in $|V|$. Rounding to the closest integer yields $\alpha_w(G)$. ∎

To obtain a maximum-weight stable set explicitly, we again need perfection of the graph:

Corollary 67.5b. *A maximum-weight stable set in a perfect graph can be found in polynomial time.*

Proof. Let $G = (V, E)$ be a perfect graph and $w : V \to \mathbb{Z}_+$. Iteratively, for each $v \in V$, replace G by $G - v$ if $\alpha_w(G - v) = \alpha_w(G)$. By the perfection of G, we can calculate these values in polynomial time, by Corollary 67.5a.

We end up with a graph that forms a maximum-weight stable set in the original graph. ∎

As perfection is closed under taking complements, also a maximum-weight clique in a perfect graph can be found in polynomial time. So for any $w : V \to \mathbb{Z}_+$, we can determine

(67.18) $\omega_w(G) :=$maximum of $w(C)$ over cliques C of G

in polynomial time.

Moreover, a minimum weighted colouring of a perfect graph can be found in polynomial time (again, we follow the method given in Grötschel, Lovász, and Schrijver [1988]):

Corollary 67.5c. *Given a perfect graph $G = (V, E)$ and a weight function $w : V \to \mathbb{Z}_+$, a minimum weighted colouring can be found in polynomial time.*

Proof. Let $G = (V, E)$ be a perfect graph and let $w : V \to \mathbb{Z}_+$. As in the proof of Corollary 67.2c, we can find a stable set S intersecting each maximum-weight clique in G, as follows. Starting with $t = 0$, we iteratively extend a list of maximum-weight cliques $K_1, \ldots, K_t$. First find a stable set S intersecting each of $K_1, \ldots, K_t$. This can be done by considering

(67.19) $$c := \chi^{K_1} + \cdots + \chi^{K_t},$$

and finding a stable set S maximizing $c(S)$. This can be found by replacing each vertex v by $c(v)$ nonadjacent vertices (adjacent to the new vertices that replace vertices adjacent to v), and finding a maximum-size stable set in the new graph. This gives a stable set S maximizing $c(S)$.

Necessarily, $c(S) = t$, since G has a stable set intersecting each maximum-weight clique (as G_w is perfect). So S intersects each K_i.

If $\omega_w(G-S) < \omega_w(G)$, then S intersects each maximum-weight clique in G, and we have the required S. If $\omega_w(G-S) = \omega_w(G)$, we find a maximum-weight clique K_{t+1} in $G-S$, add it to our list, and iterate.

The number of iterations is bounded by $|V|$, since in each iteration the dimension of the space L_t of vector $x \in \mathbb{R}^V$ with $x(K_i) = 1$ for each i drops, since for the S found we have $\chi^S \in L_t$ and $\chi^S \notin L_{t+1}$.

This describes the method to find a stable set intersecting all maximum-weight cliques. To find a minimum weighted colouring, we iteratively find stable sets $S_1, \ldots, S_i$, $\lambda_1, \ldots, \lambda_i \in \mathbb{Z}_+$, and a weight function w_i as follows. Set $w_1 := w$. Next iteratively for $i = 1, 2, \ldots$, as long as $w_i \neq \mathbf{0}$, find a stable set S_i intersecting all cliques C maximizing $w_i(C)$, calculate

(67.20) $$\lambda_i := \omega_{w_i}(G) - \omega_{w_i}(G - S_i),$$

and set $w_{i+1} := w_i - \lambda_i \chi^{S_i}$.

Then the λ_i, S_i form a minimum weighted colouring, since

(67.21) $$\sum_i \lambda_i \chi^{S_i} = w \text{ and } \sum_i \lambda_i = \omega_w(G) = \chi_w(G).$$

To prove this, we first show:

(67.22) $$\omega_{w_{i+1}}(G) = \omega_{w_{i+1}}(G - S_i) = \omega_{w_i}(G - S_i) = \omega_{w_i}(G) - \lambda_i.$$

Here the second equality is trivial (since w_i and w_{i+1} coincide outside S_i). The third inequality follows from definition (67.20) of λ_i. For the first equality, $\geq$ is trivial. To see $\leq$, consider a clique C intersecting S_i. Then $w_{i+1}(C) = w_i(C) - \lambda_i|C \cap S_i| \leq \omega_{w_i}(G) - \lambda_i$. This proves (67.22), which implies the second equality in (67.21).

Moreover, the number of iterations is at most $|V|$, since in each iteration the face of the clique polytope spanned by the cliques C maximizing $w_i(C)$, increases in dimension: each clique C in G maximizing $w_i(C)$ also maximizes $w_{i+1}(C)$ (since $w_{i+1}(C) \geq w_i(C) - \lambda_i = \omega_{w_i}(G) - \lambda_i = \omega_{w_{i+1}}(G)$, by (67.22)), and there is a clique C maximizing $w_{i+1}(C)$ but not $w_i(C)$ (namely any clique C of $G - S_i$ maximizing $w_i(C)$, since $w_{i+1}(C) = w_i(C) = \omega_{w_i}(G - S_i) = \omega_{w_{i+1}}(G)$). ∎

67.3. Strong polynomial-time solvability

In the previous section we showed the polynomial-time solvability of the weighted versions of the stable set and colouring problems in perfect graphs. By Theorem 5.11 of Frank and Tardos [1985,1987], this can be strengthened to *strong* polynomial-time solvability.

Theorem 67.6. *A maximum-weight clique and a minimum weighted colouring in a perfect graph can be found in strongly polynomial time.*

Proof. A maximum-weight clique can be found in strongly polynomial time by Theorem 5.11, since the class of clique polytopes of perfect graphs is polynomial-time solvable by Corollary 67.5b.

Next, a minimum weighted colouring can be found with the method described in the proof of Corollary 67.5c: it is strongly polynomial-time because we can find (by the above) a maximum-weight clique in strongly polynomial time. ∎

This implies:

Corollary 67.6a. *A maximum-weight stable set and a minimum-weight vertex cover in a perfect graph can be found in strongly polynomial time.*

Proof. Directly from Theorem 67.6, since stable sets in a perfect graph are precisely the cliques in the complementary graph, which is again perfect. Moreover, the vertex covers are precisely the complements of stable sets. ∎

67.4. Further results and notes

67.4a. Further on $\vartheta(G)$

In this section we give some further results on the function $\vartheta(G)$, and we consider the related convex body TH(G). We use the following notation, for vector $a, b \in \mathbb{R}_+^V$:

(67.23) b/a is the vector in $\mathbb{R}^V$ with vth entry $b(v)/a(v)$,
$\sqrt{b} = b^{\frac{1}{2}}$ is the vector in $\mathbb{R}^V$ with vth entry $b(v)^{\frac{1}{2}}$,
$b^{-\frac{1}{2}}$ is the vector in $\mathbb{R}^V$ with vth entry $b(v)^{-\frac{1}{2}}$,
Δ_b is the $V \times V$ diagonal matrix with diagonal b.

We set $(b/a)_v := 0$ if $a_v = 0$ and $(b^{-\frac{1}{2}})_v := 0$ if $b_v = 0$. (This will turn out not to harm the consistency.)

Moreover, we define, for any graph $G = (V, E)$ and any symmetric matrix M:

(67.24) $\mathcal{L}_G :=$ the set of symmetric $V \times V$ matrices A with $A_{u,v} = 0$ if $u = v$ or u and v are nonadjacent;
$\Lambda(M) :=$ the largest eigenvalue of M,
PSD := the set of symmetric positive semidefinite matrices.

We usually restrict PSD to appropriate dimensions, like $V \times V$. We define for any two matrices X, Y (of equal dimensions) the 'inner product' $X \bullet Y$ by

(67.25) $$X \bullet Y := \mathrm{Tr}(XY^{\mathsf{T}}).$$

So if $X \in \mathcal{M}_G$ and $Y \in \mathcal{L}_G$, then $X \bullet Y = 0$.

A min-max relation for $\vartheta_w(G)$

$\vartheta_w(G)$ is defined as a maximum. Applying convex duality, we can describe $\vartheta_w(G)$ alternatively as a minimum (Lovász [1979d]):

Theorem 67.7. *For each $w \in \mathbb{R}_+^V$:*

(67.26) $$\vartheta_w(G) = \min\{\Lambda(W + A) \mid A \in \mathcal{L}_G\},$$

where $W := \sqrt{w}\sqrt{w}^{\mathsf{T}}$.

Proof. Let M maximize $\sqrt{w}^{\mathsf{T}} M \sqrt{w}$ over $\mathrm{PSD} \cap \mathcal{M}_G$. So $\vartheta_w(G) = \sqrt{w}^{\mathsf{T}} M \sqrt{w}$.

To prove $\leq$ in (67.26), let $A \in \mathcal{L}_G$ attain the minimum in (67.26) and let $\lambda := \Lambda(W + A)$. Then $Y := \lambda I - W - A$ is positive semidefinite, and hence

(67.27) $$\begin{aligned} 0 \leq Y \bullet M &= (\lambda I - W - A) \bullet M = \lambda \mathrm{Tr} M - W \bullet M = \lambda - \sqrt{w}^{\mathsf{T}} M \sqrt{w} \\ &= \Lambda(W + A) - \vartheta_w(G). \end{aligned}$$

To prove $\geq$ in (67.26), we use convexity theory. Since M maximizes $W \bullet M$ over the intersection of the convex sets PSD and $\mathcal{M}_G$, there exist supporting hyperplanes $\{X \mid C \bullet X = \gamma\}$ of PSD and $\{X \mid D \bullet X = \delta\}$ of $\mathcal{M}_G$ such that

(67.28) $$\mathrm{PSD} \subseteq \{X \mid C \bullet X \geq \gamma\},\ \mathcal{M}_G \subseteq \{X \mid D \bullet X \geq \delta\},\ C \bullet M = \gamma,\ D \bullet M = \delta, \text{ and } W = C + D.$$

Since PSD and $\mathcal{M}_G$ consist of symmetric matrices only, we can assume that C and D are symmetric (we can replace them by $\frac{1}{2}(C + C^{\mathsf{T}})$ and $\frac{1}{2}(D + D^{\mathsf{T}})$).

Since PSD is a convex cone, we have $\gamma = 0$. Then $C \in \mathrm{PSD}$, as $xx^{\mathsf{T}} \in \mathrm{PSD}$ for each $x \in \mathbb{R}^V$, hence $x^{\mathsf{T}} C x = C \bullet (xx^{\mathsf{T}}) \geq 0$.

Since $\mathcal{M}_G$ is an affine space and since $D \bullet M = \delta$, we have $\mathcal{M}_G \subseteq \{X \mid D \bullet X = \delta\}$. This implies that $D = \delta \cdot I - A$ for some $A \in \mathcal{L}_G$ (since each symmetric $0,1$ matrix containing precisely one 1 belongs to $\mathcal{M}_G$; the matrix remains to belong to $\mathcal{M}_G$ after putting a nonzero entry in any nonadjacent position and its transpose). So

(67.29) $$\delta = D \bullet M = (W - C) \bullet M = W \bullet M.$$

As C is positive semidefinite, $\delta \cdot I - W - A$ is positive semidefinite. Hence

(67.30) $$\Lambda(W + A) \leq \delta = W \bullet M = \vartheta_w(G).$$ ∎

The product $\vartheta(G)\vartheta(\overline{G})$ is at least $|V|$

For perfect graphs $G = (V, E)$, we have $\alpha(G)\omega(G) \geq |V|$, and hence $\vartheta(G)\vartheta(\overline{G}) \geq |V|$. The latter inequality holds for any graph G. To prove it, we use the following fact from matrix theory:

(67.31) If X and Y are symmetric positive semidefinite $n \times n$ matrices, then also $X * Y$ is positive definite,

where $X * Y$ is the $n \times n$ matrix given by: $(X * Y)_{i,j} = X_{i,j} Y_{i,j}$. (67.31) follows from the fact that there exist vectors $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ with $X_{i,j} = u_i^{\mathsf{T}} u_j$ and $Y_{i,j} = v_i^{\mathsf{T}} v_j$ for all i, j. Hence $(X * Y)_{i,j} = (u_i \circ v_i)^{\mathsf{T}} (u_j \circ v_j)$ for all i, j, where $\circ$ denotes tensor product[13]. So $X * Y$ is positive semidefinite.

Theorem 67.8. *$\vartheta(G)\vartheta(\overline{G}) \geq |V|$ for each graph $G = (V, E)$.*

Proof. By (67.26), there exist $A \in \mathcal{L}_G$ and $B \in \mathcal{L}_{\overline{G}}$ with

(67.32) $$\vartheta(G) = \Lambda(J + A) \text{ and } \vartheta(\overline{G}) = \Lambda(J + B).$$

So $C := \vartheta(G) \cdot I - J - A$ and $D := \vartheta(\overline{G}) \cdot I - J - B$ are positive semidefinite. Now

(67.33) $$\begin{aligned} &C * D + C * J + J * D = (C + J) * (D + J) - J * J \\ &= (\vartheta(G) \cdot I - A) * (\vartheta(\overline{G}) \cdot I - B) - J = \vartheta(G)\vartheta(\overline{G}) \cdot I - J \end{aligned}$$

(as $A * I = I * B = A * B$ is the all-zero matrix). By (67.31), the first matrix in (67.33) is positive semidefinite, hence also the last. So

(67.34) $$0 \leq \mathbf{1}^{\mathsf{T}} (\vartheta(G)\vartheta(\overline{G}) \cdot I - J)\mathbf{1} = \vartheta(G)\vartheta(\overline{G})|V| - |V|^2,$$

implying the theorem. ∎

The convex body TH(G)

The function $\vartheta_w(G)$ is related to a convex body TH(G) defined in Grötschel, Lovász, and Schrijver [1986]. The following equivalent representation of TH(G) was given by Lovász and Schrijver [1991].

For any symmetric matrix A, define the matrix $R(A)$ by:

(67.35) $$R(A) := \begin{pmatrix} 1 & a^{\mathsf{T}} \\ a & A \end{pmatrix},$$

where $a := \text{diag}A$ (the diagonal vector of A; that is, $a_i = A_{i,i}$ for each coordinate i).

Given a graph $G = (V, E)$, consider the collection $\mathcal{R}_G$ of symmetric $V \times V$ matrices A with $R(A)$ positive semidefinite and with $A_{u,v} = 0$ for distinct adjacent u, v. Then define:

(67.36) $$\text{TH}(G) = \{\text{diag}A \mid A \in \mathcal{R}_G\}.$$

Theorem 67.9. TH(G) *is convex and down-monotone in $\mathbb{R}_+^V$.*

Proof. TH(G) is convex, as it is a projection of the convex set $\mathcal{R}_G$. Moreover, if $a \in \text{TH}(G)$ and $\mathbf{0} \leq b \leq a$, then $b \in \text{TH}(G)$. Indeed, since $a \in \text{TH}(G)$, there exists a matrix $A \in \mathcal{R}_G$ with $a = \text{diag}A$. Then the matrix

(67.37) $$\begin{pmatrix} 1 & 0 \\ 0 & \Delta_{b/a} \end{pmatrix} \begin{pmatrix} 1 & a^{\mathsf{T}} \\ a & A \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \Delta_{b/a} \end{pmatrix} = \begin{pmatrix} 1 & b^{\mathsf{T}} \\ b & \Delta_{b/a} A \Delta_{b/a} \end{pmatrix}$$

[13] The *tensor product* of vectors $x \in \mathbb{R}^U$ and $y \in \mathbb{R}^V$ is the vector $x \circ y$ in $\mathbb{R}^{U \times V}$ defined by: $(x \circ y)_{(u,v)} := x_u y_v$ for $u \in U$ and $v \in V$.

is positive semidefinite. As the vth entry on the diagonal of $\Delta_{b/a}A\Delta_{b/a}$ is equal to $b(v)^2/a(v)$ (or 0 if $a(v)=0$), which is at most $b(v)$, we have that

(67.38) $$\Delta_{b/a}A\Delta_{b/a}+(\Delta_{b-b^2/a})$$

belongs to $\mathcal{R}_G$ and has diagonal equal to b. This proves that $b\in \mathrm{TH}(G)$, and hence $\mathrm{TH}(G)$ is down-monotone. ∎

To obtain a relation of $\mathrm{TH}(G)$ with the function $\vartheta_w(G)$, we first show the following, where for $x,y\in\mathbb{R}^V$, $x*y$ is the vector in $\mathbb{R}^V$ defined by:

(67.39) $$(x*y)_v := x_v y_v \text{ for } v\in V.$$

Theorem 67.10. *Let M maximize $\sqrt{w}^{\mathsf{T}}M\sqrt{w}$ over $\mathrm{PSD}\cap\mathcal{M}_G$. Then*

(67.40) $$M\sqrt{w}=\vartheta_w(G)\cdot b*w^{-\frac{1}{2}},$$

where $b:=\mathrm{diag}M$.

Proof. The maximum of

(67.41) $$\sqrt{w}^{\mathsf{T}}\Delta_x M\Delta_x\sqrt{w}$$

over $x\in\mathbb{R}^V$ satisfying $x^{\mathsf{T}}\Delta_b x=1$, is attained by $x=\mathbf{1}$. (Otherwise we can replace M by $\Delta_x M\Delta_x$ to increase $\sqrt{w}^{\mathsf{T}}M\sqrt{w}$.) Now (67.41) is equal to

(67.42) $$x^{\mathsf{T}}\Delta_{\sqrt{w}}M\Delta_{\sqrt{w}}x.$$

So the maximum of (67.42) over $x\in\mathbb{R}^V$ satisfying $x^{\mathsf{T}}\Delta_b x=1$, is attained by $x=\mathbf{1}$. Hence, by Lagrange's theorem, there exists a $\mu\in\mathbb{R}$ with

(67.43) $$\Delta_{\sqrt{w}}M\Delta_{\sqrt{w}}\mathbf{1}=\mu\cdot\Delta_b\mathbf{1}=\mu\cdot b.$$

Then

(67.44) $$\vartheta_w(G)=\sqrt{w}^{\mathsf{T}}M\sqrt{w}=\mathbf{1}^{\mathsf{T}}\Delta_{\sqrt{w}}M\Delta_{\sqrt{w}}\mathbf{1}=\mu\mathbf{1}^{\mathsf{T}}b=\mu\mathrm{Tr}M=\mu.$$

(67.43) and (67.44) give

(67.45) $$M\sqrt{w}=M\Delta_{\sqrt{w}}\mathbf{1}=\mu\cdot w^{-\frac{1}{2}}*b=\vartheta_w(G)\cdot b*w^{-\frac{1}{2}},$$

which is (67.40). ∎

Now the relation of $\mathrm{TH}(G)$ with $\vartheta_w(G)$ is:

Theorem 67.11. *For each $w\in\mathbb{R}_+^V$:*

(67.46) $$\vartheta_w(G)=\max\{w^{\mathsf{T}}x \mid x\in\mathrm{TH}(G)\}.$$

Proof. I. We first show $\leq$ in (67.46). Let M be a matrix maximizing $\sqrt{w}^{\mathsf{T}}M\sqrt{w}$ over the positive semidefinite matrices $M\in\mathcal{M}_G$. It suffices to show that the matrix

(67.47) $$A:=\vartheta_w(G)\cdot\Delta_{w^{-\frac{1}{2}}}M\Delta_{w^{-\frac{1}{2}}}$$

belongs to $\mathcal{R}_G$, since $w^{\mathsf{T}}\mathrm{diag}A=\vartheta_w(G)\mathrm{Tr}M=\vartheta_w(G)$.

Trivially, $A_{u,v}=0$ for distinct adjacent u,v (since $M_{u,v}=0$ for distinct adjacent u,v). To see that $R(A)$ is positive semidefinite, write $a:=\mathrm{diag}A$, $b:=\mathrm{diag}M$, and $\vartheta:=\vartheta_w(G)$. By (67.40) we have $M\sqrt{w}=\vartheta\cdot b*w^{-\frac{1}{2}}$. So $\Delta_{w^{-\frac{1}{2}}}M\sqrt{w}=\vartheta\cdot\Delta_{w^{-\frac{1}{2}}}(b*w^{-\frac{1}{2}})=\vartheta\cdot(b/w)$. Hence

(67.48) $$R(A) = \begin{pmatrix} 1 & a^{\mathsf{T}} \\ a & A \end{pmatrix} = \begin{pmatrix} 1 & \vartheta \cdot (b/w)^{\mathsf{T}} \\ \vartheta \cdot (b/w) & \vartheta \cdot \Delta_{w^{-\frac{1}{2}}} M \Delta_{w^{-\frac{1}{2}}} \end{pmatrix}$$
$$= \begin{pmatrix} \vartheta^{-1}\sqrt{w}^{\mathsf{T}} M \sqrt{w} & \sqrt{w}^{\mathsf{T}} M \Delta_{w^{-\frac{1}{2}}} \\ \Delta_{w^{-\frac{1}{2}}} M \sqrt{w} & \vartheta \cdot \Delta_{w^{-\frac{1}{2}}} M \Delta_{w^{-\frac{1}{2}}} \end{pmatrix} = \vartheta^{-1} \cdot U^{\mathsf{T}} M U,$$

where U is the matrix given by

(67.49) $$U := (\sqrt{w} \quad \vartheta \cdot \Delta_{w^{-\frac{1}{2}}}).$$

So $R(A)$ is positive semidefinite.

II. To see $\geq$ in (67.46), let $A \in \mathcal{R}_G$ maximize $w^{\mathsf{T}}\mathrm{diag}A$. Define $a := \mathrm{diag}A$, $\eta := w^{\mathsf{T}}a$, and

(67.50) $$M := \eta^{-1} \cdot \Delta_{w^{\frac{1}{2}}} A \Delta_{w^{\frac{1}{2}}}.$$

Trivially, M is positive semidefinite and belongs to $\mathcal{M}_G$. Also

(67.51) $$0 \leq (\eta, -w^{\mathsf{T}}) \begin{pmatrix} 1 & a^{\mathsf{T}} \\ a & A \end{pmatrix} \begin{pmatrix} \eta \\ -w \end{pmatrix} = \eta^2 - 2\eta \cdot w^{\mathsf{T}} a + w^{\mathsf{T}} A w$$
$$= \eta \sqrt{w}^{\mathsf{T}} M \sqrt{w} - \eta^2.$$

Therefore $\sqrt{w}^{\mathsf{T}} M \sqrt{w} \geq \eta$, which proves $\geq$ in (67.46). ∎

(67.46) implies that $\vartheta_w(G)$ is a convex function of w and that

(67.52) $$\mathrm{TH}(G) = \{x \in \mathbb{R}^V_+ \mid w^{\mathsf{T}} x \leq \vartheta_w(G) \text{ for each } w \in \mathbb{R}^V_+\}.$$

By (67.8),

(67.53) $$\alpha_w(G) \leq \vartheta_w(G) \leq \overline{\chi}^*_w(G).$$

This gives:

Corollary 67.11a. *For each graph $G = (V, E)$:*

(67.54) $$P_{\text{stable set}}(G) \subseteq \mathrm{TH}(G) \subseteq A(P_{\text{clique}}(G)).$$

Proof. This follows directly from Theorem 67.11 with the inequalities (67.53), since for each $w \in \mathbb{R}^V_+$:

(67.55) $$\alpha_w(G) = \max\{w^{\mathsf{T}} x \mid x \in P_{\text{stable set}}(G)\},$$
$$\vartheta_w(G) = \max\{w^{\mathsf{T}} x \mid x \in \mathrm{TH}(G)\},$$
$$\overline{\chi}^*_w(G) = \max\{w^{\mathsf{T}} x \mid x \in A(P_{\text{clique}}(G))\}.$$ ∎

The antiblocking body of TH(G)

It turns out that taking the antiblocking body $A(\mathrm{TH}(G))$ of $\mathrm{TH}(G)$ corresponds to replacing G by its complement (Grötschel, Lovász, and Schrijver [1986]). We first observe that

(67.56) $$A(\mathrm{TH}(G)) = \{w \in \mathbb{R}^V_+ \mid \vartheta_w(G) \leq 1\},$$

since for each $w : V \to \mathbb{R}_+$: $w \in A(\mathrm{TH}(G)) \iff \max\{w^\mathsf{T}x \mid x \in \mathrm{TH}(G)\} \le 1 \iff \vartheta_w(G) \le 1$.

Theorem 67.12. *$A(\mathrm{TH}(G)) = \mathrm{TH}(\overline{G})$.*

Proof. I. We first show $A(\mathrm{TH}(G)) \subseteq \mathrm{TH}(\overline{G})$. Let $w \in A(\mathrm{TH}(G))$; that is (by (67.56)), $\vartheta_w(G) \le 1$. To show that w belongs to $\mathrm{TH}(\overline{G})$ we should show by (67.52) that

(67.57) $$w^\mathsf{T}a \le \vartheta_a(\overline{G})$$

for each $a \in \mathbb{R}^V_+$.

By (67.26), there exist $A \in \mathcal{L}_G$ and $B \in \mathcal{L}_{\overline{G}}$ such that

(67.58) $$\vartheta_w(G) = \Lambda(\sqrt{w}\sqrt{w}^\mathsf{T} + A) \text{ and } \vartheta_a(\overline{G}) = \Lambda(\sqrt{a}\sqrt{a}^\mathsf{T} + B).$$

So $C := \vartheta_w(G) \cdot I - \sqrt{w}\sqrt{w}^\mathsf{T} - A$ and $D := \vartheta_a(\overline{G}) \cdot I - \sqrt{a}\sqrt{a}^\mathsf{T} - B$ are positive semidefinite. Therefore, the matrix

(67.59) $$\vartheta_w(G)\vartheta_a(\overline{G})\cdot I - \sqrt{w*a}\sqrt{w*a}^\mathsf{T} = C*D + C*(\sqrt{a}\sqrt{a}^\mathsf{T}) + (\sqrt{w}\sqrt{w}^\mathsf{T})*D$$

is positive semidefinite by (67.31) (note that $A * I = I * B = A * B$ is the all-zero matrix). Hence

(67.60) $$\begin{aligned} 0 &\le \sqrt{w*a}^\mathsf{T}(\vartheta_w(G)\vartheta_a(\overline{G}) \cdot I - \sqrt{w*a}\sqrt{w*a}^\mathsf{T})\sqrt{w*a} \\ &= \vartheta_w(G)\vartheta_a(\overline{G})\sqrt{w*a}^\mathsf{T}\sqrt{w*a} - \sqrt{w*a}^\mathsf{T}\sqrt{w*a}\sqrt{w*a}^\mathsf{T}\sqrt{w*a} \\ &= \vartheta_w(G)\vartheta_a(\overline{G})w^\mathsf{T}a - (w^\mathsf{T}a)^2, \end{aligned}$$

implying (67.57).

II. To prove $\mathrm{TH}(\overline{G}) \subseteq A(\mathrm{TH}(G))$, let $w \in \mathrm{TH}(\overline{G})$. By (67.56) we should prove $\vartheta_w(G) \le 1$.

Let B maximize $\sqrt{w}^\mathsf{T}B\sqrt{w}$ over $\mathrm{PSD} \cap \mathcal{M}_G$. Let $b := \mathrm{diag}B$ and define

(67.61) $$C := \Delta_{\sqrt{w/b}} B \Delta_{\sqrt{w/b}}.$$

Then, with (67.40),

(67.62) $$C\sqrt{b} = \Delta_{\sqrt{w/b}} B\sqrt{w} = \mu \cdot \Delta_{\sqrt{w/b}} b * w^{-\frac{1}{2}} = \mu \cdot \sqrt{b},$$

where $\mu := \vartheta_w(G)$. So C has $\sqrt{b}$ as eigenvector, with eigenvalue μ. Since C is positive semidefinite, also the matrix

(67.63) $$C - \mu(\sqrt{b}\sqrt{b}^\mathsf{T})$$

is positive semidefinite. Hence the matrix

(67.64) $$\Delta_{w^{-\frac{1}{2}}}(C - \mu \cdot \sqrt{b}\sqrt{b}^\mathsf{T})\Delta_{w^{-\frac{1}{2}}} = \Delta_{b^{-\frac{1}{2}}} B \Delta_{b^{-\frac{1}{2}}} - \mu \cdot \sqrt{b/w}\sqrt{b/w}^\mathsf{T}$$

is positive semidefinite.

Define $A := I - \Delta_{b^{-\frac{1}{2}}} B \Delta_{b^{-\frac{1}{2}}}$ and $z := b/w$. So $A \in \mathcal{L}_{\overline{G}}$ and $\mu \cdot \sqrt{z}\sqrt{z}^\mathsf{T} + A$ has largest eigenvalue at most 1. Hence $\vartheta_z(\overline{G}) \le \mu^{-1}$, and so

(67.65) $$\vartheta_w(G)\vartheta_z(\overline{G}) = \mu\vartheta_z(\overline{G}) \le 1 = \mathrm{Tr}B = b^\mathsf{T}\mathbf{1} = w^\mathsf{T}z \le \vartheta_z(\overline{G}),$$

where the last inequality holds as $w \in \mathrm{TH}(\overline{G})$. Hence $\vartheta_w(G) \le 1$. ∎

Facets of $\mathrm{TH}(\boldsymbol{G})$

A subset F of $\mathrm{TH}(G)$ is called a *facet* of $\mathrm{TH}(G)$ if there is an inequality $c^{\mathsf{T}}x \leq \gamma$ (with $c \neq \mathbf{0}$) which is valid for $\mathrm{TH}(G)$, such that F is the set of vectors in $\mathrm{TH}(G)$ having equality and such that F has dimension $|V|-1$. Then (Grötschel, Lovász, and Schrijver [1986]):

Theorem 67.13. *For each graph $G=(V,E)$, each facet F of $\mathrm{TH}(G)$ is determined by an inequality $x_v \geq 0$ for some $v \in V$ or by $x(C) \leq 1$ for some clique C of G.*

Proof. Let F be determined by the inequality $c^{\mathsf{T}}x \leq \gamma$. If there is a $v \in V$ with $x_v = 0$ for each $x \in F$, then F is determined by the inequality $x_v \geq 0$. So we can assume that $x > 0$ for some $x \in F$. Since $\mathrm{TH}(G) = A(\mathrm{TH}(\overline{G}))$, there is a $w \in \mathbb{R}_+^V$ with $\vartheta_w(\overline{G}) = 1$ and F is determined by $w^{\mathsf{T}}x \leq 1$. So $w \in \mathrm{TH}(\overline{G})$, and therefore there is a matrix $A \in \mathcal{R}_{\overline{G}}$ with $\mathrm{diag}A = w$. As $A \in \mathcal{R}_{\overline{G}}$, the matrix $R(A)$ is positive semidefinite. Hence there exist linearly independent vectors $\begin{pmatrix}\alpha_i\\ a_i\end{pmatrix}$ $(i=1,\dots,k)$ such that

(67.66) $$\begin{pmatrix}1 & w^{\mathsf{T}}\\ w & A\end{pmatrix} = R(A) = \sum_{i=1}^{k}\begin{pmatrix}\alpha_i\\ a_i\end{pmatrix}(\alpha_i, a_i^{\mathsf{T}}).$$

We can assume that $\alpha_i \geq 0$ for each $i=1,\dots,k$. Now

(67.67) $$a_i^{\mathsf{T}}x = \alpha_i \text{ for each } x \in F \text{ and each } i=1,\dots,k.$$

To see this, choose $x \in F$. As $x \in \mathrm{TH}(G)$, there is a matrix $B \in \mathcal{R}_G$ with $\mathrm{diag}B = x$. Since $R(B)$ is positive semidefinite, also the matrix

(67.68) $$B' := \begin{pmatrix}1 & -x^{\mathsf{T}}\\ -x & B\end{pmatrix}$$

is positive semidefinite. We therefore have (where again $X \bullet Y := \mathrm{Tr}(XY^{\mathsf{T}})$):

(67.69) $$\sum_{i=1}^{k}(\alpha_i, a_i^{\mathsf{T}})B'\begin{pmatrix}\alpha_i\\ a_i\end{pmatrix} = R(A)\bullet B' = 1-2w^{\mathsf{T}}x+A\bullet B = 1-2w^{\mathsf{T}}x+w^{\mathsf{T}}x = 0.$$

(Here $A \bullet B = w^{\mathsf{T}}x$ follows from the fact that $A \in \mathcal{R}_G$, $B \in \mathcal{R}_{\overline{G}}$, $\mathrm{diag}A = w$, and $\mathrm{diag}B = x$.)

Since B' is positive semidefinite, (67.69) implies that, for each $i=1,\dots,k$:

(67.70) $$(\alpha_i, a_i^{\mathsf{T}})B'\begin{pmatrix}\alpha_i\\ a_i\end{pmatrix} = 0,$$

and therefore

(67.71) $$B'\begin{pmatrix}\alpha_i\\ a_i\end{pmatrix} = \mathbf{0}.$$

In particular,

(67.72) $$(1, -x^{\mathsf{T}})\begin{pmatrix}\alpha_i\\ a_i\end{pmatrix} = 0,$$

that is, $a_i^{\mathsf{T}} x = \alpha_i$, proving (67.67).

Since F is a facet, and since the $\begin{pmatrix} \alpha_i \\ a_i \end{pmatrix}$ are linearly independent, we know $k = 1$. So

$$(67.73) \qquad \begin{pmatrix} 1 & w^{\mathsf{T}} \\ w & A \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ a_1 \end{pmatrix} (\alpha_1, a_1^{\mathsf{T}}).$$

Since $\alpha_1 \geq 0$, this implies $\alpha_1 = 1$ and $a_1 = w$. Since $\text{diag}A = w$, we know $w(v)^2 = w(v)$ for each $v \in V$, and so $w \in \{0,1\}^V$. Hence $A = \chi^C (\chi^C)^{\mathsf{T}}$ for some $C \subseteq V$. As $A_{u,v} = 0$ for distinct nonadjacent u, v, we know that C is a clique. ∎

This gives as consequence:

Corollary 67.13a. TH(G) *is a polytope if and only if* G *is perfect.*

Proof. If G is perfect, we have

$$(67.74) \qquad P_{\text{stable set}}(G) \subseteq \text{TH}(G) \subseteq A(P_{\text{clique}}(G)) = P_{\text{stable set}}(G),$$

implying that TH$(G) = P_{\text{stable set}}(G)$, and therefore is a polytope.

To see the reverse implication, if TH(G) is a polytope, by (67.54) and Theorem 67.13, TH(G) is fully determined by the nonnegativity and clique inequalities; that is,

$$(67.75) \qquad \text{TH}(G) = A(P_{\text{clique}}(G)).$$

Since also $A(\text{TH}(G)) = \text{TH}(\overline{G})$ is a polytope, we know similarly that $\text{TH}(\overline{G}) = A(P_{\text{clique}}(\overline{G}))$. Hence

$$(67.76) \qquad \text{TH}(G) = A(\text{TH}(\overline{G})) = P_{\text{clique}}(\overline{G}) = P_{\text{stable set}}(G).$$

(67.75) and (67.76) imply that $P_{\text{stable set}}(G) = A(P_{\text{clique}}(G))$, and therefore G is perfect by Corollary 65.2e. ∎

Characterizing perfection by $\vartheta(G)$

Lovász [1983b] showed that perfection can be characterized by the function $\vartheta(G)$. To this end, Lovász first proved:

Theorem 67.14. *If* G *is a partitionable graph, then*

$$(67.77) \qquad \alpha(G) < \vartheta(G) < \overline{\chi}^*(G).$$

Proof. Let M be the incidence matrix of the maximum-size stable sets in G and let N be the incidence matrix of the maximum-size cliques of G. Define $n := |VG|$, $\alpha := \alpha(G)$, and $\omega := \omega(G)$. We first show the second inequality.

Let λ be the smallest eigenvalue of $N^{\mathsf{T}} N$. Since N is nonsingular (Theorem 65.9), we know $\lambda > 0$, and since $\text{Tr}(N^{\mathsf{T}} N) = n\omega$ and $N^{\mathsf{T}} N \mathbf{1} = \omega^2 \cdot \mathbf{1}$, we know $\lambda < \omega$ (otherwise $\text{Tr}(N^{\mathsf{T}} N) \geq \omega^2 + (n-1)\omega > n\omega$). So

$$(67.78) \qquad N^{\mathsf{T}} N - \lambda I - \frac{\omega^2 - \lambda}{n} J$$

is positive semidefinite, and therefore

$$\frac{n(\omega-\lambda)}{\omega^2-\lambda}I - J + \frac{n}{\omega^2-\lambda}(N^{\mathsf{T}}N - \omega I) \tag{67.79}$$

is positive semidefinite. So (using (67.26) and (65.24))

$$\vartheta(G) \le \Lambda(J - \frac{n}{\omega^2-\lambda}(N^{\mathsf{T}}N - \omega I)) \le \frac{n(\omega-\lambda)}{\omega^2-\lambda} < \frac{n}{\omega} = \chi^*(\overline{G}). \tag{67.80}$$

So we have the second inequality in (67.77), which implies the first, since:

$$\vartheta(G) \ge \frac{n}{\vartheta(\overline{G})} > \frac{n}{\chi^*(G)} = \alpha, \tag{67.81}$$

by (65.24) and Theorem 67.8. ∎

This implies a characterization of perfect graphs:

Corollary 67.14a. *For any graph G, the following are equivalent:*

(67.82)
(i) *G is perfect,*
(ii) *$\alpha(H) = \vartheta(H)$ for each induced subgraph H of G,*
(iii) *$\vartheta(H) = \overline{\chi}^*(H)$ for each induced subgraph H of G,*
(iv) *$\vartheta(H)$ is an integer for each induced subgraph H of G.*

Proof. Directly from Theorem 67.14, using (65.24). ∎

67.4b. The Shannon capacity $\Theta(G)$

Shannon [1956] introduced the following parameter $\Theta(G)$, now called the Shannon capacity of a graph G.

The *strong product* $G \cdot H$ of graphs G and H is the graph with vertex set $VG \times VH$, with two distinct vertices (u, v) and (u', v') adjacent if and only if u and u' are equal or adjacent in G and v and v' are equal or adjacent in H.

The strong product of k copies of G is denoted by G^k. Then the *Shannon capacity* $\Theta(G)$ of G is defined by:

$$\Theta(G) = \sup_k \sqrt[k]{\alpha(G^k)}. \tag{67.83}$$

(The interpretation is that if V is an alphabet, and adjacency means 'confusable', then $\alpha(G^k)$ is the maximum number of k-letter words any two of which have unequal and inconfusable letters in at least one position. Then $\Theta(G)$ is the maximum possible 'information rate'.)

Since $\alpha(G^{k+l}) \ge \alpha(G^k)\alpha(G^l)$, we know by Fekete's lemma (Corollary 2.2a) that

$$\Theta(G) = \lim_{k\to\infty} \sqrt[k]{\alpha(G^k)}. \tag{67.84}$$

Guo and Watanabe [1990] showed that there exist graphs G for which $\Theta(G)$ is not achieved by a finite product (that is, $\sqrt[k]{\alpha(G^k)} < \Theta(G)$ for each k).

Since $\alpha(G^k) \ge \alpha(G)^k$, we have

$$\alpha(G) \le \Theta(G), \tag{67.85}$$

while strict inequality may hold: the 5-circuit C_5 has $\alpha(C_5) = 2$ and $\alpha(C_5^2) = 5$. (If C_5 has vertices $1, \ldots, 5$ and edges 12, 23, 34, 45, and 51, then $(1,1)$, $(2,3)$, $(3,5)$, $(4,2)$, $(5,4)$ is a stable set in C_5^2.) So $\Theta(C_5) \geq \sqrt{5}$, and Shannon [1956] raised the question if equality holds here. Shannon proved $\Theta(C_5) \leq \frac{5}{2}$; more generally, he proved, for any graph G:

(67.86) $\qquad \Theta(G) \leq \overline{\chi}^*(G)$,

where $\overline{\chi}^*(G)$ is the fractional clique cover number. This bound can be proved by showing that

(67.87) $\qquad \overline{\chi}^*(G \cdot H) \leq \overline{\chi}^*(G)\overline{\chi}^*(H)$.

This follows from the fact that if C and D are cliques of G and H respectively, then $C \times D$ is a clique of $G \cdot H$; hence if $\lambda : \mathcal{C} \to \mathbb{R}_+$ and $\mu : \mathcal{D} \to \mathbb{R}_+$ are minimum fractional clique covers for G and H respectively, where $\mathcal{C}$ and $\mathcal{D}$ denote the collections of cliques of G and H respectively, then (where $\circ$ denotes tensor product — see footnote on page 1161, and $\mathbf{1}_U$ denotes the all-one vector in $\mathbb{R}^U$, for any set U)

(67.88)
$$\begin{aligned}\sum_{C \in \mathcal{C}} \sum_{D \in \mathcal{D}} \lambda_C \mu_D \chi^{C \times D} &= \sum_{C \in \mathcal{C}} \sum_{D \in \mathcal{D}} \lambda_C \mu_D (\chi^C \circ \chi^D) \\ &= \Big(\sum_{C \in \mathcal{C}} \lambda_C \chi^C \Big) \circ \Big(\sum_{D \in \mathcal{D}} \mu_D \chi^D \Big) = \mathbf{1}_{VG} \circ \mathbf{1}_{VH} = \mathbf{1}_{VG \times VH}\end{aligned}$$

and hence

(67.89)
$$\overline{\chi}^*(G \cdot H) \leq \sum_{C \in \mathcal{C}} \sum_{D \in \mathcal{D}} \lambda_C \mu_D = \Big(\sum_{C \in \mathcal{C}} \lambda_C \Big) \Big(\sum_{D \in \mathcal{D}} \mu_D \Big) = \overline{\chi}^*(G)\overline{\chi}^*(H).$$

This proves (67.87) (in (67.112) we show equality).

(67.87) implies (67.86), since

(67.90)
$$\sqrt[k]{\alpha(G^k)} \leq \sqrt{\overline{\chi}^*(G^k)} \leq \sqrt{\overline{\chi}^*(G)^k} = \overline{\chi}^*(G).$$

This bound was improved by Lovász [1979d] as follows (which will imply that $\Theta(C_5) = \sqrt{5}$):

Theorem 67.15. *$\Theta(G) \leq \vartheta(G)$ for each graph G.*

Proof. Since $\alpha(G) \leq \vartheta(G)$, it suffices to show that for each k: $\alpha(G^k) \leq \vartheta(G)^k$. For this it suffices to show that

(67.91) $\qquad \vartheta(G \cdot H) \leq \vartheta(G)\vartheta(H)$

for any graphs G and H.

By (67.26), there exist matrices $A \in \mathcal{L}_G$ and $B \in \mathcal{L}_H$ such that

(67.92) $\qquad \vartheta(G) = \Lambda(J_{VG} + A)$ and $\vartheta(H) = \Lambda(J_{VH} + B)$,

where J_U denotes the $U \times U$ all-one matrix, for any set U. Hence the matrices

(67.93) $\qquad C := \vartheta(G) \cdot I_{VG} - J_{VG} - A$ and $D := \vartheta(H) \cdot I_{VH} - J_{VH} - B$

are positive semidefinite, where I_U denotes the $U \times U$ identity matrix, for any set U.

Therefore, also the following matrix[14] is positive semidefinite:

[14] The *tensor product* of a $W \times X$ matrix M and a $Y \times Z$ matrix N (where W, X, Y, Z are sets), is the $(W \times Y) \times (X \times Z)$ matrix $M \circ N$ defined by

(67.94) $$\begin{aligned} C \circ D + C \circ J_{VH} + J_{VG} \circ D &= (C + J_{VG}) \circ (D + J_{VH}) - J_{VG} \circ J_{VH} \\ &= (\vartheta(G) \cdot I_{VG} - A) \circ (\vartheta(H) \cdot I_{VH} - B) - J_{VG \times VH} \\ &= \vartheta(G)\vartheta(H) \cdot I_{VG \times VH} - J_{VG \times VH} - M, \end{aligned}$$

where $M := \vartheta(G) \cdot I_{VG} \circ B + \vartheta(H) A \circ I_{VH} - A \circ B$. Since $I_{VG} \circ B$, $A \circ I_{VH}$, and $A \circ B$ belong to $\mathcal{L}_{G \cdot H}$,[15] also M belongs to $\mathcal{L}_{G \cdot H}$. Therefore,

(67.95) $$\vartheta(G \cdot H) \leq \Lambda(J_{VG \times VH} + M) \leq \vartheta(G)\vartheta(H),$$

giving (67.91). ∎

This proof consists of showing the inequality (67.91) for any two graphs G and H. In fact, equality holds (Lovász [1979d]):

(67.96) $$\vartheta(G \cdot H) = \vartheta(G)\vartheta(H).$$

Indeed, let M and N attain the maximum in definition (67.1) for $\vartheta(G)$ and $\vartheta(H)$ respectively. Then $M \circ N \in \mathcal{M}_{G \cdot H}$, and hence

(67.97) $$\begin{aligned} \vartheta(G \cdot H) &\geq \mathbf{1}^{\mathsf{T}}_{VG \times VH}(M \circ N)\mathbf{1}_{VG \times VH} = (\mathbf{1}^{\mathsf{T}}_{VG} M \mathbf{1}_{VG})(\mathbf{1}^{\mathsf{T}}_{VH} N \mathbf{1}_{VH}) \\ &= \vartheta(G)\vartheta(H). \end{aligned}$$

Theorem 67.15 implies that $\Theta(C_5) = \sqrt{5}$. One may give an explicit construction to prove this, but it also follows from the following general result (Lovász [1979d]):[16]

Theorem 67.16. *For each graph $G = (V, E)$: $\vartheta(G)\vartheta(\overline{G}) \geq |V|$, with equality if G is vertex-transitive.*

Proof. The inequality is Theorem 67.8. If G is vertex-transitive, then $\mathbf{1}^{\mathsf{T}} x$ is maximized over $\mathrm{TH}(G)$ at a vector $x = \mu \cdot \mathbf{1}$ for some $\mu \in \mathbb{R}$, since if it is maximized at x we can replace it by

(67.98) $$\frac{1}{|\Gamma|} \sum_{P \in \Gamma} Px,$$

where Γ is the group of permutation matrices representing automorphisms of G. (This follows from the fact that $Px \in \mathrm{TH}(G)$ and $\mathbf{1}^{\mathsf{T}} Px = \mathbf{1}^{\mathsf{T}} x$.)

As the maximum value is equal to $\vartheta := \vartheta(G)$, we know $\mathbf{1}^{\mathsf{T}} x = \vartheta$, and so $\mu = \vartheta/n$, where $n := |V|$. Since $x \in \mathrm{TH}(G) = A(\mathrm{TH}(\overline{G}))$ (by Theorem 67.12), we have $\vartheta_x(\overline{G}) \leq 1$; hence (as $x = \mu \cdot \mathbf{1}$) $\vartheta(\overline{G}) \leq \mu^{-1} = n/\vartheta$. This shows $\vartheta(G)\vartheta(\overline{G}) \leq n$. ∎

$$(M \circ N)_{(w,y),(x,z)} := M_{w,x} N_{y,z}$$

for $w \in W$, $x \in X$, $y \in Y$, $z \in Z$. If M and N are symmetric positive semidefinite matrices, then $M \circ N$ is symmetric and positive semidefinite again, since if $M = U^{\mathsf{T}} U$ and $N = V^{\mathsf{T}} V$, then $M \circ N = (U \circ V)^{\mathsf{T}} (U \circ V)$.

[15] To see this, let (u, v) and (u', v') be equal or nonadjacent. Then (by definition of $G \cdot H$) $u = u'$ and $v = v'$, or $u \neq u'$ and u and u' are nonadjacent, or $v \neq v'$ and v and v' are nonadjacent. Hence $(I_{VG})_{u,u'} = 0$ or $B_{v,v'} = 0$, and $A_{u,u'} = 0$ or $(I_{VH})_{v,v'} = 0$, and $A_{u,u'} = 0$ or $B_{v,v'} = 0$.

[16] An *automorphism* of a graph $G = (V, E)$ is a permutation $\pi : V \to V$ with $E = \{\{\pi(u), \pi(v)\} \mid \{u, v\} \in E\}$. The graph G is *vertex-transitive* if for all $u, v \in V$ there exists an automorphism π with $\pi(u) = v$.

Since $\overline{C}_5$ is isomorphic to C_5, Theorem 67.16 gives $\vartheta(C_5) = \sqrt{5}$. So $\Theta(G) \leq \sqrt{5}$. As $\Theta(G) \geq \sqrt{\alpha(C_5^2)} = \sqrt{5}$, one has $\Theta(G) = \sqrt{5}$.

Another consequence of Theorem 67.16 is that for any vertex-transitive graph G: $\Theta(G \cdot \overline{G}) = |VG|$, since the pairs (v, v) for $v \in VG$ form a stable set in $G \cdot \overline{G}$ (so $\Theta(G \cdot \overline{G}) \geq |VG|$), and since $\Theta(G \cdot \overline{G}) \leq \vartheta(G \cdot \overline{G}) = \vartheta(G)\vartheta(\overline{G}) = |VG|$. If moreover G is self-complementary (like C_5), then $\Theta(G) = \sqrt{|VG|}$.

For graphs that are not vertex-transitive, $\vartheta(G)\vartheta(\overline{G}) > |VG|$ may hold, even $\alpha(G)\alpha(\overline{G}) > |VG|$, for instance for $G = K_{1,2}$.

Lovász [1979d] also gave the value of $\vartheta(C_n)$ for any odd circuit C_n:

(67.99) $$\vartheta(C_n) = \frac{n \cos(\pi/n)}{1 + \cos(\pi/n)} \text{ for odd } n.$$

For odd $n \geq 7$, it is unknown if this is the value of $\Theta(C_n)$. Since each C_n is vertex-transitive, by Theorem 67.16 we can derive from (67.99) the value of $\vartheta(\overline{C}_n)$ for odd n.

Lovász asked the question if $\Theta(G) = \vartheta(G)$ for each graph G. This was answered in the negative by Haemers [1979], by giving the following alternative upper bound on the Shannon capacity of a graph $G = (V, E)$. Let $\eta(G)$ be the minimum rank of a $V \times V$ matrix M (over any field) such that $M_{v,v} = 1$ for each $v \in V$ and $M_{u,v} = 0$ for distinct nonadjacent u and v. Then

(67.100) $$\Theta(G) \leq \eta(G).$$

This follows from the facts that $\alpha(G) \leq \eta(G)$ (since any stable set S in G gives an $S \times S$ identity submatrix of M), and that $\eta(G \cdot H) \leq \eta(G)\eta(H)$ (since $\text{rank}(M \circ N) = \text{rank}(M)\text{rank}(N)$ for any two matrices (over the same field)). Moreover, one has $\eta(G) \leq \overline{\chi}(G)$ (by considering, for any clique cover of G, the $\{0,1\}$ matrix M with $M_{u,v} = 1$ if and only if u and v belong to some clique in the clique cover).

Haemers gave a graph G on 27 vertices (the complement of the 'Schläfli graph') with $\eta(G) \leq 7$ and $\vartheta(G) = 9$, implying $\Theta(G) \leq 7 < \vartheta(G)$. Since $\vartheta(\overline{G}) = 3$, this also gives an example of a graph G satisfying $\Theta(G)\Theta(\overline{G}) < |VG|$ and (hence) $\Theta(G)\Theta(\overline{G}) < \Theta(G \cdot \overline{G})$. (This disproves the conjecture of Shannon [1956] that $\Theta(G)\Theta(H) = \Theta(G \cdot H)$ for all graphs G, H, and answering to the negative the question of Lovász [1979d] whether $\Theta(G)\Theta(\overline{G}) \geq |VG|$ for all graphs G.)

It is unknown if Haemers' bound $\eta(G)$ can be computed in polynomial time. (Peeters [1996] reports results on this. More work on Haemers' bound in Haemers [1981].)

The following bound follows with a method of Rosenfeld [1967]:

(67.101) $$\alpha(G \cdot H) \leq \overline{\chi}^*(G)\alpha(H).$$

To see this, let $C_1, \ldots, C_k$ be cliques in G and $\lambda_1, \ldots, \lambda_k \geq 0$ be such that

(67.102) $$\lambda_1 \chi^{C_1} + \cdots + \lambda_k \chi^{C_k} = \mathbf{1}_{VG} \text{ and } \lambda_1 + \cdots + \lambda_k = \overline{\chi}^*(G).$$

Let $S \subseteq VG \times VH$ be a stable set in $G \cdot H$ of size $\alpha(G \cdot H)$. For each $u \in VG$, let $S_u := \{v \in VH \mid (u, v) \in S\}$. Then S_u is a stable set of H, and if u and u' are adjacent vertices of G, then $S_u \cap S_{u'} = \emptyset$. For each $i = 1, \ldots, k$, let

(67.103) $$T_i := \{v \in VH \mid \exists u \in C_i : (u, v) \in S\} = \bigcup_{u \in C_i} S_u.$$

Since C_i is a clique in G, T_i is a stable set in H, and $|T_i| = \sum_{u \in C_i} |S_u|$. Hence

(67.104) $$|S| = \sum_{u \in VG} |S_u| = \sum_{i=1}^{k} \lambda_i \sum_{u \in C_i} |S_u| = \sum_{i=1}^{k} \lambda_i |T_i| \leq \sum_{i=1}^{k} \lambda_i \alpha(H) = \overline{\chi}^*(G)\alpha(H).$$

This shows (67.101).

Rosenfeld [1967] showed that for each graph G:

(67.105) $$\alpha(G \cdot H) = \alpha(G)\alpha(H) \text{ for each graph } H \iff \alpha(G) = \overline{\chi}^*(G).$$

Here $\Longleftarrow$ follows from (67.101). To see $\Longrightarrow$, let $x \in \mathbb{Q}_+^{VG}$ be a vector satisfying $x(C) \leq 1$ for each clique C, and $\mathbf{1}^\mathsf{T} x = \overline{\chi}^*(G)$. Let K be a positive integer such that $w := K \cdot x$ is integer. Let G^w be the graph obtained from G by replacing each vertex u by a clique C_u of size $w(u)$ (where vertices in distinct $C_u, C_{u'}$ are adjacent if and only if u and u' are adjacent). Then $\omega(G^w) \leq K$. Hence for $H := \overline{G^w}$ we have $\alpha(H) \leq K$.

Now let

(67.106) $$S := \{(u, v) \mid u \in VG, v \in C_u\}.$$

Then S is a stable set in $G \cdot H$, since if (u, v) and (u', v') are distinct elements in S, then, if $u = u'$, v and v' belong to C_u and hence are nonadjacent in H, and, if $u \neq u'$, u and u' are nonadjacent in G or v and v' are nonadjacent in H.

So $|S| \leq \alpha(G \cdot H) = \alpha(G)\alpha(H)$. Hence

(67.107) $$\overline{\chi}^*(G) = \mathbf{1}^\mathsf{T} x = \frac{1}{K}\mathbf{1}^\mathsf{T} w = \frac{1}{K}|S| \leq \frac{1}{K}\alpha(G)\alpha(H) \leq \alpha(G).$$

Hence $\overline{\chi}^*(G) = \alpha(G)$.

More results on the stable set number of products of graphs are given by Vizing [1963], Barnes and Mackey [1978], and Jha and Slutzki [1994].

The stable set number of products of circuits

The following equality was given by Baumert, McEliece, Rodemich, Rumsey, Stanley, and Taylor [1971] and Markosyan [1971]:

(67.108) $$\alpha(C_{2k+1}^2) = k^2 + \lfloor \tfrac{1}{2}k \rfloor.$$

$\leq$ directly follows from (67.101), since $\alpha(C_{2k+1}) = k$ and $\overline{\chi}^*(C_{2k+1}) = k + \frac{1}{2}$. To see $\geq$, we may assume that the vertices of C_{2k+1} are $0, 1, \ldots, 2k$, in order. Then the pairs $(2i, \lfloor 2i/k \rfloor)$, for $i = 1, \ldots, k^2 + \lfloor \frac{1}{2}k \rfloor$, where we take integers mod $2k+1$, form a stable set of size $k^2 + \lfloor \frac{1}{2}k \rfloor$ in C_{2k+1}^2.

Baumert, McEliece, Rodemich, Rumsey, Stanley, and Taylor [1971] showed moreover the following inequalities (next to several other estimates for $\alpha(C_n^k)$):

(67.109) $$\alpha(C_{n+2}^k) \geq 1 + \frac{(n+2)^k - 2^k}{n^k}\alpha(C_n^k),$$
$$\alpha(C_n^k) \leq \frac{n^k - n^{k-1}}{2^k},$$
$$\alpha(C_5^3) = 10,\ \alpha(C_5^4) = 25,\ \alpha(C_7^3) = 33.$$

Hales [1973] extended (67.108) to:

(67.110) $$\alpha(C_{2k+1} \cdot C_{2l+1}) = kl + \lfloor \tfrac{1}{2}\min\{k, l\} \rfloor.$$

Related results on the stable set number of products of circuits are given by Sonnemann and Krafft [1974], Stein [1977], Hell and Roberts [1982], Mead and Narkiewicz [1982], Vesel [1998], and Vesel and Žerovnik [1998].

67.4c. Clique cover numbers of products of graphs

As for the analogue of the Shannon capacity for clique cover numbers, McEliece and Posner [1971] showed that it gives no new parameter. We follow the proof of Lovász [1975c].

Theorem 67.17. *For any graph G:*

(67.111) $$\inf_k \sqrt[k]{\overline{\chi}(G^k)} = \lim_{k\to\infty} \sqrt[k]{\overline{\chi}(G^k)} = \overline{\chi}^*(G).$$

Proof. We first show that for any two graphs G, H:

(67.112) $$\overline{\chi}^*(G \cdot H) = \overline{\chi}^*(G)\overline{\chi}^*(H).$$

Here $\leq$ follows from (67.87). To see $\geq$, choose vectors $x : VG \to \mathbb{R}_+$ with $x(C) \leq 1$ for each clique, and with $x(VG) = \overline{\chi}^*(G)$, and $z : VH \to \mathbb{R}_+$ with $z(C) \leq 1$ for each clique, and with $z(VH) = \overline{\chi}^*(H)$. Define $y : VG \times VH \to \mathbb{R}_+$ by

(67.113) $$y(u, v) := x(u)z(v)$$

for $(u, v) \in VG \times VH$. Then $y(C) \leq 1$ for each clique C of $G \cdot H$, since there are cliques C' and C'' of G and H, respectively, such that $C \subseteq C' \times C''$; then $y(C) \leq y(C' \times C'') = x(C')z(C'') \leq 1$.

Hence

(67.114) $$\overline{\chi}^*(G \cdot H) \geq y(VG \times VH) = x(VG)z(VH) = \overline{\chi}^*(G)\overline{\chi}^*(H).$$

This proves (67.112).

To prove (67.111), the first equality follows from Fekete's lemma (Corollary 2.2a), since $\overline{\chi}(G^{k+l}) = \overline{\chi}(G^k) \cdot \overline{\chi}(G^l)$. Also we have by (67.112):

(67.115) $$\inf_k \sqrt[k]{\overline{\chi}(G^k)} \geq \inf_k \sqrt[k]{\overline{\chi}^*(G^k)} = \overline{\chi}^*(G),$$

So it suffices to prove the reverse inequality in (67.115). Since $\omega(G^k) = \omega(G)^k$ and since $\overline{\chi}^*(G^k) = \overline{\chi}^*(G)^k$, we have by Theorem 64.13 (applied to $\overline{G^k}$):

(67.116) $$\inf_k \sqrt[k]{\overline{\chi}(G^k)} \leq \inf_k \sqrt[k]{(1 + \ln \omega(G^k))\overline{\chi}^*(G^k)} = \inf_k \sqrt[k]{(1 + k \ln \omega(G))}\overline{\chi}^*(G) = \overline{\chi}^*(G),$$

as required. ∎

An alternative proof was given by Hell and Roberts [1982]. A related information-theoretic characterization of perfect graphs was given by Csiszár, Kőrner, Lovász, Marton, and Simonyi [1990] (proving a conjecture of Kőrner and Marton [1988]). More on the colouring number of products of graphs can be found in Borowiecki [1972], Greenwell and Lovász [1974], Vesztergombi [1980,1981], Turzík [1983], Duffus, Sands, and Woodrow [1985], El-Zahar and Sauer [1985], Puš [1988], Soukop [1988], Linial and Vazirani [1989], and Klavžar [1996] (survey).

Hales [1973] showed that for all graphs G, H:

(67.117) $$\overline{\chi}(G \cdot H) \geq \overline{\chi}^*(G)\overline{\chi}(H),$$

and

(67.118) $$\overline{\chi}(C_{2k+1} \cdot C_{2l+1}) = (k + 1)(l + 1) - \lceil \tfrac{1}{2} \min\{k, l\}\rceil.$$

McEliece and Taylor [1973] showed that $\overline{\chi}(C^2_{n,t}) = \lceil n/t\lceil n/t\rceil\rceil$, where $C_{n,t}$ is the graph obtained from the circuit C_n by adding all chords connecting vertices at distance less than t in C_n.

67.4d. A sharper upper bound $\vartheta'(G)$ on $\alpha(G)$

McEliece, Rodemich, and Rumsey [1978] and Schrijver [1979a] gave the following sharper bound $\vartheta'(G)$ on the stable set number $\alpha(G)$, generally sharper than $\vartheta(G)$. Again, let $\mathcal{M}_G$ be the collection of symmetric $V \times V$ matrices satisfying $M_{u,v} = 0$ for any two distinct adjacent vertices u and v, and $\mathrm{Tr}M = 1$. (Here $\mathrm{Tr}M$ is the trace of M (sum of diagonal elements).) Then define

(67.119) $\vartheta'(G) := \max\{\mathbf{1}^{\mathsf{T}}M\mathbf{1} \mid M \in \mathcal{M}_G$ nonnegative and positive semidefinite$\}$.

Here $\mathbf{1}$ denotes the all-one vector in $\mathbb{R}^V$. Similarly to $\vartheta(G)$, the value of $\vartheta'(G)$ can be calculated in polynomial time. Moreover

(67.120) $\alpha(G) \leq \vartheta'(G) \leq \vartheta(G)$

for each graph G. The first inequality is proved similarly to the proof of the first inequality in Theorem 67.1, while the second inequality follows from the fact that the range of the maximization problem for $\vartheta'(G)$ is contained in that for $\vartheta(G)$.

$\vartheta'(G)$ indeed can be a sharper upper bound on the stable set number than $\vartheta(G)$, as M.R. Best (cf. Schrijver [1979a]) found the following example of a graph G with $\vartheta'(G) < \vartheta(G)$. The vertex set is $\{0,1\}^6$, two vectors being adjacent if and only if their Hamming distance[17] is at most 3. Then $\vartheta'(G) = 4$ whereas $\vartheta(G) = 16/3$.

Schrijver [1979a] gave relations of $\vartheta'(G)$ with the linear programming bound for codes of Delsarte [1973]. Related work can be found in Schrijver [1981a] and Miklós [1996]. (The polynomial-time computable upper bound for $\alpha(G)$ given by Luz [1995] is at least $\vartheta'(G)$ for all graphs G.)

67.4e. An operator strengthening convex bodies

The matrix method describing $\mathrm{TH}(G)$ given in Section 67.4a can be seen as a special case of a method of improving approximations of the stable set polytope — in fact, of any polytope with $\{0,1\}$ vertices (Lovász and Schrijver [1989,1991]).

Let K be a convex set, let $R(A)$ be defined as in (67.35), and define

(67.121) $\mathcal{N}_K :=$ the collection of symmetric $n \times n$ matrices A with $R(A)$ positive semidefinite, and with $A_i \in A_{i,i} \cdot K$ and $\mathrm{diag}A - A_i \in (1 - A_{i,i}) \cdot K$ for each $i = 1, \ldots, n$,

where A_i denotes the ith column of A.

Define the following new convex set $N_+(K)$:

(67.122) $N_+(K) := \{\mathrm{diag}A \mid A \in \mathcal{N}_K\}$.

Then $N_+(K) \subseteq [0,1]^n$, since $R(A)$ is positive semidefinite. The ellipsoid method gives, for any collection $\mathcal{K}$ of convex sets:

(67.123) if the optimization problem over K is polynomial-time solvable for each $K \in \mathcal{K}$, then also the optimization problem over $N_+(K)$ is polynomial-time solvable for each $K \in \mathcal{K}$.

[17] The *Hamming distance* of two vectors of equal dimension is equal to the number of coordinates in which they differ.

Indeed, if the optimization problem over K is polynomial-time solvable, then the membership problem over K is polynomial-time solvable. Hence the membership problem over $\mathcal{N}_K$ is polynomial-time solvable, implying that the optimization problem over $\mathcal{N}_K$ is polynomial-time solvable. Therefore, the optimization problem over $N_+(K)$ is polynomial-time solvable. (Cf. Chapter 4 of Grötschel, Lovász, and Schrijver [1988].)

Before proving further properties of the operator N_+, we note that it commutes with the following reflection. Define $r : \mathbb{R}^n \to \mathbb{R}^n$ by $r(x)_1 := 1 - x_1$ and $r(x)_i := x_i$ for $i = 2, \ldots, n$, for $x \in \mathbb{R}^n$: Then

(67.124) $$N_+(r(K)) = r(N_+(K)).$$

To see this, let, for any $n \times n$ matrix A, the matrix A' be defined by:

(67.125) $$A'_{1,1} := 1 - A_{1,1};\ A'_{1,i} := A'_{i,1} := A_{i,i} - A_{i,1} \text{ for } i = 2, \ldots, n;$$
$$A'_{i,j} := A_{i,j} \text{ for } i, j = 2, \ldots, n.$$

Then $R(A)$ is positive semidefinite if and only if $R(A')$ is positive semidefinite, since

(67.126) $$R(A') = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & I \end{pmatrix} R(A) \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & I \end{pmatrix}.$$

Moreover, $A \in \mathcal{N}_K \iff A' \in \mathcal{N}_{r(K)}$ and $\mathrm{diag}A' = r(\mathrm{diag}A)$. This gives (67.124).

From this one can derive, if K is compact and convex and intersects $[0,1]^n$:

(67.127) $$N_+(K) \subseteq K.$$

For let $A \in \mathcal{N}_K$ and define $a := \mathrm{diag}A$. If $a \notin K$, there exists a $w \in \mathbb{R}^n$ and $\beta \in \mathbb{R}$ with $w^{\mathsf{T}}x \le \beta$ for each $x \in K$ and $w^{\mathsf{T}}a > \beta$. Since by (67.124) we can flip signs if necessary, we can assume $w \ge \mathbf{0}$. Then, since for each i the vector A_i belongs to $A_{i,i} \cdot K$,

(67.128) $$w^{\mathsf{T}}Aw = \sum_i w_i \big(\sum_j w_j A_{i,j}\big) = \sum_i w_i (w^{\mathsf{T}}A_i) \le \sum_i w_i A_{i,i}\beta = \beta w^{\mathsf{T}}a.$$

Hence

(67.129) $$0 \le (w^{\mathsf{T}}a, -w^{\mathsf{T}}) \begin{pmatrix} 1 & a^{\mathsf{T}} \\ a & A \end{pmatrix} \begin{pmatrix} w^{\mathsf{T}}a \\ -w \end{pmatrix} = (w^{\mathsf{T}}a)^2 - 2(w^{\mathsf{T}}a)^2 + w^{\mathsf{T}}Aw$$
$$\le -(w^{\mathsf{T}}a)^2 + \beta \cdot w^{\mathsf{T}}a = (\beta - w^{\mathsf{T}}a)w^{\mathsf{T}}a < 0,$$

since $\beta - w^{\mathsf{T}}a < 0$ and $w^{\mathsf{T}}a > \beta \ge 0$ (since $\beta \ge w^{\mathsf{T}}x \ge 0$ for any $x \in K \cap [0,1]^n$). This is a contradiction, showing (67.127).

Moreover, if $K \subseteq [0,1]^n$, then $N_+(K)$ remains to contain the integer hull of K:

(67.130) $$(N_+(K))_{\mathrm{I}} = K_{\mathrm{I}}.$$

To see this, it suffices to show that $x \in N_+(K)$ for each 0,1 vector x in K. Obviously, $A := xx^{\mathsf{T}}$ belongs to $\mathcal{N}_K$. Hence $x = \mathrm{diag}A$ belongs to $N_+(K)$. This proves (67.130).

Finally, if $K \subseteq [0,1]^n$, then repeated application of the N_+ operator gives the integer hull K_{I} of K. In fact, one has:

(67.131) $$N_+^n(K) = K_{\mathrm{I}}.$$

This follows from the fact that for each $j = 1, \ldots, n$:

(67.132) $\quad N_+(K) \subseteq \text{conv.hull}\{x \in K \mid x_j \in \{0,1\}\}.$

To see this, we may assume that $j = n$. Let $a \in N_+(K)$, with $a = \text{diag}A$ and $A \in \mathcal{N}_K$. Then $A_n \in a_n \cdot K$ and $(a - A_n) \in (1 - a_n) \cdot K$. If $a_n \in \{0,1\}$, then a belongs to the right-hand side of (67.132). So we can assume that $0 < a_n < 1$. Set

(67.133) $\quad a' := \frac{1}{a_n} A_n$ and $a'' := \frac{1}{1-a_n}(a - A_n)$.

Then a' and a'' belong to K, and $a'_n = 1$, $a''_n = 0$. As $a = a_n \cdot a' + (1 - a_n) \cdot a''$, we have that a belongs to the right-hand side of (67.132). This proves (67.132).

(67.131) implies that, when starting with $K := \text{TH}(G)$, we can obtain better and better approximations of $P_{\text{stable set}}(G)$ by applying the N_+ operator. After any fixed number of iterations, we can optimize over the convex body in polynomial time, by (67.123).

Stephen and Tunçel [1999] showed that for the line graph $G = L(K_{2n+1})$ of the complete graph K_{2n+1}, when starting with the polytope determined by the non-negativity and edge constraints ((64.10) in Section 64.5), the number of iterations is precisely n. Related results were given by Cook and Dash [2001].

Leaving out the positive semidefiniteness condition in $\mathcal{N}_K$ yields a weaker operator $N(K)$, which however still satisfies a number of the above properties, including (67.131). The operator $N(K)$ is a special case of a more general operator introduced by Sherali and Adams [1990].

Results relating a related operator to perfection of graphs were given by Aguilera, Escalante, and Nasini [2002].

67.4f. Further notes

Juhász [1982] showed that for a random graph G on n vertices, $\vartheta(G)$ is of the order $\sqrt{n}$, while $\Theta(G)$ is 'likely' to be of the order $\log n$. Knuth [1994] asked if there is a constant c such that $\vartheta(G) \leq c\sqrt{n}\alpha(G)$ for each graph G. This was answered negatively by Feige [1995,1997], who showed that there is a constant $c > 0$ such that

(67.134) $\quad \vartheta(G) > \alpha(G)n/2^{c\sqrt{\log n}}$

for infinitely many graphs G (where $n := |VG|$).

The results of Kashin and Konyagin [1981] and Konyagin [1981] imply that if $\alpha(G) \leq 2$, then $\vartheta(G) \leq 2^{\frac{2}{3}} n^{\frac{1}{3}}$ and (in the worst case) $\vartheta(G) = \Omega(n^{\frac{1}{3}}/\sqrt{\log n})$.

Karger, Motwani, and Sudan [1994,1998] showed the existence of a constant $c > 0$ such that

(67.135) $\quad \overline{\chi}(G) \leq n^{1-\frac{c}{\vartheta(G)}}$

for each graph G (where $n := |VG|$). More on approximating $\alpha(G)$ or $\overline{\chi}(G)$ by $\vartheta(G)$ can be found in Szegedy [1994] and Charikar [2002].

Kleinberg and Goemans [1998] observed that for any graph G:

(67.136) $\quad \tau(G) \leq 2(|V| - \vartheta(G)) \leq 2\tau(G)$

(where $\tau(G)$ is the vertex cover number of G), and they showed that the factor 2 cannot be improved. Thus the factor 2 as relative error of $\nu(G)$ for approximating $\tau(G)$ is not improved by $2(|V| - \vartheta(G))$.

Fast practical algorithms to compute $\vartheta(G)$, based on interior-point methods, were developed by Alizadeh [1991,1995]. The latter paper also gives a survey on applying semidefinite programming to combinatorial optimization.

A colouring algorithm for perfect graphs based on decomposition was described by Hsu [1986]. An on-line colouring algorithm for perfect graphs (not necessarily yielding an optimum colouring) was given by Kierstead and Kolossa [1996]. An algorithm for colouring some perfect graphs was given by Aït Haddadène, Gravier, and Maffray [1998]. Kratochvil and Sebő [1997] studied the complexity of colouring a perfect graph if some vertices are pre-coloured. Brandstädt [1987] showed the NP-completeness of several optimization problems for special classes of perfect graphs, like finding a minimum feedback vertex set or a minimum dominating set.

Introductory surveys were given by Knuth [1994] and Goemans [1997] on $\vartheta(G)$, by Grötschel, Lovász, and Schrijver [1984c] on polynomial-time algorithms for clique and colouring problems in perfect graphs, and by Reed [2001a] on semi-definite programming in relation to perfect graphs. Another characterization of perfection in terms of $\mathrm{TH}(G)$ was given by Shepherd [2001].

A generalization of $\vartheta(G)$ was given by Narasimhan and Manber [1990]. A generalization of the Shannon capacity to directed graphs was studied by Bidamon and Meyniel [1985]. An analogue of the Shannon capacity based on the 'independent domination number' of a graph, was investigated by Farber [1986]. The Shannon capacity of probabilistic graphs was investigated by Marton [1993].

Further investigations of eigenvalue methods to bound the Shannon capacity are reported by Haemers [1995] and Fiol [1999]. Further convex programming duality phenomena for perfect graphs were found by Wei [1988].

67.4g. Historical notes on perfect graphs

Shannon

As Berge [1997] mentioned, the perfect graph conjectures root in work of Shannon [1956] concerning the 'zero error capacity of a noisy channel'. It amounts to a study of what we now call the Shannon capacity of a graph. Shannon gave the example of C_5 where $\alpha(C_5) = 2$ and $\alpha(C_5^2) = 5$, implying $\Theta(C_5) \geq \sqrt{5} > \alpha(C_5)$. Denoting the logarithm of the Shannon capacity by C_0, Shannon remarked:

> No method has been found for determining C_0 for the general discrete channel, and this we propose as an interesting problem in coding theory.

Shannon proved the following lower and upper bounds on the Shannon capacity $\Theta(G)$ of a graph $G = (V, E)$. First:

$$(67.137) \qquad \max_p \left(\sum (p_u p_v \mid u, v \in V, u = v \text{ or } uv \in E) \right)^{-1} \leq \Theta(G),$$

where p ranges over all $p \in \mathbb{R}_+^V$ with $\sum_{v \in V} p(v) = 1$. It was observed by Korn [1968] that this lower bound (and also the lower bound given by Gallager [1965]) is equal to the stable set number $\alpha(G)$: if $p_u > 0$ and $p_v > 0$ for two adjacent vertices u and v, either resetting $p_u := p_u + p_v$ and $p_v := 0$, or resetting $p_v := p_u + p_v$ and $p_u := 0$, would increase the value in (67.137), a contradiction. So the set $S := \{v \mid p_v > 0\}$ is a stable set. Then the value in (67.137) is maximized by taking $p_v := 1/|S|$ for $v \in S$. (As we saw in Section 64.9c, this also follows from a theorem of Motzkin and Straus [1965].)

The upper bound given by Shannon [1956] amounts to:

(67.138) $\Theta(G) \leq \overline{\chi}^*(G)$.

Shannon formulated and proved this upper bound in terms of information theory as follows. Let V be an alphabet, let Σ be a set of 'signals', and for $v \in V$ and $\sigma \in \Sigma$, let $p_{v,\sigma}$ be the probability that when transmitting symbol v, signal σ is received. So $\sum_{\sigma\in\Sigma} p_{v,\sigma} = 1$ for each $v \in V$. Let G be the graph on V where two elements $u, v \in V$ are adjacent if and only if there is a signal σ with $p_{u,\sigma} > 0$ and $p_{v,\sigma} > 0$. For each $\sigma \in \Sigma$, define the clique $K_\sigma := \{v \in V \mid p_{v,\sigma} > 0\}$ and the real number $\lambda_\sigma := \max\{p_{v,\sigma} \mid v \in V\}$. So

(67.139) $$\sum_{\sigma\in\Sigma} \lambda_\sigma \chi^{K_\sigma} \geq \mathbf{1}.$$

Hence, by definition of $\overline{\chi}^*(G)$,

(67.140) $$\overline{\chi}^*(G) \leq \sum_{\sigma\in\Sigma} \lambda_\sigma.$$

Moreover, for any fixed G, the minimum of the right-hand side in (67.140) is equal to the left-hand side.

For any $v = (v_1, \ldots, v_k) \in V^k$ and $s = (s_1, \ldots, s_k) \in \Sigma^k$ define

(67.141) $$p_{v,s} := \prod_{i=1}^{k} p_{v_i,s_i} \text{ and } \lambda_s := \prod_{i=1}^{k} \lambda_{s_i}.$$

So $p_{v,s}$ is the probability that transmitted word v is received as word s.

Now consider any nonempty 'code' $C \subseteq V^k$. The 'error probability' of C is equal to

(67.142) $$q(C) := \min_{\phi} \frac{1}{|C|} \sum_{v\in C} \sum (p_{v,s} \mid s \in \Sigma^k, \phi(s) \neq v),$$

where ϕ ranges over all functions $\phi : \Sigma^k \to C$. So it is the minimum error probability taken over all possible 'decoding schemes' ϕ. Trivially, this minimum is attained by the function ϕ with $\phi(s)$ equal to any $v \in C$ maximizing $p_{v,s}$ over $v \in C$. So

(67.143) $$1 - q(C) = \frac{1}{|C|} \sum_{s\in\Sigma^k} \max_{v\in C} p_{v,s} \leq \frac{1}{|C|} \sum_{s\in\Sigma^k} \lambda_s = \frac{1}{|C|} \Big(\sum_{\sigma\in\Sigma} \lambda_\sigma \Big)^k.$$

Therefore,

(67.144) $$\sqrt[k]{|C|} \leq \frac{\sum_{\sigma\in\Sigma} \lambda_\sigma}{\sqrt[k]{1 - q(C)}}.$$

Now $q(C) = 0$ if and only if C is stable in G^k. Minimizing over all Σ and probability distributions $p_{v,\sigma}$ then yields

(67.145) $$\sqrt[k]{|C|} \leq \overline{\chi}^*(G).$$

So this gives (67.138).

Shannon next observed that if a graph $G = (V, E)$ has a function $f : V \to V$ such that $f(u) \neq f(v)$ for any distinct nonadjacent vertices u and v, and such that $f(V)$ is a stable set, then $\Theta(G) = \alpha(G)$. The condition clearly is equivalent to: $\alpha(G) = \overline{\chi}(G)$. Shannon noticed that this yields the value of $\Theta(G)$ for all graphs G

with at most 5 vertices, except for C_5, for which he derived $\sqrt{5} \le \Theta(C_5) \le \frac{5}{2}$ from (67.138). Shannon observed that on 6 vertices all but four graphs have $\alpha(G) = \overline{\chi}(G)$, and that the Shannon capacity of these four graphs can be expressed in terms of $\Theta(C_5)$. On 7 vertices, he stated that 'at least one new situation arises', namely C_7.

Shannon proved that if G and H are disjoint graphs, then $\Theta(G + H) \ge \Theta(G)+\Theta(H)$ and $\Theta(G\cdot H) \ge \Theta(G)\cdot\Theta(H)$, and that equality holds if $\alpha(G) = \overline{\chi}(G)$. Moreover, he conjectured equality for all G, H, but for the product this was disproved by Haemers [1979], and for the sum by Alon [1998].

Berge

As remarked, in developing the concept of perfect graph Berge was motivated by Shannon's problem on the capacity of graphs. We quote from the article 'Motivations and history of some of my conjectures' of Berge [1997]:

> *June 1957:* When he heard that I was writing a book on graph theory, my friend M.P. Schützenberger drew my attention on an interesting paper of Shannon [51] which was presented at a meeting for engineers and statisticians, but which could have been missed by mathematicians working in algebra or combinatorics.

(Berge's reference [51] is Shannon [1956].)

In his book 'Théorie des graphes' (Theory of Graphs), Berge [1958b] called a function $\sigma : VG \to VG$ a *preserving function* ('application préservante'), if for any two distinct nonadjacent vertices u, v, also $\sigma(u)$ and $\sigma(v)$ are distinct and nonadjacent. Then, like Shannon, he considered graphs G having a preserving function σ mapping VG to a stable subset of VG. Clearly, these are exactly the graphs with $\alpha(G) = \overline{\chi}(G)$.

Berge [1958b] also mentioned that M.P. Schützenberger conjectured that

$$(67.146) \qquad \Theta(G) = \lim_{k\to\infty} \sqrt[k]{\alpha(G^k)},$$

which was shown by Lyubich [1964] to follow directly from Fekete's lemma (Corollary 2.2a).

According to Berge [1997], the problem of finding the minimal graphs G with $\alpha(G) < \Theta(G)$ was discussed in January 1960 at the Seminar of R. Fortet, where he asked (prompted by graphs found by A. Ghouila-Houri) if it is true that each graph G not having an odd hole or odd antihole satisfies $\alpha(G) = \Theta(G)$:

> This conjecture, somewhat weaker than the Perfect Graph Conjecture, was motivated by the remark that for the most usual channels, the graphs representing the possible confusions between a set of signals (in particular the interval graphs) have no odd holes and no odd antiholes, and are *optimal in the sense of Shannon.*

At the first international meeting on graph theory held at Dobogókő (Hungary) in October 1959, Hajnal and Surányi [1958] presented the result that $\alpha(G) = \chi(\overline{G})$ for each chordal graph G. This motivated Berge to show that the same holds for complements of chordal graphs. This result was announced, with partial proof, in the paper Berge [1960a], which moreover mentions that several known results yield other classes of graphs G with $\omega(G) = \chi(G)$. In particular, it is observed that theorems of Kőnig imply that $\omega(G) = \chi(G)$ if G or $\overline{G}$ is the line graph of a bipartite graph — 'propriétés remarquables' (remarkable properties) according to Berge.

These results were presented at the Second International Symposium on Graph Theory at the Martin-Luther-Universität in Halle an der Saale (German Democratic Republic) in April 1960. In his memoirs, Berge [1997] mentioned that[18]

> At that time, we were pretty sure that there were no other minimal obstructions; for that reason, at the end of my talk in Halle, I proposed the following open problem: *If a graph G and its complement are semi-Gallai graphs, is it ture that $\gamma(G) = \omega(G)$?*

where a graph is *semi-Gallai* if it has no odd hole, and where $\gamma(G)$ is Berge's notation for the colouring number of G.

So, according to Berge, the strong perfect graph conjecture was stated in 1960 in Halle. It seems however that Berge was hesitating in putting the conjecture in print. It is not quoted in the written abstract of the talk (Berge [1961]), which in this respect only says that

> Angesichts einer solchen Menge von Beispielen könnte man vermuten, daß für jeden semi-Gallaischen Graphen G die Beziehung $\omega(G) = \gamma(G)$ gilt. Aber das stimmt nicht, wie das folgende, von einem unserer Schüler, Herrn Ghouila-Houri, angegebene Gegenbeispiel zeigt:
> G ist ein Graph mit den Knoten a, b, c, d, e, f, g und den Kanten ac, ad, ae, af, bd, be, bf, bg, ce, cf, cg, df, dg, eg. Man kann leicht zeigen, daß G ein semi-Gallaischer Graph ist mit $\omega(G) = 3$, aber $\gamma(G) = 4$ (siehe Abbildung 1).[19]

(This example ($\overline{C}_7$) was also given by Shannon [1956].) Incidentally, in this paper, Berge called graphs G satisfying $\alpha(G) = \overline{\chi}(G)$ *perfect graphs of Shannon* ('vollkommenen Graphen von Shannon').

About the strong perfect graph conjecture, Berge and Chvátal [1984] wrote:

> An early effort of Alain Ghouila-Houri failed to produce a counterexample to this conjecture. Despite this encouraging sign, Berge felt that the conjecture might be too ambitious. Therefore he restricted himself to a weaker conjecture in the hope that it might be easier to settle.

According to Berge and Chvátal [1984] (where a triangulated graph is a chordal graph),

> After the meeting at Halle an der Saale in 1960, the Strong Perfect Graph Conjecture received the enthusiastic support of G. Hajós and T. Gallai. In fact, Gallai provided further evidence in support of the conjecture by strengthening the results on triangulated graphs: he proved that a graph is α-perfect and γ-perfect whenever each of its odd cycles of length at least five has at least two non-crossing chords.

In Gallai [1962], only a proof of $\alpha(G) = \overline{\chi}(G)$ is given, for graphs G in which any odd circuit of length at least 5 has two noncrossing chords. Berge [1997] reported that Gallai informed him in a letter that he knew that also $\omega(G) = \chi(G)$ holds for such graphs. However, Gallai's paper does not mention this, and no reference is made to Berge's conjectures.

Berge and Chvátal [1984] continued:

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[18] As we aim at verbatim quotations, we leave the typo unchanged.

[19] In view of such a multitude of examples one could conjecture that for each semi-Gallai graph G the relation $\omega(G) = \gamma(G)$ holds. But that does not hold, as the following counterexample, presented by one of our students, Mr Ghouila-Houri, shows:
G is a graph with nodes a, b, c, d, e, f, g and edges ac, ad, ae, af, bd, be, bf, bg, ce, cf, cg, df, dg, eg. One can easily show, that G is a semi-Gallai graph with $\omega(G) = 3$, but $\gamma(G) = 4$ (see Figure 1).

> Nevertheless, Berge still felt that the weak conjecture was more promising. At a conference at Rand Corporation in the summer of 1961, he had fruitful discussions with Alan Hoffman, Ray Fulkerson and others. Later on, discussions between Alan Hoffman and Paul Gilmore led Gilmore to a rediscovery of the Strong Perfect Graph Conjecture and to an attempt to axiomatize the relevant properties of cliques in perfect graphs.

Berge [1997] wrote that the discussions at the RAND Corporation with Alan Hoffman encouraged him to write a paper 'in English'. This paper might have been the first version of the paper 'Some classes of perfect graphs' (Berge [1963a]), published in a booklet 'Six Papers on Graph Theory' by the Indian Statistical Institute in Calcutta, which Berge visited in March-April 1963 and where he gave a series of lectures. The booklet contains no year of publication, and the preface mentions that it is intended for private circulation, and that the papers will be given for publication by journals.

The paper contains as new results that $\omega(G) = \chi(G)$ for unimodular graphs and their complements, and also a full proof that it holds for chordal graphs (announced earlier). The paper seems to be the first written account of the concept of perfect graph, and of the perfect graph conjectures, in the last section of the paper:

> V. CONJECTURES
>
> The problem of characterizing α-perfect and γ-perfect graphs seems difficult, but the preceding results enable us to state several conjectures. For instance :
>
> Conjecture 1. A graph is α-perfect if and only if it is γ-perfect
>
> Conjecture 2. A graph is γ-perfect if and only if it does not contain an elementary odd cycle of one of the following types :
>
> type 1 : the cycle is of length greater than 3 and does not possess any chord ;
>
> type 2 : the cycle is of length greater than 3, and does not possess any triangular chord, but possesses all its non-triangular chords (a chord is triangular if it determines a triangle with the edges of the cycle)
>
> Conjecture 3. A graph is α-perfect if and only if it does not contain an elementary odd cycle of type 1 or 2.
>
> It is easy to show that conjecture 2 is equivalent to conjecture 3, and implies conjecture 1. It is also easy to show that if a graph is γ-perfect (or α-perfect), then it does not contain an elementary odd cycle of type 1 or 2.

At the General Assembly of the U.R.S.I. (Union Radio Scientifique Internationale) in Tokyo in September 1963, Berge developed further on the relations between perfection and optimum codes in the sense of Shannon. We quote the abstract (Berge [1963b]):

> 3. Claude Berge : *Sur une conjecture relative au problème des codes optimaux de Shannon*, on considère un émetteur qui peut émettre un ensemble de signaux, par suite du bruit chaque signal peut donner plusieurs interprétations à la réception. On trace le graphe dont les sommets représentent les différents signaux, deux points étant liés par une arête si les signaux correspondants peuvent être confondus à la réception. Le problème essentiel est de caractériser les graphes que l'on peut enrichir, on aboutit ainsi à une conjecture que l'on démontre pour certaines classes particulières.[20]

[20] 3. Claude Berge : *On a conjecture related to the problem of the optimal codes of Shannon*, we consider a transmitter that can transmit a set of signals, as a consequence of noise each signal can give several interpretations at the reception. We make the graph

Berge [1997] wrote that the paper Berge [1963a] was distributed to all participants of the U.R.S.I. meeting in 1963, and that a French version of it was published as Berge [1966], added with some new results and an appendix with some results proved in Berge [1967], in order to make the conjecture more plausible and more interesting.

The paper Berge [1966] is more descriptive, but gives more relations to the Shannon problem, and also mentions the strong perfect graph conjecture, attributing it jointly to P.C. Gilmore. After remarking that $\alpha(G) \neq \overline{\chi}(G)$ for odd circuits of length at least 5 and their complements, the paper states:

> Nous nous sommes proposés de voir si la réciproque était vraie, et sommes arrivés à la conjecture suivante avec P. GILMORE:
> **Conjecture.** *Soit G un graphe de signaux; il est parfait si et seulement s'il ne contient pas un cycle impair sans cordes (de longueur > 3), ni le complémentaire d'un cycle impair sans cordes (de longueur > 3).*[21]

Berge [1966] also claimed, without proof, that $\Theta(G) = \alpha(G)$ if and only if $\overline{\chi}(G) = \alpha(G)$:

> On voit aussi *que la condition nécessaire et suffisante pour que la capacité du graphe de signaux G soit égale à $\alpha(G)$ est que $\alpha(G) = \theta(G)$.*[22]

(Italics of Berge, who denoted the clique cover number $\overline{\chi}(G)$ of G by $\theta(G)$.) However, the line graph $L(K_6)$ of K_6 is a counterexample to this (it has $\alpha = \Theta = \overline{\chi}^* = 3$ and $\overline{\chi} = 4$).

The paper 'Some classes of perfect graphs' was published again in a book on *Graph Theory and Theoretical Physics* edited by F. Harary (Berge [1967]). According to Berge [1997], this paper is 'a final version' of the manuscript, with suggestions by Hoffman, and was handed over to Harary at the end of a NATO Advanced Study Institute on Graph Theory in Frascati, Italy in March-April 1964. Compared with Berge [1963a], the paper contains no new results, and moreover the last section with the perfect graph conjectures (quoted above) has been omitted.

This paper was published also in the Proceedings of a Conference on Combinatorial Mathematics and Its Applications at the University of North Carolina at Chapel Hill, 10–14 April 1967. It is followed by a 'Discussion on Professor Berge's Paper' by M.E. Watkins stating that 'it seems likely that G is perfect if and only if $\overline{G}$ is perfect'. Berge [1996] mentioned that this addendum

> contributed to make the perfect graph conjecture popular. Before the Chapel Hill conference, I did not get much interest for my problems from the mathematics community; the first symposium lecture about perfect graphs from other mathematicians was delivered by Horst Sachs [20] at the Calgary conference in 1969.

(Berge's reference [20] is Sachs [1970].)

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the vertices of which represent the different signals, two points being connected by an edge if the corresponding signals can be confused at the reception. The essential problem is of characterizing the graphs that one can enrich, we arrive this way at a conjecture that we prove for certain particular classes.

[21] We have resolved to see if the reverse would be true, and have arrived at the following conjecture with P. GILMORE:

Conjecture. *Let G be a graph of signals; it is perfect if and only if it neither contains an odd circuit without chords (of length > 3), nor the complement of an odd circuit without chords (of length > 3).*

[22] One also sees *that the necessary and sufficient condition for that the capacity of the graph of signals G is equal to $\alpha(G)$ is that $\alpha(G) = \theta(G)$.*

Fulkerson

The results on perfect graphs obtained until then being restricted to specific classes of graphs, the first serious dent in solving the perfect graph conjectures in general was made by Fulkerson in a RAND Report of 1970 on antiblocking polyhedra. They led Fulkerson to prove a 'pluperfect graph theorem', but also to doubt the validity of the weak perfect graph conjecture, which blocked him finishing it off.

The RAND Report (Fulkerson [1970c]) was published as Fulkerson [1972a], and the results were presented at the Second Chapel Hill Conference on Combinatorial Mathematics and Its Applications at the University of North Carolina at Chapel Hill in May 1970 (Fulkerson [1970d]), and at the 7th International Mathematical Programming Symposium in 1970 in The Hague, for which a survey paper on blocking and antiblocking pairs of polyhedra was written (Fulkerson [1970a,1971a]).

Fulkerson called a graph G *γ-pluperfect* if $\chi(H) = \omega(H)$ for each graph H obtained from G by deleting and replicating vertices. In particular, if G is γ-pluperfect, then G is γ-perfect.

What Fulkerson [1970a,1971a] proved is that:

(67.147) $\quad G$ is γ-pluperfect $\iff$ $\overline{G}$ is γ-pluperfect.

The proof is not hard, but is based on a series of pioneering observations and general polyhedral insights that are now fundamental in polyhedral combinatorics. It uses the linear programming duality equality

(67.148) $$\max\{w^{\mathsf{T}}x \mid x \geq \mathbf{0}, Mx \leq \mathbf{1}\} = \min\{y^{\mathsf{T}}\mathbf{1} \mid y \geq \mathbf{0}, y^{\mathsf{T}}M \geq w^{\mathsf{T}}\},$$

where M is the incidence matrix of the stable sets of G and where $w : V \to \mathbb{R}_+$. Then:

(67.149) $\quad G = (V, E)$ is γ-pluperfect
$\overset{1}{\iff}$ $\forall w : V \to \mathbb{Z}_+$, both optima in (67.148) are attained by integer solutions x and y
$\overset{2}{\iff}$ $\forall w : V \to \mathbb{Z}_+$, the maximum in (67.148) is attained by an integer solution x
$\overset{3}{\iff}$ $\forall w : V \to \mathbb{Q}_+$, the maximum in (67.148) is attained by an integer solution x
$\overset{4}{\iff}$ $\forall w : V \to \mathbb{R}_+$, the maximum in (67.148) is attained by an integer solution x
$\overset{5}{\iff}$ each vertex of the polytope $\{x \mid x \geq \mathbf{0}, Mx \leq \mathbf{1}\}$ is integer
$\overset{6}{\iff}$ the clique polytope of G is determined by the nonnegativity and stable set constraints.

The first equivalence in (67.149) follows by observing that a weight $w(v)$ of a vertex v corresponds to replacing v by a clique of size $w(v)$; this is equivalent to duplicating v $w(v)-1$ times, or, if $w(v) = 0$, deleting v. The second equivalence can be derived by considering, for any $w : V \to \mathbb{Z}_+$ an inequality $x(S) \leq 1$ in $Mx \leq \mathbf{1}$ satisfied with equality by all optimum solutions; hence replacing w by $w - \chi^S$ the maximum decreases, hence by at least 1 (as it has an integer value); as the minimum decreases by at most 1, we obtain an integer optimum dual solution by induction. The third and fourth equivalences follow by scaling w and by continuity. The fifth equivalence is general polyhedral theory, and the sixth one follows by observing that the integer solutions of $x \geq \mathbf{0}$, $Mx \leq \mathbf{1}$ are precisely the incidence vectors of cliques.

Now by Fulkerson's theory of antiblocking polyhedra, the last statement in (67.149) is invariant under interchanging 'clique' and 'stable set'; that is, under replacing G by the complementary graph $\overline{G}$. Hence the same holds for the first statement.

Fulkerson [1970c,1970a,1971a,1972a] gave another, symmetrical characterization of γ-pluperfect graphs:

(67.150) a graph $G = (V, E)$ is γ-pluperfect if and only if for all $l, w : V \to \mathbb{Z}_+$, the maximum of $l(S)w(C)$ over all stable sets S and cliques C is at least $\sum_v l(v)w(v)$.

For this, Fulkerson was inspired by the length-width inequality for blocking pairs of hypergraphs given in a 1965 preprint of Lehman [1965,1979].

The weak perfect graph conjecture implies that each perfect graph G is γ-pluperfect, since trivially if $\chi(\overline{H}) = \omega(\overline{H})$ for each induced subgraph H of G, then $\chi(\overline{H}) = \omega(\overline{H})$ for each H obtained from G by deleting and replicating vertices. (Note that $\chi(\overline{H}) = \chi(\overline{G})$ and $\omega(\overline{H}) = \omega(\overline{G})$ if H arises from G by duplicating a vertex.)

So, as Fulkerson [1970a,1971a] remarked ('theorem 14'), the perfect graph conjecture is equivalent to: each γ-perfect graph is γ-pluperfect; or: γ-perfection is maintained under duplicating vertices (later called the *replication lemma*):

> Thus to prove the perfect graph conjecture, it would suffice to prove that γ-perfection implies γ-pluperfection. For this it would suffice to show that if G is γ-perfect, and if we duplicate an arbitrary vertex v in G and join v to its duplicate vertex, the new graph G' is again γ-perfect.

Another way of stating it is: if for each $w : V \to \{0, 1\}$ both optima in (67.148) have integer solutions, then likewise for each $w : V \to \mathbb{Z}_+$. This might seem too strong from a general polyhedral point of view, and it made Fulkerson [1970a,1971a] mistrust the conjecture:

> It is our feeling that theorem 14 casts some doubt on the validity of the perfect graph conjecture.

Lovász

The weak perfect graph conjecture was finally proved by Lovász [1972c], stating:

> Fulkerson [5] reduced the problem to the following conjecture, using the theory of antiblocking polyhedra:
> *Duplicating an arbitrary vertex of a perfect graph and joining the obtained two vertices by an edge, the arising graph is perfect.*
> In §1 we prove a theorem which contains this conjecture.

(Reference [5] is Fulkerson [1972a].) Lovász also wrote:

> It should be pointed out that thus the proof consists of two steps and the more difficult second step was done first by Fulkerson.

With respect to this, Fulkerson [1973] remarked in his comments 'On the perfect graph theorem':

> Concerning this proof, Lovász states: "It should be pointed out that thus the proof consists of two steps, and the most difficult second step was done first by Fulkerson." I would be less than candid if I did not say that I agree with this remark, at least in retrospect. But the fact remains that, while part of my aim in developing the anti-blocking theory had been to settle the perfect graph conjecture, and that while I had succeeded via this theory in reducing the conjecture to a simple lemma about graphs [3,4] (the "replication lemma", a proof of which is given in this paper) and had developed other seemingly more complicated equivalent versions of the conjecture [3,4,5], I eventually began to feel that the conjecture was probably false and thus spent several fruitless months trying to construct a counterexample. It is not altogether clear to me now just why I felt the conjecture was false, but I think it was due mainly to one equivalent version I had found [4,5], a version that does not explicitly mention graphs at all.

(The references [3,4,5] correspond to Fulkerson [1972a,1971a,1970d].)

In the preprint of this article, Fulkerson [1972b] wrote moreover, after stating the replication lemma:

> Actually I knew more: Namely that the truth or falsity of the perfect graph conjecture rested entirely on the truth or falsity of the replication lemma. I tried for awhile to prove this lemma, without success, and then, as was mentioned earlier, became convinced on other grounds that the perfect graph conjecture was probably false, and began to look for a graph that was perfect but not pluperfect. (I knew that it would do no good to look at known classes of perfect graphs, since I had been able to prove that all of these were pluperfect.) The fact is that such graphs don't exist, of course. After some months of sporadic effort along these lines, I quit working on the perfect graph conjecture, thinking that I would come back to it later. There were other aspects of anti-blocking pairs of polyhedra, and of blocking pairs of polyhedra, that I wanted to study, and, in any event, I felt that the pluperfect graph theorem was a beautiful result in its own right.
> In the spring of 1971 I received a postcard from Berge, who was then visiting the University of Waterloo, saying that he had just heard that Lovász had a proof of the perfect graph conjecture. This immediately rekindled my interest, naturally, and so I sat down at my desk and thought again about the replication lemma. Some four or five hours later, I saw a simple proof of it.

After having given a simple proof of the replication lemma, Fulkerson [1972b] continued:

> As can be seen, there is nothing deep or complicated about the proof of this lemma. Perhaps the fact that I saw a proof of it only after knowing it had to be true may say something about the psychology of invention (or, better yet, anti-invention) in mathematics, at least for me.

This is indeed an instructive illustration that believing a conjecture may help in proving it.

In a subsequent paper, Lovász [1972a] proved more strongly that a graph G is perfect if and only if $\alpha(H)\omega(H) \geq |VH|$ for each induced subgraph H of G. This generalizes the perfect graph theorem, and was suggested by A. Hajnal. It also sharpens Fulkerson's result (67.150), implying that one may restrict l and w to $\{0, 1\}$-valued functions with $l = w$.

The problem of Shannon [1956] concerning the Shannon capacity of C_5 was solved by Lovász [1979d].

In May 2002, M. Chudnovsky, N. Robertson, P.D. Seymour, and R. Thomas announced that they found a proof of the strong perfect graph conjecture, by proving a number of deep results, and building on and inspired by earlier results of, among others, V. Chvátal, M. Conforti, G. Cornuéjols, W.H. Cunningham, A. Kapoor, F. Roussel, P. Rubio, N. Sbihi, K. Vušković, and G. Zambelli.

More historical notes are given by Berge and Ramírez Alfonsín [2001] and Reed [2001b].

Chapter 68

T-perfect graphs

The class of t-perfect graphs is defined polyhedrally: the stable set polytope should be determined by the nonnegativity, edge, and odd circuit constraints. It implies that a maximum-weight stable set in such graphs can be found in polynomial time. LP duality gives a min-max relation for the maximum-weight of a stable set in t-perfect graphs.
A characterization of t-perfect graphs is not known. The widest class of t-perfect graphs known consists of those not containing certain subdivisions of K_4 as subgraph.

68.1. T-perfect graphs

A graph $G = (V, E)$ is called *t-perfect*[23] if the stable set polytope of G is determined by

(68.1)
(i) $0 \leq x_v \leq 1$ for each $v \in V$,
(ii) $x_u + x_v \leq 1$ for each edge $uv \in E$,
(iii) $x(VC) \leq \lfloor \frac{1}{2}|VC| \rfloor$ for each odd circuit C.

A prominent non-t-perfect graph is K_4. Below we shall see that, on the other hand, if K_4 does not occur in a graph in a certain way, then the graph is t-perfect. But no exact characterization of t-perfection is known.

A motivation for studying t-perfection is algorithmic, since the definition implies:

Theorem 68.1. *A maximum-weight stable set in a t-perfect graph can be found in strongly polynomial time.*

Proof. By Theorems 5.10 and 5.11, it suffices to show that the separation problem over the stable set polytope is polynomial-time solvable. Conditions (i) and (ii) in (68.1) can be tested one by one. If they are satisfied, define a function $y : E \to \mathbb{R}_+$ by:

(68.2) $y_e := 1 - x_u - x_v$

for each $e = uv \in E$. Then condition (iii) is equivalent to:

[23] t stands for 'trou' (French for 'hole').

(68.3) $y(EC) \geq 1$ for each odd circuit C

(since $y(EC) = |EC| - 2x(VC)$). The latter condition can be checked in polynomial time: Consider y as a length function, and for each $u \in V$, find an odd circuit C through u with $y(EC)$ minimal. This can be done by replacing each vertex v by two vertices v', v'', and each edge $e = vw$ by two edges $v'w''$ and $v''w'$, each of length y_e; then a shortest path from u' to u'' gives the required circuit.

If $y(EC) < 1$, we have a violated inequality. ∎

A combinatorial polynomial-time algorithm to find the stable set number of a t-perfect graph was given by Eisenbrand, Funke, Garg, and Könemann [2002]. It is based on finding (by a greedy method similar to that used in the proof of Theorem 64.13) an approximative fractional dual solution to the problem of maximizing $\mathbf{1}^{\mathsf{T}}x$ over (68.1). with relative error less than $1/|V|$. Rounding then gives the stable set number. Applying this iteratively gives an explicit maximum-size stable set.

Notes. The construction given in the proof of Theorem 68.1 shows that the maximum-weight stable set problem in a t-perfect graph can be described by a 'compact' linear programming: the stable set polytope is the projection of a polytope whose dimension and number of facets are polynomially bounded. To see this, introduce, next to the variables $y \in \mathbb{R}^E_+$, a variable $z_{u,v}$ for each $u, v \in V$. Requiring:

(68.4)
$$\begin{aligned} &z_{v,v} \geq 1 && \text{for each } v \in V,\\ &z_{u,v} \leq y_{uv} && \text{for each edge } uv \in E,\\ &z_{t,w} \leq z_{t,u} + y_{uv} + y_{vw} && \text{for all } t,u,v,w \in V \text{ with } uv, vw \in E, \end{aligned}$$

is equivalent to the odd circuit constraints. (In fact, one can do without the variables y_e, as they can be expressed in the x_v.) So a maximum-weight stable set in a t-perfect graph can be found in polynomial time with any polynomial-time linear programming algorithm.

T-perfection can also be characterized in terms of the vertex cover polytope:

Theorem 68.2. *A graph $G = (V, E)$ is t-perfect if and only if the vertex cover polytope of G is determined by:*

(68.5)
$$\begin{aligned} &\text{(i)} && 0 \leq x_v \leq 1 && \text{for each } v \in V,\\ &\text{(ii)} && x_u + x_v \geq 1 && \text{for each edge } uv \in E,\\ &\text{(iii)} && x(VC) \geq \lceil \tfrac{1}{2}|VC| \rceil && \text{for each odd circuit } C. \end{aligned}$$

Proof. System (68.5) arises from (68.1) by the reflection $x \to \mathbf{1} - x$. So integrality of the two polytopes is equivalent. ∎

68.2. Strongly t-perfect graphs

A graph $G = (V, E)$ is called *strongly t-perfect* if system (68.1) is totally dual integral. So each strongly t-perfect graph is t-perfect (Theorem 5.22). It is unknown if the reverse implication holds:

(68.6) Is every t-perfect graph strongly t-perfect?

Strong t-perfection can be characterized by the weighted version of the stable set number and a certain weighted 'edge and circuit' cover number. Let $G = (V, E)$ be a graph and let $w : V \to \mathbb{Z}_+$. In this chapter, a *w-cover* is a family of vertices, edges, and odd circuits covering each vertex v at least $w(v)$ times. By definition, the *cost* of a vertex or edge is 1, and the *cost* of an odd circuit C is $\lfloor\frac{1}{2}|VC|\rfloor$. The *cost* of a w-cover $\mathcal{F}$ is the sum of the costs of the elements of $\mathcal{F}$. Define

(68.7) $\alpha_w(G) :=$ the maximum weight of a stable set in G,
$\tilde{\rho}_w(G) :=$ the minimum cost of a w-cover.

Obviously, $\alpha_w(G) \leq \tilde{\rho}_w(G)$ for any graph G. Moreover:

(68.8) G is strongly t-perfect $\iff$ $\alpha_w(G) = \tilde{\rho}_w(G)$ for each $w : V \to \mathbb{Z}_+$.

This follows directly from a combinatorial interpretation of total dual integrality.

Notes. W.R. Pulleyblank (cf. Gerards [1989a]) observed that, even for $w = \mathbf{1}$, determining $\tilde{\rho}_w(G)$ is NP-complete, since the vertex set of a graph G can be partitioned into triangles if and only if $\tilde{\rho}_w(G) = \frac{1}{3}|V|$ where $w = \mathbf{1}$. The problem of partitioning a graph into triangles is NP-complete. Since partitioning into triangles remains to be NP-complete for planar graphs (Dyer and Frieze [1986]), even determining $\tilde{\rho}_w(G)$ for planar graphs is NP-complete.

Again, strong t-perfection is equivalent to the total dual integrality of the vertex cover constraints (68.5).

68.3. Strong t-perfection of odd-K_4-free graphs

K_4 is the smallest graph that is not t-perfect. Gerards and Schrijver [1986] showed that any graph not containing an 'odd K_4-subdivision' is t-perfect — in fact, as Gerards [1989a] showed, strongly t-perfect. We will prove this in this section (with a method inspired by Geelen and Guenin [2001]).

Call a subdivision of K_4 *odd* if each triangle of K_4 has become an odd circuit — equivalently, if the evenly subdivided edges of K_4 form a cut of K_4. We say that a graph *contains no* odd K_4-subdivision if it has no subgraph which is an odd K_4-subdivision.

Theorem 68.3. *A graph containing no odd K_4-subdivision is strongly t-perfect.*

Proof. Let $G = (V, E)$ be a counterexample with $|V| + |E|$ minimum. Then G has no isolated vertices. So we can assume that any minimum-cost w-cover contains no vertices (for any w).

For any weight function $w : V \to \mathbb{Z}_+$, denote $\alpha_w := \alpha_w(G)$ and $\tilde{\rho}_w := \tilde{\rho}_w(G)$. As G is a counterexample, there exists a $w : V \to \mathbb{Z}_+$ with $\alpha_w < \tilde{\rho}_w$.

For any such w we have, for each edge $e = uv$,

(68.9) if S maximizes $w(S)$ over stable sets S of $G - e$, then S contains u and v.

Otherwise, S is a stable set of G, implying that (by the minimality of $|V| + |E|$):

(68.10) $$\alpha_w(G) \geq \alpha_w(G - e) = \tilde{\rho}_w(G - e) \geq \tilde{\rho}_w(G),$$

a contradiction.

This implies

(68.11) $$w \geq \mathbf{1},$$

since if $w(v) = 0$ for some vertex v, then for any edge e incident with v there is a stable set S of $G - e$ maximizing $w(S)$ and not containing v (since deleting v from S does not decrease $w(S)$). This contradicts (68.9).

We next show that we can assume w to have some additional properties (for an edge $e = uv$, χ^e is the incidence vector of the set $\{u, v\}$, that is, it is the $0, 1$ vector in $\mathbb{R}^V$ having 1's in positions u and v):

Claim 1. There exist $w : V \to \mathbb{Z}_+$ and $f \in E$ such that

(68.12) $$\tilde{\rho}_{w+\chi^f} = \alpha_w + 1 = \tilde{\rho}_w = \alpha_{w+\chi^f}$$

and such that

(68.13) $$\alpha_{w-\chi^{VC}} = \tilde{\rho}_{w-\chi^{VC}}$$

for each odd circuit C.

Proof of Claim 1. As G is not bipartite (by Theorem 19.7) and not just an odd circuit (as this is trivially strongly t-perfect), we know that H has a chordless odd circuit C_0 that has at least one vertex of degree at least 3. Let v be such a vertex, and let e be an edge incident with v but which is not on C_0.

Let $B := VC_0 \setminus \{v\}$. We choose w such that $w(V \setminus B)$ is minimal. There exists a $k \in \mathbb{Z}_+$ such that for $w' := w + k \cdot \chi^B$, each stable set S of $G - e$ maximizing $w'(S)$ satisfies $|S \cap B| = \frac{1}{2}|B|$. Hence no such set S contains v, and therefore, by (68.9), $\alpha_{w'} = \tilde{\rho}_{w'}$.

Now let M be the perfect matching in $C_0 - v$. For $y : M \to \mathbb{Z}_+$ define

(68.14) $$w^y := w + \sum_{f \in M} y_f \chi^f.$$

As $\alpha_{w'} = \tilde{\rho}_{w'}$, there exists a $y : M \to \mathbb{Z}_+$ such that

(68.15) $$\alpha_{w^y} = \tilde{\rho}_{w^y}.$$

We choose such a y with $\sum_{f\in M} y_f$ minimal. Since $\alpha_w < \tilde{\rho}_w$, there exists an $f \in M$ with $y_f \geq 1$. Then, by the minimality of y, we have $\alpha_{w^y-\chi^f} < \tilde{\rho}_{w^y-\chi^f}$. So we can assume that $y_f = 1$ and $y_{f'} = 0$ for each $f' \in M \setminus \{f\}$. We show that w and f are as required.

To show (68.12), we have $\alpha_{w+\chi^f} \leq \alpha_w + 1$, since any stable set S satisfies $(w + \chi^f)(S) = w(S) + |f \cap S| \leq w(S) + 1$. This implies

(68.16) $$\alpha_w + 1 \leq \tilde{\rho}_w \leq \tilde{\rho}_{w+\chi^f} = \alpha_{w+\chi^f} \leq \alpha_w + 1,$$

implying (68.12).

Next, consider any odd circuit C in G. Then $(w-\chi^{VC})(V\setminus B) < w(V\setminus B)$, since VC is not contained in B. Therefore, by the choice of w, we have (68.13).

End of Proof of Claim 1

As from now we fix w and f satisfying (68.12) and (68.13). Let f connect vertices u and u'. Since by the minimality of G, G has no isolated vertices, there exists a minimum-cost $w+\chi^f$-cover $\mathcal{F}$ consisting only of edges and odd circuits, say, $e_1, \ldots, e_t$, $C_1, \ldots, C_k$. We choose them such that

(68.17) $$|VC_1| + \cdots + |VC_k|$$

is as small as possible. Then:

(68.18) at least two of the C_i traverse f.

To see this, let $G' := G - f$ (the graph obtained by deleting edge f). If $\alpha_w(G') = \alpha_w(G)$, then by the minimality of G, G' has a w-cover of cost α_w. As this is a w-cover in G as well, this would imply $\alpha_w = \tilde{\rho}_w$, a contradiction.

So $\alpha_w(G') > \alpha_w(G)$. That is, there exists a stable set S in G' with $w(S) > \alpha_w$. Necessarily, S contains both u and u'. Then for any circuit C traversing f:

(68.19) $$|VC \cap S| \leq \lfloor \tfrac{1}{2}|VC| \rfloor + 1.$$

Also, f is not among $e_1, \ldots, e_t$, since otherwise $\mathcal{F} \setminus \{f\}$ is a w-cover of cost $\tilde{\rho}_{w+\chi^f} - 1 = \tilde{\rho}_w - 1$, contradicting the definition of $\tilde{\rho}_w$. Setting l to the number of C_i traversing f, we obtain:

(68.20) $$\begin{aligned} &\tilde{\rho}_{w+\chi^f} \leq \alpha_w + 1 \leq w(S) = (w + \chi^f)(S) - 2 \\ &\leq -2 + \sum_{j=1}^{t} |e_j \cap S| + \sum_{i=1}^{k} |VC_i \cap S| \leq -2 + t + \sum_{i=1}^{k} \lfloor \tfrac{1}{2}|VC_i| \rfloor + l \\ &= \tilde{\rho}_{w+\chi^f} + l - 2. \end{aligned}$$

So $l \geq 2$, which is (68.18).

By (68.18) we can assume that C_1 and C_2 traverse f. It is convenient to assume that $EC_1 \setminus \{f\}$ and $EC_2 \setminus \{f\}$ are disjoint; this can be achieved by adding parallel edges. So $EC_1 \cap EC_2 = \{f\}$.

Then:

(68.21) if C is an odd circuit with $EC \subseteq EC_1 \cup EC_2$, then $f \in EC$ and $EC_1 \triangle EC_2 \triangle EC$ is again an odd circuit.

Indeed, as $EC_1 \triangle EC_2 \triangle EC$ is an odd cycle, it can be decomposed into circuits $C'_2, \ldots, C'_p$, with $C'_2, \ldots, C'_q$ odd and $C'_{q+1}, \ldots, C'_p$ even ($q \geq 2$). Then

(68.22)
$$\sum_{i=2}^{p} |EC'_i| = |EC_1 \triangle EC_2 \triangle EC|$$
$$= |EC_1| + |EC_2| - |EC| - 2|\{f\} \setminus EC|.$$

Choose for each $i = q+1, \ldots, p$ a perfect matching M_i in C'_i. Let $e'_1, \ldots, e'_r$ be the edges in the matchings M_i and in $\{f\} \setminus EC$. Then, defining $C'_1 := C$,

(68.23)
$$\chi^{VC_1} + \chi^{VC_2} = \sum_{i=1}^{q} \chi^{VC'_i} + \sum_{j=1}^{r} \chi^{e'_j}$$

and (using (68.22))

(68.24)
$$\lfloor \tfrac{1}{2}|VC_1| \rfloor + \lfloor \tfrac{1}{2}|VC_2| \rfloor = \tfrac{1}{2}|EC_1| + \tfrac{1}{2}|EC_2| - 1$$
$$= -1 + |\{f\} \setminus EC| + \tfrac{1}{2}\sum_{i=1}^{p} |EC'_i| = -1 + r + \tfrac{1}{2}\sum_{i=1}^{q} |EC'_i|$$
$$\geq r + \sum_{i=1}^{q} \lfloor \tfrac{1}{2}|VC'_i| \rfloor.$$

So replacing C_1, C_2 by $C'_1, \ldots, C'_q$ and adding $e'_1, \ldots, e'_r$ to $e_1, \ldots, e_t$, gives again a $w + \chi^f$-cover of cost at most $\tilde{\rho}_{w+\chi^f}$. This also implies $q = 2$, since otherwise we have strict inequality in (68.24), and we would obtain a w-cover of cost less than $\tilde{\rho}_w$.

If $f \notin EC$, then f is among $e'_1, \ldots, e'_r$. Hence deleting f gives a w-cover of cost at most $\tilde{\rho}_{w+\chi^f} - 1 \leq \alpha_w$, contradicting (68.12). So $f \in EC$. As this is true for any odd circuit in $EC_1 \cup EC_2$ we know that $f \in EC'_i$ for $i = 1, 2$.

If $p \geq 3$ or $r \geq 1$, then $|EC'_1| + |EC'_2| < |EC_1| + |EC_2|$, contradicting the minimality of (68.17). So $p = q = 2$ and $r = 0$, which proves (68.21).

First, it implies

(68.25) a circuit in $EC_1 \cup EC_2$ is odd if and only if it traverses f.

A second consequence is as follows. Let P_i be the $u - u'$ path $C_i \setminus \{f\}$. Orient the edges occurring in the path $P_i := C_i \setminus \{f\}$ in the direction from u to u', for $i = 1, 2$. Then

(68.26) the orientation is acyclic.

For suppose that it contains a directed circuit C. Then $(EC_1 \cup EC_2) \setminus EC$ contains a directed $u{-}u'$ path, and hence an odd circuit C'. Hence by (68.21), $EC_1 \triangle EC_2 \triangle EC'$ is an odd circuit, however containing the even circuit EC, a contradiction.

Define

(68.27) $\qquad W := VP_1 \cup VP_2$ and $F := EP_1 \cup EP_2$.

Consider the graph (W, F). It is bipartite, as it contains no odd circuits by (68.25). Moreover, u and u' belong to the same colour class. Let A and B be the colour classes of (W, F), such that $u, u' \in A$. So

(68.28)
$$A := \{v \in W \mid \text{there exists an even-length directed } u - v \text{ path}\},$$
$$B := \{v \in W \mid \text{there exists an odd-length directed } u - v \text{ path}\}.$$

(Here and below, when speaking of a directed path, it is assumed to use only the edges in $EP_1 \cup EP_2$.) Define

(68.29) $\qquad X := VP_1 \cap VP_2$ and
$$U := \{v \in V \mid w(v) = \sum_{j=1}^{t} |e_j \cap \{v\}| + \sum_{j=1}^{k} |VC_j \cap \{v\}|\}.$$

So $u, u' \notin U$, $u, u' \in X$, and $X \setminus \{u, u'\}$ is the set of vertices in W having degree 4 in the graph (W, F).

We next show the following technical, but straightforward to prove, claim:

Claim 2. Let $z \in A$, let Q be an even-length directed $u - z$ path, and let S be a stable set in G. Then

(68.30) $\qquad (w - \chi^{VQ})(S) \geq \alpha_w - \lfloor \frac{1}{2}|VQ| \rfloor + 1$

if and only if

(68.31)
(i) $|e_j \cap S| = 1$ *for each* $j = 1, \ldots, t$,
(ii) $|VC_j \cap S| = \lfloor \frac{1}{2}|VC_j| \rfloor$ *for* $j = 3, \ldots, k$,
(iii) $S \subseteq U$,
(iv) *S contains $B \setminus VQ$ and is disjoint from $A \setminus VQ$,*
(v) *S contains $B \cap X$ and is disjoint from $A \cap X$.*

Proof of Claim 2. By rerouting C_1 and C_2, we can assume that $EQ \subseteq EC_1$. Define $Z := VC_1 \setminus VQ$. So $|Z|$ is even. Consider the following sequence of (in)equalities:

(68.32)
$$\begin{aligned}
(w - \chi^{VQ})(S) &= w(S) - |VQ \cap S| \\
&\leq \sum_{j=1}^{t} |e_j \cap S| + \sum_{j=1}^{k} |VC_j \cap S| - |VQ \cap S| \\
&= \sum_{j=1}^{t} |e_j \cap S| + \sum_{j=2}^{k} |VC_j \cap S| + |Z \cap S| \leq t + \sum_{j=2}^{k} \lfloor \tfrac{1}{2}|VC_j| \rfloor + |Z \cap S| \\
&= \tilde{\rho}_{w+\chi^f} - \lfloor \tfrac{1}{2}|VC_1| \rfloor + |Z \cap S| \leq \tilde{\rho}_{w+\chi^f} - \lfloor \tfrac{1}{2}|VC_1| \rfloor + \tfrac{1}{2}|Z| \\
&= \alpha_w + 1 - \lfloor \tfrac{1}{2}|VQ| \rfloor.
\end{aligned}$$

Hence (68.30) holds if and only if equality holds throughout in (68.32), which is equivalent to (68.31). Note that (68.31)(iv) and (v) are equivalent to: S contains $VC_2 \cap B$ and is disjoint from $VC_2 \cap A$, and S contains $Z \cap B$ and is disjoint from $Z \cap A$. Hence it is equivalent to (as $u, u' \notin S$ by (68.31)(iii)): $|VC_2 \cap S| = \lfloor \frac{1}{2}|VC_2| \rfloor$ and $|Z \cap S| = \frac{1}{2}|Z|$. *End of Proof of Claim 2*

By (68.26), we can order the vertices in X as $x_0 = u, x_1, \ldots, x_s = u'$ such that both P_1 and P_2 traverse them in this order. For $j = 0, \ldots, s$, let $\mathcal{P}_j$ be the collection of directed $u - x$ paths, where $x = x_j$ if $x_j \in A$, and x is an inneighbour of x_j if $x_j \in B$. So $x \in A$ and each path in each $\mathcal{P}_j$ has even length.

Let j be the largest index for which there exists a path $Q \in \mathcal{P}_j$ with

(68.33) $\quad \alpha_{w-\chi^{VQ}} \leq \alpha_w - \lfloor \frac{1}{2}|VQ| \rfloor.$

Such a j exists, since (68.33) holds for the trivial directed $u - u$ path, as $\alpha_{w-\chi^u} \leq \alpha_w$. Also, $j < s$, since otherwise $VQ = VC$ for some odd circuit C, and hence, with (68.13) we have

(68.34) $\quad \tilde{\rho}_w \leq \tilde{\rho}_{w-\chi^{VC}} + \lfloor \frac{1}{2}|VC| \rfloor = \alpha_{w-\chi^{VC}} + \lfloor \frac{1}{2}|VC| \rfloor \leq \alpha_w,$

contradicting (68.12).

Let Q_1 and Q_2 be the two paths in $\mathcal{P}_{j+1}$ that extend Q. By the maximality of j, we know

(68.35) $\quad \alpha_{w-\chi^{VQ_i}} \geq \alpha_w - \lfloor \frac{1}{2}|VQ_i| \rfloor + 1.$

So there exist stable sets S_1 and S_2 with

(68.36) $\quad (w - \chi^{VQ_i})(S_i) \geq \alpha_w - \lfloor \frac{1}{2}|VQ_i| \rfloor + 1$

for $i = 1, 2$. So for $i = 1, 2$, (68.31) holds for Q_i, S_i. By (68.31)(iv), S_1 and S_2 coincide on $W \setminus (VQ_1 \cup VQ_2)$, and they coincide on X. In other words:

(68.37) $\quad (S_1 \triangle S_2) \cap W \subseteq (VQ_1 \cup VQ_2) \setminus X.$

Let H be the subgraph of G induced by $S_1 \triangle S_2$. So H is a bipartite graph, with colour classes $S_1 \setminus S_2$ and $S_2 \setminus S_1$. Define

(68.38) $\quad Y_i := VQ_i \setminus VQ$

for $i = 1, 2$. Then

(68.39) $\quad H$ contains a path connecting Y_1 and Y_2.

For suppose not. Let K be the union of the components of H that intersect Y_1. So K is disjoint from Y_2. Define $S := S_1 \triangle K$. Then $S \cap Y_1 = S_2 \cap Y_1$ and $S \cap Y_2 = S_1 \cap Y_2$. This implies that Q, S satisfy (68.31). Hence (68.30) holds, contradicting (68.33). This proves (68.39).

Let C be the (even) circuit formed by the two directed $x_j - x_{j+1}$ paths. So Y_1 and Y_2 are subsets of VC. Let R be a shortest path in H that connects Y_1 and Y_2; say it connects $y_1 \in Y_1$ and $y_2 \in Y_2$.

Since $y_1, y_2 \in S_1 \triangle S_2$, we know by (68.37) that $y_1, y_2 \notin X$. By (68.31)(iv), if $y_1 \in S_1 \setminus S_2$, then $y_1 \in A$ (since if $y_1 \in B$, then $y_1 \in B \setminus VQ_2$, and so $y_1 \in S_2$), and if $y_1 \in S_2 \setminus S_1$, then $y_1 \in B$ (since if $y_1 \in A$, then $y_1 \in A \setminus VQ_2$, and so $y_1 \notin S_2$). Similarly, if $y_2 \in S_2 \setminus S_1$, then $y_2 \in A$ and if $y_2 \in S_1 \setminus S_2$, then $y_2 \in B$.

So if R is even, then y_1 and y_2 belong to the same set among $S_1 \setminus S_2$, $S_2 \setminus S_1$, and hence they belong to different sets A, B. Similarly, if R is odd, then y_1 and y_2 belong to the same set among A, B. Hence R forms with part of C an odd circuit.

By (68.37), there exist a directed $u - x_j$ path N' and a directed $x_{j+1} - u'$ path N'' that are (vertex-)disjoint from $S_1 \triangle S_2$. Concatenating N', f, and N'' makes an $x_{j+1} - x_j$ path N. Then N, R, and C make an odd K_4-subdivision, with 3-valent vertices x_j, x_{j+1}, y_1, y_2. ∎

(The above proof of Claim 1 was given by D. Gijswijt.)

Notes. Theorem 68.3 includes the t-perfection of series-parallel graphs (conjectured by Chvátal [1975a], and proved by M.J. Clancy in 1977 and by Mahjoub [1988]), the strong t-perfection of series-parallel graphs (Boulala and Uhry [1979], who also gave a polynomial-time algorithm to find a maximum-weight stable set in series-parallel graphs), the t-perfection of almost bipartite graphs — graphs G having a vertex v with $G - v$ bipartite (Fonlupt and Uhry [1982]), the strong t-perfection of almost bipartite graphs (this is implicit in Sbihi and Uhry [1984]), and the t-perfection of odd-K_4-free graphs (Gerards and Schrijver [1986]).

68.4. On characterizing t-perfection

The problem if a given graph $G = (V, E)$ is t-perfect, belongs to co-NP: non-t-perfection can be certified by a noninteger vertex x^* of the polytope determined by (68.1), together with a nonsingular system of constraints that are tight for x^*. One must check that x^* satisfies all constraints among (68.1) — this can be done in polynomial time by the methods described in the proof of Theorem 68.1. A polynomial-time algorithm for, or a combinatorial certificate of, non-t-perfection is not known.

T-perfection and strong t-perfection are not closed under taking subgraphs, as is shown by Figure 68.1. However, t-perfection is closed under taking induced subgraphs. This is easy to check, as well as that it is closed under the following operation:

(68.40) choose a vertex v with $N(v)$ a stable set, and contract all edges in $\delta(v)$.

So one may ask for the minimally non-t-perfect graphs — minimal with respect to taking induced subgraphs and applying operation (68.40). Known minimal graphs include the wheels[24] with an even number of vertices and the graphs consisting of a circuit of length $4k$ and all chords connecting a

[24] A *wheel* is a graph obtained from a circuit C by adding a new vertex, adjacent to all vertices in C.

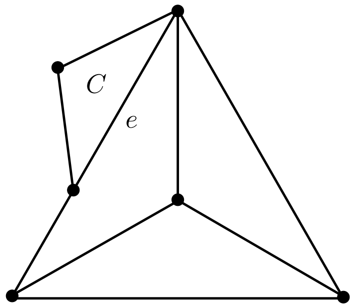


Figure 68.1

A strongly t-perfect graph G with $G - e$ not t-perfect. The strong t-perfection of G can be derived from the fact that each inclusionwise maximal stable set intersects the triangle C. Hence for any integer weight function, subtracting the incidence vector of VC, reduces the maximum weight of a stable set by 1. We therefore can assume that at least one of the vertices of C has weight 0, and hence we can delete it. We are left with a graph containing no odd K_4-subdivision — hence being strongly t-perfect (Theorem 68.3).

vertex with its opposite vertex ($k \geq 1$). Also strong t-perfection is closed under taking induced subgraphs and the operation (68.40). So one may ask a similar question for strong t-perfection.

A characterization that has been achieved is of those graphs for which each, also noninduced, subgraph is t-perfect. Here subdivisions of K_4 play a role. Call a subdivision of K_4 *bad* if it is not t-perfect.

It has been shown by Gerards and Shepherd [1998] that any graph without bad K_4-subdivision is t-perfect. Hence, each subgraph of a graph G is t-perfect if and only if G contains no bad K_4-subdivision. This was extended to: any graph without bad K_4-subdivision is strongly t-perfect (Schrijver [2002b]). So each subgraph of a graph is t-perfect if and only if each subgraph is strongly t-perfect.

The K_4-subdivisions that are bad have been characterized by Barahona and Mahjoub [1994c]. They showed that a K_4-subdivision is not t-perfect if and only if it is an odd K_4-subdivision such that the following does *not* hold: the edges of K_4 that have become an even path, form a 4-cycle in K_4, while the two other edges of K_4 are not subdivided. One may check that this is equivalent to the fact that one cannot obtain K_4 by the operations (68.40). So necessity in this characterization follows from the closedness of t-perfection under operation (68.40).

68.5. A combinatorial min-max relation

A subdivision of K_4 is called *totally odd* if it arises from K_4 be replacing each edge by an odd-length path. So a totally odd K_4-subdivision is an odd K_4-subdivision. A graph containing no totally odd K_4-subdivision need not be t-perfect (see Figure 68.2, from Chvátal [1975a]). However, Sewell and Trotter [1990,1993] showed that for weight function $w = \mathbf{1}$, the min-max relation is maintained for totally odd K_4-free graphs.

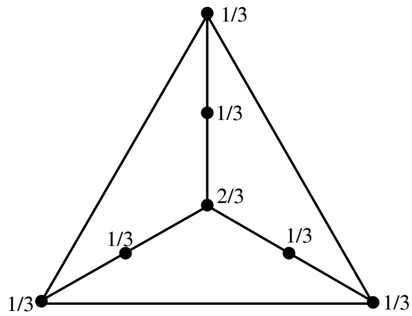


Figure 68.2
A graph containing no totally odd K_4-subdivision and not being t-perfect. The values at the vertices represent a vector satisfying (68.1) but not belonging to the stable set polytope.

This can be formulated in terms of the nonweighted version $\tilde{\rho}(G)$ of $\tilde{\rho}_w(G)$ defined in (68.7):

(68.41) $\tilde{\rho}(G) :=$ the minimum cost of a family of vertices, edges, and odd circuits covering V.

One easily checks that the minimum is attained by a vertex-disjoint family.

Obviously, for any graph G,

(68.42) $\alpha(G) \leq \tilde{\rho}(G)$.

So Sewell and Trotter [1990,1993] showed that equality holds for graphs without totally odd K_4-subdivision (generalizing a result of Gerards [1989a], who proved it for graphs without odd K_4-subdivision — a consequence of Theorem 68.3; Chvátal [1975a] proved it for series-parallel graphs).

Theorem 68.4. *For any graph G containing no totally odd K_4-subdivision, the stable set number $\alpha(G)$ is equal to $\tilde{\rho}(G)$.*

Proof. Let $G = (V, E)$ be a counterexample with $|V| + |E|$ minimal. Set $\alpha := \alpha(G)$. Then G is connected, and

(68.43) $\alpha(G-v)=\alpha$ for each $v\in V$ and $\alpha(G-e)>\alpha$ for each $e\in E$,

since otherwise $G-v$ or $G-e$ would be a smaller counterexample.

Hence for each vertex v, there exists a vertex-disjoint collection of vertices, edges, and odd circuits, covering $V\setminus\{v\}$ and of cost α. Let F_v be the set of edges contained in this collection or in one of the circuits in it. Let G_v be the graph $(V\setminus\{v\},F_v)$. So

(68.44) $\alpha(G_v)=\alpha$,

and G_v has maximum degree at most 2. Moreover, the minimality of G implies:

(68.45) $F_u\cup F_v=E\setminus\{uv\}$ for each edge uv.

To see this, trivially, $uv\notin F_u\cup F_v$. Suppose that $e\neq uv$ is an edge not contained in $F_u\cup F_v$. As $\alpha(G-e)>\alpha$, $G-e$ has a stable set S of size $\alpha+1$. By symmetry, we can assume that $v\notin S$. Then S is a stable set in the graph G_v, contradicting (68.44).

This proves (68.45), which gives:

(68.46) for each edge uv, each edge $e\neq uv$ incident with u belongs to F_v.

This follows directly from (68.45), since $e\notin F_u$ (as $u\in e$).

This is used in proving:

(68.47) G is 3-regular.

For if vertex v has degree 1, with neighbour u, then $\alpha(G-v-u)<\alpha$; since moreover $\tilde\rho(G)\leq\tilde\rho(G-u-v)+1$ (since we can add edge uv to any collection attaining the minimum for $G-u-v$), we have $\alpha(G)\geq\alpha(G-u-v)+1=\tilde\rho(G-u-v)+1\geq\tilde\rho(G)$. This contradicts the fact that G is a counterexample.

If v has degree 2, let G' be the graph obtained by contracting the edges incident with v. Then G' contains no totally odd K_4-subdivision. Moreover, it is straightforward to check that $\alpha(G)\geq\alpha(G')+1$ and $\tilde\rho(G)\leq\tilde\rho(G')+1$. As G' is smaller than G, we have $\tilde\rho(G')=\alpha(G')$. Hence $\alpha(G)\geq\alpha(G')+1=\tilde\rho(G')+1\geq\tilde\rho(G)$. Again, this contradicts the fact that G is a counterexample.

So v has degree at least 3. Let u be one of its neighbours. Then $\delta(v)\subseteq F_u\cup\{uv\}$ by (68.46). As G_u has maximum degree at most 2, we have $|\delta(v)|=3$. This proves (68.47).

By (68.45) and (68.47),

(68.48) for each edge uv of G, u is traversed by an odd circuit in F_v.

Moreover:

(68.49) Let uv be an edge of G and let C be the odd circuit in F_v traversing u. Consider any edge $e=xy$ on C with $e\notin F_u$. Then both x and y have even distance from u along $C-e$.

Let S be a stable set of $G-e$ of size $\alpha+1$. So $x, y \in S$. Moreover, $u \in S$, since otherwise $\alpha(G_u) > \alpha$ (since $e \notin F_u$), contradicting (68.44). So $v \notin S$, and hence $S \setminus \{y\}$ is a maximum-size stable set of $G-v$. Hence, $S \setminus \{y\}$ intersects C in $\lfloor \frac{1}{2}|VC| \rfloor$ vertices. Therefore, S intersects C in $\lceil \frac{1}{2}|VC| \rceil$ vertices. As $x, y \in S$, (68.49) follows.

Now choose a vertex, r say, and its neighbours, u_1, u_2, u_3 say. For each $i \in \{1,2,3\}$, F_{u_i} contains an odd circuit C_i traversing r (by (68.48)), and hence traversing the edges ru_{i+1} and ru_{i+2} (taking indices mod 3). We will construct a totally odd K_4-subdivision from them, which contradicts the condition of the theorem.

For $i = 1,2,3$, let P_i be the path in C_i from u_{i+1} to u_{i+2} obtained by deleting vertex r from C_i. Since $\alpha(G - ru_i) > \alpha$, G has a stable set S_i of size α, intersecting $\{r, u_1, u_2, u_3\}$ precisely in $\{u_i\}$. Then for all distinct $i, j \in \{1,2,3\}$:

(68.50) S_j contains all vertices along P_i at even distance from u_j.

To see this, we may assume that $i = 1$, $j = 2$. Since S_2 is a maximum-size stable set in $G - u_1$, it intersects C_1 in $\lfloor \frac{1}{2}|VC_1| \rfloor$ vertices. Since $r, u_3 \notin S_2$, S_2 contains all vertices along P_1 at even distance from u_2, proving (68.50).

This implies, for distinct $i, j, k \in \{1,2,3\}$:

(68.51) $VP_i \subseteq S_j \triangle S_k$.

One similarly shows, for distinct $i, j, k \in \{1,2,3\}$:

(68.52) P_i contains an edge that splits P_i into two even-length paths $P_{i,j}$ (containing u_j) and $P_{i,k}$ (containing u_k), in such a way that S_i contains all vertices along $P_{i,j}$ at odd distance from u_j and all vertices along $P_{i,k}$ at odd distance from u_k.

To prove this, we may assume that $i = 1$, $j = 2$, $k = 3$. Since $S := S_1 \setminus \{u_1\} \cup \{r\}$ is a maximum-size stable set in $G - u_1$, it intersects C_1 in $\lfloor \frac{1}{2}|VC_1| \rfloor$ vertices. Since S contains r, there is precisely one edge on P_1 not intersected by S. This gives the edge as required in (68.52).

This implies, for distinct $i, j \in \{1,2,3\}$:

(68.53) $VP_{i,j} = VP_i \cap (S_i \triangle S_j)$,

and hence, for distinct $i, j, k \in \{1,2,3\}$:

(68.54) $VP_i \cap VP_j = VP_{i,k} \cap VP_{j,k}$,

since

(68.55) $VP_{i,k} \cap VP_{j,k} = VP_i \cap (S_i \triangle S_k) \cap VP_j \cap (S_j \triangle S_k) = VP_i \cap VP_j$

(using (68.51)).

For each $i = 1,2,3$, vertex u_{i+2} is on P_i and P_{i+1}. Hence there is a first vertex v_i on P_i (starting from u_{i+1}), that also belongs to P_{i+1}. By (68.54), v_i occurs after v_{i+2} along P_i (seen from u_{i+1}), since $v_i \in VP_i \cap VP_{i+1} \subseteq VP_{i,i+2}$ and $v_{i+2} \in VP_{i+2} \cap VP_i \subseteq VP_{i,i+1}$. Moreover, (68.56) v_i has even distance from u_{i+2} along P_i and along P_{i+1}.

To prove this, we may assume that $i = 1$. Suppose that v_1 has odd distance from u_3 along P_1. Let f and e be the previous and next edge along P_1 (seen from u_2) and let g be the third edge incident with v_1. Since v_1 is the first vertex along P_1 belonging to P_2, we know that f is not on P_2. So $f \notin F_{u_2}$, and hence (by (68.45)) $f \in F_r$. Since g is not on P_1, we have $g \notin F_{u_1}$, and hence (again by (68.45)) $g \in F_r$. Therefore (as F_r has maximum degree at most degree 2), $e \notin F_r$. Then (68.49) implies that v_1 has even distance from u_3 along P_1. Hence $v_1 \in S_3$, and so v_1 has also even distance from u_3 along P_2 (by (68.50)). This proves (68.56).

For $i = 1, 2, 3$, let Q_i be the $u_{i+1} - v_i$ part of P_i. Then Q_i and Q_{i+1} intersect each other only in v_i (since v_i is the first vertex along P_i that is on P_{i+1}). This implies that Q_1, Q_2, Q_3 together with the edges ru_1, ru_2, and ru_3, form a totally odd K_4-subdivision, a contradiction. ∎

Recall that a graph is bipartite if and only if for each subgraph H, the stable set number $\alpha(H)$ is equal to the edge cover number $\rho(H)$. An extension of this is implied by the theorem above:

Corollary 68.4a. *A graph G contains no totally odd K_4-subdivision if and only $\alpha(H) = \tilde{\rho}(H)$ for each subgraph H of G.*

Proof. Necessity follows from Theorem 68.4. Sufficiency follows from the fact that if G is a totally odd K_4-subdivision, then $\alpha(G) < \tilde{\rho}(G)$. This can be seen by induction on $|VG|$. If $|VG| = 4$, then $G = K_4$, and $\alpha(G) = 1$, $\tilde{\rho}(G) > 1$. If $|VG| > 4$, G has a vertex v of degree 2. Let G' arise by contracting the two edges incident with v. Then, using the induction hypothesis, $\alpha(G) \leq \alpha(G') + 1 < \tilde{\rho}(G') + 1 \leq \tilde{\rho}(G)$. ∎

Theorem 68.4 also implies (in fact, is equivalent to) the following. A graph $G = (V, E)$ is called *α-critical* if $\alpha(G - e) > \alpha(G)$ for each $e \in E$. Then each connected α-critical graph is either K_1, or K_2, or an odd circuit, or contains a totally odd K_4-subdivision (answering a question of Chvátal [1975a]).

We note that Theorem 68.4 implies that the stable set number $\alpha(G)$ of a graph G without totally odd K_4-subdivision can be determined in polynomial time, as $\alpha(G)$ is equal to the maximum of $\mathbf{1}^{\mathsf{T}}x$ over (68.1) (since the separation problem is polynomial-time solvable — see Theorem 68.1). This implies that an explicit maximum-size stable set can be found in polynomial time (just by deleting vertices as long as the stable set number does not decrease).

The vertex cover number. Another consequence of Theorem 68.4 concerns the vertex cover number $\tau(G)$ of a graph $G = (V, E)$. Trivially, $\tau(G) + \alpha(G) = |V|$. Define the *profit* of an edge to be 1, and the *profit* of a circuit C to be $\lceil \frac{1}{2}|VC| \rceil$. The *profit* of a family of edges and circuits is equal to the sum of the profits of its elements. Let $\tilde{\nu}(G)$ denote the maximum profit of a collection of pairwise vertex-disjoint edges and odd circuits in G. Then there is the following analogue to Gallai's theorem (Theorem 19.1):

Theorem 68.5. *For any graph* $G = (V, E)$*:* $\tilde{\nu}(G) + \tilde{\rho}(G) = |V|$.

Proof. Define the profit of any vertex to be 0. Then $\tilde{\nu}(G)$ is equal to the maximum profit of a collection of vertices, edges, and circuits partitioning V. Similarly, $\tilde{\rho}(G)$ is equal to the minimum cost of a collection of vertices, edges, and circuits partitioning V. Now for any collection $\mathcal{C}$ of vertices, edges, and circuits partitioning V we have cost$(\mathcal{C})$+profit$(\mathcal{C}) = |V|$. Hence the minimum cost over all such collections equals $|V|$ minus the maximum profit over all such collections. This gives the required equality. ∎

With Theorem 68.4, this implies a min-max relation for the vertex cover number of totally-odd-K_4-free graphs:

Corollary 68.5a. *For any graph G containing no totally odd K_4-subdivision, the vertex cover number $\tau(G)$ is equal to $\tilde{\nu}(G)$.*

Proof. Directly from Theorems 68.4 and 68.5, and from the fact that $\alpha(G)+\tau(G) = |V|$ for any graph G. ∎

68.6. Further results and notes

68.6a. The w-stable set polyhedron

The t-perfection of odd-K_4-free graphs can be extended to apply to w-stable sets. Given a graph $G = (V, E)$ and a function $w : E \to \mathbb{Z}_+$, a *w-stable set* is a function $x : V \to \mathbb{Z}_+$ such that $x_u + x_v \leq w_e$ for each edge $e = uv$. So if $w = \mathbf{1}$ and G has no isolated vertices, w-stable sets are the incidence vectors of stable sets. The *w-stable set polyhedron* is the convex hull of the w-stable sets.

Theorem 68.3 implies a characterization of the w-stable set polyhedron of odd-K_4-free graphs. Consider the following system:

$$(68.57)\quad \begin{array}{rll} \text{(i)} & x_v \geq 0 & \text{for each } v \in V, \\ \text{(ii)} & x(e) \leq w_e & \text{for each } e \in E, \\ \text{(iii)} & x(VC) \leq \lfloor \frac{1}{2} w(EC) \rfloor & \text{for each odd circuit } C, \end{array}$$

where $x(e) = x_u + x_v$ for $e = uv$.

Theorem 68.6. *For any graph $G = (V, E)$ containing no odd K_4-subdivision and for any $w : E \to \mathbb{Z}_+$, system* (68.57) *determines the w-stable set polyhedron.*

Proof. We show that (68.57) determines an integer polyhedron, and hence is equal to the w-stable set polyhedron. Let x be a noninteger vertex of P. By resetting $w_e := w_e - \lfloor x_u \rfloor - \lfloor x_v \rfloor$ for $e = uv \in E$ and $x_v := x_v - \lfloor x_v \rfloor$ for $v \in V$, x remains a noninteger vertex of the new P. So we can assume that $0 \leq x_v < 1$ for each $v \in V$.

Let E' be the set of edges e of G with $w_e = 1$. Then $G' = (V, E')$ contains no odd K_4-subdivision, and hence is t-perfect (Theorem 68.3). So x is a convex combination of incidence vectors of stable sets of G'. As each such incidence vector satisfies (i) and (ii) of (68.57) (since $x_u + x_v \leq 1 + 1 = 2 \leq w_e$ for each edge $e = uv$ in $E \setminus E'$), it also satisfies (iii) (as it is integer). Hence x is a convex combination of integer solutions of (68.57). So P is integer. ∎

It was shown by Gijswijt and Schrijver [2002] that system (68.57) is totally dual integral for each $w : E \to \mathbb{Z}_+$ if and only if G contains no bad K_4-subdivision.

68.6b. Bidirected graphs

We saw bidirected graphs before in Chapter 36. We recall some definitions and terminology. A *bidirected graph* is a triple $G = (V, E, \sigma)$, where (V, E) is an undirected graph and where σ assigns to each $e \in E$ and each $v \in e$ a 'sign' $\sigma_{e,v} \in \{+1, -1\}$. The graph (V, E) may have loops, but we will assume that the 'two' ends of the loop have the same sign. (Other loops will be meaningless in our discussion.)

The edges e for which $\sigma_{e,v} = 1$ for each $v \in e$ are called the *positive edges*, those with $\sigma_{e,v} = -1$ for each $v \in e$ are the *negative edges*, and the remaining edges are the *directed edges*.

Clearly, undirected graphs and directed graphs can be considered as special cases of bidirected graphs. Graph terminology extends in an obvious way to bidirected graphs. The undirected graph (V, E) is called the *underlying undirected graph* of G. We also will need the *underlying signed graph* $G = (V, E, \Sigma)$, where Σ is the family of positive and negative edges. We call a circuit C in (V, E) *odd* or *even*, if $|EC \cap \Sigma|$ is odd or even, respectively.

A signed graph $G = (V, E, \Sigma)$ is called an *odd K_4-subdivision* if (V, E) is a subdivision of K_4 such that each triangle has become an odd circuit (with respect to Σ). A bidirected graph is called an *odd K_4-subdivision* if its underlying signed graph is an odd K_4-subdivision.

The $E \times V$ *incidence matrix* of a bidirected graph $G = (V, E, \sigma)$ is the $E \times V$ matrix M defined by, for $e \in E$ and $v \in V$:

(68.58) $$M_{e,v} := \begin{cases} \sigma_{e,v} & \text{if } e \text{ is not a loop,} \\ 2\sigma_{e,v} & \text{if } e \text{ is a loop,} \end{cases}$$

setting $\sigma_{e,v} := 0$ if $v \notin e$.

For $b \in \mathbb{Z}^E$, we consider integer solutions of the system $Mx \leq b$. To this end, define for any circuit C (in the undirected graph (V, E)) and any vertex v:

(68.59) $$a_{C,v} := \tfrac{1}{2} \sum_{e \in EC} M_{e,v} \text{ and } d_C := \lfloor \tfrac{1}{2} \sum_{e \in EC} b_e \rfloor.$$

As C is a circuit, $a_{C,v}$ is an integer. Hence each integer solution x of $Mx \leq b$ satisfies

(68.60) $$\sum_{v \in V} a_{C,v} x_v = \tfrac{1}{2} \sum_{e \in EC} \sum_{v \in V} M_{e,v} x_v \leq \lfloor \tfrac{1}{2} \sum_{e \in EC} b_e \rfloor = d_C.$$

Therefore, each integer solution of $Mx \leq b$ satisfies:

(68.61) (i) $Mx \leq b$,
(ii) $\sum_{v \in V} a_{C,v} x_v \leq d_C$ for each odd circuit C.

(Again, 'odd' is with respect to Σ.) Then Theorem 68.6 implies:

Corollary 68.6a. *If a bidirected graph G contains no odd K_4-subdivision, then system* (68.61) *determines an integer polyhedron.*

Proof. Make from the bidirected graph $G = (V, E, \sigma)$ the following auxiliary undirected graph $G' = (V', E')$. For each $e \in E$ which is not a positive loop, let $c_e := 1$ if e is positive, $c_e := 2$ if e is directed, and $c_e := 3$ if e is negative. Then replace e by a path P_e of length c_e connecting the two vertices in V incident with e. Let $\tilde{e}$ be the unique edge on P_e that is not incident with a vertex v of G with $\sigma_{e,v} = -1$.

If e is a positive loop at v, make a circuit P_e of length 3 starting and ending at v. Let $\tilde{e}$ be one of the two edges on P_e incident with v.

Let F be the set of edges f of G' that are on P_e for some $e \in E$ and satisfy $f \neq \tilde{e}$. As G has no odd K_4-subdivision (as a bidirected graph), G' has no odd K_4-subdivision (as an undirected graph). Hence by Theorem 68.6, the following system (in $x \in \mathbb{R}^{V'}$) determines an integer polyhedron:

(68.62)
$$\begin{array}{rll} \text{(i)} & x(\tilde{e}) \leq b_e & \text{for each edge } e \in E, \\ \text{(ii)} & x(f) = 0 & \text{for each edge } f \in F, \\ \text{(iii)} & x(VC) \leq \lfloor \frac{1}{2}|VC| \rfloor & \text{for each odd circuit } C \text{ in } G'. \end{array}$$

(Here 'odd' refers to the length of the circuit. As usual, $x(f) := x_u + x_v$ where u and v are the ends of f for $f \in E'$.) This implies that system (68.61) determines an integer polyhedron, since the conditions (68.62)(ii) allow elimination of the variables x_v for $v \in V' \setminus V$. ∎

This theorem may be used to characterize odd-K_4-free bidirected graphs. Let $G = (V, E, \sigma)$ be a bidirected graph, with $E \times V$ incidence matrix M. For $a, b \in \mathbb{Z}^E$ consider integer solutions of

(68.63) $$a \leq Mx \leq b.$$

As the matrix

(68.64) $$\begin{pmatrix} M \\ -M \end{pmatrix}$$

is again the incidence matrix of some bidirected graph, we can consider the inequalities (68.61)(ii) corresponding to matrix (68.64). They amount to:

(68.65) $$\sum_{v \in V} \tfrac{1}{2}\Big(\sum_{e \in F} M_{e,v} - \sum_{e \in EC \setminus F} M_{e,v}\Big)x_v \leq \Big\lfloor \tfrac{1}{2}\Big(\sum_{e \in F} b_e - \sum_{e \in EC \setminus F} a_e\Big)\Big\rfloor \text{ for each}$$
odd circuit C and each $F \subseteq EC$.

To describe the characterization, we define 'minor' of a signed graph $G = (V, E, \Sigma)$. For $e \in E$, *deletion* of e means resetting E and Σ to $E \setminus \{e\}$ and $\Sigma \setminus \{e\}$. *Deletion* of a vertex v means deleting all edges incident with v, and deleting v from V. If e is not a loop, *contraction* of e means the following. Let e have ends u and v. If $e \in \Sigma$, reset $\Sigma := \Sigma \triangle \delta(u)$. Otherwise, let Σ be unchanged. Next contract e in (V, E). This definition depends on the choice of the end u of e, but for the application below this will be irrelevant. A *resigning* means choosing $U \subseteq V$ and resetting Σ to $\Sigma \triangle \delta(U)$. A signed graph H is called a *minor* of a signed graph G if

H arises from G by a series of deletions of edges and vertices, contractions of edges, and resignings.

Then we have the following characterization (Gerards and Schrijver [1986]), where odd-K_4 stands for the signed graph (VK_4, EK_4, EK_4).

Corollary 68.6b. *For any bidirected graph G the following are equivalent:*

(68.66) (i) *G contains no odd K_4-subdivision as subgraph;*
(ii) *the signed graph underlying G has no odd-K_4 minor;*
(iii) *for all integer vectors a, b, system* (68.63)(68.65) *determines a box-integer polyhedron.*

Proof. The implication (ii)⇒(i) follows from the easy fact that any odd K_4-subdivision in G would yield an odd-K_4 minor of the signed graph underlying G.

The implication (i)⇒(iii) can be derived from Corollary 68.6a as follows. Replace any 'box' constraint $d_v \leq x_v \leq c_v$ by $2d_v \leq 2x_v \leq 2c_v$, and incorporate it into M, by adding loops at v. Then the constraint (68.65) corresponding to such a loop C at v is $x_v \leq c_v$ or $-x_v \leq -d_v$. This gives a reduction to Corollary 68.6a.

To see the implication (iii)⇒(ii), note that (iii) is invariant under deleting rows of M and under multiplying rows or columns by -1. It is also closed under contractions of any edge e, as it amounts to taking $a_e = b_e = 0$ in (68.63). So, in proving (iii)⇒(ii), if the signed graph underlying G has an odd-K_4 minor, we may assume that it is odd-K_4. By multiplying rows and columns of M by -1, we may assume that M is nonnegative. Then we do not have an integer polytope for $a = \mathbf{0}$, $b = \mathbf{1}$, $d = \mathbf{0}$, $c = \mathbf{1}$. ∎

In other words, the bidirected graphs without odd K_4-subdivision are precisely those whose $E \times V$ incidence matrix has strong Chvátal rank at most 1 (cf. Section 36.7a, where it is shown that the transpose of *each* such matrix has strong Chvátal rank at most 1).

68.6c. Characterizing odd-K_4-free graphs by mixing stable sets and vertex covers

A similar characterization can be formulated in terms of just undirected graphs, by mixing stable sets and vertex covers. Call a graph H an *odd minor* of a graph G if H arises from G by deleting edges and vertices, and by contracting all edges in some cut $\delta(U)$ (in the graph without the deleted edges). The following is easy to show:

(68.67) A graph G contains an odd K_4-subdivision $\iff$ G contains K_4 as odd minor.

For a graph $G = (V, E)$ and $F \subseteq E$, a subset U of V is called *F-stable* if U is a stable set of the graph (V, F). U is called an *F-cover* if U is a vertex cover of (V, F). Let F_1 and F_2 be disjoint subsets of E, and consider the system:

(68.68)
$$\begin{array}{ll} 0 \le x_v \le 1 & \text{for } v \in V, \\ x(e) \le 1 & \text{for } e \in F_1, \\ x(e) \ge 1 & \text{for } e \in F_2, \\ \displaystyle\sum_{e \in EC \cap F_1} x(e) - \sum_{e \in EC \cap F_2} x(e) \le |EC \cap F_1| - |EC \cap F_2| - 1 & \\ & \text{for each odd circuit } C \text{ with } EC \subseteq F_1 \cup F_2 \end{array}$$

(where $x(e) := x_u + x_v$ for $e = uv \in E$).

Corollary 68.6c. *For any graph $G = (V, E)$ the following are equivalent:*

(68.69) (i) *G contains no odd K_4-subdivision;*
(ii) *for all disjoint $F_1, F_2 \subseteq E$, the convex hull of the incidence vectors of the F_1-stable F_2-covers is determined by* (68.68).

Proof. The implication (i)⇒(ii) follows from Corollary 68.6a. To see (ii)⇒(i), we first show that (ii) is maintained under taking odd minors. Maintenance under deletion of edges or vertices is trivial. To see that it is maintained under contraction of cuts, let $U \subseteq V$ and let $G' = (V', E')$ be the contracted graph. Let F_1' and F_2' be disjoint subsets of E', and let x' satisfy (68.68) for G', F_1', F_2'. Define $x : V \to \mathbb{R}$ as follows, where, for $v \in V$, v' denotes the vertex of G' to which v is contracted:

(68.70)
$$x_v := \begin{cases} x'_{v'} & \text{if } v \in U, \\ 1 - x'_{v'} & \text{if } v \in V \setminus U. \end{cases}$$

Moreover, define F_1 and F_2 by:

(68.71)
$$\begin{array}{l} F_1 := (F_1' \cap E[U]) \cup (F_2' \cap E[V \setminus U]) \cup \delta(U), \\ F_2 := (F_2' \cap E[U]) \cup (F_1' \cap E[V \setminus U]). \end{array}$$

Then x satisfies (68.68) with respect to G, F_1, F_2. Hence x is a convex combination of integer solutions of (68.68). Applying the construction in reverse to (68.70), we obtain x' as a convex combination of integer solutions of (68.68) with respect to G', F_1', F_2'.

This shows that (68.69)(ii) is maintained under taking odd minors. Moreover, K_4 violates the condition (taking $F_1 := E$, $F_2 := \emptyset$, $x_v := \frac{1}{3}$ for each $v \in V$). This shows sufficiency of the condition. ∎

68.6d. Orientations of discrepancy 1

A directed graph $D = (V, A)$ is said to have *discrepancy k* if for each (undirected) circuit, the number of forward arcs differs by at most k from the number of backward arcs.

The proof of Gerards [1989a] of the strong t-perfection of odd-K_4-free graphs (Theorem 68.3) is by showing that each such graph can be decomposed into graphs that have an orientation of discrepancy 1, using a characterization of Gerards [1994] of orientability of discrepancy 1 and a decomposition theorem of Gerards, Lovász, Schrijver, Seymour, Shih, and Truemper [1993] (cf. Gerards [1990]). As the graphs having an orientation of discrepancy 1 can be shown to be strongly t-perfect with minimum-cost flow techniques (see Theorem 68.7 below), and as the composition maintains total dual integrality of (68.1), the required result follows.

It is not difficult to show that the underlying undirected graph of any digraph of discrepancy 1, contains no odd K_4-subdivision. So, by Theorem 68.3, any undirected graph having an orientation of discrepancy 1, is strongly t-perfect. Gerards gave a direct proof of the strong t-perfection of such graphs, based on the following minimum-cost circulation argument:

Lemma 68.7α. *Let $D = (V, A)$ be a directed graph and let $b : A \to \mathbb{Z}_+$. Then the following system is totally dual integral:*

$$\text{(68.72)} \qquad \begin{array}{lll} \text{(i)} & x_v \geq 0 & \text{for } v \in V, \\ \text{(ii)} & x(VC) \leq b(AC) & \text{for each directed circuit } C \text{ in } D. \end{array}$$

Proof. Choose $w : V \to \mathbb{Z}_+$. We must show that the dual of maximizing $w^{\mathsf{T}}x$ over (68.72) has an integer optimum solution.

Make another directed graph $\widetilde{D} = (\widetilde{V}, \widetilde{A})$ as follows. For each vertex v of D, make two vertices v', v'' and an arc (v', v''), and for each arc (u, v) of D, make an arc (u'', v'). This defines $\widetilde{D}$.

Define $g, f : \widetilde{A} \to \mathbb{Z}_+$ by:

$$\text{(68.73)} \qquad \begin{array}{llll} g(v', v'') := w(v) & \text{and} & f(v', v'') := 0 & \text{for } v \in V, \\ g(u'', v') := 0 & \text{and} & f(u'', v') := b(u, v) & \text{for } (u, v) \in A. \end{array}$$

Then the maximum of $w^{\mathsf{T}}x$ over (68.72) is equal to the maximum of $g^{\mathsf{T}}z$ where $z : \widetilde{A} \to \mathbb{R}_+$ satisfies

$$\text{(68.74)} \qquad z(A\widetilde{C}) \leq f(A\widetilde{C}) \text{ for each directed circuit } \widetilde{C} \text{ in } \widetilde{D}.$$

So if we consider $f - z$ as length function on $\widetilde{A}$, then (68.74) says that each directed circuit in $\widetilde{D}$ has nonnegative length. Hence, by Theorem 8.2, the maximum is equal to the maximum of $g^{\mathsf{T}}z$ over $z : \widetilde{A} \to \mathbb{R}_+$ for which there exists a $p : \widetilde{V} \to \mathbb{R}$ such that

$$\text{(68.75)} \qquad z(\tilde{a}) + p(t) - p(s) \leq f(\tilde{a}) \text{ for each } \tilde{a} = (s, t) \in \widetilde{A}.$$

The latter system has a totally unimodular constraint matrix, and hence the LP has integer optimum primal and dual solutions. The dual asks for the minimum of $y^{\mathsf{T}}f$ where $y : \widetilde{A} \to \mathbb{Z}_+$ satisfies

$$\text{(68.76)} \qquad \begin{array}{ll} y(\tilde{a}) \geq g(\tilde{a}) & \text{for each } \tilde{a} \in \widetilde{A}, \\ y(\delta^{\text{in}}(\tilde{v})) = y(\delta^{\text{out}}(\tilde{v})) & \text{for each } \tilde{v} \in \widetilde{V}. \end{array}$$

So y is a circulation in $\widetilde{D}$. Hence y is a nonnegative integer combination of incidence vectors of directed circuits $\widetilde{C}$ in $\widetilde{D}$:

$$\text{(68.77)} \qquad y = \sum_{\widetilde{C}} \lambda_{\widetilde{C}} \chi^{A\widetilde{C}}.$$

For each directed circuit $\widetilde{C}$ in $\widetilde{D}$, let C denote the corresponding directed circuit in D (obtained by contracting all arcs (v', v'') occurring in $\widetilde{C}$). Then

$$\text{(68.78)} \qquad y^{\mathsf{T}}f = \sum_{\widetilde{C}} \lambda_{\widetilde{C}} (\chi^{A\widetilde{C}})^{\mathsf{T}} f = \sum_{\widetilde{C}} \lambda_{\widetilde{C}} f(A\widetilde{C}) = \sum_{\widetilde{C}} \lambda_{\widetilde{C}} b(AC)$$

and

(68.79) $$\sum_{\widetilde{C}} \lambda_{\widetilde{C}} \chi^{VC} \geq w.$$

Hence we have obtained an integer dual solution for the problem of maximizing $w^{\mathsf{T}} x$ over (68.72). ∎

This lemma implies:

Theorem 68.7. *Let $G = (V, E)$ be an undirected graph having an orientation D of discrepancy* 1. *Then G is strongly t-perfect.*

Proof. Let $D' = (V, A')$ be the digraph obtained from D by adding a reverse arc (v, u) for each arc (u, v) of D, defining $b(u, v) := 1$ and $b(v, u) := 0$. Then the total dual integrality of (68.1) follows directly from the total dual integrality of (68.72). Note that each directed circuit C' in D' gives an undirected circuit C in D, with $b(AC')$ equal to the number of forward arcs in C. As D has discrepancy 1, $\lfloor \frac{1}{2}|VC| \rfloor$ is equal to the minimum value of $b(AC')$ and $b(AC'^{-1})$. ∎

This immediately implies the strong t-perfection of *almost bipartite* graphs — graphs having a vertex v with $G - v$ bipartite, since they have an orientation of discrepancy 1, as one easily checks.

68.6e. Colourings and odd K_4-subdivisions

Zang [1998] and Thomassen [2001] showed that any graph G without totally odd K_4-subdivision satisfies $\chi(G) \leq 3$.[25] We may interpret this in terms of the integer decomposition and rounding properties. Consider the antiblocking polytope Q of the stable set polytope of a graph $G = (V, E)$:

(68.80) $$\begin{array}{ll} x_v \geq 0 & \text{for each } v \in V, \\ x(S) \leq 1 & \text{for each stable set } S. \end{array}$$

If G is t-perfect, the vertices of Q are: the origin, the unit base vectors, the incidence vectors of the edges, and the vectors $\chi^{VC} / \lfloor \frac{1}{2}|VC| \rfloor$ where C is an odd circuit. (This follows from the definition of t-perfection with antiblocking polyhedra theory.) Hence the fractional colouring number $\chi^*(G)$ of G, which is equal to the maximum of $\mathbf{1}^{\mathsf{T}} x$ over (68.80) (cf. Section 64.8), is equal to

(68.81) $$\max\{2, \max\{\frac{|VC|}{\lfloor \frac{1}{2}|VC| \rfloor} \mid C \text{ odd circuit}\}\}$$

(assuming $E \neq \emptyset$). For nonbipartite graphs, this value is equal to 3. So for graphs G without totally odd K_4-subdivision, the colouring number $\chi(G)$ is equal to the round-up $\lceil \chi^*(G) \rceil$ of the fractional colouring number.

[25] This was conjectured by Toft [1975], and extends results of Hadwiger [1943] that a 4-chromatic graph contains a K_4-subdivision, of Catlin [1979] that it contains an odd K_4-subdivision, and of Gerards and Shepherd [1998] that it contains a bad K_4-subdivision. Zeidl [1958] showed that any vertex of a minimally 4-chromatic graph lies in a subdivided K_4 that contains an odd circuit. Other partial and related results were found by Krusensjterna-Hafstrøm and Toft [1980], Thomassen and Toft [1981], and Jensen and Shepherd [1995].

A.M.H. Gerards (personal communication 2001) showed that system (68.80) has the integer rounding property if G has no odd K_4-subdivision. It implies that the corresponding stable set polytope has the integer decomposition property. This is equivalent to:

(68.82) $$\chi_w(G) = \lceil \chi_w^*(G) \rceil$$

for each odd-K_4-free graph G and each $w : VG \to \mathbb{Z}_+$.

This does not hold for any t-perfect graph: M. Laurent and P.D. Seymour showed in 1994 that the complement of the line graph of a prism (complement of C_6) is t-perfect, but is not 3-colourable; hence its stable set polytope does not have the integer decomposition property.

68.6f. Homomorphisms

Let G and H be simple graphs. A *homomorphism* $G \to H$ is a function $\phi : VG \to VH$ such that if $uv \in EG$, then $\phi(u)\phi(v) \in EH$ (in particular, $\phi(u) \neq \phi(v)$). Obviously, if there exists a homomorphism $G \to H$, then $\chi(G) \leq \chi(H)$.

For any k, let $K_4^{(k)}$ be the graph obtained from K_4 by replacing each edge by a path of length k. Then one may check that for odd k there is no homomorphism $K_4^{(k)} \to C_{2k+1}$.

Catlin [1985] showed that this is essentially the only counterexample: if G is a connected graph of maximum degree 3 and $k \in \mathbb{Z}_+$, such that any two vertices of G of degree 3 have distance at least k, and such that there is no homomorphism $G \to C_{2k+1}$, then k is odd and $G = K_4^{(k)}$. (This extends Brooks' theorem (Theorem 64.3) for $k = 1$.)

Gerards [1988] extended this to: if a nonbipartite graph G has no odd minor equal to K_4 or to the graph obtained from the triangle by adding for each edge a new vertex adjacent to the ends of the edge, then there is a homomorphism $G \to C_t$, where t is the shortest length of an odd circuit of G. Further results are given by Catlin [1988].

68.6g. Further notes

Sbihi and Uhry [1984] call a graph $G = (V, E)$ *h-perfect*[26] if the stable set polytope is determined by

(68.83)
$$\begin{array}{rll} \text{(i)} & x_v \geq 0 & \text{for } v \in V, \\ \text{(ii)} & x(C) \leq 1 & \text{for each clique } C, \\ \text{(iii)} & x(VC) \leq \lfloor \frac{1}{2}|VC| \rfloor & \text{for each odd circuit } C. \end{array}$$

So perfect graphs and t-perfect graphs are h-perfect. Sbihi and Uhry showed that substituting bipartite graphs for edges of a series-parallel graph preserves h-perfection.

The t-perfection of line graphs, and classes of graphs that are h-perfect but not t-perfect, were studied by Cao and Nemhauser [1998]. Gerards [1990] gave a survey on signed graphs without odd K_4-subdivision.

[26] h stands for 'hole' (English for 'trou').

Chapter 69

Claw-free graphs

Claw-free graphs are graphs not having $K_{1,3}$ as induced subgraph. We show the result of Minty and Sbihi that a maximum-size stable set in a claw-free graph can be found in strongly polynomial time, and the extension of Minty to the weighted case.

69.1. Introduction

A graph $G = (V, E)$ is called *claw-free* if no induced subgraph of G is isomorphic to $K_{1,3}$. Minty [1980] and Sbihi [1980] showed that a maximum-size stable set in a claw-free graph can be found in polynomial time. Since the line graph of any graph is claw-free, this generalizes Edmonds' polynomial-time algorithm for finding a maximum-size matching in a graph.

Sbihi's algorithm is an extension of Edmonds' blossom shrinking technique, while Minty gave a reduction to the maximum-size matching problem. Minty [1980] also indicated that his algorithm can be extended to the weighted case by reduction to Edmonds' weighted matching algorithm. The final argument for this was given by Nakamura and Tamura [2001].

In Section 69.2, we describe Minty's method for finding a maximum-size stable set in claw-free graphs, and in Section 69.3 we describe the extension to the weighted case.

69.2. Maximum-size stable set in a claw-free graph

An important property of claw-free graphs is that any vertex has at most two neighbours in any stable set. This enables us to augment stable sets by S-augmenting paths, which we define now.

Let $G = (V, E)$ be a graph and let S be a stable set in G. A walk $P = (v_0, v_1, \ldots, v_k)$ (given by its vertex-sequence) is called *S-alternating* if precisely one of v_{i-1}, v_i belongs to S, for each $i = 1, \ldots, k$. It is an *S-augmenting path* if moreover P is a path, $v_0, v_k \notin S$, and $(S \setminus \{v_1, v_3, \ldots, v_{k-1}\}) \cup \{v_0, v_2, \ldots, v_k\}$ is stable. This implies that (if $k \geq 2$) each of v_0 and v_k has precisely one neighbour in S, and each of $v_2, v_4, \ldots, v_{k-2}$ precisely two.

It is easy to see that if G is claw-free, then there is a stable set larger than S if and only if there exists an S-augmenting path. Indeed, sufficiency follows from the definition of S-augmenting path. To see necessity, let S' be a stable set larger than S. Then the subgraph of G induced by $S \triangle S'$ has a component K with more vertices in S' than in S. Since G is claw-free, this subgraph has maximum degree 2, and hence K forms an S-augmenting path.

So in order to find a maximum-size stable set, it suffices to have a polynomial-time algorithm to find for given S, an S-augmenting path, if any. For this, it suffices to describe a polynomial-time algorithm to find an S-augmenting $a-b$ path for prescribed $a, b \in V \setminus S$ (if any). Varying over all $a, b \in V \setminus S$, we find an S-augmenting path (if any).

Therefore, from now on we fix $a, b \in V \setminus S$. Then we can assume:

(69.1) $a \neq b$; a and b have degree 1, each with neighbour in S, say s_a and s_b; $s_a \neq s_b$; each $v \in V \setminus S$ with $v \neq a, b$ has precisely two neighbours in S; for each $s \in S$ with $s \neq s_a, s_b$ there are at least two vertices in S at distance two from s; G is connected.

Indeed, otherwise finding an S-augmenting path is trivial, or it does not exist; moreover, we can delete all neighbours of a or b distinct from s_a or s_b, and all vertices in $S \setminus \{s_a, s_b\}$ that have less than two vertices in S at distance two.

The assumptions (69.1) imply that any S-augmenting path connects a and b. Consider an S-alternating path

(69.2) $P = (v_0, s_1, v_1, \ldots, s_k, v_k)$

(given by its vertex-sequence), with $v_0 = a$ and $v_k = b$. So $s_1 = s_a$ and $s_k = s_b$. Then (under the assumptions (69.1)):

Lemma 69.1α**.** *P is S-augmenting if and only if v_{i-1} and v_i are nonadjacent for each $i = 2, \ldots, k-1$.*

Proof. Necessity being trivial, we show sufficiency. It suffices to show that $(S \setminus \{s_1, \ldots, s_k\}) \cup \{v_0, \ldots, v_k\}$ is a stable set. Any two vertices in S are nonadjacent. All neighbours in S of any v_i are among $s_1, \ldots, s_k$. Finally, suppose that any v_i, v_j are adjacent, with $i < j$. Then $j \geq i + 2$, since v_i and v_{i+1} are nonadjacent by the condition. But then v_i is adjacent to the three pairwise nonadjacent vertices s_i, s_{i+1}, and v_j. This contradicts the claw-freeness of G. ∎

We next prove a basic lemma of Minty [1980]. Define, for $u, v \in V \setminus S$:

(69.3) $u \sim v \iff N(u) \cap S = N(v) \cap S.$

Clearly, $\sim$ is an equivalence relation. We call any equivalence class a *similarity class*, and if $u \sim v$ we say that u and v are *similar*. So for each $s \in S$, $N(s)$ is a union of similarity classes.

We call a vertex $s \in S$ *splittable* if $N(s)$ can be partitioned into two classes X, Y such that

(69.4) $uv \in E \iff u, v \in X$ or $u, v \in Y$

for all $u, v \in N(s)$ with $u \not\sim v$. If s is splittable, we call X and Y the *classes* of s. Define

(69.5) $S' := \{s \in S \mid s \text{ is splittable}\}$ and $S'' := S \setminus S'$.

Then $s_a, s_b \in S'$, since $N(s_a) \setminus \{a\}$ is a clique, as a has no neighbours in $N(s_a)$ (by assumption (69.1)) and as G is claw-free — similarly for s_b. Moreover:

Lemma 69.1β. *Each vertex $s \in S$ having at least three vertices in S at distance two, belongs to S'.*

Proof. Since $s_a, s_b \in S'$, we may assume that $s \neq s_a, s_b$. Let $G' = (N(s), F)$ be the subgraph of G with

(69.6) $F := \{uv \in E \mid u, v \in N(s), u \not\sim v\}$.

Then

(69.7) each component of G' induces a clique of G.

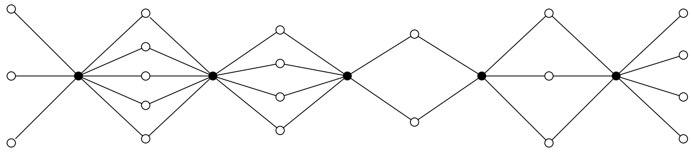
Suppose not. Let $P = (v_0, v_1, \ldots, v_k)$ be a shortest path in G' with $v_0 v_k \notin E$. If $k = 2$, then $v_0 \not\sim v_1 \not\sim v_2$, and hence v_1 has a neighbour $t \in S$ which is not a neighbour of v_0 or v_2. But then v_1 is adjacent to the pairwise nonadjacent t, v_0, v_2, contradicting the claw-freeness of G.

If $k = 3$, then as P is shortest, $v_0 v_2, v_1 v_3 \in E \setminus F$. So $v_0 \sim v_2$ and $v_1 \sim v_3$. Choose a vertex p with $p \not\sim v_0$ and $p \not\sim v_1$. (This is possible since $N(s)$ contains at least three similarity classes.) Then p has a neighbour t in S which is not a neighbour of any of v_0, v_1, v_2, v_3. Since $N(s)$ contains no three pairwise nonadjacent vertices (as G is claw-free), we know that $v_0 p \in E$ or $v_3 p \in E$. By symmetry, we can assume that $v_0 p \in E$, and hence $v_0 p \in F$. Then, by the minimality of k, we know successively that $v_1 p \in F$, $v_2 p \in F$, and $v_3 p \in F$. But then $v_0 p$ and $p v_3$ are in F, and hence, by the minimality of k, $v_0 v_3 \in E$.

If $k \geq 4$, then $v_0 v_2, v_0 v_3 \in E$, hence (since $v_2 \not\sim v_3$) $v_0 v_2 \in F$ or $v_0 v_3 \in F$, contradicting the minimality of k. This proves (69.7).

Since G is claw-free, G' has at least one component, X say, that intersects at least two of the similarity classes. If G' has at most two components, or if X contains all but at most one similarity class, we are done, taking $Y := N(s) \setminus X$. If G' has at least three components and $N(s) \setminus X$ intersects at least two similarity classes, then G' has two other components Y, Z for which there exist $x \in X$, $y \in Y$, and $z \in Z$ with $x \not\sim y \not\sim z \not\sim x$, as one easily checks[27]. But then s is adjacent to the three pairwise nonadjacent vertices x, y, z, contradicting the claw-freeness of G. ∎

[27] Let $y \in N(s) \setminus X$ be such that there exist $x', x'' \in X$ with $y \not\sim x' \not\sim x'' \not\sim y$. Let Y be the component of G' containing y. Let Z be a third component, if possible containing



• vertex in S ◦ vertex not in S

Figure 69.1
A typical bone

Now consider the subgraph

(69.8) $(V \setminus S', \delta(S''))$

of G. It is a bipartite graph, with colour classes S'' and $V \setminus S$. We call each component of this graph a *bone*. (A typical bone is depicted in Figure 69.1.) By Lemma 69.1β, each $s \in S''$ has at most two vertices in S at distance two. Hence any bone B consists of a series of vertices $s_1, \ldots, s_k$ in S'', together with disjoint nonempty sets $V_0, V_1, \ldots, V_k$ of vertices such that s_i is incident with each vertex in $V_{i-1} \cup V_i$ for each $i = 1, \ldots, k$. Moreover, B has two neighbours in S', say s and t, where s is adjacent to all vertices in V_0 and t is adjacent to all vertices in V_k. (It might be that $s = t$.) The degenerate case is that $k = 0$, where B is a singleton vertex in $V \setminus S$ with two neighbours in S'.

The relevance of bones is that if we leave out from any S-augmenting path the vertices that belong to S', we are left with a number of subpaths, each of which is an S''-augmenting path contained in some bone. So in constructing or analyzing an S-augmenting path, we can decompose it into S''-augmenting paths, glued together at vertices in S'. Here the classes of the vertices in S' come in, since the ends of the two subpaths glued together at $s \in S'$ should belong to different classes of s. This motivates the following graph H (called the *Edmonds graph* by Minty [1980])[28].

H has vertex set

(69.9) $\{(s, X) \mid s \in S',\ X \text{ class of } s\} \setminus \{(s_a, \{a\}), (s_b, \{b\})\}$

and the following edges:

(69.10) (i) $\{(s, X), (s, Y)\}$ for $s \in S' \setminus \{s_a, s_b\}$ and X, Y the classes of s;

a vertex nonsimilar to y. Then, if Z contains a vertex $z \not\sim y$, we can take for x one of x', x''. If Z contains no such vertex, let $z \in Z$. Then Y contains a vertex $y' \not\sim z$. As $z \not\sim x'$ and $z \not\sim x''$, we are done again.

[28] We note that in constructing H we could restrict S' to those vertices in S that have at least 3 vertices in S at distance two, together with s_a and s_b. However, for the extension to the weighted case, we need S' as defined above (namely, by being splittable).

(ii) $\{(s, X), (t, Y)\}$ for vertices $(s, X), (t, Y)$ of H such that there exists an S''-augmenting $X - Y$ path P.

So the path P is contained in the bone B containing x and y for some $x \in X$ and some $y \in Y$. Its existence can be checked as follows. Let $V_0, V_1, \ldots, V_k$ be as above. Make the digraph D on $V_0 \cup V_1 \cup \ldots \cup V_k$ with an arc from $u \in V_{i-1}$ to $v \in V_i$ if $uv \notin E$ (for $i = 1, \ldots, k$). Then a directed $X - Y$ path in D gives a path P as required, and conversely.

Let M be the matching of edges in (69.10)(i). So M covers all vertices of H, except the vertices $(s_a, N(s_a) \setminus \{a\})$ and $(s_b, N(s_b) \setminus \{b\})$. Then (under the assumptions (69.1)):

Lemma 69.1γ. *G has an S-augmenting path $\iff$ H has an M-augmenting path. We can obtain one from the other in polynomial time.*

Proof. Let $P = (v_0, s_1, v_1, \ldots, s_k, v_k)$ be an S-augmenting path in G, with $v_0 = a$ and $v_k = b$. Let $s_{i_1}, \ldots, s_{i_t}$ be those vertices in P that belong to S' (in order). So $i_1 = 1$ and $i_t = k$. For $j = 1, \ldots, t$, let X_j and Y_j be the classes of s_{i_j} that contain $v_{i_j - 1}$ and v_{i_j}, respectively. Then $X_j \neq Y_j$, since v_{i_j-1} and v_{i_j} are nonsimilar and nonadjacent. Moreover, the subpath of P between any two s_{i_j} and $s_{i_{j+1}}$ forms an S''-augmenting $Y_j - X_{j+1}$ path. Hence

(69.11) $$((s_{i_1}, Y_1), (s_{i_2}, X_2), (s_{i_2}, Y_2), \ldots, (s_{i_{t-1}}, X_{t-1}), (s_{i_{t-1}}, Y_{t-1}), (s_{i_t}, X_t))$$

is an M-augmenting path in H.

We can reverse this construction. Indeed, any M-augmenting path Q yields an S-alternating $a - b$ walk P in G, by inserting appropriate S''-augmenting paths.

In fact, P is a path. For suppose that P traverses some vertex u of G more than once. Then u belongs to two of the inserted paths. Necessarily, they belong to the same bone B. Hence B has a neighbour in S' that is traversed more than once. But then Q traverses some matching edge more than once, a contradiction.

So P is a path. Moreover, any two vertices at distance two in P are nonadjacent, by construction of P. So P is S-augmenting, by Lemma 69.1α. ∎

Concluding, we have obtained the result of Minty [1980] and Sbihi [1980]:

Theorem 69.1. *A maximum-size stable set in a claw-free graph can be found in polynomial time.*

Proof. From Lemma 69.1γ, since finding an M-augmenting path in H is equivalent to finding a perfect matching in M. The latter problem is polynomial-time solvable by Corollary 24.4a. ∎

69.3. Maximum-weight stable set in a claw-free graph

There is an obvious way of extending the above construction to the weighted case, but there is a catch in it. The idea was noted by Minty [1980], and finalized by Nakamura and Tamura [2001].

Let $G = (V, E)$ be a graph and let $w : V \to \mathbb{R}_+$ be a weight function. Call a stable set S *extreme* if it has maximum weight among all stable sets of size $|S|$. It suffices to describe an algorithm to derive from any extreme stable set S, an extreme set of size $|S| + 1$, if any (since then we can start with $S := \emptyset$, enumerate extreme stable sets of all possible sizes, and choose one of maximum weight among them).

The following observations are basic:

Lemma 69.2α. *Let $G = (V, E)$ be a claw-free graph, let $w : V \to \mathbb{R}_+$, and let S be an extreme stable set. Then:*

(69.12) (i) *each S-alternating chordless circuit satisfies $w(VC \setminus S) \leq w(VC \cap S)$;*
(ii) *if P is an S-augmenting path maximizing $w(VP \setminus S) - w(VP \cap S)$, then $S \triangle VP$ is an extreme stable set of size $|S| + 1$.*

Proof. (i) follows from the fact that $S \triangle VC$ is a stable set of size $|S|$, and hence $w(S) \geq w(S \triangle VC) = w(S) + w(VC \setminus S) - w(VC \cap S)$.

(ii) can be seen as follows. Let $\widetilde{S}$ be an extreme stable set of size $|S|+1$. The subgraph induced by $S \triangle \widetilde{S}$ has a component K with $|K \cap \widetilde{S}| > |K \cap S|$. Since G is claw-free, K has maximum degree at most 2. So K is an S-augmenting path, and hence $|K \cap \widetilde{S}| = |K \cap S| + 1$. Let $L := (S \triangle \widetilde{S}) \setminus K$. Then $S \triangle L$ and $\widetilde{S} \triangle L$ are stable sets of size $|S|$ and $|S| + 1$ respectively. Since S is extreme, $w(L \cap \widetilde{S}) \leq w(L \cap S)$. Hence $w(\widetilde{S} \triangle L) \geq w(\widetilde{S})$. So $\widetilde{S} \triangle L$ is extreme again. Hence we can assume that $L = \emptyset$. Then, since K is an S-augmenting path:

(69.13)
$$
\begin{aligned}
w(S \triangle VP) &= w(S) + w(VP \setminus S) - w(VP \cap S) \\
&\geq w(S) + w(K \setminus S) - w(K \cap S) = w(\widetilde{S}).
\end{aligned}
$$

So $S \triangle VP$ is extreme. ∎

Statement (ii) of Lemma 69.2α implies that, to find an extreme stable set of size $|S| + 1$, it suffices to find an S-augmenting path P maximizing $w(VP \setminus S) - w(VP \cap S)$. By enumerating over all pairs $a, b \in V \setminus S$, it suffices to find, for each fixed $a, b \in V \setminus S$, an S-augmenting $a - b$ path P maximizing $w(VP \setminus S) - w(VP \cap S)$ (if any). Then we can make again the assumptions (69.1), and construct the graph H. Define a weight function ω on the edges of H (following the items in (69.10)) as follows:

(69.14) (i) $\omega(\{(s, X), (s, Y)\}) := w(s)$,
(ii) $\omega(\{(s, X), (t, Y)\}) :=$ the maximum of $w(VP \setminus S'') - w(VP \cap S'')$ over all S''-augmenting $X - Y$ paths P.

The maximum in (69.14)(ii) can be found in strongly polynomial time, since it amounts to finding a longest directed $X-Y$ path in the acyclic digraph D described just after (69.10).

Now a maximum-weight perfect matching in H need not yield a maximum-weight stable set in G, as was pointed out by Nakamura and Tamura [2001], since there might exist M-alternating circuits that increase the weight of M, while they do not correspond to a chordless S-alternating circuit. However, this can be avoided by preprocessing as follows.

We can assume that for each $v \in V \setminus S$ with $v \neq a, b$:

(69.15) (i) there exist s, t, x, y such that (x, s, v, t, y) is a chordless S-alternating path and such that $N(x) \cap N(y) \cap S = \emptyset$;
(ii) there exist $s, t \in S'$ and classes X of s and Y of t such that there exists an S''-augmenting $X-Y$ path and such that each S''-augmenting $X-Y$ path P attaining the maximum in (69.14)(ii), traverses v.

Otherwise v is on no maximum-weight S-augmenting path, and hence we can delete v. The conditions (69.15) can be tested in strongly polynomial time (for (ii) using digraph D). Hence the deletions take strongly polynomial time only.

Fix for each edge e of H in (69.14)(ii), a path P_e attaining the maximum. Then we can transform any M-alternating path or circuit to an S-alternating walk or closed walk, by replacing each such edge e by P_e. We call this the *corresponding* walk or closed walk in G.

Lemma 69.2β. *Under the assumptions* (69.15)*, each M-alternating circuit C in H satisfies $\omega(EC \setminus M) \leq \omega(EC \cap M)$.*

Proof. Suppose not. Choose C maximizing $\omega(EC \setminus M) - \omega(EC \cap M)$. Let Γ be the corresponding S-alternating closed walk in G. Then Γ is not a chordless circuit, since otherwise

(69.16) $$w(V\Gamma \setminus S) - w(V\Gamma \cap S) = \omega(EC \setminus M) - \omega(EC \cap M) > 0,$$

which contradicts (i) of Lemma 69.2α.

Since each P_e is simple and chordless, it follows that $EC \setminus M$ contains distinct edges e, f for which there exist $u \in VP_e$ and $v \in VP_f$ with $u = v$ or $uv \in E$. This implies that C has length 4, and that e and f are the only edges in $EC \setminus M$. So P_e and P_f are in the same bone B. Let s and t be the neighbours of B in S'. Let s have classes Y, Z and t have classes W, X such that P_e connects Y and W and P_f connects Z and X. Write

(69.17) $$P_e = (u_0, s_1, u_1, \ldots, s_k, u_k) \text{ and } P_f = (v_0, s_1, v_1, \ldots, s_k, v_k)$$

for some $k \geq 0$ and $s_1, \ldots, s_k \in S''$, where $u_0 \in Y$, $u_k \in W$, $v_0 \in Z$, $v_k \in X$. We define $s_0 := s$ and $s_{k+1} := t$. Now

(69.18) for each $i = 1, \ldots, k$: $u_{i-1}v_i \in E$ or $v_{i-1}u_i \in E$.

Otherwise, we can 'switch' P_e and P_f at s_i to obtain the S''-augmenting paths

(69.19) $Q := (u_0, s_1, \ldots, u_{i-1}, s_i, v_i, \ldots, s_k, v_k)$ and
$R := (v_0, s_1, \ldots, v_{i-1}, s_i, u_i, \ldots, s_k, u_k)$.

Hence H has edges $\{(s, Y), (t, X)\}$ and $\{(s, Z), (t, W)\}$, and

(69.20)
$$\omega(\{(s, Y), (t, X)\}) + \omega(\{(s, Z), (t, W)\}) \geq \omega(\{(s, Y), (t, W)\}) + \omega(\{(s, Z), (t, X)\}).$$

By the choice of C, we have equality, and hence the paths Q and R attain the corresponding maxima in (69.14)(ii). It implies, by assumption (69.15)(ii), that u_{i-1}, v_{i-1}, u_i, and v_i are the only neighbours of s_i. Since none of u_{i-1}, v_{i-1} are adjacent to any of u_i, v_i, we have that s_i is splittable, that is, $s_i \in S'$, a contradiction. This proves (69.18).

Next

(69.21) $u_0v_0 \notin E$ and $u_kv_k \notin E$.

For suppose that (say) $u_0v_0 \in E$. By (69.15)(i), there exist $x, y \in V \setminus S$ such that (x, s, u_0, s_1, y) is a chordless path and such that $N(x) \cap N(y) \cap S = \emptyset$. As x is nonadjacent to u_0, and as $u_0 \in X$, we have $x \in Y$, and so (as $v_0 \in Y$) $xv_0 \in E$.

If $k = 0$, we have similarly $yv_0 \in E$. Then v_0 is adjacent to the pairwise nonadjacent x, u_0, y, a contradiction.

So $k \geq 1$. Then $y \sim u_1$ and $N(y) \cap S = \{s_1, s_2\}$. So $xs_1, xs_2 \notin E$ (since $N(x) \cap N(y) \cap S = \emptyset$). This implies $xu_1 \notin E$, since otherwise u_1 is adjacent to the pairwise nonadjacent s_1, s_2, x. Hence $v_0u_1 \notin E$, since otherwise v_0 is adjacent to the pairwise nonadjacent x, u_0, u_1. By symmetry, also $u_0v_1 \notin E$. This contradicts (69.18), and hence proves (69.21).

Moreover,

(69.22) there is an i with $0 \leq i \leq k$ and $u_iv_i \in E$,

as otherwise each circuit $(s_i, u_i, s_{i+1}, v_i, s_i)$ is S-alternating and chordless, which implies $w(u_i) + w(v_i) - w(s_i) - w(s_{i+1}) \leq 0$ by Lemma 69.2α. This gives the contradiction

(69.23)
$$\begin{aligned} 0 &< \omega(EC \setminus M) - \omega(EC \cap M) \\ &= w(VP_e \setminus S'') - w(VP_e \cap S'') + w(VP_f \setminus S'') - w(VP_f \cap S'') \\ &\quad - w(s) - w(t) = \sum_{i=0}^{k}(w(u_i) + w(v_i) - w(s_i) - w(s_{i+1})) \leq 0, \end{aligned}$$

proving (69.22).

Now let i be the smallest index with $u_iv_i \in E$. By (69.21), we know $1 \leq i \leq k-1$. By (69.18) and by symmetry we can assume that $v_iu_{i+1} \in E$. Since s_i is adjacent to u_{i-1}, v_{i-1}, and v_i, and since $u_{i-1}v_{i-1} \notin E$ and $v_{i-1}v_i \notin E$, we know $u_{i-1}v_i \in E$. Then v_i is adjacent to the pairwise nonadjacent u_{i-1}, u_i, and u_{i+1}, a contradiction. ∎

Now find a maximum-weight perfect matching N in H, with the maximum-weight perfect matching algorithm (Chapter 26). By Lemma 69.2β, we can assume that $N = M \triangle EQ$ for some M-augmenting path Q in H (since if $N \triangle M$ contains a circuit C, then $N \triangle EC$ again is a maximum-weight perfect matching in H). Then Q maximizes $\omega(EQ \setminus M) - \omega(EQ \cap M)$ over all M-augmenting paths. Let P be the corresponding path in G. Then P is an S-augmenting path in G maximizing $w(VP \setminus S) - w(VP \cap S)$, as required.

We conclude:

Theorem 69.2. *A maximum-weight stable set in a claw-free graph can be found in strongly polynomial time.*

Proof. See above. ∎

69.4. Further results and notes

69.4a. On the stable set polytope of a claw-free graph

The polynomial-time solvability of the maximum-weight stable set problem for claw-free graphs implies that the optimization problem over the stable set polytope $P_{\text{stable set}}(G)$ of a claw-free graph $G = (V, E)$ is polynomial-time solvable. Hence also the separation problem is polynomial-time solvable (with the ellipsoid method (Theorem 5.10)). It implies (cf. Theorem 5.11) that, given a vector $x \in \mathbb{Q}^V$, one can decide in strongly polynomial time if x belongs to $P_{\text{stable set}}(G)$, and if not, find a facet-inducing inequality violated by x.

So in this respect, the stable set polytope of a claw-free graph is under control. However, no explicit description is known of a system that determines $P_{\text{stable set}}(G)$. As we saw in Section 25.2, such a description is known for the special case where G is the line graph of some graph H — that is, for the matching polytope of H. In this special case, each facet can be described by an inequality with coefficients in $\{0, 1\}$.

The latter fact does not generalize to claw-free graphs. Giles and Trotter [1981] showed that for each $k \in \mathbb{Z}_+$ there exists a claw-free graph such that its stable set polytope has a facet that is described by a linear inequality with coefficients k and $k + 1$. (This refutes a conjecture of Sbihi [1978].)

Galluccio and Sassano [1997] characterized those facets of the stable set polytope of a claw-free graph that can be described by an inequality with all coefficients in $\{0, 1\}$ (the *rank facets*).

More on facets of the stable set polytope of special classes of claw-free graphs can be found in Ben Rebea [1981] and Oriolo [2002] (for graphs such that for each vertex v, the graph induced by $N(v)$ is the complement of a bipartite graph) and Pulleyblank and Shepherd [1993] (for claw-free graphs such that no vertex has three pairwise nonadjacent vertices at distance two).

69.4b. Further notes

Minty [1980] observed that finding a maximum-size stable set in a graph without induced $K_{1,4}$ is NP-complete. This follows from the fact that the 3-dimensional assignment problem can be reduced to it (its intersection graph has no induced $K_{1,4}$).

Poljak [1974] showed that finding a maximum-size stable set in a triangle-free graph is NP-complete. It implies that finding a maximum-size clique in a claw-free graph is NP-complete.

Shepherd [1995] characterized the stable set polytope of *near-bipartite graphs*, that is, graphs with $G - N(v)$ bipartite for each $v \in VG$. They include the complements of line graphs, and the complement of any near-bipartite graph is claw-free.

Ben Rebea [1981] showed that each connected claw-free graph G with $\alpha(G) \geq 3$ not containing an induced C_5, contains no odd antihole. This was extended by Fouquet [1993] who showed that each connected claw-free graph G with $\alpha(G) \geq 4$ contains no odd antihole with at least 7 vertices.

Lovász and Plummer [1986] gave a variant of Minty's reduction of the maximum-size stable set problem in claw-free graphs to the maximum-size matching problem.

Beineke [1970] (for simple graphs), N. Robertson (unpublished), Hemminger [1971] (abstract only), and Bermond and Meyer [1973] characterized line graphs by means of forbidden induced subgraphs (six graphs next to $K_{1,3}$).

The polynomial-time solvability of the weighted stable set problem for claw-free graphs was extended to claw-free bidirected graphs by Nakamura and Tamura [1998]. A linear-time algorithm for 'triangulated' bidirected graphs was given by Nakamura and Tamura [2000].