Part V

Trees, Branchings, and Connectors

Part V: Trees, Branchings, and Connectors

This part focuses on structures that are defined by connecting several pairs of vertices simultaneously, with most basic structure that of a *spanning tree*. A spanning tree can be characterized as a minimal set of edges that connects each pair of vertices by at least one path — that is, a minimal *connector*. Alternatively, it can be characterized as a maximal set of edge that connects each pair of vertices by at most one path — that is, a maximal *forest*.

Finding a shortest spanning tree belongs to classical combinatorial optimization, with lots of applications in planning road, energy, and communication networks, in chip design, and in clustering data in areas like biology, taxonomy, archeology, and, more generally, in any large data base. Spanning trees are well under control polyhedrally and algorithmically, both as to *shortest* and as to *disjoint* spanning trees. They form a prime area of application of matroid theory.

There are several variations and generalizations of the notion of spanning tree that are also well under control, like arborescences, branchings, biconnectors, bibranchings, directed cut covers, and matching forests.

An illustrious variant that is worse under control is the Hamiltonian circuit — in other words, the traveling salesman tour — which (in the directed case) can be considered as a smallest strongly connected subgraph. The traveling salesman problem is NP-complete and no complete polyhedral characterization is known. It implies that more general optimization problems like finding a shortest strong connector or a cheapest connectivity augmentation also are NP-complete. In this part we will however come across some special cases that *are* well-solvable and well-characterized.

In this part we also discuss the powerful framework designed by Edmonds and Giles, based on defining the concept of a *submodular flow* in a directed graph with a submodular function on its vertex set. It unifies several of the results and techniques of the present part and of the previous part on matroids and submodular functions.

Chapters:

Chapter 50

Shortest spanning trees

In this chapter we consider shortest spanning trees in undirected graphs. We show that the greedy algorithm finds a shortest spanning tree in a graph, and moreover yields min-max relations and polyhedral characterizations. These are special cases of results on matroids discussed in Chapter 40, but deserve special consideration since the graph framework allows a number of additional viewpoints and opportunities.

We recall some terminology and elementary facts. A graph $G = (V, E)$ is called a *tree* if G is connected and contains no circuit. For any graph $G = (V, E)$, a subset F of E is called:

- a *spanning tree* if (V, F) is a tree,
- a *forest* if F contains no circuit,
- a *maximal forest* if F is an inclusionwise maximal forest,
- a *connector* if (V, F) is connected.

A graph G has a spanning tree if and only if G is connected. For any connected graph $G = (V, E)$, each of the following characterizes a subset F of E as a spanning tree:

- F is a maximal forest;
- F is an inclusionwise minimal connector;
- F is a forest with $|F| = |V| - 1$;
- F is a connector with $|F| = |V| - 1$.

In any graph $G = (V, E)$, a maximal forest has $|V| - k$ edges, where k is the number of components of G; it forms a spanning tree in each of the components of G. So each inclusionwise maximal forest is a maximum-size forest; that is, each forest is contained in a maximum-size forest. Similarly, each connector contains a minimum-size connector.

50.1. Shortest spanning trees

Let $G = (V, E)$ be a connected graph and let $l : E \to \mathbb{R}$ be a function, called the *length* function. For any subset F of E, the *length* $l(F)$ of F is, by definition:

(50.1) $$l(F) := \sum_{e \in F} l(e).$$

In this section we consider the problem of finding a shortest spanning tree in G — that is, one of minimum length.

While this is a special case of finding a minimum-weight base in a matroid, and hence can be solved with the greedy algorithm (Section 40.1), spanning trees allow some variation on the method, essentially because we can exploit the presence of the vertex set (graphic matroids are defined on the edge set only).

Also these variants of the greedy method will be called greedy. Such methods go back to Borůvka [1926a]. The correctness of each of the variants follows from the following basic phenomenon.

Call a forest F *good* if there exists a shortest spanning tree T of G that contains F. (So we are out for a good spanning tree.) Then:

Theorem 50.1. *Let F be a good forest and let e be an edge not in F. Then $F \cup \{e\}$ is a good forest if and only if*

(50.2) *there exists a cut C disjoint from F such that e is shortest among the edges in C.*

Proof. To see necessity, let T be a shortest spanning tree containing $F \cup \{e\}$. Let C be the unique cut disjoint from $T \setminus \{e\}$. Then e is shortest in C, since if $f \in C$, then $T' := (T \setminus \{e\}) \cup \{f\}$ is again a spanning tree. As $l(T') \geq l(T)$ we have $l(f) \geq l(e)$.

To see sufficiency, let T be a shortest spanning tree containing F. Let P be the path in T between the ends of e. Then P contains at least one edge f that belongs to C. Then $T' := (T \setminus \{f\}) \cup \{e\}$ is a spanning tree again. By assumption, $l(e) \leq l(f)$ and hence $l(T') \leq l(T)$. Hence T' is a shortest spanning tree again. As $F \cup \{e\}$ is contained in T', it is a good forest. ∎

(The idea of this proof is in Jarník [1930].)

This theorem offers us a framework for an algorithm: starting with $F := \emptyset$, iteratively extend F by an edge e satisfying (50.2). We end up with a shortest spanning tree.

Rule (50.2) was formulated by Tarjan [1983], and is the most liberal rule in obtaining greedily a shortest spanning tree. The variants of the greedy method are obtained by specifying how to choose edge e.

The first variant, the *tree-growing method*, was given by Jarník [1930] (and by Kruskal [1956], Prim [1957], Dijkstra [1959]). It is also called the *Jarník-Prim method* or *Prim's method* (Prim was the first giving an $O(n^2)$ implementation):

(50.3) Fix a vertex r. Set $F := \emptyset$. As long as F is not a spanning tree, let K be the component of F containing r, let e be a shortest edge leaving K, and reset $F := F \cup \{e\}$.

Corollary 50.1a. *Prim's method yields a shortest spanning tree.*

Proof. Directly from Theorem 50.1, by taking $C := \delta(K)$. ∎

A second variant, the *forest-merging method* or *Kruskal's method*, is due to Kruskal [1956] (and to Loberman and Weinberger [1957] and Prim [1957]):

(50.4) Set $F := \emptyset$. As long as F is not a spanning tree, choose a shortest edge e for which $F \cup \{e\}$ is a forest, and reset $F := F \cup \{e\}$.

(So this version is the true specialization of the greedy algorithm for matroids to graphs.)

Corollary 50.1b. *Kruskal's method yields a shortest spanning tree.*

Proof. Again directly from Theorem 50.1, as e is shortest in the cut $\delta(K)$ for each of the two components K of F incident with e. ∎

Prim [1957] and Loberman and Weinberger [1957] observed that the optimality of the greedy method implies that each length function which gives the same order of the edges (like the logarithm or square of the lengths), has the same collection of shortest spanning trees. Similarly, the shortest spanning tree minimizes the *product* of the lengths (if nonnegative).

In a similar way one finds a *longest* spanning tree. The maximum length of a forest and the minimum length of a connector can also be found with the greedy method.

Note that the greedy method is flexible: We can change our rule of choosing the new edge e at any time throughout the algorithm, as long as at any choice of e, (50.2) is satisfied.

As Prim [1957] and Dijkstra [1959] remark, the value of any variant of the greedy method depends on its implementation. One should have efficient ways to store and update information on the components of F, and on finding an edge satisfying (50.2). We now consider such implementations for Prim's and for Kruskal's method.

50.2. Implementing Prim's method

Prim [1957] and Dijkstra [1959] described implementations of Prim's method that run in time $O(n^2)$. (Here we assume without loss of generality that the graph is simple.)

To this end, we indicate at any vertex v, whether or not v belongs to the component K containing r of the current forest F, and in case $v \notin K$, we store at v a shortest edge e_v connecting v with K (void if there is no such edge). Then at each iteration, we scan all vertices, and select one, v say, for which $v \notin K$ and e_v is shortest. We add e_v to F, and v to K, and for each edge vu incident with v, we replace e_u by vu if $u \notin K$ and vu is shorter than e_u (or if e_u is void).

As each iteration takes $O(n)$ time and as there are $n - 1$ iterations we have the result stated by Dijkstra [1959]:

Theorem 50.2. *A shortest spanning tree can be found in time $O(n^2)$.*

Proof. See above. ∎

In fact, by applying 2-heaps (Section 7.3) one can obtain a running time bound of $O(m \log n)$ (E.L. Johnson, cf. Kershenbaum and Van Slyke [1972]), and with Fibonacci heaps (Section 7.4) one obtains (Fredman and Tarjan [1984,1987]):

Theorem 50.3. *A shortest spanning tree can be found in time $O(m + n \log n)$.*

Proof. Directly by applying Fibonacci heaps as described in Section 7.4. ∎

50.3. Implementing Kruskal's method

Bottleneck in implementing Kruskal's method is the necessity to scan the edges sorted by length. As the best bound for sorting is $O(m \log n)$, we cannot hope for implementations of Kruskal's method faster than that.

However, the bound $O(m \log n)$ is easy to achieve. In fact, as was noticed by Kershenbaum and Van Slyke [1972] (using ideas of Van Slyke and Frank [1972]), it is easy to implement Kruskal's method such that the time *after sorting* is $O(m + n \log n)$. This can be done with elementary data-structures like lists; no heaps are needed.

Indeed, it is not hard to design a simple data structure that tests in constant time if the ends of any edge belong to different components of the current forest F, and that merges components in time linear in the size of the smaller component[1].

Then the iterations take $O(m + n \log n)$ time, since checking if the ends of an edge belong to different components takes $O(m)$ time overall, while merging takes $O(n \log n)$ time overall: any vertex v belongs at most $\log_2 n$ times to the smaller component when merging, as, at any such event, the component containing v at least doubles in size.

[1] Consider any forest F. Represent each component K by a (singly) linked list. For any vertex v, let $r(v)$ be the first vertex of the list L_v containing v.

Initially, for each vertex v, $r(v) = v$, as $L_v = \{v\}$. At any iteration, the edge $e = uv$ considered connects different components of F if and only if $r(u) \neq r(v)$. Checking this takes constant time.

If $r(u) \neq r(v)$, we can determine which of the lists L_u, L_v is smallest in time $O(\min\{|L_u|, |L_v|\})$ (by scanning them in parallel, starting at $r(u)$ and $r(v)$). Assume without loss of generality that $|L_u| \leq |L_v|$. Then we reset $r(u') := r(v)$ for all u' in L_u, and we insert L_u into L_v directly after v. This can be done in time $O(|L_u|)$.

Tarjan [1983] showed that if the edges are presorted, a minimum spanning tree can be found in time $O(m\alpha(m,n))$ (where $\alpha(m,n)$ is the 'inverse Ackermann function — see Section 50.6a).

50.3a. Parallel forest-merging

A variant that suggests parallel implementation was given by Borůvka [1926a,1926b] — the *parallel forest-merging method* or *Borůvka's method*. (This method was also given by Choquet [1938] (without proof) and Florek, Łukaszewicz, Perkal, Steinhaus, and Zubrzycki [1951a].) It assumes that all edge lengths are different:

(50.5) Set $F := \emptyset$. As long as F is not a spanning tree do the following: choose for each component K of F the shortest edge leaving K, and add all chosen edges to F.

Theorem 50.4. *Assuming that all edge lengths are different, the parallel forest-merging variant yields a shortest spanning tree.*

Proof. We show that F remains a good forest throughout the iterations. Consider some iteration, and let F be the good forest at the start of the iteration. Let $e_1,\ldots,e_k$ be the edges added in the iteration, indexed such that $l(e_1) < l(e_2) < \cdots < l(e_k)$. By the selection rule (50.5), for each $i = 1,\ldots,k$, e_i is the shortest edge leaving some component K of F. Then K is also a component of $F \cup \{e_1,\ldots,e_{i-1}\}$, as none of $e_1,\ldots,e_{i-1}$ leave K (since e_i is shortest leaving K). Hence for each $i = 1,\ldots,k$, $F \cup \{e_1,\ldots,e_i\}$ is a good forest (by induction on i). Concluding, the iteration yields a good forest. ∎

50.3b. A dual greedy algorithm

We can consider a dual approach by iteratively decreasing a connector, instead of iteratively growing a forest. The analogy can be exhibited as follows.

Let $G = (V,E)$ be a connected graph and let $l : E \to\in \mathbb{R}$ be a length function. Call a connector $K \subseteq E$ *good* if K contains a shortest spanning tree. Then we have:

Theorem 50.5. *Let K be a good connector and let $e \in K$. Then $K \setminus \{e\}$ is a good connector if and only if*

(50.6) *K contains a circuit C such that e is a longest edge in C.*

Proof. To see necessity, let T be a shortest spanning tree contained in $K \setminus \{e\}$. Let C be the unique circuit contained in $T \cup \{e\}$. Then e is longest in C, since if $f \in C$, then $T' := (T \setminus \{f\}) \cup \{e\}$ is again a spanning tree. As $l(T') \geq l(T)$ we have $l(e) \geq l(f)$.

To see sufficiency, let T be a shortest spanning tree contained in K. If $e \notin T$, then also $K \setminus \{e\}$ contains T, and hence $K \setminus \{e\}$ is a good connector. So we can assume that $e \in T$. Let D be the cut determined by $T - e$. Then the circuit C contains at least one edge $f \neq e$ that belongs to D. So $T' := (T \setminus \{e\}) \cup \{f\}$ is a spanning tree again. By assumption, $l(e) \geq l(f)$ and hence $l(T') \leq l(T)$. Hence T'

is a shortest spanning tree again. It is contained in $K \setminus \{e\}$, which therefore is a good connector. ∎

So we can formulate the *dual greedy algorithm*: starting with $K := E$, iteratively remove from K an edge e satisfying (50.6). We end up with a shortest spanning tree.

A special case is the following algorithm, proposed by Kruskal [1956]: iteratively delete a longest edge e that is not a bridge. We end up with a shortest spanning tree.

50.4. The longest forest and the forest polytope

The greedy algorithm can be easily adapted so as to give:

Theorem 50.6. *A longest forest can be found in strongly polynomial time.*

Proof. It suffices to find a longest spanning tree in any component. This can be done with the greedy method. ∎

As Edmonds [1971] noticed, it is easy to derive with the greedy method a min-max relation for the maximum length of a forest in a graph $G = (V, E)$. This is similar to the results of Section 40.2.

Theorem 50.7. *Let $G = (V, E)$ be a graph and let $l \in \mathbb{Z}_+^E$. Then the maximum length of a forest is equal to the minimum value of*

$$(50.7)\qquad \sum_{U \in \mathcal{P}(V) \setminus \{\emptyset\}} y_U(|U| - 1),$$

where $y \in \mathbb{Z}_+^{\mathcal{P}(V)\setminus\{\emptyset\}}$ satisfies

$$(50.8)\qquad \sum_{U \in \mathcal{P}(V) \setminus \{\emptyset\}} y_U \chi^{E[U]} \geq l.$$

Proof. The maximum cannot be larger than the minimum, since for any forest F and any $y \in \mathbb{Z}_+^{\mathcal{P}(V)\setminus\{\emptyset\}}$ satisfying (50.8) one has:

$$(50.9)\qquad l(F) \leq \sum_{U \in \mathcal{P}(V) \setminus \{\emptyset\}} y_U |E[U] \cap F| \leq \sum_{U \in \mathcal{P}(V) \setminus \{\emptyset\}} y_U(|U| - 1).$$

To see equality, let $k := \max\{l(e) \mid e \in E\}$, and let E_i be the set of edges e with $l(e) \geq i$, for $i = 0, 1, \ldots, k$. For each $U \in \mathcal{P}(V) \setminus \{\emptyset\}$, let y_U be the number of $i \in \{1, \ldots, k\}$ such that U is a component of the graph (V, E_i). Then it is easy to see that y satisfies (50.8).

We can find a sequence of forests $F_k \subseteq \cdots \subseteq F_1 \subseteq F_0$, where for $i = 0, 1, \ldots, k$, F_i is a maximal forest in (V, E_i) containing F_{i+1}, setting $F_{k+1} := \emptyset$.

Then for $F := F_0$ we have:

$$(50.10) \qquad l(F) = \sum_{i=0}^{k} i|F_i \setminus F_{i+1}| = \sum_{i=1}^{k} |F_i| = \sum_{i=1}^{k} (|V| - \kappa(V, E_i)) \\ = \sum_{U \in \mathcal{P}(V) \setminus \{\emptyset\}} y_U(|U| - 1),$$

where $\kappa(V, E_i)$ denotes the number of components of the graph (V, E_i). ∎

(The series of forests $F_k \subseteq F_{k-1} \subseteq \cdots \subseteq F_1 \subseteq F_0$, corresponds to the greedy method.)

Note that this theorem gives, if G is connected, a min-max relation for the maximum length of a spanning tree.

For any graph $G = (V, E)$, let the *forest polytope* of G, denoted by $P_{\text{forest}}(G)$, be the convex hull of the incidence vectors (in $\mathbb{R}^E$) of the forests of G. The following characterization of the forest polytope is (in matroid terms) due to Edmonds [1971] (announced in Edmonds [1967a]):

Corollary 50.7a. *The forest polytope of a graph G is determined by*

$$(50.11) \qquad \begin{array}{lll} \text{(i)} & x_e \geq 0 & \text{for } e \in E, \\ \text{(ii)} & x(E[U]) \leq |U| - 1 & \text{for nonempty } U \subseteq V. \end{array}$$

Proof. Trivially, the incidence vector of any forest satisfies (50.11), and hence the forest polytope is contained in the polytope determined by (50.11). Suppose now that the latter polytope is larger. Then (since both polytopes are rational and down-monotone in $\mathbb{R}^E_+$) there exists a vector $l \in \mathbb{Q}^E_+$ such that the maximum value of $l^{\mathsf{T}}x$ over (50.11) is larger than the maximum of $l(F)$ over forests F. We can assume that l is integer. However, by Theorem 50.7, the maximum of $l(F)$ is at least the minimum value of the problem dual to maximizing $l^{\mathsf{T}}x$ over (50.11), a contradiction. ∎

Theorem 50.7 can be stated equivalently in TDI terms as follows:

Corollary 50.7b. *System* (50.11) *is totally dual integral.*

Proof. This follows from Theorem 50.7, by the definition of total dual integrality. ∎

Having a description of the forest polytope, we can derive a description of the *spanning tree polytope* $P_{\text{spanning tree}}(G)$ of a graph $G = (V, E)$, which is the convex hull of the incidence vectors of the spanning trees in G.

Corollary 50.7c. *The spanning tree polytope of a graph $G = (V, E)$ is determined by*

$$\text{(50.12)}\qquad \begin{array}{rll} \text{(i)} & x_e \geq 0 & \text{for } e \in E, \\ \text{(ii)} & x(E[U]) \leq |U| - 1 & \text{for nonempty } U \subseteq V, \\ \text{(iii)} & x(E) = |V| - 1. & \end{array}$$

Proof. Directly from Corollary 50.7a, since the spanning trees are exactly the forests of size $|V| - 1$, and since there exist no forests larger than that. ∎

One also directly has a TDI result:

Corollary 50.7d. *System* (50.12) *is totally dual integral.*

Proof. Directly from Corollary 50.7b, since (50.12) arises from (50.11) by setting an inequality to equality (cf. Theorem 5.25). ∎

Theorem 40.5 implies that (if G is loopless) an inequality (50.12)(ii) is facet-inducing if and only if $|U| \geq 2$ and U induces a 2-connected subgraph of G (cf. Grötschel [1977a]).

In Section 51.4 we consider the problem of testing membership of the forest polytope.

50.5. The shortest connector and the connector polytope

The greedy method also provides a min-max relation for the minimum length of a connector in a graph $G = (V, E)$. Let Π denote the collection of partitions of V into nonempty subsets. For any partition $\mathcal{P}$ of V, let $\delta(\mathcal{P})$ denote the set of edges connecting two different classes of $\mathcal{P}$. So any connector contains at least $|\mathcal{P}| - 1$ edges in $\delta(\mathcal{P})$.

Theorem 50.8. *Let $G = (V, E)$ be a connected graph and let $l \in \mathbb{Z}_+^E$. Then the minimum length of a spanning tree is equal to the maximum value of*

$$\text{(50.13)}\qquad \sum_{\mathcal{P} \in \Pi} y_{\mathcal{P}}(|\mathcal{P}| - 1),$$

where $y \in \mathbb{Z}_+^\Pi$ such that

$$\text{(50.14)}\qquad \sum_{\mathcal{P} \in \Pi} y_{\mathcal{P}} \chi^{\delta(\mathcal{P})} \leq l.$$

Proof. The minimum cannot be smaller than the maximum, since for any spanning tree T and any $y \in \mathbb{Z}_+^\Pi$ satisfying (50.14) one has:

$$\text{(50.15)}\qquad l(T) \geq \sum_{\mathcal{P} \in \Pi} y_{\mathcal{P}} \chi^{\delta(\mathcal{P})}(T) = \sum_{\mathcal{P} \in \Pi} y_{\mathcal{P}} |\delta(\mathcal{P}) \cap T| \geq \sum_{\mathcal{P} \in \Pi} y_{\mathcal{P}}(|\mathcal{P}| - 1).$$

To see equality, define $k := \max\{l(e) \mid e \in E\}$ and for $i = 0, 1, \ldots, k$, let E_i be the set of edges e with $l(e) \leq i$. For each $\mathcal{P} \in \Pi$, let $y_{\mathcal{P}}$ be the number of

$i \in \{1, \ldots, k\}$ such that $\mathcal{P}$ is the collection of components of (V, E_i). Then it is easy to see that y satisfies (50.14).

We can find a sequence of forests $F_0 \subseteq F_1 \subseteq \cdots F_{k-1} \subseteq F_k$, where F_0 is a maximal forest in (V, E_0), and where for $i = 0, \ldots, k$, F_i is a maximal forest in (V, E_i) containing F_{i-1}, setting $F_{-1} := \emptyset$.

Then for $T := F_k$ we have:

$$(50.16) \qquad l(T) = \sum_{i=0}^{k} i|F_i \setminus F_{i-1}| = k|T| - \sum_{i=0}^{k-1} |F_i| = \sum_{i=0}^{k-1}(|V| - 1 - |F_i|)$$
$$= \sum_{i=1}^{k-1}(\kappa(V, E_i) - 1) = \sum_{\mathcal{P} \in \Pi} y_{\mathcal{P}}(|\mathcal{P}| - 1),$$

where $\kappa(V, E_i)$ denotes the number of components of the graph (V, E_i). ∎

For any graph $G = (V, E)$, let the *connector polytope* of G, denoted by $P_{\text{connector}}(G)$, be the convex hull of the incidence vectors (in $\mathbb{R}^E$) of the connectors of G. The following characterization can be derived from Edmonds [1970b], and was stated explicitly by Fulkerson [1970b]:

Corollary 50.8a. *The connector polytope of a graph G is determined by*

$$(50.17) \qquad \begin{array}{lll} \text{(i)} & 0 \leq x_e \leq 1 & \text{for } e \in E, \\ \text{(ii)} & x(\delta(\mathcal{P})) \geq |\mathcal{P}| - 1 & \text{for } \mathcal{P} \in \Pi. \end{array}$$

Proof. Trivially, the incidence vector of any connector satisfies (50.17), and hence the connector polytope is contained in the polytope determined by (50.17). Suppose now that the latter polytope is larger. Then (since both polytopes are rational and up-monotone in $[0, 1]^E$) there exists a vector $l \in \mathbb{Q}_+^E$ such that the minimum value of $l^\mathsf{T} x$ over (50.17) is smaller than the minimum of $l(C)$ over connectors C. We can assume that l is integer. However, by Theorem 50.8, the minimum of $l(C)$ is at most the maximum value of the problem dual to minimizing $l^\mathsf{T} x$ over (50.17), a contradiction. ∎

Theorem 50.8 can be stated equivalently in TDI terms as follows:

Corollary 50.8b. *System* (50.17) *is totally dual integral.*

Proof. This follows from Theorem 50.8, by the definition of total dual integrality. ∎

Chopra [1989] described the facets of the connector polytope. In Section 51.4 we consider the problem of testing membership of the connector polytope.

50.6. Further results and notes

50.6a. Complexity survey for shortest spanning tree

	$O(nm)$	Jarník [1930]
	$O(n^2)$	Prim [1957], Dijkstra [1959]
	$O(m \log n)$	Kershenbaum and Van Slyke [1972], E.L. Johnson (cf. Kershenbaum and Van Slyke [1972])
	$O(m \log_{m/n} n)$	D.B. Johnson [1975b]
	$O(m\sqrt{\log n})$	R.E. Tarjan (cf. Yao [1975])
	$O(m \log\log n)$	Yao [1975]
	$O(m \log\log_{m/n} n)$	Cheriton and Tarjan [1976], Tarjan [1983]
$*$	$O((m + n\log L)\log\log L)$	D.B. Johnson [1977b]
	$O(m + n\log n)$	Fredman and Tarjan [1984,1987]
	$O(m\beta(m,n))$	Fredman and Tarjan [1984,1987]
	$O(m \log\beta(m,n))$	Gabow, Galil, Spencer, and Tarjan [1986] (cf. Gabow, Galil, and Spencer [1984])
	$O(m(\log_n L + \alpha(m,n)))$	Gabow [1983b,1985b]
	$O(m\alpha(m,n)\log\alpha(m,n))$	Chazelle [1997]
$*$	$O(m\alpha(m,n))$	Chazelle [2000]

As before, $*$ indicates an asymptotically best bound in the table. Moreover, $\beta(m,n) := \min\{i \mid \log_2^{(i)} n \leq m/n\}$ and $L := \max\{l(e) \mid e \in E\}$ (assuming l nonnegative integer). The function $\alpha(m,n)$ is the *inverse Ackermann function*, defined as follows. For $i,j \geq 1$, the *Ackermann function* $A(i,j)$ is defined recursively by:

$$
\begin{aligned}
&A(1,j) = 2^j && \text{for } j \geq 1,\\
&A(i,1) = A(i-1,2) && \text{for } i \geq 2,\\
&A(i,j) = A(i-1,A(i,j-1)) && \text{for } i,j \geq 2.
\end{aligned}
\tag{50.18}
$$

Next, for $m \geq n \geq 1$,

$$\alpha(m,n) := \min\{i \geq 1 \mid A(i,\lfloor m/n \rfloor) > \log_2 n\}. \tag{50.19}$$

The function $\alpha(m,n)$ is extremely slowly growing.

Fredman and Willard [1990,1994] gave a ‘strongly trans-dichotomous’ linear-time minimum spanning tree algorithm (where capabilities of random access machines, like addressing, can be used). Based on sampling, Karger [1993,1998] found a simple linear-time approximative spanning tree algorithm, and an $O(m+n\log n)$-time minimum spanning tree algorithm not using Fibonacci heaps.

Katoh, Ibaraki, and Mine [1981] gave an algorithm to find the Kth shortest spanning tree in time $O(Km + \min\{n^2, m\log\log n\})$ (improving slightly Gabow

[1977]). They also gave an algorithm to find the second shortest spanning tree in time $O(\min\{n^2, m\alpha(m,n)\})$.

Pettie and Ramachandran [2000,2002a] showed that a shortest spanning tree can be found in time $O(\mathcal{T}^*(m,n))$, where $\mathcal{T}^*(m,n)$ is the minimum number of edge length comparisons needed to determine the solution.

Frederickson [1983a,1985] gave an $O(\sqrt{m})$-time algorithm to update a shortest spanning tree (and the data-structure) if one edge changes length. Spira and Pan [1973,1975] and Chin and Houck [1978] gave fast algorithms to update a shortest spanning tree if vertices are added or removed. More on sensitivity and most vital edges can be found in Tarjan [1982], Hsu, Jan, Lee, Hung, and Chern [1991], Dixon, Rauch, and Tarjan [1992], Iwano and Katoh [1993], Lin and Chern [1993], and Frederickson and Solis-Oba [1996,1999].

Tarjan [1979] showed that the minimality of a given spanning tree can be checked in time $O(m\alpha(m,n))$ (cf. Dixon, Rauch, and Tarjan [1992]). Komlós [1984, 1985] showed that the minimality of a given spanning tree can be checked by $O(m)$ comparisons of edge lengths. King [1997] gave a linear-time implementation in the unit-cost RAM model. A randomized linear-time algorithm was given by Klein and Tarjan [1994], and Karger, Klein, and Tarjan [1995].

Gabow and Tarjan [1984] (cf. Gabow and Tarjan [1979]) showed that the problem of finding a shortest spanning tree with a prescribed number of edges incident with a (one) given vertex r, is linear-time equivalent to the (unconstrained) shortest spanning tree problem. They also showed that if the edges of a graph are coloured red and blue, a shortest spanning tree having exactly k red edges (for given k) can be found in time $O(m \log\log_{2+\frac{m}{n}} n + n \log n)$.

Brezovec, Cornuéjols, and Glover [1988] gave an efficient algorithm to find a shortest spanning tree in a coloured graph with, for each colour, an upper and a lower bound on the number of edges in the tree of that colour.

Camerini [1978] showed that a spanning tree minimizing $\max_{e\in T} l(e)$ can be found in $O(m)$ time.

Geometric spanning trees (on vertices in Euclidean space, with Euclidean distance as length function) were considered by Bentley, Weide, and Yao [1980], Yao [1982], Supowit [1983], Clarkson [1984,1989], and Agarwal, Edelsbrunner, Schwarzkopf, and Welzl [1991].

50.6b. Characterization of shortest spanning trees

The following theorem is implicit in Kalaba [1960]:

Theorem 50.9. *Let $G = (V,E)$ be a graph, let $l \in \mathbb{R}^E$ be a length function, and let T be a spanning tree in G. Then T is a shortest spanning tree if and only if $l(f) \geq l(e)$ for all $e \in T$ and $f \in E \setminus T$ with $T - e + f$ a spanning tree.*

Proof. Necessity being trivial, we show sufficiency. Let the condition be satisfied, and suppose that T is not a shortest spanning tree. Choose a shorter spanning tree T' with $|T' \setminus T|$ minimal. Let $f \in T' \setminus T$. Let e be an edge on the circuit in $T \cup \{f\}$ with $e \neq f$, such that e connects the two components of $T' \setminus \{f\}$. Then $(T\setminus\{e\})\cup\{f\}$ is a spanning tree, and hence $l(f) \geq l(e)$. Define $T'' := (T'\setminus\{f\})\cup\{e\}$. Then $l(T'') \leq l(T') < l(T)$ and $|T'' \setminus T| < |T' \setminus T|$, contradicting our minimality assumption. ∎

This theorem gives a good characterization of the minimum length of a spanning tree. (As Kalaba [1960] pointed out, it also gives an algorithm to find a shortest spanning tree (by iteratively exchanging one edge for another if it makes the tree shorter), but it is not polynomial-time.)

Recall that a forest is called *good* if it is contained in a shortest spanning tree.

Corollary 50.9a. *Let $G = (V, E)$ be a connected graph, let $l \in \mathbb{R}^E$ be a length function, and let F be a forest. Then F is good if and only if for each $e \in F$ there exists a cut C with $C \cap F = \{e\}$ and with e shortest in C.*

Proof. To see necessity, let F be good and let $e \in F$. So there exists a shortest spanning tree T containing F. By Theorem 50.9, e is a shortest edge connecting the two components of $T - e$. This gives the required cut C.

Sufficiency is shown by induction on $|F|$, the case $F = \emptyset$ being trivial. Choose $e \in F$. By induction, $F \setminus \{e\}$ is good (as the condition is maintained for $F \setminus \{e\}$). The condition implies that (50.2) is satisfied, and hence F is good by Theorem 50.1. ∎

50.6c. The maximum reliability problem

Often, in designing a network, one is not primarily interested in minimizing the total length, but rather in maximizing 'reliability' (for instance when designing energy or communication networks).

Let $G = (V, E)$ be a connected graph and let $r : E \to \mathbb{R}_+$ be a function. Let us call $r(e)$ the *reliability* of edge e. For any path P in G, the *reliability* of P is, by definition, the minimum reliability of the edges occurring in P. The *reliability* $r_G(s, t)$ of two vertices s and t is equal to the maximum reliability of P where P ranges over all $s - t$ paths. That is,

(50.20) $$r_G(s, t) := \max_P \min_{e \in EP} r(e),$$

where the maximum ranges over all $s - t$ paths P. (The value of $r_G(s, t)$ can be found with the method described in Section 8.6e.)

The problem now is to find a minimal subgraph H of G having the same reliability as G; that is, with $r_H = r_G$. Hu [1961] observed that there is a spanning tree carrying the reliability of G. More precisely, Hu showed that any spanning tree T of maximum total reliability is such a tree (also shown by Kalaba [1964]):

Corollary 50.9b. *Let $G = (V, E)$ be a graph, let $r \in \mathbb{R}^E$, and let T be any spanning tree. Then $r_T(s, t) = r_G(s, t)$ for all s, t if and only if T is a spanning tree in G maximizing $r(T)$.*

Proof. To see sufficiency, let T maximize $r(T)$. Choose $s, t \in V$, and let P be a path in G attaining maximum (50.20). Let e be an edge on the $s - t$ path in T with minimum $r(e)$. Then P contains an edge f connecting the two components of $T - e$. As T maximizes $r(T)$ we have $r(f) \leq r(e)$. Hence

(50.21) $$r_T(s, t) = r(e) \geq r(f) \geq r_G(s, t).$$

Since trivially $r_T(s,t) \leq r_G(s,t)$, this shows sufficiency.

To see necessity, we apply Theorem 50.9. Choose $e \in T$, and suppose that there is an edge f connecting the components of $T - e$, with $r(f) > r(e)$. Then for the ends s, t of f we have

$$r_G(s,t) \geq r(f) > r(e) \geq r_T(s,t), \tag{50.22}$$

a contradiction. ∎

Corollary 50.9b implies:

Corollary 50.9c. *Let $G = (V,E)$ be a complete graph and let $l : E \to \mathbb{R}_+$ be a length function satisfying*

$$l(uw) \geq \min\{l(uv), l(vw)\} \tag{50.23}$$

for all distinct $u, v, w \in V$. Let T be a longest spanning tree in G. Then for all $u, w \in V$, $l(uw)$ is equal to the minimum length of the edges in the $u - w$ path in T.

Proof. Note that (50.23) implies that $l(uw)$ is equal to the reliability $r_G(u,w)$ of u and w, taking $r := l$. So the corollary follows from Corollary 50.9b. ∎

This implies the following. Let $G = (V,E)$ be a graph and let $c : E \to \mathbb{R}_+$ be a capacity function. Let K be the complete graph on V. For each edge st of K, let the length $l(st)$ be the minimum capacity of any $s - t$ cut in G. (An $s - t$ cut is any subset $\delta(W)$ with $s \in W, t \notin W$.)

Let T be a longest spanning tree in K. Then for all $s, t \in V$, $l(st)$ is equal to the minimum length of the edges of T in the $s - t$ path in T.

(This tree need not be a Gomory-Hu tree, as is shown by the complete graph on vertices $1, 2, 3$ and $c(12) = 1$ and $c(13) = c(23) = 2$. Then edges 12 and 13 form a tree as above, but it is not a Gomory-Hu tree.)

50.6d. Exchange properties of forests

The following fundamental property of forests in fact is the basis of most theorems in this chapter. It is the 'exchange property' that makes the collection of forests into a matroid.

Theorem 50.10. *Let $G = (V,E)$ be a graph and let F and F' be forests with $|F| < |F'|$. Then $F \cup \{e\}$ is a forest for some $e \in F' \setminus F$.*

Proof. We can assume that $E = F \cup F'$. If no such edge e exists, then F is a maximal forest in G. This however implies that $|F| \geq |F'|$, a contradiction. ∎

Call a forest F *extreme* if $l(F') \geq l(F)$ for each forest F' satisfying $|F'| = |F|$. The forests made iteratively in Kruskal's method all are extreme, since:

Corollary 50.10a. *Let F be an extreme forest and let e be a shortest edge with $e \notin F$ and $F \cup \{e\}$ a forest. Then $F \cup \{e\}$ is extreme again.*

Proof. Let F' be an extreme forest with $|F'| = |F| + 1$. By Theorem 50.10, there exists an $e' \in F' \setminus F$ such that $F \cup \{e'\}$ is a forest. As F is extreme we have $l(F' \setminus \{e'\}) \geq l(F)$. Hence $l(F \cup \{e'\}) \leq l(F')$. Also, by the choice of e, $l(e) \leq l(e')$. So $l(F \cup \{e\}) \leq l(F')$. Concluding, $F \cup \{e\}$ is extreme (as F' is extreme). ∎

The following corollary is due to Florek, Łukaszewicz, Perkal, Steinhaus, and Zubrzycki [1951a]. Recall that a forest is called *good* if it is contained in a shortest spanning tree.

Corollary 50.10b. *Each extreme forest is good.*

Proof. Directly from Corollary 50.10a, since it implies that each extreme forest is contained in an extreme maximal forest, and hence in a shortest maximal forest; so it is good. ∎

We also can derive a 'slice-integrality' result:

Corollary 50.10c. *Let $G = (V, E)$ be a graph and let $k, l \in \mathbb{Z}_+$. Then the convex hull of the incidence vectors of forests F with $k \leq |F| \leq l$ is equal to the intersection of the forest polytope of G with $\{x \in \mathbb{R}^E \mid k \leq x(E) \leq l\}$.*

Proof. Let x be in the forest polytope with $k \leq x(E) \leq l$. Let $x = \sum_F \lambda_F \chi^F$, where F ranges over all forests and where the λ_F are nonnegative reals with $\sum_F \lambda_F = 1$. Choose the λ_F with

$$(50.24) \qquad \sum_F \lambda_F |F|^2$$

minimal. Then

$$(50.25) \qquad |F'| \leq |F| + 1 \text{ for all } F, F' \text{ with } \lambda_F > 0 \text{ and } \lambda_{F'} > 0.$$

Otherwise we can choose $e \in F' \setminus F$ such that $F \cup \{e\}$ is a forest (by Theorem 50.10). Let $\alpha := \min\{\lambda_F, \lambda_{F'}\}$. Then decreasing λ_F and $\lambda_{F'}$ by α and increasing $\lambda_{F \cup \{e\}}$ and $\lambda_{F' \setminus \{e\}}$ by α, decreases sum (50.24). This contradicts our assumption, and proves (50.25).

It implies that $k \leq |F| \leq l$ for each F with $\lambda_F > 0$, and we have the corollary. ∎

50.6e. Uniqueness of shortest spanning tree

Kotzig [1961b] characterized when there is a unique shortest spanning tree:

Theorem 50.11. *Let $G = (V, E)$ be a graph, let $l \in \mathbb{R}^E$ be a length function, and let T be a spanning tree in G. Then T is a unique shortest spanning tree if and only if $l(f) > l(e)$ for all $e \in T$ and $f \in E \setminus T$ such that $T - e + f$ is a spanning tree.*

Proof. As the proof of Theorem 50.9. ∎

This implies a sufficient condition given by Borůvka [1926a]:

Corollary 50.11a. *Let $G = (V, E)$ be a graph and let $l \in \mathbb{R}^E$ be a length function with $l(e) \neq l(f)$ if $e \neq f$. Then there is a unique shortest spanning tree.*

Proof. Directly from Theorem 50.11. ∎

Let $G = (V, E)$ be a connected graph and let $l \in \mathbb{R}^E$ be a length function, with $l(e) \neq l(f)$ if $e \neq f$. Define

(50.26) $\quad T := \{e \in E \mid \exists \text{ cut } C \text{ such that } e \text{ is the shortest edge of } C\}.$

Then

(50.27) $\quad E \setminus T = \{e \in E \mid \exists \text{ circuit } D \text{ such that } e \text{ is the longest edge in } D\}.$

This is easy, since if some edge e is contained in some cut C and some circuit D, then there exists an edge $f \neq e$ in $C \cap D$. If $l(f) < l(e)$, then e is not shortest in C, and if $l(f) > l(e)$, then e is not longest in D. Moreover, for any $e \in E$, if no circuit D as in (50.27) exists, then each circuit D containing e contains an edge f with $l(f) > l(e)$. Hence the set of edges f with $l(f) \geq l(e)$ contains a cut C containing e. This C is as in (50.26).

Now (Dijkstra [1960], Rosenstiehl [1967]):

(50.28) $\quad T$ is the unique shortest spanning tree in G.

Indeed, T is a forest, since each circuit D intersects $E \setminus T$ (namely, in the longest edge of D). Moreover, T is a connector, since each cut C intersects T (namely, in the shortest edge of C). T is the unique shortest spanning tree. This follows from Theorem 50.11, since for each $e \in T$ and each $f \notin T$, if $(T \setminus \{e\}) \cup \{f\}$ is a spanning tree, then $l(e) < l(f)$ as e is the shortest edge in the cut determined by $T - e$.

50.6f. Forest covers

Let $G = (V, E)$ be an undirected graph. A subset F of E is called a *forest cover* if F is both a forest and an edge cover. Forest covers turn out to be interesting algorithmically and polyhedrally.

As Gamble and Pulleyblank [1989] point out, White [1971] showed:

Theorem 50.12. *Given a graph $G = (V, E)$ and a weight function $w \in \mathbb{Q}^E$, a minimum-weight forest cover can be found in strongly polynomial time.*

Proof. Let E_- be the set of edges of negative weight and let V_- be the set of vertices covered by E_-. Let $V_+ := V \setminus V_-$. First find a subset F' of $E[V_+] \cup \delta(V_+)$ covering V_+, of minimum weight. This can be done in strongly polynomial time, by a variation of the strongly polynomial-time algorithm for the minimum weight edge cover problem. (In fact, it is a special case of Theorem 34.4.)

Next find a forest F'' in $E[V_-]$ of minimum weight. Again, this can be done in strongly polynomial time, by Theorem 50.6.

We can assume that any proper subset of F' does not cover V_+. It implies that F' is a forest and that for any vertex $v \in V_+$ incident with some edge e in F' with $e \in \delta(V_+)$, e is the only edge in F' incident with v.

This implies that $F' \cup F''$ is a forest. Moreover, it is an edge cover, since F' covers V_+ and F'' covers V_-, since any vertex in V_- is incident with an edge of negative weight.

So $F' \cup F''$ is a forest cover. To see that it has minimum weight, let $B \subseteq E$ be any forest cover. Let $B'' := B \cap E[V_-]$ and $B' := B \setminus B''$. Then $w(B') \geq w(F')$, since B' covers V_+. Also, $w(B'') \geq w(F'')$, since B'' is a forest. So $w(B) \geq w(F)$. ∎

Gamble and Pulleyblank [1989] showed that White's method implies a characterization of the *forest cover polytope* $P_{\text{forest cover}}(G)$ of a graph G, which is the convex hull of the incidence vectors of forest covers in G. It turns out to be equal to the intersection of the forest polytope (characterized in Corollary 50.7a) and the edge cover polytope (characterized in Corollary 27.3a):

Theorem 50.13. *For any undirected graph $G = (V, E)$:*

(50.29) $$P_{\text{forest cover}}(G) = P_{\text{forest}}(G) \cap P_{\text{edge cover}}(G).$$

Proof. The inclusion $\subseteq$ is trivial, as any forest cover is both a forest and an edge cover. Suppose that the reverse inclusion does not hold, and let x be a vertex of $P_{\text{forest}}(G) \cap P_{\text{edge cover}}(G)$ which is not in $P_{\text{forest cover}}(G)$. Let $w \in \mathbb{Q}^E$ be a weight function such that x uniquely minimizes $w^{\mathsf{T}}x$ over $P_{\text{forest}}(G) \cap P_{\text{edge cover}}(G)$. We can assume that $w(e) \neq 0$ for each edge e (as we can perturb w slightly).

Again let E_- be the set of edges of negative weight, V_- be the set of vertices covered by E_-, and $V_+ := V \setminus V_-$. Since x is in the edge cover polytope, there exists a subset F' of $E[V_+] \cup \delta(V_+)$ covering V_+ with

(50.30) $$w(F') \leq \sum_{e \in E[V_+] \cup \delta(V_+)} w(e) x_e.$$

Similarly, since x is in the forest polytope, there is a forest F'' in $E[V_-]$ with

(50.31) $$w(F'') \leq \sum_{e \in E[V_-]} w(e) x_e.$$

Now, as in the proof of Theorem 50.12, $F := F' \cup F''$ is a forest cover. Since $w(F) \leq w^{\mathsf{T}}x$, this contradicts our assumptions on x and w. ∎

White [1971] also considered the problem of finding a minimum weight forest cover of given size k. Gamble and Pulleyblank [1989] showed that the convex hull of the incidence vectors of forest covers of size k is equal to the intersection of the forest cover polytope with the hyperplane $\{x \in \mathbb{R}^E \mid x(E) = k\}$.

Cerdeira [1994] related forest covers to matroid intersection.

50.6g. Further notes

Let $G = (V, E)$ be a graph. Call a subset U of V *circuit-free* if U spans no circuit; that is, it induces a forest as subgraph of G. Ding and Zang [1999] characterized the graphs G for which the convex hull of the incidence vectors of circuit-free sets is determined by

$$\text{(50.32)} \qquad \begin{array}{ll} 0 \leq x_v \leq 1 & \text{for each vertex } v, \\ x(VC) \leq |VC| - 1 & \text{for each circuit } C. \end{array}$$

Their characterization implies that (50.32) is totally dual integral as soon as it determines an integer polytope.

Goemans [1992] studied the convex hull of the incidence vectors of (not necessarily spanning) subtrees of a graph.

Brennan [1982] reported on good experimental results with an implementation of Kruskal's method by only partially sorting the edges until the successive shortest edges to be added to the current forest can be identified.

Győri [1978] and Lovász [1977a] showed that if $G = (V, E)$ is k-connected and $v_1, \ldots, v_k$ are distinct vertices, and $n_1, \ldots, n_k$ are positive integers with $n_1 + \cdots + n_k = |V|$, then G contains a forest F such that the component containing v_i has size n_i $(i = 1, \ldots, k)$. For $k = 2$, Győri's proof gives an $O(nm)$-time algorithm. A linear-time algorithm for $k = 2$ was given by Suzuki, Takahashi, and Nishizeki [1990]. More can be found in Győri [1981].

Khuller, Raghavachari, and Young [1993,1995b] considered spanning trees that belance between shortest spanning trees and shortest paths trees.

Books covering shortest spanning trees include Even [1973,1979], Christofides [1975], Lawler [1976b], Minieka [1978], Hu [1982], Papadimitriou and Steiglitz [1982], Smith [1982], Aho, Hopcroft, and Ullman [1983], Sysło, Deo, and Kowalik [1983], Tarjan [1983], Gondran and Minoux [1984], Nemhauser and Wolsey [1988], Chen [1990], Cormen, Leiserson, and Rivest [1990], Lengauer [1990]. Ahuja, Magnanti, and Orlin [1993], Cook, Cunningham, Pulleyblank, and Schrijver [1998], Jungnickel [1999], and Korte and Vygen [2000]. Pierce [1975] and Golden and Magnanti [1977] gave bibliographies on algorithms for shortest spanning tree.

50.6h. Historical notes on shortest spanning trees

We refer to Graham and Hell [1985] for an extensive historical survey of shortest tree algorithms, with several quotes (with translations) from old papers. Our notes below have profited from their investigations.

We recall some terminology for a shortest spanning tree algorithm. We call it *tree-growing* if we keep a tree on a subset of the vertices, and iteratively extend it by adding an edge joining the tree with a vertex outside of the tree. It is *forest-merging* if we keep a forest, and iteratively merge two components by joining them by an edge. It is called *parallel forest-merging* if forest-merging is performed in parallel, by connecting each component to its nearest neighbouring component (assuming all lengths are distinct).

Borůvka: parallel forest-merging

Borůvka [1926a] described the problem of finding a shortest spanning tree as follows (the paper is in Czech; we quote from its German summary; for quotes from Czech with translation, see Graham and Hell [1985]):

> In dieser Arbeit löse ich folgendes Problem:
> Es möge eine Matrix der bis auf die Bedingungen $r_{\alpha\alpha} = 0$, $r_{\alpha\beta} = r_{\beta\alpha}$ positiven und von einander verschiedenen Zahlen $r_{\alpha\beta}$ $(\alpha, \beta = 1, 2, \ldots n; n \geq 2)$ gegeben sein.

> Aus dieser ist eine Gruppe von einander und von Null verschiedener Zahlen auszuwählen, so dass
> 1° in ihr zu zwei willkürlich gewählten natürlichen Zahlen p_1, p_2 ($\leq n$) eine Teilgruppe von der Gestalt
>
> $$r_{p_1c_2}, r_{c_2c_3}, r_{c_3c_4}, \ldots r_{c_{q-2}c_{q-1}}, r_{c_{q-1}p_2}$$
>
> existiere,
> 2° die Summe ihrer Glieder kleiner sei als die Summe der Glieder irgendeiner anderen, der Bedingung 1° genügenden Gruppe von einander und von Null verschiedenen Zahlen.[2]

So Borůvka stated that the spanning tree found is the unique shortest. He assumed that all edge lengths are different.

Borůvka next described parallel forest-merging, in a somewhat complicated way. (He did not have the language of graph theory at hand.) The idea is to update a number of vertex-disjoint paths $P_1, \ldots, P_k$ (initially $k = 0$). Along any P_i, the edge lengths are decreasing. Let v be the last vertex of P_k and let e be the edge of shortest length incident with v. If the other end vertex of e is not yet covered by any P_i, we extend P_k with e, and iterate. Otherwise, if not all vertices are covered yet by the P_i, we choose such a vertex v, and start a new path P_{k+1} at v. If all vertices are covered by the P_i, we shrink each of the P_i to one vertex, and iterate. At the end, the edges chosen throughout the iterations form a shortest spanning tree. It is easy to see that this in fact is 'parallel forest-merging'.

The interest of Borůvka in this problem came from a question of the Electric Power Company of Western Moravia in Brno, at the beginning of the 1920s, asking for the most economical construction of an electric power network (see Borůvka [1977]).

In a follow-up paper, Borůvka [1926b] gave a simple explanation of the method by means of an example. We refer to Nešetřil, Milková, and Nešetřilova [2001] for translations of and comments on the two papers of Borůvka.

Jarník: tree-growing

In a reaction to Borůvka's work, Jarník wrote on 12 February 1929 a letter to Borůvka in which he described a 'new solution of a minimal problem discussed by Mr Borůvka'. This 'new solution' is the tree-growing method. An extract of the letter was published as Jarník [1930]. We quote from the German summary:

> a_1 ist eine beliebige unter den Zahlen $1, 2, \ldots, n$.
> a_2 ist durch

[2] In this work, I solve the following problem:
A matrix may be given of positive distinct numbers $r_{\alpha\beta}$ ($\alpha, \beta = 1, 2 \ldots n$; $n \geq 2$), up to the conditions $r_{\alpha\alpha} = 0$, $r_{\alpha\beta} = r_{\beta\alpha}$.
From this, a group of numbers, different from each other and from zero, should be selected such that
1° for arbitrarily chosen natural numbers p_1, p_2 ($\leq n$) a subgroup of it exists of the form

$$r_{p_1c_2}, r_{c_2c_3}, r_{c_3c_4}, \ldots r_{c_{q-2}c_{q-1}}, r_{c_{q-1}p_2},$$

2° the sum of its members be smaller than the sum of the members of any other group of numbers different from each other and from zero, satisfying condition 1°.

$$r_{a_1,a_2} = \min_{\begin{pmatrix} l = 1,2,\dots,n \\ l \neq a_1 \end{pmatrix}} r_{a_1,l}$$

> definiert.
> Wenn $2 \leq k < n$ und wenn $[a_1, a_2], \dots, [a_{2k-3}, a_{2k-2}]$ bereits bestimmt sind, so wird $[a_{2k-1}, a_{2k}]$ durch
> $$r_{a_{2k-1},a_{2k}} = \min r_{i,j},$$
> definiert, wo i alle Zahlen $a_1, a_2, \dots, a_{2k-2}$, j aber alle übrigen von den Zahlen $1, 2, \dots, n$ durchläuft.[3]

Again, Jarník assumed that all lengths are distinct and showed that then the shortest spanning tree is unique. For a detailed discussion and translation of the article of Jarník [1930] (and of Jarník and Kössler [1934] on the Steiner tree problem), see Korte and Nešetřil [2001].

Other discoveries of parallel forest-merging

Parallel forest-merging was described also by Choquet [1938] (without proof), who gave as motivation the construction of road systems:

> Étant donné n villes du plan, il s'agit de trouver un réseau de routes permettant d'aller d'une quelconque de ces villes à une autre et tel que:
> 1° la longueur globale du réseau soit minimum;
> 2° exception faite des villes, on ne peut partir d'aucun point dans plus de deux directions, afin d'assurer la sûreté de la circulation; ceci entraîne, par exemple, que lorsque deux routes semblent se croiser en un point qui n'est pas une ville, elles passent en fait l'une au-dessus de l'autre et ne communiquent pas entre elles en ce point, qu'on appellera faux-croisement.[4]

He was one of the first concerned on the complexity of the method:

> Le réseau cherché sera tracé après $2n$ opérations élémentaires au plus, en appelant opération élémentaire la recherche du continu le plus voisin d'un continu donné.[5]

---

[3] a_1 is an arbitrary one among the numbers $1, 2, \dots, n$.
a_2 is defined by
$$r_{a_1,a_2} = \min_{\begin{pmatrix} l = 1,2,\dots,n \\ l \neq a_1 \end{pmatrix}} r_{a_1,l}.$$
If $2 \leq k < n$ and if $[a_1, a_2], \dots, [a_{2k-3}, a_{2k-2}]$ are determined already, then $[a_{2k-1}, a_{2k}]$ is defined by
$$r_{a_{2k-1},a_{2k}} = \min r_{i,j},$$
where i runs through all numbers $a_1, a_2, \dots, a_{2k-2}$, j however through all remaining of the numbers $1, 2, \dots, n$.

[4] Being given n cities of the plane, the point is to find a network of routes allowing to go from an arbitrary of these cities to another and such that:
1° the global length of the network be minimum;
2° except for the cities, one cannot depart from any point in more than two directions, in order to assure the certainty of the circulation; this entails, for instance, that when two routes seem to cross each other in a point which is not a city, they pass in fact one above the other and do not communicate among them in this point, which we shall call a false crossing.

[5] The network looked for will be traced after at most $2n$ elementary operations, calling the search for the continuum closest to a given continuum an elementary operation.

Also Florek, Łukaszewicz, Perkal, Steinhaus, and Zubrzycki [1951a,1951b] described parallel forest-merging. They were motivated by clustering in anthropology, taxonomy, etc. In the latter paper, they apply the method to:

> 1° the capitals of Poland's provinces, 2° two collections of excavated skulls, 3° 42 archeological finds, 4° the liverworts of Silesian Beskid mountains with forests as their background, and to the forests of Silesian Beskid mountains with the liverworts appearing in them as their background.

Kruskal

Kruskal [1956] was motivated by Borůvka's first paper and by the application to the traveling salesman problem, described as follows (where [1] refers to Borůvka [1926a]):

> Several years ago a typewritten translation (of obscure origin) of [1] raised some interest. This paper is devoted to the following theorem: If a (finite) connected graph has a positive real number attached to each edge (the *length* of the edge), and if these lengths are all distinct, then among the spanning trees (German: Gerüst) of the graph there is only one, the sum of whose edges is a minimum; that is, the shortest spanning tree of the graph is unique. (Actually in [1] this theorem is stated and proved in terms of the "matrix of lengths" of the graph, that is, the matrix $\|a_{ij}\|$ where a_{ij} is the length of the edge connecting vertices i and j. Of course, it is assumed that $a_{ij} = a_{ji}$ and that $a_{ii} = 0$ for all i and j.)
> The proof in [1] is based on a not unreasonable method of constructing a spanning subtree of minimum length. It is in this construction that the interest largely lies, for it is a solution to a problem (Problem 1 below) which on the surface is closely related to one version (Problem 2 below) of the well-known traveling salesman problem.
> PROBLEM 1. Give a practical method for constructing a spanning subtree of minimum length.
> PROBLEM 2. Give a practical method for constructing an unbranched spanning subtree of minimum length.
> The construction in [1] is unnecessarily elaborate. In the present paper I give several simpler constructions which solve Problem 1, and I show how one of these constructions may be used to prove the theorem of [1]. Probably it is true that any construction which solves Problem 1 may be used to prove this theorem.

Kruskal described three algorithms: Construction A: iteratively choose the shortest edge that can be added (forest-merging); Construction B: fix a nonempty set U of vertices, and choose iteratively the shortest edge leaving some component intersecting U (a generalization of tree-growing); Construction A′: iteratively remove the longest edge that can be removed without making the graph disconnected. He proved that Construction A implies the uniqueness of shortest spanning tree if all lengths are distinct.

In his reminiscences, Kruskal [1997] wrote about Borůvka's method:

> In one way, the method of construction was very elegant. In another way, however, it was unnecessarily complicated. A goal which has always been important to me is to find simpler ways to describe complicated ideas, and that is all I tried to do here. I simplified the construction down to its essence, but it seems to me that the idea of Professor Borůvka's method is still present in my version.

Prim

Prim [1957] gave the following motivation:

> A problem of inherent interest in the planning of large-scale communication, distribution and transportation networks also arises in connection with the current rate structure for Bell System leased-line services.

He described the following algorithm: choose a component of the current forest, and connect it to the nearest component. He observed that Kruskal's constructions A and B are special cases of this.

Prim noticed that in fact only the order of the lengths determines if a spanning tree is shortest:

> The *shortest spanning subtree* of a connected labelled graph also minimizes all increasing symmetric functions, and maximizes all decreasing symmetric functions, of the edge "lengths."

Prim preferred starting at a vertex and growing a tree for computational reasons:

> This computational procedure is easily programmed for an automatic computer so as to handle quite large-scale problems. One of its advantages is its avoidance of checks for closed cycles and connectedness. Another is that it never requires access to more than two rows of distance data at a time — no matter how large the problem.

The implementation described by Prim has $O(n^2)$ running time.

Loberman and Weinberger

Loberman and Weinberger [1957] gave minimizing wire connections as motivation:

> In the construction of a digital computer in which high-frequency circuitry is used, it is desirable and often necessary when making connections between terminals to minimize the total wire length in order to reduce the capacitance and delay-line effects of long wire leads.

They described two methods: tree-growing and forest-merging. Only after they had designed their algorithms, they discovered that their algorithms were given earlier by Kruskal [1956].

> However, it is felt that the more detailed implementation and general proofs of the procedures justify this paper.

They next described how to implement Kruskal's method, in particular, how to merge forests. They also observed that the minimality of a spanning tree depends only on the order of the lengths, and not on their specific values:

> After the initial sorting into a list where the branches are of monotonically increasing length, the actual value of the length of any branch no longer appears explicitly in the subsequent manipulations. As a result, some other parameter such as the square of the length could have been used. More generally, the same minimum tree will persist for all variations in branch lengths that do not disturb the original relative order.

Dijkstra

Dijkstra [1959] gave again the tree-growing method, which he preferred (for computational reasons) above the forest-merging method of Kruskal and Loberman and Weinberger (overlooking the fact that these authors also gave the tree-growing method):

> The solution given here is to be preferred to the solution given by J.B. KRUSKAL [*1*] and those given by H. LOBERMAN and A. WEINBERGER [*2*]. In their solutions all the — possibly $\frac{1}{2}n(n-1)$ — branches are first of all sorted according to length. Even if the length of the branches is a computable function of the node coordinates, their methods demand that data for all branches are stored simultaneously. Our method requires the simultaneous storing of the data for at most n branches, ...

Dijkstra described an $O(n^2)$ implementation.

Dijkstra [1960] gave the following alternative shortest spanning tree method: order edges arbitrarily, find the first edge that forms a circuit with previous edges; delete the longest edge from this circuit, and continue. (This method was also found by Rosenstiehl [1967].) This generalizes both forest-merging and tree-growing, by choosing the order appropriately.

Further work

Kalaba [1960] proposed the method of first choosing a spanning tree arbitrarily, and next adding, iteratively, an edge and removing the longest edge in the circuit arising.

Kotzig [1961b] gave again Kruskal's Algorithm A' (a referee pointed Kruskal's work out to Kotzig). Kotzig moreover showed that there is a unique minimum spanning tree T if and only if for each edge e not in T, e is the unique longest edge in the circuit in $T \cup \{e\}$.

As mentioned, Graham and Hell [1985] give an extensive survey on the history of the minimum spanning tree (and minimum Steiner tree) problem. See also Nešetřil [1997] for additional notes on the history of the minimum spanning tree problem.

Chapter 51

Packing and covering of trees

The basic facts on packing and covering of trees follow directly from those on matroid union. In this chapter we check what these results amount to in terms of graphs, and we give some more direct algorithms.

51.1. Unions of forests

For any graph $G = (V, E)$ and any partition $\mathcal{P}$ of V, let $\delta(\mathcal{P})$ denote the set of edges connecting distinct classes of $\mathcal{P}$. From the following consequence of the matroid union theorem we will derive other results on tree packing and covering:

Theorem 51.1. *Let $G = (V, E)$ be an undirected graph and let $k \in \mathbb{Z}_+$. Then the maximum size of the union of k forests is equal to the minimum value of*

$$|\delta(\mathcal{P})| + k(|V| - |\mathcal{P}|) \tag{51.1}$$

taken over all partitions $\mathcal{P}$ of V into nonempty classes.

Proof. This follows directly from the matroid union theorem (Corollary 42.1a) applied to the cycle matroid M of G. Indeed, by Corollary 42.1b, the maximum size of the union of k forests is equal to the minimum value of

$$|E \setminus F| + kr_M(F), \tag{51.2}$$

where $r_M(F)$ is the maximum size of a forest contained in F. We can assume that each component of (V, F) is an induced subgraph of G. So taking $\mathcal{P}$ equal to the set of components of (V, F), we see that $r_M(F) = |V| - |\mathcal{P}|$, and hence that the minimum of (51.2) is equal to the minimum of (51.1). ∎

51.2. Disjoint spanning trees

Theorem 51.1 has a number of consequences. First we have the following tree packing result of Tutte [1961a] and Nash-Williams [1961b]:

Corollary 51.1a (Tutte-Nash-Williams disjoint trees theorem)**.** *A graph $G = (V, E)$ contains k edge-disjoint spanning trees if and only if*

(51.3) $\quad |\delta(\mathcal{P})| \geq k(|\mathcal{P}| - 1)$

for each partition $\mathcal{P}$ of V into nonempty classes.

Proof. To see necessity of (51.3), each spanning tree contains at least $|\mathcal{P}| - 1$ edges in $\delta(\mathcal{P})$. To show sufficiency, it is equivalent to show that there exist $k(|V| - 1)$ edges that can be covered by k forests. By Theorem 51.1, this is indeed possible, since

(51.4) $\quad |\delta(\mathcal{P})| + k(|V| - |\mathcal{P}|) \geq k(|\mathcal{P}| - 1) + k(|V| - |\mathcal{P}|) = k(|V| - 1)$

for each partition $\mathcal{P}$ of V into nonempty sets. ∎

Gusfield [1983] observed that the Tutte-Nash-Williams disjoint trees theorem (Corollary 51.1a) implies that each $2k$-edge-connected undirected graph has k edge-disjoint spanning trees (since $|\delta(\mathcal{P})| \geq k|\mathcal{P}| \geq k(|\mathcal{P}| - 1)$).

Similarly to the line pursued in Section 42.2, Corollary 51.1a can be formulated equivalently in polyhedral terms:

Corollary 51.1b. *The connector polytope of a graph has the integer decomposition property.*

Proof. Similar to the proof of Corollary 42.1e. ∎

For any connected graph $G = (V, E)$, define the *strength* of G by:

(51.5)
$$\begin{aligned} \text{strength}(G) &:= \max\{\lambda \mid \mathbf{1} \in \lambda \cdot P_{\text{connector}}(G)\} \\ &= \max\{\sum_T \lambda_T \mid \lambda_T \geq 0, \sum_T \lambda_T \chi^T \leq \mathbf{1}\}, \end{aligned}$$

where T ranges over the spanning trees of G, and where $\mathbf{1}$ denotes the all-1 vector in $\mathbb{R}^E$.

The Tutte-Nash-Williams disjoint trees theorem is equivalent to: the maximum number of disjoint spanning trees in a graph $G = (V, E)$ is equal to $\lfloor \text{strength}(G) \rfloor$. Similarly, the capacitated version of the Tutte-Nash-Williams theorem is equivalent to the integer rounding property of the system (cf. Section 42.2):

(51.6)
$$\begin{aligned} x_e &\geq 0 && \text{for } e \in E, \\ x(T) &\geq 1 && \text{for each spanning tree } T. \end{aligned}$$

51.3. Covering by forests

Dual to Corollary 51.1a is the following forest covering theorem of Nash-Williams [1964], where $E[U]$ denotes the set of edges contained in U. (The theorem is also a consequence of a theorem of Horn [1955] on covering vector sets by linearly independent sets, since each graphic matroid is linear.)

Corollary 51.1c (Nash-Williams' covering forests theorem)**.** *The edge set of a graph $G = (V, E)$ can be covered by k forests if and only if*

$$|E[U]| \leq k(|U| - 1) \tag{51.7}$$

for each nonempty subset U of V.

Proof. Since any forest has at most $|U| - 1$ edges contained in U, we have necessity of (51.7). To see sufficiency, notice that (51.7) implies

$$|E| - |\delta(\mathcal{P})| = \sum_{U \in \mathcal{P}} |E[U]| \leq \sum_{U \in \mathcal{P}} k(|U| - 1) = k(|V| - |\mathcal{P}|) \tag{51.8}$$

for any partition $\mathcal{P}$ of V into nonempty sets. So $|\delta(\mathcal{P})| + k(|V| - |\mathcal{P}|) \geq |E|$, and hence Theorem 51.1 implies that there exist k forests covering E. ∎

(Nash-Williams [1964] derived Corollary 51.1c from Corollary 51.1a.)

Again, this corollary can be formulated in terms of the integer decomposition property:

Corollary 51.1d. *For any graph G, the forest polytope has the integer decomposition property.*

Proof. Similar to the proof of Corollary 42.1e. ∎

These results are equivalent to: the minimum number of forests needed to cover the edges of a graph $G = (V, E)$ is equal to

$$\lceil \min\{\lambda \mid \mathbf{1} \in \lambda \cdot P_{\text{forest}}(G)\} \rceil, \tag{51.9}$$

where $\mathbf{1}$ denotes the all-one vector in $\mathbb{R}^E$. A similar relation holds for the capacitated case, which is equivalent to the integer rounding property of the system:

$$\begin{aligned} x_e &\geq 0 && \text{for } e \in E, \\ x(F) &\leq 1 && \text{for each forest } F. \end{aligned} \tag{51.10}$$

The minimum number of forests needed to cover the edges of a graph G is called the *arboricity* of G.

51.4. Complexity

The complexity results on matroid union in Sections 40.3, 42.3 and 42.4 imply that these packing and covering problems for forests and trees are solvable in polynomial time:

Theorem 51.2. *For any graph $G = (V, E)$, a maximum number of edge-disjoint spanning trees and a minimum number of forests covering E can be found in polynomial time.*

Proof. See Section 42.3. ∎

Also weighted versions of it can be solved in strongly polynomial time, for instance, finding a maximum-weight union of k forests in a graph. We give in this section some direct proofs.

To study the complexity of the capacitated and fractional cases, we first observe the following auxiliary result, that is (when applied to undirected graphs) at the base of several algorithms on the forest and connector polytopes, and was observed by Rhys [1970], Picard and Ratliff [1975,1978], Picard [1976], Trubin [1978], Picard and Queyranne [1982a], Padberg and Wolsey [1983], and Cunningham [1985a]. (It also follows from the strong polynomial-time solvability of submodular function minimization, but there is an easier direct method.)

Theorem 51.3. *Given a digraph $D = (V, A)$, $x \in \mathbb{Q}_+^A$, $y \in \mathbb{Q}^V$, and disjoint subsets S and T, we can find a set U with $T \subseteq U \subseteq V \setminus S$ minimizing*

$$x(\delta^{\text{in}}(U)) + y(U) \tag{51.11}$$

in strongly polynomial time.

Proof. Extend D by two new vertices s and t, and arcs (s, v) for $v \in V$ with $y_v > 0$ and (v, t) for $v \in V$ with $y_v < 0$. This gives the digraph $D' = (V \cup \{s, t\}, A')$. Define a capacity function c on A' by:

$$\begin{array}{ll} c(u,v) := x(u,v) & \text{for } (u,v) \in A, \\ c(s,v) := y_v & \text{if } (s,v) \in A', \\ c(v,t) := -y_v & \text{for } (v,t) \in A'. \end{array} \tag{51.12}$$

Let $\kappa := -c(\delta^{\text{in}}_{A'}(t))$ (the sum of the negative y_v's). Then

$$\begin{aligned} c(\delta^{\text{in}}_{A'}(U \cup \{t\})) &= x(\delta^{\text{in}}_A(U)) + \sum_{\substack{v \in U \\ y_v > 0}} y_v - \sum_{\substack{v \in V \setminus U \\ y_v > 0}} y_v \\ &= x(\delta^{\text{in}}_A(U)) + \sum_{v \in U} y_v - \sum_{\substack{v \in V \\ y_v < 0}} y_v = x(\delta^{\text{in}}_A(U)) + y(U) - \kappa \end{aligned} \tag{51.13}$$

for any $U \subseteq V$. Thus minimizing $x(\delta^{\text{in}}_A(U)) + y(U)$ is reduced to finding a minimum-capacity $(S \cup \{s\}) - (T \cup \{t\})$ cut in D'. ∎

Testing membership and finding most violated inequalities

A first consequence of Theorem 51.3 is that we can test membership, and find a most violated inequality, for the forest polytope (Picard and Queyranne [1982b] (suggested by W.H. Cunningham) and Padberg and Wolsey [1983]).

Corollary 51.3a. *Given a graph $G = (V, E)$ and $x \in \mathbb{Q}_+^E$, we can decide if x belongs to $P_{\text{forest}}(G)$, and if not, find a most violated inequality among* (50.11)*, in strongly polynomial time.*

Proof. Define $y_v := 2 - x(\delta(v))$ for $v \in V$. Then

$$(51.14) \qquad 2(x(E[U]) - |U|) = \sum_{v \in U} x(\delta(v)) - x(\delta(U)) - 2|U| \\ = -x(\delta(U)) - y(U).$$

So any nonempty $U \subseteq V$ minimizing $x(\delta(U)) + y(U)$, maximizes $x(E[U]) - |U|$. By Theorem 51.3, we can find such a U in strongly polynomial time. If $x(E[U]) \leq |U| - 1$, x belongs to $P_{\text{forest}}(G)$, and otherwise U gives a most violated inequality. ∎

A similar result holds for the up hull of the connector polytope, which we show with a method of Jünger and Pulleyblank [1995]:

Corollary 51.3b. *Given a graph $G = (V, E)$ and $x \in \mathbb{Q}_+^E$, we can find a partition $\mathcal{P}$ of V into nonempty sets minimizing*

$$(51.15) \qquad x(\delta(\mathcal{P})) - |\mathcal{P}|$$

in strongly polynomial time.

Proof. We first construct a vector $y \in \mathbb{Q}^V$, by updating a vector y. Throughout, y satisfies

$$(51.16) \qquad y(U) \leq x(\delta(U)) - 2 \text{ for each nonempty } U \subseteq V.$$

Start with $y_v := -2$ for all $v \in V$. Successively, for each $v \in V$, reset y_v to $y_v + \alpha$, where α is the minimum value of

$$(51.17) \qquad x(\delta(U)) - 2 - y(U)$$

taken over all $U \subseteq V$ containing v. Such a U can be found in strongly polynomial time by Theorem 51.3.

We end up with a y satisfying (51.16). Moreover, each $v \in V$ is contained in some set U with $y(U) = x(\delta(U)) - 2$.

Let $\mathcal{P}$ be the inclusionwise maximal sets U satisfying $y(U) = x(\delta(U)) - 2$. Then $\mathcal{P}$ is a partition of V, since if $T, U \in \mathcal{P}$ and $T \cap U \neq \emptyset$, then (by the submodularity of $x(\delta(Y))$) $y(T \cup U) = x(\delta(T \cup U)) - 2$, and hence $T = U = T \cup U$.

This $\mathcal{P}$ is as required, since for each partition $\mathcal{Q}$ of V into nonempty sets we have

$$(51.18) \qquad 2x(\delta(\mathcal{Q})) - 2|\mathcal{Q}| = \sum_{U \in \mathcal{Q}} (x(\delta(U)) - 2) \geq \sum_{U \in \mathcal{Q}} y(U) = y(V),$$

with equality if $\mathcal{Q} = \mathcal{P}$. ∎

(This method is analogous to calculating the Dilworth truncation as discussed in Theorem 48.4.)

Corollary 51.3b implies for finding the most violated inequality:

Corollary 51.3c. *Given a graph $G = (V, E)$ and $x \in \mathbb{Q}_+^E$, we can decide if x belongs to $P^{\uparrow}_{\text{connector}}(G)$, and if not, find a most violated inequality among* (50.17)(ii)*, in strongly polynomial time.*

Proof. By Corollary 51.3b, we can find a partition $\mathcal{P}$ of V into nonempty sets, minimizing $x(\delta(\mathcal{P})) - |\mathcal{P}|$. If this value is at least -1, then x belongs to the up hull of the connector polytope, while otherwise $\mathcal{P}$ gives a most violated inequality among (50.17)(ii). ∎

Barahona [1992] showed that membership in the connector polytope can be tested by solving $O(n)$ maximum flow computations (improving Cunningham [1985c]).

Fractional decomposition into trees

By definition, any vector in $P_{\text{forest}}(G)$ or $P_{\text{connector}}(G)$ can be decomposed as a convex combination of incidence vectors of forests or of connectors. These decompositions can be found in strongly polynomial time, a result due to Cunningham [1984] and Padberg and Wolsey [1984] (for the forest polytope).

In order to decompose a vector in the forest polytope as a convex combination of forests, by the following theorem it suffices to have a method to decompose a vector in the spanning tree polytope as a convex combination of spanning trees:

Theorem 51.4. *Given a connected graph $G = (V, E)$ and $x \in P_{\text{forest}}(G)$, we can find a $z \in P_{\text{spanning tree}}(G)$ with $x \leq z$ in strongly polynomial time.*

Proof. We reset x successively for each edge $e = uv$ of G as follows. Reset x_e to $x_e + \alpha$, where α is the largest value such that x remains to belong to $P_{\text{forest}}(G)$. That is, α equals the minimum value of

$$(51.19) \qquad |U| - 1 - x(E[U]) = |U| - 1 - \tfrac{1}{2}\sum_{v \in U} x(\delta(v)) + \tfrac{1}{2}x(\delta(U)),$$

taken over subsets U of V with $u, v \in U$. Such a U can be found in strongly polynomial time by Theorem 51.3.

As $P_{\text{forest}}(G) = P^{\downarrow}_{\text{spanning tree}}(G) \cap \mathbb{R}_+^E$, the final x is a z as required. ∎

Hence, to decompose a vector in the forest polytope, we can do with decomposing vectors in the spanning tree polytope:

Theorem 51.5. *Given a graph $G = (V, E)$ and $y \in P_{\text{spanning tree}}(G)$, we can find spanning trees $T_1, \ldots, T_k$ and $\lambda_1, \ldots, \lambda_k \geq 0$ satisfying*

(51.20) $$y = \lambda_1 \chi^{T_1} + \cdots + \lambda_k \chi^{T_k}$$

and $\lambda_1 + \cdots + \lambda_k = 1$, in strongly polynomial time.

Proof. Iteratively resetting y, we keep an integer weight function w such that y maximizes $w^{\mathsf{T}} y$ over the spanning tree polytope. Initially, $w := \mathbf{0}$. We describe the iteration.

Let T be a spanning tree in G with $T \subseteq \text{supp}(y)$, maximizing $w(T)$. Let $a := y - \chi^T$. If $a = \mathbf{0}$ we stop; then $y = \chi^T$. If $a \neq \mathbf{0}$, let λ be the largest rational such that

(51.21) $$\chi^T + \lambda \cdot a$$

belongs to $P^{\uparrow}_{\text{spanning tree}}(G)$.

We describe an inner iteration to find λ. We iteratively consider vectors y along the halfline $L := \{\chi^T + \lambda \cdot a \mid \lambda \geq 0\}$. Note that the function $w^{\mathsf{T}} x$ is constant on L. First we let λ be the largest rational such that $\chi^T + \lambda \cdot a$ is nonnegative, and set $z := \chi^T + \lambda \cdot a$.

We iteratively reset z. We check if z belongs to the spanning tree polytope, and if not, we find a constraint among (50.12) most violated by z. That is, we find a nonempty subset U of V minimizing $|U| - 1 - z(E[U])$. Let z' be the vector on L attaining $x(E[U]) \leq |U| - 1$ with equality.

Consider any inequality $x(E[U']) \leq |U'| - 1$ violated by z'. Then

(51.22) $$|U'| - 1 - |T \cap E[U']| < |U| - 1 - |T \cap E[U]|.$$

This can be seen by considering the function $d(x) := (|U| - 1 - x(E[U])) - (|U'| - 1 - x(E[U']))$. We have $d(z) \leq 0$ and $d(z') > 0$, and hence, as d is linear, $d(\chi^T) > 0$; that is, we have (51.22). So resetting $z := z'$, there are at most $|V|$ inner iterations.

Let y' be the final z found. Since $\lambda \geq 1$ (as $y \in P_{\text{spanning tree}}(G)$) and $y = \lambda^{-1} \cdot y' + (1 - \lambda^{-1}) \cdot \chi^T$, any convex decomposition of y' into incidence vectors of spanning trees, yields such a decomposition of y. We show that this recursion terminates.

If we apply no iteration, then $\text{supp}(y') \subset \text{supp}(y)$. So replacing y, w by y', w gives a reduction.

If we do at least one iteration, we find a U such that y' satisfies $y'(E[U]) = |U| - 1$ while $|T \cap E[U]| < |U| - 1$. In this case we replace y, w by $y', w' := 2w + \chi^{E[U]}$.

Then y' maximizes ${w'}^{\mathsf{T}} x$ over the spanning tree polytope. Indeed, for any x in the spanning tree polytope, we have

(51.23) $$\begin{aligned} {w'}^{\mathsf{T}} x = 2w^{\mathsf{T}} x + x(E[U]) \leq 2w^{\mathsf{T}} y + |U| - 1 = 2w^{\mathsf{T}} y' + y'(E[U]) \\ = {w'}^{\mathsf{T}} y'. \end{aligned}$$

Moreover, each tree T' maximizing $w'(T')$ also maximizes $w(T')$ (by the greedy method: for any ordering of V for which w' is nondecreasing, also w is nondecreasing). However, T does not maximize $w'(T)$, since $w'(T) = 2w(T) + |T \cap E[U]| < 2w(T) + |U| - 1 = 2w^{\mathsf{T}}y + |U| - 1 = w'^{\mathsf{T}}y'$. So the dimension of the face of vectors x maximizing $w'^{\mathsf{T}}x$ is less than the dimension of the face of vectors x maximizing $w^{\mathsf{T}}x$.

So the number of iterations is at most $|E|$. This shows that the method is strongly polynomial-time. ∎

Now we can derive from the previous two theorems, an algorithmic result for fractional forest decomposition:

Corollary 51.5a. *Given a graph $G = (V, E)$ and $y \in P_{\text{forest}}(G)$, we can find forests $F_1, \ldots, F_k$ and $\lambda_1, \ldots, \lambda_k \geq 0$ satisfying*

(51.24) $$y = \lambda_1 \chi^{F_1} + \cdots + \lambda_k \chi^{F_k}$$

and $\lambda_1 + \cdots + \lambda_k = 1$, in strongly polynomial time.

Proof. We can assume that G is connected, as we can consider each component of G separately. By Theorem 51.4, we can find a $z \in P_{\text{spanning tree}}(G)$ with $y \leq z$ in strongly polynomial time. By Theorem 51.5, we can decompose z as a convex combination of incidence vectors of spanning trees in strongly polynomial time. By restricting the spanning trees to subforests if necessary, we obtain a convex decomposition of y into incidence vectors of forests. ∎

We can proceed similarly for decomposing a vector in the connector polytope. To this end, we show the analogue for connectors of Theorem 51.4:

Theorem 51.6. *Given a graph $G = (V, E)$ and $x \in P^{\uparrow}_{\text{connector}}(G)$, we can find a $z \in P_{\text{spanning tree}}(G)$ with $x \geq z$, in strongly polynomial time.*

Proof. The method described in the proof of Corollary 51.3b gives a vector $y \in \mathbb{Q}^V$ satisfying

(51.25) $$y(U) \leq x(\delta(U)) - 2 \text{ for each nonempty } U \subseteq V,$$

and a partition $\mathcal{P}$ of V into nonempty sets with $y(U) = x(\delta(U)) - 2$ for each $U \in \mathcal{P}$. Hence

(51.26) $$y(V) = \sum_{U \in \mathcal{P}} (x(\delta(U)) - 2) = 2x(\delta(\mathcal{P})) - 2|\mathcal{P}| \geq -2.$$

By decreasing components of y appropriately, we can achieve that $y(V) = -2$, while maintaining (51.25).

We are going to modify y and x, maintaining (51.25) and $y(V) = -2$. For each $u, v \in V$ with $e = uv \in E$, we do the following. Let α be the minimum value of $x(\delta(U)) - 2 - y(U)$ taken over subsets U of V with $u \in U$, $v \notin U$. So $\alpha \geq 0$. Let $\beta := \min\{x_e, \frac{1}{2}\alpha\}$ and reset

(51.27) $\qquad x_e := x_e - \beta,\ y_u := y_u + \beta,\ y_v := y_v - \beta.$

Then (51.25) is maintained, and the collection $\mathcal{C}$ of subsets U having equality in (51.25) is not reduced. Moreover, in the new situation, $x_e = 0$ or there is a $U \in \mathcal{C}$ with $u \in U$ and $v \notin U$. Also, the new x belongs to $P^{\uparrow}_{\text{connector}}(G)$, as for any partition $\mathcal{Q}$ of V into nonempty sets we have

(51.28) $$\sum_{U \in \mathcal{Q}} x(\delta(U)) \geq y(V) + 2|\mathcal{Q}| = 2|\mathcal{Q}| - 2.$$

Doing this for each edge e (in both directions), we end up with x, y satisfying (51.25) such that

(51.29) for all adjacent u, v, if $x_{uv} > 0$, then there is a $U \in \mathcal{C}$ with $u \in U$ and $v \notin U$.

This implies that

(51.30) $$y_u = x(\delta(u)) - 2$$

for each $u \in V$. Indeed, $\mathcal{C}$ is closed under unions and intersections of intersecting sets. Let U be the smallest set in $\mathcal{C}$ containing u. (This exists, since $V \in \mathcal{C}$.) To show (51.30), we must show $U = \{u\}$. Suppose therefore that $U \neq \{u\}$. By (51.29), there is no edge e connecting u and $U \setminus \{u\}$ with $x_e > 0$. Hence

(51.31) $$\begin{aligned} y(U) &= y_u + y(U \setminus \{u\}) \leq x(\delta(u)) - 2 + x(\delta(U \setminus \{u\})) - 2 \\ &= x(\delta(U)) - 4 < x(\delta(U)) - 2, \end{aligned}$$

contradicting the fact that $U \in \mathcal{C}$. This proves (51.30).

Hence

(51.32) $$2x(E) = \sum_{u \in V} x(\delta(u)) = y(V) + 2|V| = 2(|V| - 1),$$

and so $x(E) = |V| - 1$. This implies that x belongs to the spanning tree polytope. ∎

This implies for fractional connector decomposition:

Corollary 51.6a. *Given a graph $G = (V, E)$ and $x \in P_{\text{connector}}(G)$, we can find connectors $C_1, \ldots, C_k$ and $\lambda_1, \ldots, \lambda_k \geq 0$ satisfying*

(51.33) $$x = \lambda_1 \chi^{C_1} + \cdots + \lambda_k \chi^{C_k}$$

and $\lambda_1 + \cdots + \lambda_k = 1$, in strongly polynomial time.

Proof. By Theorem 51.6, we can find a $z \in P_{\text{spanning tree}}(G)$ with $x \geq z$ in strongly polynomial time. By Theorem 51.5, we can decompose z as a convex combination of incidence vectors of spanning trees in strongly polynomial time. This gives a decomposition as required. ∎

Fractionally packing and covering trees and forests

We now consider the problem of finding a maximum fractional packing of spanning trees subject to a given capacity function, and its dual, finding a minimum fractional covering by forests of a given demand function.

Since we have proved above that convex decompositions can be found in strongly polynomial time, we only need to give a method to find the optimum *values* of the fractional packing and covering.

The method is a variant of a 'fractional programming method' initiated by Isbell and Marlow [1956], and developed by Dinkelbach [1967], Schaible [1976], Picard and Queyranne [1982a], Padberg and Wolsey [1984], and Cunningham [1985c].

It implies the following result of Picard and Queyranne [1982a] and Padberg and Wolsey [1984]:

Theorem 51.7. *Given a graph $G = (V, E)$ and $y \in \mathbb{Q}_+^E$, we can find the minimum λ such that $y \in \lambda \cdot P_{\text{forest}}(G)$, in strongly polynomial time.*

Proof. We can assume that y does not belong to the forest polytope. (Otherwise multiply y by a sufficiently large scalar.) Let L be the line through $\mathbf{0}$ and y. We iteratively reset y as follows. Find a nonempty subset U of V minimizing $|U| - 1 - y(E[U])$. Let y' be the vector on L with $|U| - 1 - y'(E[U]) = 0$.

Now if y' violates $x(E[U']) \leq |U'| - 1$ for some U', then $|U'| < |U|$, since the function $d(x) := (|U| - 1 - x(E[U])) - (|U'| - 1 - x(E[U']))$ is nonpositive at y and positive at y', implying that it is positive at $\mathbf{0}$ (as d is linear in x).

We reset $y := y'$ and iterate, until y belongs to $P_{\text{forest}}(G)$. So after at most $|V|$ iterations the process terminates, with a y on the boundary of $P_{\text{forest}}(G)$. Comparing the final y with the original y gives the required λ. ∎

Hence we have for fractional forest covering:

Corollary 51.7a. *Given a graph $G = (V, E)$ and $y \in \mathbb{Q}_+^E$, we can find forests $F_1, \ldots, F_k$ and rationals $\lambda_1, \ldots, \lambda_k \geq 0$ such that*

$$y = \lambda_1 \chi^{F_1} + \cdots + \lambda_k \chi^{F_k} \tag{51.34}$$

with $\lambda_1 + \ldots + \lambda_k$ minimal, in strongly polynomial time.

Proof. By Theorem 51.7, we can find the minimum value of λ such that y belongs to $\lambda \cdot P_{\text{forest}}(G)$. If $\lambda = 0$, then $y = \mathbf{0}$, and (51.34) is trivial. If $\lambda > 0$, then by Corollary 51.5a we can decompose $\lambda^{-1} \cdot y$ as a convex combination of incidence vectors of forests. This gives the required decomposition of y. ∎

Similar results holds for fractional tree packing (Cunningham [1984, 1985c]). First one has:

Theorem 51.8. *Given a connected graph $G = (V, E)$ and $y \in \mathbb{Q}_+^E$, we can find the maximum λ such that $y \in \lambda \cdot P_{\text{connector}}(G)$, in strongly polynomial time.*

Proof. If $\text{supp}(y)$ is not a connector, then $\lambda = 0$. So we may assume that $\text{supp}(y)$ is a connector. We can also assume that $y \notin P_{\text{connector}}(G)$. Let L be the line through $\mathbf{0}$ and y. We iteratively reset y as follows. Find a partition $\mathcal{P}$ of V into nonempty sets minimizing $y(\delta(\mathcal{P})) - (|\mathcal{P}| - 1)$ (by Corollary 51.3b). Let y' be the vector on L with $y'(\delta(\mathcal{P})) = |\mathcal{P}| - 1$.

Now if y' violates $x(\delta(\mathcal{P}')) \geq |\mathcal{P}'| - 1$ for some partition $\mathcal{P}'$ of V into nonempty sets, then $|\mathcal{P}'| < |\mathcal{P}|$, since the function $d(x) := (x(\delta(\mathcal{P})) - |\mathcal{P}| + 1) - (x(\delta(\mathcal{P}')) - |\mathcal{P}'| + 1)$ is nonpositive at y and positive at y', implying that it is negative at $\mathbf{0}$ (as d is linear in x).

We reset $y := y'$ and iterate, until y belongs to $P_{\text{connector}}(G)$. So after at most $|V|$ iterations the process terminates, in which case we have a y on the boundary of $P_{\text{connector}}(G)$. Comparing the final y with the original y gives the required λ. ∎

This implies for fractional tree packing:

Corollary 51.8a. *Given a connected graph $G = (V, E)$ and $x \in \mathbb{Q}_+^E$, we can find spanning trees $T_1, \ldots, T_k$ and rationals $\lambda_1, \ldots, \lambda_k \geq 0$ such that*

(51.35) $$x \geq \lambda_1 \chi^{T_1} + \cdots + \lambda_k \chi^{T_k}$$

with $\lambda_1 + \ldots + \lambda_k$ maximal, in strongly polynomial time.

Proof. By Theorem 51.8, we can find the maximum value of λ such that x belongs to $\lambda \cdot P_{\text{connector}}(G)$. If $\lambda = 0$, we take $k = 0$. If $\lambda > 0$, by Corollary 51.6a we can decompose $\lambda^{-1} \cdot x$ as a convex combination of incidence vectors of connectors. This gives the required decomposition of x. ∎

Integer packing and covering of trees

It is not difficult to derive integer versions of the above algorithms, but they are not strongly polynomial-time, as we round numbers in it. In fact, an integer packing or covering cannot be found in strongly polynomial time, as it would imply a strongly polynomial-time algorithm for testing if an integer k is even (which algorithm does not exist[6]): k is even if and only if K_3 has $\frac{3}{2}k$ spanning trees containing each edge at most k times.

[6] For any strongly polynomial-time algorithm with one integer k as input, there is a number L and a rational function $q : \mathbb{Z} \to \mathbb{Q}$ such that if $k > L$, then the output equals $q(k)$. (This can be proved by induction on the number of steps of the algorithm.) However, there do not exist a rational function q and a number L such that for $k > L$, $q(k) = 0$ if k is even, and $q(k) = 1$ if k is odd.

Weakly polynomial-time algorithms follow directly from the fractional case with the help of the theorems of Nash-Williams and Tutte on disjoint trees and covering forests.

Theorem 51.9. *Given a graph $G = (V, E)$ and $y \in \mathbb{Z}_+^E$, we can find forests $F_1, \ldots, F_t$ and integers $\lambda_1, \ldots, \lambda_t \geq 0$ such that*

(51.36) $$y = \lambda_1 \chi^{F_1} + \cdots + \lambda_t \chi^{F_t}$$

with $\lambda_1 + \ldots + \lambda_t$ minimal, in polynomial time.

Proof. First find $F_1, \ldots, F_k$ and $\lambda_1, \ldots, \lambda_k$ as in Corollary 51.7a. We can assume that $k \leq |E|$ (by Carathéodory's theorem). Let

(51.37) $$y' := \sum_{i=1}^{k} (\lambda_i - \lfloor \lambda_i \rfloor) \chi^{F_i} = y - \sum_{i=1}^{k} \lfloor \lambda_i \rfloor \chi^{F_i}.$$

So y' is integer.

Replace each edge e by y'_e parallel edges, making G'. By Theorem 51.2, we can find a minimum number of forests partitioning the edges of G', in polynomial time (as $y'_e \leq |E|$ for each $e \in E$). This gives forests $F_{k+1}, \ldots, F_t$ in G.

Setting $\lambda_i := 1$ for $i = k+1, \ldots, t$, we show that this gives a solution of our problem. Trivially, (51.36) is satisfied (with λ_i replaced by $\lfloor \lambda_i \rfloor$). By Nash-Williams' covering forests theorem (Theorem 51.1c), using (51.37),

(51.38) $$t - k \leq \lceil \sum_{i=1}^{k} (\lambda_i - \lfloor \lambda_i \rfloor) \rceil.$$

Therefore,

(51.39) $$\sum_{i=1}^{t} \lfloor \lambda_i \rfloor = (t - k) + \sum_{i=1}^{k} \lfloor \lambda_i \rfloor \leq \lceil \sum_{i=1}^{k} \lambda_i \rceil,$$

proving that the decomposition is optimum. ∎

One similarly shows for tree packing:

Theorem 51.10. *Given a connected graph $G = (V, E)$ and $y \in \mathbb{Z}_+^E$, we can find spanning trees $T_1, \ldots, T_t$ and integers $\lambda_1, \ldots, \lambda_t \geq 0$ such that*

(51.40) $$y \geq \lambda_1 \chi^{T_1} + \cdots + \lambda_t \chi^{T_t}$$

with $\lambda_1 + \ldots + \lambda_t$ maximal, in polynomial time.

Proof. First find $T_1, \ldots, T_k$ and $\lambda_1, \ldots, \lambda_k$ as in Corollary 51.8a. We can assume that $k \leq |E|$ (by Carathéodory's theorem). Let

(51.41) $$y' := \lceil \sum_{i=1}^{k} (\lambda_i - \lfloor \lambda_i \rfloor) \chi^{T_i} \rceil.$$

Replace each edge e by y'_e parallel edges, making G'. By Theorem 51.2, we can find a maximum number of edge-disjoint spanning trees in G', in polynomial time (as $y'_e \leq |E|$ for each $e \in E$). This gives spanning trees $T_{k+1}, \ldots, T_t$ in G.

Setting $\lambda_i := 1$ for $i = k+1, \ldots, t$, we show that this gives a solution of our problem. Trivially, (51.40) is satisfied (with λ_i replaced by $\lfloor \lambda_i \rfloor$). By the Tutte-Nash-Williams disjoint trees theorem using (51.41),

$$t - k \geq \lfloor \sum_{i=1}^{k} (\lambda_i - \lfloor \lambda_i \rfloor) \rfloor. \tag{51.42}$$

Therefore,

$$\sum_{i=1}^{t} \lfloor \lambda_i \rfloor = (t-k) + \sum_{i=1}^{k} \lfloor \lambda_i \rfloor \geq \lfloor \sum_{i=1}^{k} \lambda_i \rfloor, \tag{51.43}$$

proving that the decomposition is optimum. ∎

51.5. Further results and notes

51.5a. Complexity survey for tree packing and covering

Complexity survey for finding a maximum number of (or k) disjoint spanning trees ($*$ indicates an asymptotically best bound in the table):

	$O(m^2 \log n)$	Imai [1983a]
	$O(m^2)$	Roskind and Tarjan [1985] (announced by Tarjan [1976]) *for simple graphs*
$*$	$O(m\sqrt{\frac{m}{n}}(m + n \log n) \log \frac{m}{n})$	Gabow and Westermann [1988,1992]
$*$	$O(nm \log \frac{m}{n})$	Gabow and Westermann [1988,1992]
$*$	$O(kn\sqrt{m + kn \log n})$	Gabow [1991a] (announced)

Complexity survey for finding a minimum number of forests covering all edges of the graph:

$O(n^4)$	Picard and Queyranne [1982a] (finding the number) *for simple graphs*
$O(n^2 m \log^2 n)$	Picard and Queyranne [1982a] (finding the number) *for simple graphs*

≫

continued

	$O(m^2)$	Imai [1983a], Roskind and Tarjan [1985] (announced by Tarjan [1976]) *for simple graphs*
*	$O(nm \log n)$	Gabow and Westermann [1988,1992]
	$O(m(m(m + n \log n) \log m)^{1/3})$	Gabow and Westermann [1988,1992]
*	$O(m^{3/2} \log(n^2/m))$	Gabow [1995b,1998]

Liu and Wang [1988] gave an $O(k^2n^2m(m + kn^2))$-time algorithm to find a minimum-weight union F of k edge-disjoint spanning trees in a graph $G = (V, E)$, where E is partitioned into classes $E_1, \ldots, E_t$, such that $a_i \leq |F \cap E_i| \leq b_i$ for each i, given a partition $E_1, \ldots, E_t$ of E and numbers a_i and b_i for all i.

Complexity survey for finding a maximum-size union of k forests:

	$O(k^2n^2)$	Imai [1983a], Roskind and Tarjan [1985] (announced by Tarjan [1976]) *for simple graphs*
	$O(k^{3/2}\sqrt{nm(m + n \log n)})$	Gabow and Stallmann [1985]
*	$O(k^{3/2}n\sqrt{m + n \log n})$	Gabow and Westermann [1988,1992]
*	$O(k^{1/2}m\sqrt{m + n \log n})$	Gabow and Westermann [1988,1992]
*	$O(kn^2 \log k)$	Gabow and Westermann [1988,1992]
*	$O(\frac{m^2}{k} \log k)$	Gabow and Westermann [1988,1992]

Algorithms for finding a maximum-size union of two forests were given by Kishi and Kajitani [1967,1968,1969] and Kameda and Toida [1973].

Complexity survey for finding a maximum-weight union of k forests:

	$O(k^2n^2 + m \log m)$	Roskind and Tarjan [1985] *for simple graphs*
*	$O(kn^2 \log k + m \log m)$	Gabow and Westermann [1988,1992]
*	$O(\frac{m^2}{k} \log k + m \log m)$	Gabow and Westermann [1988,1992]

Roskind and Tarjan [1985] (cf. Clausen and Hansen [1980]) gave an $O(k^2n^2 + m \log m)$-time algorithm for finding a maximum-weight union of k disjoint spanning trees, in a simple graph.

As for **the capacitated case**, the methods given in Section 51.4 indicate that packing and covering problems on forests and trees can be solved by a series of minimum-capacity cut problem (as they reduce to Theorem 51.3). A parametric minimum cut method designed by Gallo, Grigoriadis, and Tarjan [1989] allows to combine several consecutive minimum cut computations, improving the efficiency of the corresponding tree packing and covering problem, as was done by Gusfield [1991].

The published algorithms for integer packing and coverings of trees all are based on rounding the fractional version, not increasing the complexity of the problem, except that rounding is included as an operation. This blocks these algorithms from being strongly polynomial-time: as we saw in Section 51.4, it can be proved that there exists no strongly polynomial-time algorithm for finding an optimum integer packing of spanning trees under a given capacity (similarly, for integer covering by forests).

The following table gives a complexity survey for finding a maximum fractional packing of spanning trees subject to a given integer capacity function c, or a minimum fractional covering by forests subject to a given demand function c. Here it seems that the optimum value can be found faster than an explicit fractional packing or covering. The problems of finding an optimum fractional packing of trees is close to that of finding an optimum fractional covering of forests (or trees), so we present their complexity in one survey.

For any graph $G = (V, E)$ and $c : E \to \mathbb{R}_+$, the *strength* is the maximum value of λ such that c belongs to $\lambda \cdot P_{\text{connector}}(G)$. It is equal to the maximum size of a fractional packing of spanning trees subject to c. The *fractional arboricity* is the minimum value of λ such that c belongs to $\lambda \cdot P_{\text{forest}}(G)$. This is equal to the minimum size of a fractional c-covering by forest.

	$O(nm^8)$	Cunningham [1984]: finding an optimum fractional packing of trees
	$O(nm \cdot \mathrm{MF}(n, n^2))$	Cunningham [1985c]: computing strength
	$O(n^4 m^2 \log^2 C)$	Gabow [1991a] (announced): computing strength
*	$O(n^3 m)$	Gusfield [1991]: computing strength
	$O(nm^2 \log(n^2/m))$	Gusfield [1991]: computing strength
	$O(n^3 \cdot \mathrm{MF}(n, m))$	Trubin [1991]: finding an optimum fractional packing of trees
	$O(n^2 \cdot \mathrm{MF}(n, n^2))$	Barahona [1992]: computing strength
	$O(n^2 \cdot \mathrm{MF}(n, n^2))$	Barahona [1995]: finding optimum fractional packing of trees
*	$O(n \cdot \mathrm{MF}(n, m))$	Cheng and Cunningham [1994]: computing strength
*	$O(n \cdot \mathrm{MF}(n, m))$	Gabow [1995b,1998]: computing strength and fractional arboricity
*	$O(n^3 m \log(n^2/m))$	Gabow and Manu [1995,1998]: finding an optimum fractional packing of trees and an optimum fractional covering by forests
*	$O(n^2 m \log C \log(n^2/m))$	Gabow and Manu [1995,1998]: finding an optimum fractional packing of trees and an optimum fractional covering by forests

Here $\mathrm{MF}(n, m)$ is the complexity of finding a maximum-value $s - t$ flow subject to c in a digraph with n vertices and m arcs, and $C := \|c\|_{\max}$ (assuming c integer).

51.5b. Further notes

A special case of a question asked by A. Frank (cf. Schrijver [1979b], Frank [1995]) amounts to the following:

(51.44) (?) Let $G = (V, E)$ be an undirected graph and let $s \in V$. Suppose that for each vertex $t \neq s$, there exist k internally vertex-disjoint $s-t$ paths. Then G has k spanning trees such that for each vertex $t \neq s$, the $s-t$ paths in these trees are internally vertex-disjoint. (?)

(The spanning trees need not be edge-disjoint — otherwise $G = K_3$ would form a counterexample.) For $k = 2$, (51.44) was proved by Itai and Rodeh [1984,1988], and for $k = 3$ by Cheriyan and Maheshwari [1988] and Zehavi and Itai [1989].

Peng, Chen, and Koh [1991] showed that for any undirected graph $G = (V, E)$ and any $p, k \in \mathbb{Z}_+$, there exist k disjoint forests each with p components if and only if

(51.45) $$|\delta(\mathcal{P})| \geq k(|\mathcal{P}| - p)$$

for each partition $\mathcal{P}$ of V into nonempty sets. This in fact is the matroid base packing theorem (Corollary 42.1d) applied to the $(|V| - p)$-truncation of the cycle matroid of G.

Theorem 42.10 of Seymour [1998] implies that if the edges of a graph $G = (V, E)$ can be partitioned into k forests and if for each $e \in E$ a subset L_e of $\{1, 2, \ldots\}$ with $|L_e| = k$ is given, then we can partition E into forests $F_1, F_2, \ldots$ such that $j \in L_e$ for each $j \geq 1$ and each $e \in F_j$.

Henneberg [1911] and Laman [1970] characterized those graphs which have, after adding any edge, two edge-disjoint spanning trees. This was extended to k edge-disjoint spanning trees by Frank and Szegő [2001].

Farber, Richter, and Shank [1985] showed the following. Let $G = (V, E)$ be an undirected graph. Let $\mathcal{V}$ be the collection of pairs (T_1, T_2) of edge-disjoint spanning trees T_1 and T_2 in G. Call two pairs (T_1, T_2) and (S_1, S_2) in $\mathcal{V}$ *adjacent* if $|(T_1 \cup T_2) - (S_1 \cup S_2)| = 2$. Then this determines a connected graph on $\mathcal{V}$.

Cunningham [1985c] gave a strongly polynomial-time algorithm ($O(nm \min\{n^2, m \log n\})$) to find a minimum-cost set of capacities to be added to a capacitated graph so as to create the existence of k edge-disjoint spanning trees; that is, given $G = (V, E)$ and $c, k \in \mathbb{Z}_+^E$, solving

(51.46) $$\sum_{e \in E} k(e) x_e$$

where $x \in \mathbb{Z}_+^E$ satisfies

(51.47) $$(c + x)(\delta(\mathcal{P})) \geq k(|\mathcal{P}| - 1)$$

for each partition $\mathcal{P}$ of V into nonempty sets. (It amounts to finding a minimum-cost integer vector in a contrapolymatroid.) Related work can be found in Baïou, Barahona, and Mahjoub [2000].

Chapter 52

Longest branchings and shortest arborescences

We next consider trees in directed graphs. We recall some terminology and facts. Let $D = (V, A)$ be a digraph. A *branching* is a subset B of A such that B contains no undirected circuit and such that for each vertex v there is at most one arc in B entering v. A *root* of B is a vertex not entered by any arc in B. For any branching B, each weak component of (V, B) contains a unique root.
A branching B is called an *arborescence* if the digraph (V, B) is weakly connected; equivalently, if (V, B) is a rooted tree. So each arborescence B has a unique root r. We say that B is *rooted at* r, and we call B an r*-arborescence*. An r-arborescence can be characterized as a directed spanning tree B such that each vertex is reachable in B from r. A digraph $D = (V, A)$ contains an r-arborescence if and only if each vertex of D is reachable from r.

52.1. Finding a shortest r-arborescence

Let be given a digraph $D = (V, A)$, a vertex r, and a length function $l : A \to \mathbb{Q}_+$. We consider the problem of finding a shortest (= minimum-length) r-arborescence.

We cannot apply here the greedy method of starting at the root r and iteratively extending an r-arborescence on a subset U of V, by the shortest arc leaving U. This is shown by the example of Figure 52.1.

The following algorithm was given by Chu and Liu [1965], Edmonds [1967a], and Bock [1971]:

Algorithm to find a shortest r-arborescence. Let $A_0 := \{a \in A \mid l(a) = 0\}$. If A_0 contains an r-arborescence B, then B is a shortest r-arborescence. If A_0 contains no r-arborescence, there is a strong component K of (V, A_0) with $r \notin K$ and with $l(a) > 0$ for each $a \in \delta^{\text{in}}(K)$. Let $\alpha := \min\{l(a) \mid a \in \delta^{\text{in}}(K)\}$. Set $l'(a) := l(a) - \alpha$ if $a \in \delta^{\text{in}}(K)$ and $l'(a) := l(a)$ otherwise.

Find (recursively) a shortest r-arborescence B with respect to l'. As K is a strong component of (V, A_0), we can choose B such that $|B \cap \delta^{\text{in}}(K)| = 1$

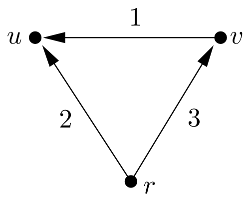

Figure 52.1
In a greedy method one would first choose the shortest arc leaving r, which is (r,u). This arc however is not contained in the shortest r-arborescence.

(since if $|B \cap \delta^{\text{in}}(K)| \geq 2$, then there exists an $a \in B \cap \delta^{\text{in}}(K)$ such that the set $(B \setminus \{a\}) \cup A_0$ contains an r-arborescence, say B', with $l'(B') \leq l'(B) - l'(a) \leq l'(B)$).

Then B is also a shortest r-arborescence with respect to l, since for any r-arborescence B':

(52.1) $$l(B') = l'(B') + \alpha|B' \cap \delta^{\text{in}}(K)| \geq l'(B') + \alpha \geq l'(B) + \alpha = l(B).$$

Since the number of iterations is at most m (as in each step A_0 increases), we have:

Theorem 52.1. *A shortest r-arborescence can be found in strongly polynomial time.*

Proof. See above. ∎

In fact, direct analysis gives the following result of Chu and Liu [1965], Edmonds [1967a], and Bock [1971]:

Theorem 52.2. *A shortest r-arborescence can be found in time $O(nm)$.*

Proof. First note that there are at most $2n$ iterations. This can be seen as follows. Let k be the number of strong components of (V, A_0), and let k_0 be the number of strong components K of (V, A_0) with $d^{\text{in}}_{A_0}(K) = 0$. Then at any iteration, the number $k + k_0$ decreases. Indeed, if the strong component K selected remains a strong component, then $d^{\text{in}}_{A_0}(K) \neq 0$ in the next iteration; so k_0 decreases. Otherwise, k decreases. Hence there are at most $2n$ iterations.

Next, each iteration can be performed in time $O(m)$. Indeed, in time $O(m)$ we can identify the set U of vertices not reachable in (V, A_0) from r. Next, by Theorem 6.6 one can identify the strong components of the subgraph of (V, A_0) induced by U, in time $O(m)$. Moreover, by Theorem 6.5 we can order the vertices in U pre-topologically. Then the first vertex in this order belongs to a strong component K such that each arc a entering K has $l(a) > 0$. ∎

Tarjan [1977] showed that this algorithm has an $O(\min\{n^2, m \log n\})$-time implementation.

52.1a. r-arborescences as common bases of two matroids

Let $D = (V, A)$ be a digraph and let $r \in V$. The r-arborescences can be considered as the common bases in two matroids on A: M_1 is the cycle matroid of the underlying undirected graph, and M_2 is the partition matroid on A induced by the sets $\delta^{\text{in}}(v)$ for $v \in V \setminus \{r\}$. We assume without loss of generality that no arc of D enters r.

Then the r-arborescences are exactly the common bases of M_1 and M_2. This gives us a reduction of polyhedral and algorithmic results to matroid intersection. In particular, Theorem 52.1 follows from the strong polynomial-time solvability of weighted matroid intersection.

52.2. Related problems

The complexity results of Section 52.1 immediately imply similar results for finding optimum branchings and arborescences without specifying a root. First we note:

Corollary 52.2a. *Given a digraph $D = (V, A)$, $r \in V$, and a length function $l : A \to \mathbb{Q}$, a longest r-arborescence can be found in $O(nm)$ time.*

Proof. Define $L := \max\{l(a) \mid a \in A\}$ and $l'(a) := L - l(a)$ for each $a \in A$. Then an r-arborescence B minimizing $l'(B)$ is an r-arborescence maximizing $l(B)$. ∎

Then we have for longest branching:

Corollary 52.2b. *Given a digraph $D = (V, A)$ and a length function $l \in \mathbb{Q}^A$, a longest branching can be found in time $O(nm)$.*

Proof. We can assume that l is nonnegative, by deleting all arcs of negative length. Extend D by a new vertex r and new arcs (r, v) for all $v \in V$, each of length 0. Let B be a longest r-arborescence in D' (this can be found in $O(nm)$-time by Corollary 52.2a). Then trivially $B \cap A$ is a longest branching in D. ∎

Similarly, for finding a shortest arborescence, without prescribing a root:

Corollary 52.2c. *Given a digraph $D = (V, A)$ and a length function $l \in \mathbb{Q}^A_+$, a shortest arborescence can be found in time $O(nm)$.*

Proof. Extend D by a new vertex r and arcs (r, v) for each $v \in V$, giving digraph D'. Let $l(r, v) := Ln$, where $L := \max\{l(a) \mid a \in A\}$. If D has an arborescence, then a shortest r-arborescence in D' has only one arc leaving r, and deleting this arc gives a shortest arborescence in D. ∎

52.3. A min-max relation for shortest r-arborescences

We now characterize the minimum length of an r-arborescence. Let $D = (V, A)$ be a digraph and let $r \in V$. Call a set C of arcs an *r-cut* if there exists a nonempty subset U of $V \setminus \{r\}$ with

(52.2) $C = \delta^{\text{in}}(U)$.

It is not difficult to show that

(52.3) the collection of inclusionwise minimal arc sets intersecting each r-arborescence is equal to the collection of inclusionwise minimal r-cuts,

and

(52.4) the collection of inclusionwise minimal arc sets intersecting each r-cut is equal to the collection of r-arborescences.

The following theorem follows directly from the method of Edmonds [1967a], and was stated explicitly by Bock [1971] (and also by Fulkerson [1974]):

Theorem 52.3 (optimum arborescence theorem)**.** *Let $D = (V, A)$ be a digraph, let $r \in V$, and let $l : A \to \mathbb{Z}_+$. Then the minimum length of an r-arborescence is equal to the maximum size of a family of r-cuts such that each arc a is in at most $l(a)$ of them.*

Proof. Clearly, the maximum is not more than the minimum, as each r-cut intersects each r-arborescence.

We prove the reverse inequality by induction on $\sum_{a \in A} l(a)$. Let $A_0 := \{a \in A \mid l(a) = 0\}$. If A_0 contains an r-arborescence, the minimum is 0, while the maximum is at least 0.

If A_0 contains no r-arborescence, there exists a strong component K of the digraph (V, A_0) with $r \notin K$ and with $l(a) > 0$ for each $a \in \delta^{\text{in}}(K)$. Define $l' := l - \chi^{\delta^{\text{in}}(K)}$. By induction there exist an r-arborescence B and r-cuts $C_1, \ldots, C_t$ such that each arc a is in at most $l'(a)$ of the C_i and such that $l'(B) = t$. We may assume that $|B \cap \delta^{\text{in}}(K)| = 1$, since if $|B \cap \delta^{\text{in}}(K)| \geq 2$, then for each $a \in B \cap \delta^{\text{in}}(K)$, $(B \setminus \{a\}) \cup A_0$ contains an r-arborescence, say B', with $l'(B') \leq l'(B) - l'(a) \leq l'(B)$.

It follows that $l(B) = t + 1$. Moreover, taking $C_{t+1} := \delta^{\text{in}}(K)$, each arc a is in at most $l(a)$ of the $C_1, \ldots, C_{t+1}$. ∎

Note that if B is a shortest r-arborescence, then $|B \cap C| = 1$ for any r-cut C in the maximum-size family. Moreover, for any $a \in B$, $l(a)$ is equal to the number of r-cuts C chosen with $a \in C$.

52.4. The r-arborescence polytope

Given a digraph $D = (V, A)$ and a vertex $r \in V$, the *r-arborescence polytope* is defined as the convex hull of the incidence vectors (in $\mathbb{R}^A$) of the r-arborescences; that is,

(52.5) $$P_{r\text{-arborescence}}(D) := \text{conv.hull}\{\chi^B \mid B \ r\text{-arborescence}\}.$$

Theorem 52.3 implies that the r-arborescence polytope of D is determined by:

(52.6)
$$\begin{array}{rll} \text{(i)} & x_a \geq 0 & \text{for } a \in A, \\ \text{(ii)} & x(C) \geq 1 & \text{for each } r\text{-cut } C, \\ \text{(iii)} & x(\delta^{\text{in}}(v)) = 1 & \text{for } v \in V \setminus \{r\}. \end{array}$$

To prove this, we first characterize the up hull of the r-arborescence polytope, where as usual the up hull of the r-arborescence polytope is defined as

(52.7) $$P^{\uparrow}_{r\text{-arborescence}}(D) := P_{r\text{-arborescence}}(D) + \mathbb{R}^A_+.$$

Corollary 52.3a. $P^{\uparrow}_{r\text{-arborescence}}(D)$ *is determined by*

(52.8)
$$\begin{array}{rll} \text{(i)} & x_a \geq 0 & \text{for } a \in A, \\ \text{(ii)} & x(C) \geq 1 & \text{for each } r\text{-cut } C. \end{array}$$

Proof. The incidence vector of any r-arborescence trivially satisfies (52.8); hence $P^{\uparrow}_{r\text{-arborescence}}(D)$ is contained in the polyhedron Q determined by (52.8).

Suppose that the reverse inclusion does not hold. Then there exists a rational length function $l \in \mathbb{Q}^A_+$ such that the minimum value of $l^{\mathsf{T}}x$ over Q is less than the minimum length of an r-arborescence. We can assume that l is integer. However, the minimum value of $l^{\mathsf{T}}x$ over Q cannot be less than the maximum described in Theorem 52.3. So we have a contradiction. ▮

Since the r-arborescence polytope is a face of its up hull, this implies:

Corollary 52.3b. *The r-arborescence polytope is determined by* (52.6).

Proof. Directly from Corollary 52.3a. ▮

Corollary 52.3a also implies for the restriction to the unit cube:

Corollary 52.3c. *The convex hull of incidence vectors of arc sets containing an r-arborescence is determined by*

(52.9) (i) $0 \leq x_a \leq 1$ for $a \in A$,
(ii) $x(C) \geq 1$ for each r-cut C.

Proof. Directly from Corollary 52.3a with Theorem 5.19. ∎

Theorem 52.3 can be reformulated in TDI terms as:

Corollary 52.3d. *System* (52.8) *is TDI.*

Proof. Choose a length function $l \in \mathbb{Z}_+^A$, and consider the dual problem of minimizing $l^{\mathsf{T}}x$ over (52.8). For each r-cut C, let y_C be the number of times C is chosen in the maximum family in Theorem 52.3. Moreover, let B be a shortest r-arborescence. Then by Theorem 52.3, $x := \chi^B$ and the y_C form a dual pair of optimum solutions. As the y_C are integer, it follows that (52.8) is TDI. ∎

This in turn implies for the r-arborescence polytope:

Corollary 52.3e. *System* (52.6) *is TDI.*

Proof. Directly from Corollary 52.3d, with Theorem 5.25, since (52.6) arises from (52.8) by setting some of the inequalities to equality. ∎

For the intersection with the unit cube it gives:

Corollary 52.3f. *System* (52.9) *is TDI.*

Proof. Directly from Corollary 52.3d, with Theorem 5.23. ∎

In fact, (poly)matroid intersection theory gives the box-total dual integrality of (52.8):

Theorem 52.4. *System* (52.8) *is box-TDI.*

Proof. Let M_1 be the cycle matroid of the undirected graph underlying $D = (V, A)$, and let M_2 be the partition matroid induced by the sets $\delta^{\text{in}}(v)$ for $v \in V \setminus \{r\}$. By Corollary 46.1d, the system

(52.10) $x(B) \geq |V| - 1 - r_{M_i}(A \setminus B)$ for $i = 1, 2$ and $B \subseteq A$,

is box-TDI. Now any inequality in (52.10) is a nonnegative integer combination of inequalities (52.8).

Indeed, if $i = 1$, then $r_{M_1}(A \setminus B)$ is equal to $|V|$ minus the number of weak components of the digraph $(V, A \setminus B)$. So the inequality in (52.10) states that $x(B)$ is at least the number of weak components of $(V, A \setminus B)$ not containing r.

Hence it is a sum of the inequalities $x(\delta^{\text{in}}(K)) \geq 1$ for each weak component K of $(V, A \setminus B)$ not containing r, and of $x_a \geq 0$ for all $a \in B$ not entering any of these components.

If $i = 2$, then $r_{M_2}(A \setminus B)$ is equal to the number of $v \neq r$ entered by at least one arc in $A \setminus B$. So the inequality in (52.10) states that $x(B)$ is at least the number of $v \neq r$ with $\delta^{\text{in}}(v) \subseteq B$. It therefore is a sum of the inequalities $x(\delta^{\text{in}}(v)) \geq 1$ for these v, and $x_a \geq 0$ for all $a \in B$ not entering any of these vertices.

So Corollary 46.1d implies that (52.8) is box-TDI. ∎

52.4a. Uncrossing cuts

Edmonds and Giles [1977] and Frank [1979b] gave the following procedure of proving that system (52.8) is box-TDI (cf. Corollary 52.3b). The proof is longer than that given above, but it is a special case of a far more general approach (to be discussed in Chapter 60), and is therefore worth noting at this point.

System (52.8) is equivalent to:

(52.11) (i) $x_a \geq 0$ for $a \in A$,
(ii) $x(\delta^{\text{in}}(U)) \geq 1$ for $\emptyset \neq U \subseteq V \setminus \{r\}$

Consider any length function $l \in \mathbb{R}_+^A$. Let y_U form an optimum solution to the problem dual to minimizing $l^{\mathsf{T}}x$ over (52.11):

(52.12)
$$\begin{array}{ll} \text{maximize} & \sum_U y_U \\ \text{subject to} & y_U \geq 0 \text{ for all } U, \\ & \sum_U y_U \chi^{\delta^{\text{in}}(U)} \leq l, \end{array}$$

where U ranges over the nonempty subsets of $V \setminus \{r\}$.

Choose the y_U in such a way that

(52.13)
$$\sum_U y_U |U||V \setminus U|$$

is as small as possible. Then the collection

(52.14)
$$\mathcal{F} := \{U \mid y_U > 0\}$$

is laminar; that is,

(52.15)
$$U \cap W = \emptyset \text{ or } U \subseteq W \text{ or } W \subseteq U \text{ for all } U, W \in \mathcal{F}.$$

For suppose not. Let $\alpha := \min\{y_U, y_W\}$. Decrease y_U and y_W by α, and increase $y_{U \cap W}$ and $y_{U \cup W}$ by α. Then y remains a feasible dual solution, since

(52.16)
$$\chi^{\delta^{\text{in}}(U \cap W)} + \chi^{\delta^{\text{in}}(U \cup W)} \leq \chi^{\delta^{\text{in}}(U)} + \chi^{\delta^{\text{in}}(W)}.$$

Moreover, y remains trivially optimum. However, sum (52.13) decreases (by Theorem 2.1), contradicting our assumption. So $\mathcal{F}$ is laminar.

Now the $\mathcal{F} \times A$ matrix M with

(52.17)
$$M_{U,a} := \begin{cases} 1 & \text{if } a \in \delta^{\text{in}}(U), \\ 0 & \text{otherwise,} \end{cases}$$

is totally unimodular. In fact, it is a network matrix. For make a directed tree T as follows. The vertex set of T is the set $\mathcal{F}' := \mathcal{F} \cup \{V\}$, while for each $U \in \mathcal{F}$ there is an arc a_U from W to U where W is the smallest set in $\mathcal{F}'$ with $W \supset U$. This is in fact an arborescence with root V.

We also define a digraph $\widetilde{D} = (\mathcal{F}', \widetilde{A})$. For each arc $a = (u, v)$ of D, let $\tilde{a}$ be an arc from the smallest set in $\mathcal{F}'$ containing both u and v, to the smallest set in $\mathcal{F}'$ containing v. Let $\widetilde{A} := \{\tilde{a} \mid a \in A\}$.

Identifying any set U in $\mathcal{F}$ with the arc a_U of T, the network matrix generated by directed tree T and digraph $\widetilde{D}$ is an $\mathcal{F} \times \widetilde{A}$ matrix which is the same as M. So M is totally unimodular. Therefore, by Theorem 5.35, (52.11) is box-TDI.

52.5. A min-max relation for longest branchings

We now consider longest branchings. Characterizing the maximum size of a branching is easy:

Theorem 52.5. *Let $D = (V, A)$ be a digraph. Then the maximum size of a branching is equal to $|V|$ minus the number of strong components K of D with $d^{\text{in}}_A(K) = 0$.*

Proof. The theorem follows directly from: (i) each branching has at least one root in any strong component K of D with $d^{\text{in}}_A(K) = 0$, and (ii) if a set R intersects each such K, then there is a branching with root set R (since each vertex of D is reachable from R). ∎

From Theorem 52.3 one can derive a min-max relation for the maximum length of a branching in a digraph. The reduction is similar to the reduction of the algorithmic problem of finding a longest branching to that of finding a shortest r-arborescence.

However, a direct proof can be derived from matroid intersection. Consider the system:

(52.18)	(i)	$x_a \geq 0$	for $a \in A$,
	(ii)	$x(\delta^{\text{in}}(v)) \leq 1$	for $v \in V$,
	(iii)	$x(A[U]) \leq \lvert U\rvert - 1$	for $U \subseteq V$, $U \neq \emptyset$.

Theorem 52.6. *System* (52.18) *is TDI.*

Proof. Directly from Theorem 41.12, applied to the cycle matroid M_1 of the undirected graph underlying $D = (V, A)$, and the partition matroid M_2 induced by the sets $\delta^{\text{in}}(v)$ for $v \in V$. Then each inequality $x(B) \leq r_{M_1}(B)$ is the sum of the inequalities $x(A[U]) \leq |U| - 1$ for the weak components U of (V, B), and $-x_a \leq 0$ for those arcs $a \in A \setminus B$ contained in any weak component of (V, B). Each inequality $x(B) \leq r_{M_2}(B)$ is the sum of the inequalities $x(\delta^{\text{in}}(v)) \leq 1$ for those v entered by at least one arc in B, and

$-x_a \leq 0$ for those arcs $a \in A \setminus B$ that enter a vertex v entered by at least one arc in B. ∎

52.6. The branching polytope

The previous corollary immediately implies a description of the *branching polytope* $P_{\text{branching}}(D)$ of D, which is the convex hull of the incidence vectors of branchings in D (stated by Edmonds [1967a]):

Corollary 52.6a. *The branching polytope of $D = (V, A)$ is determined by* (52.18).

Proof. Directly from Theorem 52.6, since the integer solutions of (52.18) are the incidence vectors of the branchings. ∎

Also the following theorem of Edmonds [1967a] follows from matroid intersection theory:

Corollary 52.6b. *Let $D = (V, A)$ be a digraph and let $k \in \mathbb{Z}_+$. Then the convex hull of the incidence vectors of branchings of size k is equal to the intersection of the branching polytope of D with the hyperplane $\{x \mid x(A) = k\}$.*

Proof. This is the common base polytope of the k-truncations of the matroids M_1 and M_2 defined in the proof of Theorem 52.6. ∎

In Corollary 53.3a we shall see that the convex hull of the incidence vectors of branchings of size k has the integer decomposition property (McDiarmid [1983]).

Giles and Hausmann [1979] characterized which pairs of branchings give adjacent vertices of the branching polytope, and Giles [1975,1978b] and Grötschel [1977a] characterized the facets of the branching polytope.

52.7. The arborescence polytope

The results on branchings in the previous section can be specialized to arborescences (without prescribed root). Given a digraph $D = (V, A)$, the *arborescence polytope* of D, denoted by $P_{\text{arborescence}}(D)$, is the convex hull of the incidence vectors of arborescences.

Corollary 52.6c. *The arborescence polytope is determined by*

$$\begin{array}{llll}
(52.19) & \text{(i)} & x_a \geq 0 & \text{for } a \in A, \\
& \text{(ii)} & x(\delta^{\text{in}}(v)) \leq 1 & \text{for } v \in V, \\
& \text{(iii)} & x(A[U]) \leq |U| - 1 & \text{for } U \subseteq V,\ U \neq \emptyset, \\
& \text{(iv)} & x(A) = |V| - 1. &
\end{array}$$

Proof. Directly from Corollary 52.6a, since $P_{\text{arborescence}}(D)$ is the face of $P_{\text{branching}}(D)$ determined by the hyperplane $x(A) = |V| - 1$. ∎

One similarly obtains from Theorem 52.6 the following, which yields a min-max relation for the minimum length of an arborescence:

Corollary 52.6d. *System* (52.19) *is TDI.*

Proof. From Theorem 52.6, with Theorem 5.25. ∎

52.8. Further results and notes

52.8a. Complexity survey for shortest r-arborescence

	$O(nm)$	Chu and Liu [1965], Edmonds [1967a], Bock [1971]
	$O(n^2)$	Tarjan [1977] (cf. Camerini, Fratta, and Maffioli [1979])
	$O(m \log n)$	Tarjan [1977] (cf. Camerini, Fratta, and Maffioli [1979])
	$O(n \log n + m \log\log\log_{m/n} n)$	Gabow, Galil, and Spencer [1984]
*	$O(m + n \log n)$	Gabow, Galil, Spencer, and Tarjan [1986]

As before, $*$ indicates an asymptotically best bound in the table.

X. Guozhi (see Guan [1979]), Gabow and Tarjan [1979,1984], and Gabow, Galil, Spencer, and Tarjan [1986] studied the problem of finding a shortest r-arborescence with exactly k arcs leaving r, yielding an $O(m+n \log n)$-time algorithm. Hou [1996] gave an $O(k^3m^3)$-time algorithm to find the k shortest r-arborescences in a digraph.

Gabow and Tarjan [1988a] gave $O(m+n \log n)$- and $O(m \log^* n)$-time algorithms for the *bottleneck* r-arborescence problem (that is, minimizing the maximum arc cost), improving the $O(m \log n)$-time algorithm of Camerini [1978]. (Here $\log^* n$ is the minimum i with $\log_2^{(i)} n \leq 1$.)

52.8b. Concise LP-formulation for shortest r-arborescence

Wong [1984] and Maculan [1986] observed that the problem of finding a shortest r-arborescence can be formulated as a concise linear programming problem. In fact, the dominant $P^{\uparrow}_{r\text{-arborescence}}(D)$ of the r-arborescence polytope is the projection of a polyhedron in nm dimensions determined by at most $n(2m+n)$ constraints.

Theorem 52.7. *Let $D = (V, A)$ be a digraph and let $r \in V$. Then $P^{\uparrow}_{r\text{-arborescence}}(D)$ is equal to the set Q of all vectors $x \in \mathbb{R}^A_+$ such that for each $u \in V \setminus \{r\}$ there exists an $r-u$ flow f_u of value 1 satisfying $f_u \leq x$.*

Proof. Since the incidence vector $x = \chi^B$ of any r-arborescence satisfies the constraints, we know that $P^{\uparrow}_{r\text{-arborescence}}(D)$ is contained in Q.

To see the reverse inclusion, let $x \in Q$. Then for each nonempty subset U of $V \setminus \{r\}$ one has

(52.20) $$x(\delta^{\text{in}}(U)) \geq f_u(\delta^{\text{in}}(U)) \geq 1,$$

where u is any vertex in U and where f_u is an $r-u$ flow of value 1 with $f_u \leq x$. So by Corollary 52.3a, x belongs to $P^{\uparrow}_{r\text{-arborescence}}(D)$. ∎

This implies that a shortest r-arborescence can be found by solving a linear programming problem of polynomial size:

Corollary 52.7a. *Let $D = (V, A)$ be a digraph and let $r \in V$ and $l \in \mathbb{R}^A_+$. Then the length of a shortest r-arborescence is equal to the minimum value of*

(52.21) $$\sum_{a \in A} l(a) x_a,$$

where $x \in \mathbb{R}^A$ is such that for each $u \in V \setminus \{r\}$ there exists an $r-u$ flow f_u of value 1 with $f_u \leq x$.

Proof. Directly from Theorem 52.7. ∎

52.8c. Further notes

Frank [1979b] showed the following. Let $D = (V, A)$ be a digraph and let $r \in V$. Then a subset A' of A is contained in an r-arborescence if and only if $|\mathcal{U}| \leq |V| - 1 - |A'|$ for each laminar collection $\mathcal{U}$ of nonempty subsets of $V \setminus \{r\}$ such that each arc of D enters at most one set in $\mathcal{U}$ and no arc in A' enters any set in $\mathcal{U}$.

Goemans [1992,1994] studied the convex hull of (not necessarily spanning) partial r-arborescences.

Karp [1972a] gave a shortening of the proof of Edmonds [1967a] of the correctness of the shortest r-arborescence algorithm.

Books covering shortest arborescences include Minieka [1978], Papadimitriou and Steiglitz [1982], and Gondran and Minoux [1984].

Chapter 53

Packing and covering of branchings and arborescences

Packing arborescences is a special case of packing common bases in two matroids. However, no general matroid theorem is known that covers this case. In Section 42.6c the maximum number of common bases in two strongly base orderable matroids was characterized, but this does not apply to packing arborescences, as graphic matroids are generally not strongly base orderable. Yet, min-max relations and polyhedral characterizations can be proved for packing arborescences, and similarly for covering by branchings.

53.1. Disjoint branchings

Edmonds [1973] gave the following characterization of the existence of disjoint branchings in a given directed graph $D = (V, A)$. We give the proof of Lovász [1976c]. The *root set* of a branching B is the set of roots of B, that is, the set of sources of the digraph (V, B).

Theorem 53.1 (Edmonds' disjoint branchings theorem)**.** *Let $D = (V, A)$ be a digraph and let $R_1, \ldots, R_k$ be subsets of V. Then there exist disjoint branchings $B_1, \ldots, B_k$ such that B_i has root set R_i (for $i = 1, \ldots, k$) if and only if*

(53.1) $\qquad d^{\text{in}}(U) \geq |\{i \mid R_i \cap U = \emptyset\}|,$

for each nonempty subset U of V.

Proof. Necessity being trivial, we show sufficiency, by induction on $|V \setminus R_1| + \cdots + |V \setminus R_k|$. If $R_1 = \cdots = R_k = V$, the theorem is trivial, so we can assume that $R_1 \neq V$. For each $U \subseteq V$, define

(53.2) $\qquad g(U) := |\{i \mid R_i \cap U = \emptyset\}|.$

Let W be an inclusionwise minimal set with the properties that $W \cap R_1 \neq \emptyset$, $W \setminus R_1 \neq \emptyset$, and $d^{\text{in}}(W) = g(W)$. Such a set exists, since $W = V$ would qualify.

Then

(53.3) $$d^{\mathrm{in}}(W \setminus R_1) \geq g(W \setminus R_1) > g(W) = d^{\mathrm{in}}(W),$$

and hence there exists an arc $a = (u, v)$ in A with $u \in W \cap R_1$ and $v \in W \setminus R_1$. It suffices to show that (53.1) is maintained after resetting $A := A \setminus \{a\}$ and $R_1 := R_1 \cup \{v\}$, since after resetting we can apply induction, and assign a to B_1.

To see that (53.1) is maintained, suppose that to the contrary there is a $U \subseteq V$ violating the condition after resetting. Then in resetting, $d^{\mathrm{in}}(U)$ decreases by 1 while $g(U)$ is unchanged. So a enters U, and, before resetting we had $d^{\mathrm{in}}(U) = g(U)$ and $U \cap R_1 \neq \emptyset$. This implies (before resetting):

(53.4) $$\begin{aligned} d^{\mathrm{in}}(U \cap W) &\leq d^{\mathrm{in}}(U) + d^{\mathrm{in}}(W) - d^{\mathrm{in}}(U \cup W) \\ &\leq g(U) + g(W) - g(U \cup W) \leq g(U \cap W). \end{aligned}$$

So we have equality throughout. Hence $d^{\mathrm{in}}(U \cap W) = g(U \cap W)$ and $R_1 \cap (U \cap W) \neq \emptyset$ (as $R_1 \cap W \neq \emptyset$ and $R_1 \cap U \neq \emptyset$, and $g(U \cap W) = g(U) + g(W) - g(U \cup W)$). Also $(U \cap W) \setminus R_1 \neq \emptyset$ (since $v \in U \cap W$) and $U \cap W \subset W$ (as $u \notin U \cap W$). This contradicts the minimality of W. ∎

(Also the method of Tarjan [1974a] is based on the existence of an arc a as in this proof. Fulkerson and Harding [1976] gave another proof of the existence of such ar arc (more complicated than that of Lovász given above).)

53.2. Disjoint r-arborescences

The previous theorem implies a characterization of the existence of disjoint arborescences with prescribed roots:

Corollary 53.1a. *Let $D = (V, A)$ be a digraph and let $r_1, \ldots, r_k \in V$. Then there exist k disjoint arborescences $B_1, \ldots, B_k$, where B_i has root r_i (for $i = 1, \ldots, k$) if and only if each nonempty subset U of V is entered by at least as many arcs as there exist i with $r_i \notin U$.*

Proof. Directly from Edmonds' disjoint branchings theorem (Theorem 53.1) by taking $R_i := \{r_i\}$ for all i. ∎

If all roots are equal, we obtain the following min-max relation, announced by Edmonds [1970b]. Recall that an *r-cut* is a cut $\delta^{\mathrm{in}}(U)$ where U is a nonempty subset of $V \setminus \{r\}$.

Corollary 53.1b (Edmonds' disjoint arborescences theorem)**.** *Let $D = (V, A)$ be a digraph and let $r \in V$. Then the maximum number of disjoint r-arborescences is equal to the minimum size of an r-cut.*

Proof. Directly from Corollary 53.1a by taking k equal to the minimum size of an r-cut and $r_i := r$ for $i = 1, \ldots, k$. ∎

Note that Edmonds' disjoint arborescences theorem implies Menger's theorem: for any digraph $D = (V, A)$ and $r, s \in V$, if k is the minimum size of an $r-s$ cut, we can extend D by k parallel arcs from s to v, for each vertex $v \neq s$; in the extended graph, the minimum size of an r-cut is k, and hence it contains k arc-disjoint r-arborescences. This gives k arc-disjoint $r-s$ paths in the original graph D.

One can reformulate Edmonds' disjoint arborescences theorem in a number of ways (Edmonds [1975]):

Corollary 53.1c. *Let $D = (V, A)$ be a digraph and let $r \in V$. Then for each $k \in \mathbb{Z}_+$ the following are equivalent:*

(53.5) (i) *there exist k disjoint r-arborescences;*
(ii) *for each nonempty $U \subseteq V \setminus \{r\}$, $d^{\text{in}}(U) \geq k$;*
(iii) *for each $s \neq r$ there exist k arc-disjoint $r-s$ paths in D;*
(iv) *there exist k edge-disjoint spanning trees in the underlying undirected graph such that for each $s \neq r$ there are exactly k arcs entering s covered by these trees.*

Proof. The equivalence of (i) and (ii) follows from Edmonds' disjoint arborescences theorem (Theorem 53.1b), and the equivalence of (ii) and (iii) is a direct consequence of Menger's theorem.

The implication (i) $\Rightarrow$ (iv) is trivial. To prove (iv) $\Rightarrow$ (ii), suppose that (iv) holds, and let U be a nonempty subset of $V \setminus \{r\}$. Each spanning tree has at most $|U| - 1$ arcs contained in U. So the spanning trees of (iv) together have at most $k(|U| - 1)$ arcs contained in U. Moreover, they have exactly $k|U|$ arcs with head in U. Hence, at least k arcs enter U. ∎

An interesting consequence of Edmonds' disjoint arborescences theorem was observed by Shiloach [1979a] and concerns the arc-connectivity of a directed graph:

Corollary 53.1d. *A digraph $D = (V, A)$ is k-arc-connected if and only if for all $s_1, t_1, \ldots, s_k, t_k \in V$ there exist arc-disjoint paths $P_1, \ldots, P_k$, where P_i runs from s_i to t_i ($i = 1, \ldots, k$).*

Proof. Sufficiency follows by taking $s_1 = \cdots = s_k$ and $t_1 = \cdots = t_k$. To see necessity, extend D by a vertex r and arcs (r, s_i) for $i = 1, \ldots, k$. By Edmonds' disjoint arborescences theorem (Corollary 53.1b), the extended digraph has k disjoint r-arborescences, since each nonempty subset U of V is entered by at least k arcs of D'. Choosing the $s_i - t_i$ path in the r-arborescence containing (r, s_i), for $i = 1, \ldots, k$, we obtain paths as required. ∎

53.3. The capacitated case

The capacitated version of the min-max relation for disjoint r-arborescences reads:

Corollary 53.1e. *Let $D = (V, A)$ be a digraph, let $r \in V$, and let $c \in \mathbb{Z}_+^A$ be a capacity function. Then the minimum capacity of an r-cut is equal to the maximum value of $\sum_B \lambda_B$, where λ_B is a nonnegative integer for each r-arborescence B such that*

(53.6) $$\sum_B \lambda_B \chi^B \leq c.$$

Proof. Directly from Corollary 53.1b by replacing each arc a by $c(a)$ parallel arcs. ∎

One can equivalently formulate this in term of total dual integrality. To see this, consider the *r-cut polytope* $P_{r\text{-cut}}(D)$ of D, defined as the convex hull of the incidence vectors of the r-cuts in D. In particular, consider the up hull

(53.7) $$P^{\uparrow}_{r\text{-cut}}(D) := P_{r\text{-cut}}(D) + \mathbb{R}_+^A$$

of the r-cut polytope.

In Corollary 52.3a we saw that the up hull $P^{\uparrow}_{r\text{-arborescence}}(D)$ of the r-arborescence polytope of D is determined by:

(53.8)
(i) $x_a \geq 0$ for each arc a,
(ii) $x(C) \geq 1$ for each r-cut C.

By the theory of blocking polyhedra, this implies that $P^{\uparrow}_{r\text{-cut}}(D)$ is determined by:

(53.9)
(i) $x_a \geq 0$ for each arc a,
(ii) $x(B) \geq 1$ for each r-arborescence B.

In fact:

Corollary 53.1f. *System* (53.9) *determines $P^{\uparrow}_{r\text{-cut}}(D)$ and is TDI.*

Proof. The first part follows from the theory of blocking polyhedra applied to Corollary 52.3a, and the second part is equivalent to Corollary 53.1e. ∎

Another equivalent form is:

(53.10) For any digraph $D = (V, A)$ and $r \in V$, the r-arborescence polytope has the integer decomposition property.

By Theorem 5.30, the number of r-arborescences B with $\lambda_B \geq 1$ in Corollary 53.1e can be taken to be at most $2|A| - 1$. (This improves a result of Pevzner [1979a] giving an $O(nm)$ upper bound.) Gabow and Manu [1995, 1998] showed an upper bound of $|V| + |A| - 2$.

53.4. Disjoint arborescences

Frank [1979a,1981c] derived from Corollary 53.1a the following min-max relation for disjoint arborescences without a prescribed root. (A *subpartition* of V is a partition of a subset of V.)

Corollary 53.1g. *Let $D = (V, A)$ be a digraph and let $k \in \mathbb{Z}_+$. Then A contains k disjoint arborescences if and only if*

$$\sum_{U \in \mathcal{P}} d^{\mathrm{in}}(U) \geq k(|\mathcal{P}| - 1) \tag{53.11}$$

for each subpartition $\mathcal{P}$ of V with nonempty classes.

Proof. Necessity being easy, we show sufficiency. Choose $x \in \mathbb{Z}_+^V$ such that

$$x(U) \geq k - d^{\mathrm{in}}(U) \tag{53.12}$$

for each nonempty subset U of V, with $x(V)$ as small as possible. We show that $x(V) = k$. Since $x(V) \geq k$ by (53.12), it suffices to show $x(V) \leq k$.

Let $\mathcal{P}$ be the collection of inclusionwise maximal nonempty sets having equality in (53.12). Then $\mathcal{P}$ is a subpartition, for suppose that $U, W \in \mathcal{P}$ with $U \cap W \neq \emptyset$. Then

$$\begin{aligned} x(U \cup W) &= x(U) + x(W) - x(U \cap W) \\ &\leq (k - d^{\mathrm{in}}(U)) + (k - d^{\mathrm{in}}(W)) - (k - d^{\mathrm{in}}(U \cap W)) \\ &\leq (k - d^{\mathrm{in}}(U \cup W)), \end{aligned} \tag{53.13}$$

and hence $U \cup W \in \mathcal{P}$. So $U = W$.

Now for each $v \in V$ with $x_v > 0$ there exists a set U in $\mathcal{P}$ containing v, since otherwise we could decrease x_v. Hence

$$x(V) = \sum_{U \in \mathcal{P}} x(U) = \sum_{U \in \mathcal{P}} (k - d^{\mathrm{in}}(U)) \leq k, \tag{53.14}$$

by (53.11).

So $x(V) = k$. Now let $r_1, \ldots, r_k$ be vertices such that any vertex v occurs x_v times among the r_i. Then by Corollary 53.1a there exist disjoint arborescences $B_1, \ldots, B_k$, where B_i has root r_i. This shows the corollary. ∎

53.5. Covering by branchings

Let $A[U]$ denote the set of arcs in A with both ends in U. Frank [1979a] observed that the following min-max relation for covering by branchings can be derived from Edmonds' disjoint arborescences theorem:

Corollary 53.1h. *Let $D = (V, A)$ be a digraph and let $k \in \mathbb{Z}_+$. Then A can be covered by k branchings if and only if*

(53.15) (i) $\deg^{\text{in}}(v) \leq k$ *for each* $v \in V$,
(ii) $|A[U]| \leq k(|U| - 1)$ *for each nonempty subset* U *of* V.

Proof. Necessity being trivial, we show sufficiency. Extend D by a new vertex r, and for each $v \in V$, $k - \deg^{\text{in}}(v)$ parallel arcs from r to v. Let D' be the digraph thus arising. So each vertex in V is entered by exactly k arcs of D', and D' has $k|V|$ arcs.

Now each nonempty subset U of V is entered by at least k arcs of D', since exactly $k|U|$ arcs have their head in U and at most $k(|U|-1)$ arcs have both ends in U. So by Edmonds' disjoint arborescences theorem (Theorem 53.1b), D' has k disjoint r-arborescences. Since D' has exactly $k|V|$ arcs, these arborescences partition the arc set of D'. Hence restricting them to the arcs of the original graph D, we obtain k branchings partitioning A. ∎

(This was also shown by Markosyan and Gasparyan [1986].)

Corollary 53.1h is equivalent to:

Corollary 53.1i. *Let $D = (V, A)$ be a digraph and let $k \in \mathbb{Z}_+$. Then A can be covered by k branchings if and only if* $\deg^{\text{in}}(v) \leq k$ *for each* $v \in V$ *and* A *can be covered by k forests of the underlying undirected graph.*

Proof. Directly from Corollary 53.1h with Corollary 51.1c. ∎

Corollary 53.1h implies a polyhedral result of Baum and Trotter [1981] (attributing the proof to R. Giles):

Corollary 53.1j. *The branching polytope of a digraph $D = (V, A)$ has the integer decomposition property.*

Proof. Let $k \in \mathbb{Z}_+$ and let x be an integer vector in $k \cdot P_{\text{branching}}(D)$. Let $D' = (V, A')$ be the digraph obtained from D by replacing any arc $a = (u, v)$ by x_a parallel arcs from u to v. Then by Corollary 53.1h, A' can be partitioned into k branchings. This gives a decomposition of x as a sum of the incidence vectors of k branchings in D. ∎

53.6. An exchange property of branchings

We derive an exchange property of branchings from Edmonds' disjoint branchings theorem (Theorem 53.1). It implies that the branchings in an optimum covering can be taken of almost equal size. It will also be used in Section 59.5 on the total dual integrality of the matching forest constraints.

We first show a lemma. For any branching B, let $R(B)$ denote the set of roots of B.

Lemma 53.2α. *Let B_1 and B_2 be branchings partitioning the arc set A of a digraph $D = (V, A)$. Let R_1 and R_2 be sets with $R_1 \cup R_2 = R(B_1) \cup R(B_2)$ and $R_1 \cap R_2 = R(B_1) \cap R(B_2)$. Then A can be split into branchings B'_1 and B'_2 with $R(B'_i) = R_i$ for $i = 1, 2$ if and only if each strong component K of D with $d^{\mathrm{in}}(K) = 0$ intersects both R_1 and R_2.*

Proof. Necessity is easy, since the root set of any branching intersects any strong component K with $d^{\mathrm{in}}(K) = 0$.

To see sufficiency, by Edmonds' disjoint branchings theorem (Theorem 53.1), branchings B'_1 and B'_2 as required exist if and only if

(53.16) $$d^{\mathrm{in}}(U) \geq |\{i \in \{1, 2\} \mid U \cap R_i = \emptyset\}|$$

for each nonempty $U \subseteq V$. (Actually, Edmonds' theorem gives the existence of disjoint branchings B'_1 and B'_2 satisfying $R(B'_i) = R_i$ for $i = 1, 2$. That $B'_1 \cup B'_2 = A$ follows from the fact that $|B'_1| + |B'_2| = |B_1| + |B_2|$, as $|R(B'_1)| + |R(B'_2)| = |R(B_1)| + |R(B_2)|$.)

Suppose that inequality (53.16) does not hold. Then the right-hand side is positive. If it is 2, then U is disjoint from both R_1 and R_2, and hence from both $R(B_1)$ and $R(B_2)$ (since $R_1 \cup R_2 = R(B_1) \cup R(B_2)$), implying that both B_1 and B_2 enter U, and so $d^{\mathrm{in}}(U) \geq 2$.

So the right-hand side is 1, and hence the left-hand side is 0. We can assume that U is an inclusionwise minimal set with this property. It implies that U is a strong component of D. Then by the condition, U intersects both R_1 and R_2, contradicting the fact that the right-hand side in (53.16) is 1. ∎

First, this implies the following exchange property of branchings:

Theorem 53.2. *Let B_1 and B_2 be branchings in a digraph $D = (V, A)$. Let s be a root of B_2 and let r be the root of the arborescence in B_1 containing s. Then D contains branchings B'_1 and B'_2 satisfying*

(53.17) $$B'_1 \cup B'_2 = B_1 \cup B_1,\ B'_1 \cap B'_2 = B_1 \cap B_2,$$ $$\text{and } R(B'_1) = R(B_1) \cup \{s\} \text{ or } R(B'_1) = (R(B_1) \setminus \{r\}) \cup \{s\}.$$

Proof. We may assume that B_1, B_2 partition A, since we can delete all arcs not occurring in $B_1 \cup B_2$, and add parallel arcs for those in $B_1 \cap B_2$. We may also assume that $s \neq r$ (since the theorem is trivial if $s = r$).

Let K be the strong component of D containing s. If no arc of D enters K, then $r \in K$ (as B_1 contains a directed path from r to s), and hence r is not a root of B_2 (as otherwise no arc enters r while K is strongly connected); define $R_1 := (R(B_1) \setminus \{r\}) \cup \{s\}$ and $R_2 := (R(B_2) \setminus \{s\}) \cup \{r\}$.

Alternatively, if some arc of D enters K, define $R_1 := R(B_1) \cup \{s\}$ and $R_2 := R(B_2) \setminus \{s\}$. Then Lemma 53.2$\alpha$ implies that A can be split into branchings B'_1 and B'_2 with $R(B'_i) = R_i$ for $i = 1, 2$. ∎

The lemma also implies that a packing of branchings can be balanced in the following sense:

Theorem 53.3. *Let $D = (V, A)$ be a digraph. If A can be covered by k branchings, then A can be covered by k branchings each of size $\lfloor |A|/k \rfloor$ or $\lceil |A|/k \rceil$.*

Proof. Consider any two branchings B_1, B_2 in the covering which differ in size by at least 2. Consider the digraph $D' = (V, B_1 \cup B_2)$. We can find subsets R_1 and R_2 of V with $R_1 \cup R_2 = R(B_1) \cup R(B_2)$ and $R_1 \cap R_2 = R(B_1) \cap R(B_2)$, such that each strong component K of D' with $d^{\mathrm{in}}_{D'}(K) = 0$ intersects both R_1 and R_2, and such that R_1 and R_2 differ by at most 1 in size. (We can first include, for any such component K, one element in $K \cap R(B_1)$ in R_1, and one element in $K \cap R(B_2)$ in R_2; next we distribute the remaining elements in $R(B_1)$ and $R(B_2)$ almost equally over R_1 and R_2).

Then, by Lemma 53.2α, $B_1 \cup B_2$ can be partitioned into branchings B'_1 and B'_2 with $R(B'_i) = R_i$ for $i = 1, 2$. Then B'_1 and B'_2 differ by at most 1 in size. Replacing B_1 and B_2 in the covering by B'_1 and B'_2, and iterating this, we end up with a covering by k branchings, any two of which differ in size by at most 1. This is a covering as required. ∎

This theorem implies the integer decomposition property of the convex hull of branchings of size k (McDiarmid [1983]):

Corollary 53.3a. *Let $D = (V, A)$ be a digraph and let $k \in \mathbb{Z}_+$. Then the convex hull of the incidence vectors of the branchings of size k has the integer decomposition property.*

Proof. Choose $p \in \mathbb{Z}_+$, and let x be an integer vector in $p \cdot \mathrm{conv.hull}\{\chi^B \mid B$ branching, $|B| = k\}$. By Corollary 53.1j, x is a sum of the incidence vectors of p branchings. Let $D' = (V, A')$ be the digraph arising from D by replacing any arc a by x_a parallel arcs. Then A' can be partitioned into p branchings. Now $|A'|/p = x(A)/p = k$. So, by Theorem 53.3, we can take these branchings all of size k. Hence x is the sum of the incidence vectors of p branchings each of size k. ∎

53.7. Covering by r-arborescences

Vidyasankar [1978a] proved the following covering analogue of Edmonds' disjoint branchings theorem. (A weaker version was shown by Frank [1979a] (cf. Frank [1979b]).) For any digraph $D = (V, A)$ and $U \subseteq V$, let $H(U)$ denote the set of outneighbours of $V \setminus U$; that is, the set of the heads of the arcs entering U. So $H(U) \subseteq U$.

Theorem 53.4. *Let $D = (V, A)$ be a digraph, let $r \in V$, and let $k \in \mathbb{Z}_+$. Then A can be covered by k r-arborescences if and only if*

(53.18) $\deg^{\text{in}}(v) \leq k$ *for each* $v \in V$, *and* $\deg^{\text{in}}(r) = 0$,

and

(53.19) $$\sum_{v \in H(U)} (k - \deg^{\text{in}}(v)) \geq k - d^{\text{in}}(U)$$

for each nonempty subset U of $V \setminus \{r\}$.

Proof. Necessity of (53.18) is trivial. To see necessity of (53.19), let U be a nonempty subset of $V \setminus \{r\}$. Then each r-arborescence B intersects the set

(53.20) $$\bigcup_{v \in H(U)} \delta^{\text{in}}(v) \setminus \delta^{\text{in}}(U)$$

in at most $|H(U)| - 1$ arcs, since at least one arc of B should enter U. Hence if A can be covered by k r-arborescences, the size of set (53.20) is at most $k(|H(U)| - 1)$, implying (53.19).

To see sufficiency, we can assume that for any arc a of D, if we would add a parallel arc to a, then (53.18) or (53.19) is violated (since deleting parallel arcs does not increase the minimum number of r-arborescences needed to cover the arcs).

If $\deg^{\text{in}}(v) = k$ for each vertex $v \neq r$, then A can be decomposed into k r-arborescences by Edmonds' disjoint arborescences theorem (Corollary 53.1b), since then (53.19) implies that $d^{\text{in}}(U) \geq k$ for each nonempty subset U of $V \setminus \{r\}$.

So we can assume that there exists a vertex $u \neq r$ with $\deg^{\text{in}}(u) < k$. Consider the collection $\mathcal{C}$ of nonempty subsets U of $V \setminus \{r\}$ having equality in (53.19) and with $u \in H(U)$. Then $\mathcal{C}$ is closed under taking union and intersection. Indeed, let U and W be in $\mathcal{C}$. Then

(53.21) $$\begin{aligned} &\sum_{v \in H(U \cap W)} (k - \deg^{\text{in}}(v)) + \sum_{v \in H(U \cup W)} (k - \deg^{\text{in}}(v)) \\ &\leq \sum_{v \in H(U)} (k - \deg^{\text{in}}(v)) + \sum_{v \in H(W)} (k - \deg^{\text{in}}(v)) \\ &= (k - d^{\text{in}}(U)) + (k - d^{\text{in}}(W)) \\ &\leq (k - d^{\text{in}}(U \cap W)) + (k - d^{\text{in}}(U \cup W)). \end{aligned}$$

The first inequality follows from

(53.22) $$\begin{aligned} &H(U \cap W) \cap H(U \cup W) \subseteq H(U) \cap H(W) \text{ and} \\ &H(U \cap W) \cup H(U \cup W) \subseteq H(U) \cup H(W), \end{aligned}$$

as one easily checks.

By (53.19), (53.21) implies that we have equality throughout, As we have equality in the first inequality in (53.21), and as $k - \deg^{\text{in}}(u) > 0$, we know that $u \in H(U \cap W) \cap H(U \cup W)$. So $U \cap W$ and $U \cup W$ belong to $\mathcal{C}$.

Now for each arc a entering u, if we would add an arc parallel to a, (53.19) is violated for some U. This implies that for each arc a entering u there exists a $U \in \mathcal{C}$ such that the tail of a is in U. We can take for U the largest set in $\mathcal{C}$. Hence for each arc a entering u, the tail of a is in U. This contradicts the fact that $u \in H(U)$. ∎

Frank [1979b] showed the following consequence of this result:

Corollary 53.4a. *Let $D = (V, A)$ be a digraph, let $r \in V$, and let $k \in \mathbb{Z}_+$. Then A can be covered by k r-arborescences if and only if*

(53.23) $k \cdot s(A') \geq |A'|$

for each $A' \subseteq A$. Here $s(A')$ denotes the maximum of $|B \cap A'|$ over r-arborescences B.

Proof. As necessity is trivial, we show sufficiency, by showing that (53.23) implies (53.18) and (53.19). To see (53.18), apply (53.23) to $A' := \delta^{\text{in}}(v)$. To see (53.19), apply (53.23) to A' equal to the set (53.20). ∎

Note that for *acyclic* digraphs, the minimum number of r-arborescences needed to cover all arcs is easily characterized (Vidyasankar [1978a]):

Theorem 53.5. *Let $D = (V, A)$ be an acyclic digraph and let $r \in V$. Then A can be covered by k r-arborescences if and only if r is the only source of D and each indegree is at most k.*

Proof. Necessity being easy, we show sufficiency. Trivially, we can cover A by sets $B_1, \ldots, B_k$ such that each B_i enters each $v \neq r$ precisely once. As D is acyclic, each B_i is an r-arborescence. ∎

53.8. Minimum-length unions of k r-arborescences

Let $D = (V, A)$ be a digraph, let $r \in V$, and let $k \in \mathbb{Z}_+$. Consider the following system in the variable $x \in \mathbb{R}^A$:

(53.24)
(i) $x_a \geq 0$ for each $a \in A$,
(ii) $x(\delta^{\text{in}}(U)) \geq k$ for each nonempty $U \subseteq V \setminus \{r\}$.

The following basic result of Frank [1979b] follows from Theorem 52.4.

Theorem 53.6. *System* (53.24) *is box-TDI.*

Proof. Directly from Theorem 52.4, since if a system $Ax \leq b$ is box-TDI, then for any $k \geq 0$, the system $Ax \leq k \cdot b$ is box-TDI. ∎

This theorem has several consequences. First consider the system

(53.25) (i) $0 \le x_a \le 1$ for each $a \in A$,
(ii) $x(\delta^{\mathrm{in}}(U)) \ge k$ for each nonempty $U \subseteq V \setminus \{r\}$.

The following (cf. Frank [1979b]) implies a min-max relation for the minimum length of the union of k disjoint r-arborescences in D:

Corollary 53.6a. *System* (53.25) *is TDI, and determines the convex hull of subsets of A containing k disjoint r-arborescences.*

Proof. Directly from Theorem 53.6 and Edmonds' disjoint arborescences theorem (Corollary 53.1b). ∎

Another consequence of Theorem 53.6 is as follows. Let $D = (V, A)$ and $D' = (V, A')$ be digraphs, let $r \in V$, and let $k \in \mathbb{Z}_+$. Consider the system in the variable $x \in \mathbb{R}^A$:

(53.26) (i) $x_a \ge 0$ for each $a \in A$,
(ii) $x(\delta^{\mathrm{in}}_A(U)) \ge k - d^{\mathrm{in}}_{A'}(U)$ for each nonempty $U \subseteq V \setminus \{r\}$.

Then:

Corollary 53.6b. *System* (53.26) *is box-TDI.*

Proof. Choose $d, c \in \mathbb{Z}^A_+$. We must show that the system

(53.27) (i) $d(a) \le x_a \le c(a)$ for each $a \in A$,
(ii) $x(\delta^{\mathrm{in}}_A(U)) \ge k - d^{\mathrm{in}}_{A'}(U)$ for each nonempty $U \subseteq V \setminus \{r\}$

is TDI. Let $D'' = (V, A'')$ be the digraph with $A'' := A \cup A'$ (taking arcs multiple if they occur both in A and A'). By Theorem 53.6, the following system in the variable $x \in \mathbb{R}^{A''}$ is TDI:

(53.28) (i) $d(a) \le x_a \le c(a)$ for each $a \in A$,
(ii) $1 \le x_a \le 1$ for each $a \in A'$,
(iii) $x(\delta^{\mathrm{in}}_{A''}(U)) \ge k$ for each nonempty $U \subseteq V \setminus \{r\}$.

This implies the total dual integrality of (53.27) by Corollary 5.27a. ∎

Frank [1979a] derived the following 'rank' formula for coverings by k r-arborescences:

Corollary 53.6c. *Let $D = (V, A)$ be a digraph, let $r \in V$, and let $A' \subseteq A$. Then the maximum number of arcs in A' that can be covered by k r-arborescences is equal to the minimum value of*

$$k(|V| - 1) + \sum_{i=1}^{t} (d^{\mathrm{in}}_{A'}(V_i) - k), \tag{53.29}$$

where $V_1, \ldots, V_t$ is a laminar collection of nonempty subsets of $V \setminus \{r\}$ such that each arc in A enters at most one of these sets.

Proof. Let μ be the maximum number of arcs in A' that can be covered by k r-arborescences. Consider the system (in $x \in \mathbb{R}^A$)

$$\text{(53.30)} \quad \begin{array}{ll} \text{(i) } x_a \geq 0 & \text{for } a \in A, \\ \text{(ii) } x(\delta^{\text{in}}_A(U)) \geq k - d^{\text{in}}_{A'}(U) & \text{for each nonempty } U \subseteq V \setminus \{r\}. \end{array}$$

By Corollary 53.6b, this system is TDI. Let x be an integer vector attaining the minimum of $x(A)$ over (53.30). Then

$$\text{(53.31)} \quad \mu = k(|V| - 1) - x(A).$$

Indeed, by (53.30) and by Edmonds' disjoint arborescences theorem, there exist k r-arborescences $B_1, \ldots, B_k$ with

$$\text{(53.32)} \quad x + \chi^{A'} \geq \chi^{B_1} + \cdots + \chi^{B_k}.$$

Let A'' be the set of arcs in A' covered by no B_i. By the minimality of $x(A)$, we have that $x(A) + |A'| = k(|V| - 1) + |A''|$. As $\mu \geq |A'| - |A''|$ we have $\geq$ in (53.31). Since we can reverse this construction (starting from a set of k r-arborescences covering μ arcs in A', and making x), we have the equality in (53.31).

By the total dual integrality of (53.30), $x(A)$ is equal to the maximum value of

$$\text{(53.33)} \quad \sum_{i=1}^{t} (k - d^{\text{in}}_{A'}(V_i)),$$

taken over nonempty subsets $V_1, \ldots, V_t$ of $V \setminus \{r\}$ such that each arc in A enters at most one of these sets. If, say, $V_1 \cap V_2 \neq \emptyset$ and $V_1 \not\subseteq V_2 \not\subseteq V_1$, we can replace V_1 and V_2 by $V_1 \cap V_2$ and $V_1 \cup V_2$ without violating these conditions. Such replacements terminate by Theorem 2.1. We end up with $V_1, \ldots, V_t$ laminar as required. Therefore, with (53.31) we have the corollary. ∎

Taking $A' = A$, we get (Frank [1979b]):

Corollary 53.6d. *Let $D = (V, A)$ be a digraph and let $r \in V$. Then the maximum number of arcs that can be covered by k r-arborescences is equal to the minimum value of*

$$\text{(53.34)} \quad k(|V| - 1) + \sum_{i=1}^{t} (d^{\text{in}}(V_i) - k),$$

where $V_1, \ldots, V_t$ form a laminar collection of nonempty subsets of $V \setminus \{r\}$ such that each arc enters at most one of these sets.

Proof. This is the case $A' = A$ in Corollary 53.6c. ∎

This directly implies a min-max characterization for the minimum number of r-arborescences needed to cover all arcs. However, Theorem 53.4 gives a stronger relation.

As for unions of k branchings, Frank [1979a] derived from Corollary 53.6c:

Corollary 53.6e. *Let $D = (V, A)$ be a digraph and let $k \in \mathbb{Z}_+$. The maximum number of arcs of D that can be covered by k branchings is equal to the minimum value of*

$$k(|V| - |\mathcal{P}|) + \sum_{U \in \mathcal{P}} d^{\text{in}}(U) \tag{53.35}$$

taken over all subpartitions $\mathcal{P}$ of V with nonempty classes.

Proof. Let D' be the digraph obtained from D by adding a new vertex r and arcs (r, v) for each $v \in V$. Then the maximum number of arcs of D that can be covered by k branchings in D is equal to the maximum number of arcs in A that can be covered by k r-arborescences in D'. So Corollary 53.6c gives a min-max relation for this.

The subsets V_i form a subpartition of V since if V_i and V_j would intersect in a vertex v say, then the arc (r, v) of D' enters two sets among the V_i, contradicting the condition. ∎

As for unions of k arborescences without prescribed root, Frank [1979a] derived:

Corollary 53.6f. *Let $D = (V, A)$ be a digraph and let $k \in \mathbb{Z}_+$. Then A can be covered by k arborescences if and only if*

$$k(|V| - 1 + \lambda) \geq |A| + \sum_{i=1}^{t} (k - d^{\text{in}}(V_i)) \tag{53.36}$$

for each laminar family $(V_1, \ldots, V_t)$ of nonempty sets such that no arc enters more than one of the V_i. Here λ denotes the maximum number of V_i's having nonempty intersection.

Proof. Necessity can be seen as follows. Let A be covered by arborescences $B_1, \ldots, B_k$. For each $v \in V$, let $r(v)$ be the number of B_i having v as root. So $r(V) = k$. For each $a \in A$, let $s(a)$ be the number of B_i containing a. So $s(a) \geq 1$ for each $a \in A$. Moreover, $s(\delta^{\text{in}}(V_i)) + r(V_i) \geq k$ for each i. Hence

$$\begin{aligned} |A| + \sum_{i=1}^{t} (k - d^{\text{in}}(V_i)) &\leq |A| + \sum_{i=1}^{t} \big(r(V_i) + s(\delta^{\text{in}}(V_i)) - d^{\text{in}}(V_i)\big) \\ &\leq |A| + \sum_{a \in A} (s(a) - 1) + \sum_{i=1}^{t} r(V_i) = \sum_{a \in A} s(a) + \sum_{i=1}^{t} r(V_i) \\ &= k(|V| - 1) + \sum_{i=1}^{t} r(V_i) \leq k(|V| - 1) + k\lambda. \end{aligned} \tag{53.37}$$

The second inequality holds as each arc enters at most one of the V_i. For the last inequality, we use that $r(V) = k$ and that $V_1, \ldots, V_t$ can be partitioned into λ collections, each consisting of disjoint sets. (53.37) shows necessity.

To see sufficiency, extend D by a vertex r and by the arc set $A' := \{(r, v) \mid v \in V\}$, yielding the digraph $D' = (V \cup \{r\}, A \cup A')$. Consider the following constraints for $x \in \mathbb{R}^{A \cup A'}$:

(53.38)
$$\begin{array}{rll} \text{(i)} & x_a \geq 0 & \text{for each } a \in A \cup A', \\ \text{(ii)} & x(\delta^{\text{in}}_{D'}(U)) \geq k - d^{\text{in}}_{D}(U) & \text{for each nonempty } U \subseteq V, \\ \text{(iii)} & x(\delta^{\text{in}}_{D'}(V)) = k. & \end{array}$$

Let x attain the minimum of $x(A)$ over (53.38). Since system (53.38) is TDI by Corollary 53.6b (with Theorem 5.25), we can assume that x is integer. We show

(53.39)
$$x(A) = k(|V| - 1) - |A|.$$

First, $x(A) \geq k(|V| - 1) - |A|$, since

(53.40)
$$\begin{aligned} & x(A) + |A| + k = x(A) + |A| + x(\delta^{\text{in}}_{A'}(V)) \\ & = \sum_{v \in V} \big(x(\delta^{\text{in}}_{A'}(v)) + x(\delta^{\text{in}}_{A}(v)) + d^{\text{in}}_{A}(v)\big) \geq k|V|. \end{aligned}$$

To see the reverse inequality, $x(A)$ is equal to the optimum value μ of the problem dual to the above minimization problem: maximize

(53.41)
$$\sum_{U \in \mathcal{P}(V) \setminus \{\emptyset\}} z_U (k - d^{\text{in}}_{A}(U))$$

where $z \in \mathbb{R}_{+}^{\mathcal{P}(V) \setminus \{\emptyset\}}$ such that

(53.42)
$$\sum_{U} z_U \chi^{\delta^{\text{in}}_{D'}(U)} \leq \chi^{A}.$$

So we should prove that $\mu \leq k(|V| - 1) - |A|$.

Now let $\mathcal{U}$ be the collection of nonempty proper subsets U of V with $z_U = 1$. We may assume that $\mathcal{U}$ is laminar. Let λ be the maximum number of $U \in \mathcal{U}$ containing any vertex. Then (53.42) implies that $\lambda \leq -z_V$ (since $\chi^A(a) = 0$ for each $a = (r, v)$). Hence

(53.43)
$$\begin{aligned} & \mu = k \cdot z_V + \sum_{U \in \mathcal{U}} (k - d^{\text{in}}_{A}(U)) \leq -k\lambda + \sum_{U \in \mathcal{U}} (k - d^{\text{in}}_{A}(U)) \\ & \leq k(|V| - 1) - |A| \end{aligned}$$

by (53.36), and we have the required inequality. This proves (53.39).

Then the vector $y := x + \chi^A$ satisfies:

(53.44)
$$\begin{aligned} & y(\delta^{\text{in}}_{D'}(U)) \geq k \text{ for each nonempty } U \subseteq V, \\ & y(\delta^{\text{in}}_{D'}(V)) = k, \\ & y(A \cup A') = x(A) + x(A') + |A| = k(|V| - 1) - |A| + k + |A| = k|V|. \end{aligned}$$

So by Edmonds' disjoint arborescences theorem (Corollary 53.1b), y is the sum of the incidence vectors of k r-arborescences, each with exactly one arc leaving r. Hence (by the definition of y) A can be covered by k arborescences. ∎

53.9. The complexity of finding disjoint arborescences

By Edmonds' disjoint arborescences theorem, the maximum number of disjoint r-arborescences can be calculated in polynomial time, just by determining the minimum size of an r-cut. This can be done by determining, for each $v \in V \setminus \{r\}$, the minimum size of an $r - v$ cut, and taking the minimum of these values.

Lovász [1976c] and Tarjan [1974a] showed that actually also a maximum collection of disjoint r-arborescences can be found in polynomial time.

The proof (due to Lovász [1976c]) of Theorem 53.1 described above gives such a polynomial-time algorithm. In fact, Lovász observed that it implies the following result (obtained also by Tarjan [1974a]). Call a subset B of the arc set A of a digraph $D = (V, A)$ a *partial r-arborescence* if B is an r-arborescence for the subgraph of D induced by the set $V(B)$ of vertices covered by B. We take $V(B) := \{r\}$ if B is empty.

Theorem 53.7. *Given a digraph $D = (V, A)$ and a vertex $r \in V$, a maximum number k of disjoint r-arborescences can be found in time $O(k^2m^2)$.*

Proof. First, the number k can be determined in time $O(knm)$. Since k is equal to the minimum size of a cut $d^{\text{in}}(U)$ over nonempty subsets U of $V \setminus \{r\}$, we can determine for each $v \in V \setminus \{r\}$ a maximum set of arc-disjoint $r - v$ paths, by the augmenting path method described in Section 9.2. Actually, for $i = 1, \ldots, k$, we determine the ith augmenting paths for all $v \in V \setminus \{r\}$, before searching for the $(i+1)$th augmenting paths. In this way we can stop if for some $v \in V \setminus \{r\}$ no augmenting path exists. So in total we do at most $(n-1)(k+1)$ augmenting path searches. Thus it takes $O(knm)$ time to determine k.

Next, we can find an r-arborescence B such that

(53.45) $$d^{\text{in}}_{A \setminus B}(U) \geq k - 1 \text{ for each nonempty } U \subseteq V \setminus \{r\},$$

in time $O(km^2)$. This recursively implies the theorem.

To find B, as in the proof of Theorem 53.1, we can grow a partial r-arborescence B satisfying (53.45), starting with $B = \emptyset$. By the proof of Theorem 53.1, if $V(B) \neq V$, there exists an arc a leaving $V(B)$ such that resetting $B := B \cup \{a\}$ maintains (53.45). For any given arc a leaving $V(B)$ it amounts to testing if there exists a set $U \subseteq V \setminus \{r\}$ such that $a \in \delta^{\text{in}}(U)$ and $d^{\text{in}}_{A \setminus B}(U) = k - 1$. This can be done in $O(km)$ time with a minimum cut algorithm.

Now it is important to observe that for each arc a we need to do this test at most once: if the test result is negative, then in growing B we never have to consider arc a anymore; if the result is positive, a is added to B, and again we will not consider a again.

So to obtain an r-arborescence, we determine at most m minimum cuts, and so finding the r-arborescence B takes $O(km^2)$ time. ∎

Tong and Lawler [1983] observed that the following quite easily follows from Edmonds' disjoint arborescences theorem:

Theorem 53.8. *Given a digraph $D = (V, A)$ and a vertex $r \in V$, we can find in time $O(knm)$ a set of arcs that is the union of a maximum number k of disjoint r-arborescences.*

Proof. As in the proof of Theorem 53.7 we can determine the number k in time $O(knm)$. Now consider any vertex $v \in V$. Find k arc-disjoint $r-v$ paths in D, and delete from D each arc entering v that is on none of these paths. After that we still have $d^{\text{in}}(U) \geq k$ for any nonempty $U \subseteq V \setminus \{r\}$, since if $v \notin U$, then no arc entering U has been deleted, and if $v \in U$, then k arcs entering U are maintained, as after deletion there are still k arc-disjoint $r-v$ paths in D.

Doing this successively for all vertices $v \in V$, we are left with a digraph D with $\deg^{\text{in}}(v) = k$ if $v \neq r$ and $\deg^{\text{in}}(r) = 0$, and with $d^{\text{in}}(U) \geq k$ for each nonempty $U \subseteq V \setminus \{r\}$. So the remaining arc set is the union of k disjoint r-arborescences. As k arc-disjoint $r-v$-paths can be found in time $O(km)$, we have the required result. ∎

This implies with Theorem 53.7 a sharpening of Theorem 53.7:

Corollary 53.8a. *Given a digraph $D = (V, A)$ and a vertex $r \in V$, a maximum number k of disjoint r-arborescences can be found in time $O(knm + k^4n^2)$.*

Proof. By Theorem 53.8, we can find in time $O(knm)$ a set A' that is the union of k disjoint r-arborescences. So $m' := |A'| = k(n-1)$. Then by Theorem 53.7 we can find k disjoint r-arborescences in A', in time $O(k^2m'^2)$. Since $O(k^2m'^2) = O(k^4n^2)$, the corollary follows. ∎

Tong and Lawler [1983] in fact showed that the method of Lovász [1976c] has an $O(k^2nm)$-time implementation, yielding with Theorem 53.8 an $O(knm + k^3n^2)$-time algorithm for finding k disjoint r-arborescences.

Also the capacitated case can be solved in strongly polynomial time (Gabow [1991a,1995a]), as can be shown with the help of Edmonds' disjoint branchings theorem. (Pevzner [1979a] proved that it can be solved in *semi-strongly polynomial time*, that is, by taking rounding as one arithmetic step.)

Theorem 53.9. *Given a digraph $D = (V, A)$, $r \in V$, and a capacity function $c : A \to \mathbb{Z}_+$, we can find r-arborescences $B_1, \ldots, B_k$ and integers $\lambda_1, \cdots, \lambda_k \geq 0$ with $\sum_{i=1}^{k} \lambda_i \chi^{B_i} \leq c$ and with $\sum_{i=1}^{k} \lambda_i$ maximized, in strongly polynomial time.*

Proof. We can find the maximum value in strongly polynomial time, as it is equal to the minimum capacity of an r-cut. To find the λ_i explicitly, we show more generally that the following problem is solvable in strongly polynomial time (where $R(B)$ denotes the set of roots of B):

(53.46) given: a digraph $D = (V, A)$, a capacity function $c : A \to \mathbb{Z}_+$, a collection $\mathcal{R}$ of nonempty subsets of V, and a demand function $d : \mathcal{R} \to \mathbb{Z}_+$,
find: a collection $\mathcal{B}$ of branchings and a function $\lambda : \mathcal{B} \to \mathbb{Z}_+$, with $\sum_{B \in \mathcal{B}} \lambda_B \chi^B \leq c$ and $\sum(\lambda_B \mid B \in \mathcal{B}, R(B) = R) = d(R)$ for each $R \in \mathcal{R}$.

For any $U \subseteq V$, define

(53.47) $$g(U) := \sum(d(R) \mid R \in \mathcal{R}, R \cap U = \emptyset).$$

By replacing each arc a by $c(a)$ parallel arcs, it follows from Edmonds' disjoint branchings theorem (Theorem 53.1) that a necessary and sufficient condition for the existence of a solution of (53.46) is that

(53.48) $$c(\delta^{\text{in}}(U)) \geq g(U)$$

for each nonempty $U \subseteq V$.

We can assume that $c(a) > 0$ for each $a \in A$ and $d(R) > 0$ for each $R \in \mathcal{R}$, and that we have an $R_1 \in \mathcal{R}$ with $R_1 \neq V$.

We may also assume that (53.46) has a solution. This implies that there exists an arc $a = (u, v) \in A$ leaving R_1 and a $\mu \geq 1$ such that resetting $d(R_1) := d(R_1) - \mu$, $d(R_1 \cup \{v\}) := d(R_1 \cup \{v\}) + \mu$, $c(a) := c(a) - \mu$, maintains feasibility of (53.46). (If $R_1 \cup \{v\}$ did not belong to $\mathcal{R}$, we add it to $\mathcal{R}$.) We apply this for the maximum possible μ. This value of μ can be calculated in strongly polynomial time, as it satisfies

(53.49) $$\mu = \min\{c(a), \min\{c(\delta^{\text{in}}(W)) - g(W) \mid a \in \delta^{\text{in}}(W), W \cap R_1 \neq \emptyset\}\}$$

(for the original c and g).

To minimize $c(\delta^{\text{in}}(W)) - g(W)$ over W with $a \in \delta^{\text{in}}(W)$ and $W \cap R_1 \neq \emptyset$, add, for each $R \in \mathcal{R}$, a new vertex v_R and, for each $v \in R$, a new arc (v_R, v) of capacity $d(R)$. Moreover, add a new vertex r, and for each $R \in \mathcal{R}$, a new arc (r, v_R) of capacity $d(R)$. Let $D' = (V', A')$ be the extended digraph. With a minimum cut algorithm we can find a subset W' of $V' \setminus \{r\}$ with $a \in \delta^{\text{in}}(W)$

and $W \cap R_1 \neq \emptyset$, minimizing the capacity of $\delta^{\text{in}}_{A'}(W)$. Then $W := W' \cap V$ is a set as required.

We next apply the algorithm recursively. This describes the algorithm.

Running time. In each iteration, the number of arcs a with $c(a) > 0$ decreases or the collection $\mathcal{C} := \{U \subseteq V \mid U \neq \emptyset, c(\delta^{\text{in}}(U)) = g(U)\}$ increases. As $\mathcal{C}$ is an intersecting family, the number of times $\mathcal{C}$ increases is at most $|V|^3$ (since for each $v \in V$, the collection $\mathcal{C}_v := \{U \in \mathcal{C} \mid v \in U\}$ is a lattice family, and since each lattice family $\mathcal{L}$ is determined by the preorder $\preceq$ given by: $s \preceq t \iff$ each set in $\mathcal{L}$ containing t contains s; if $\mathcal{L}$ increases, then $\preceq$ decreases, which can happen at most $|V|^2$ times.)

So the number of iterations is at most $|A| + |V|^3$. ∎

With the reductions given earlier, this implies that the capacitated versions of packing arborescences and covering by branchings also can be solved in strongly polynomial time.

Edmonds [1975] observed that matroid intersection and union theory implies:

Theorem 53.10. *Given a digraph $D = (V, A)$, $r \in V$, $k \in \mathbb{Z}_+$, and a length function $l \in \mathbb{Q}^A$, we can find k disjoint r-arborescences $B_1, \ldots, B_k$ minimizing $l(B_1) + \cdots + l(B_k)$ in strongly polynomial time.*

Proof. This follows, with Corollary 53.1c and Theorem 53.7, from Theorem 41.8 applied to the intersection of two matroids: one being the union of k times the cycle matroid of the undirected graph underlying D; the other being the matroid in which a subset B of A is independent if and only if any $v \in V \setminus \{r\}$ is entered by at most k arcs in B. ∎

This implies:

Corollary 53.10a. *Given a digraph $D = (V, A)$, $r \in V$, $k \in \mathbb{Z}_+$, and a length function $l \in \mathbb{Q}^A$, we can find a minimum-length subset B of A with $\delta^{\text{in}}_B(U) \geq k$ for each nonempty $U \subseteq V \setminus \{r\}$ in strongly polynomial time.*

Proof. Directly from Theorem 53.10, with Edmonds' disjoint arborescences theorem (Corollary 53.1b). ∎

53.10. Further results and notes

53.10a. Complexity survey for disjoint arborescences

Finding k disjoint r-arborescences in an uncapacitated digraph ($*$ indicates an asymptotically best bound in the table):

	$O(k^2m^2)$	Lovász [1976c], Tarjan [1974a]
	$O(knm + k^3n^2)$	Tong and Lawler [1983]
*	$O(k^2n^2 + m)$	Gabow [1991a,1995a]

(As noticed by Tong and Lawler [1983], the paper of Shiloach [1979a] claiming an $O(k^2nm)$ bound, contains an essential error (the set A constructed on page 25 of Shiloach [1979a] need not have the desired properties: it is maximal under the condition that $y \notin A$, while it should be maximal under the condition that $A \cup V(T) \neq V$).)

The $O(k^2m^2)$ bound for finding k pairwise disjoint r-arborescences implies the $O(n^2\Delta^4 \log \Delta)$ bound of Markosyan and Gasparyan [1986] for finding a minimum number of branchings covering all arcs (where Δ is the maximum indegree of the vertices), by the construction given in the proof of Corollary 53.1h (as we can take $m \leq n\Delta$ and $k \leq 2\Delta$).

Tarjan [1974c] gave an $O(m + n \log n)$-time algorithm to find two disjoint r-arborescences (actually, to find two r-arborescences with smallest intersection). This was improved to $O(m\alpha(m,n))$ by Tarjan [1976] (where $\alpha(m,n)$ is the inverse Ackermann function), and to $O(m)$ by Gabow and Tarjan [1985].

Clearly, each of the bounds in the table above implies a complexity bound for the capacitated case, by replacing arcs by multiple arcs. However, this can increase the number m of arcs dramatically, and does not lead to a polynomial-time algorithm. Better bounds are given in the following table:

	$O(n^3 \cdot \mathrm{MF}(n,m))$	Pevzner [1979a] *taking rounding as one arithmetic step*
*	$O(k^2n^2 + m)$	Gabow [1991a,1995a]
*	$O(n^3m \log \frac{n^2}{m})$	Gabow and Manu [1995,1998]
*	$O(n^2m \log C \log \frac{n^2}{m})$	Gabow and Manu [1995,1998]

In these bounds, m is the number of arcs in the original graph, $\mathrm{MF}(n,m)$ denotes the time needed to solve a maximum flow problem in a digraph with n vertices and m arcs, and C is the maximum capacity (for integer capacity function).

The bounds of Gabow [1991a,1995a] and Gabow and Manu [1995,1998] in these tables also apply to the problem considered in Edmonds' disjoint branching theorem (Theorem 53.1): finding k disjoint branchings $B_1, \ldots, B_k$ where B_i has a given root set R_i ($i = 1, \ldots, k$), finding minimum coverings by branchings, and related problems. Gabow and Manu [1995,1998] also gave an $O(n^3m \log \frac{n^2}{m})$ fractional packing algorithm of r-arborescences.

Gabow [1991a,1995a] announced $O(kn(m+n \log n) \log n)$- and $O(k\sqrt{n \log n}(m+kn \log n) \log(nK))$-time algorithms to find a minimum-cost union of k disjoint r-arborescences (where K is the maximum cost, with integer cost function).

53.10b. Arborescences with roots in given subsets

Let $D = (V, A)$ be a digraph. Call a vector $x \in \mathbb{Z}_+^V$ a *root vector* if there exist disjoint arborescences such that for each $v \in V$, exactly x_v of these arborescences have root v. By Corollary 53.1a, root vectors are the integer solutions of the following system:

$$\text{(53.50)} \qquad \begin{array}{lll} \text{(i)} & x_v \geq 0 & \text{for } v \in V, \\ \text{(ii)} & x(U) \leq d^{\text{out}}(U) & \text{for each } U \subset V. \end{array}$$

This system generally does not define an integer polytope P, as is shown by the digraph with vertices u, v, w and arcs (u, v), (v, w), and (w, u), where $\frac{1}{2} \cdot \mathbf{1}$ is in P, but each integer vector x in P satisfies $\mathbf{1}^{\mathsf{T}} x \leq 1$.

Moreover, sets R of vertices for which there exist $|R|$ disjoint arborescences, rooted at distinct vertices in R, do not form the independent sets of a matroid, as is shown by the graph of Figure 53.1.

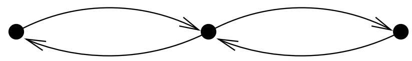

Figure 53.1

However, for any $k \in \mathbb{Z}_+$, the system

$$\text{(53.51)} \qquad x(U) \geq k - d^{\text{in}}(U) \text{ for each nonempty } U \subseteq V,$$

is box-TDI, since the right-hand side in (ii) is intersecting supermodular (cf. Sections 44.5 and 48.1).

Cai [1983] proved the following result, with a method (described below) of Frank [1981c] for proving a special case (Corollary 53.11a):

Theorem 53.11. *Let $D = (V, A)$ be a digraph such that D has k arc-disjoint arborescences. Let $l, u \in \mathbb{Z}_+^V$ with $l \leq u$. Then D has k arc-disjoint arborescences such that, for each $v \in V$, at least $l(v)$ and most $u(v)$ of these arborescences are rooted at v if and only if*

$$\text{(53.52)} \qquad u(U) + d^{\text{in}}(U) \geq k \text{ and } l(U) + \sum_{W \in \mathcal{P}} (k - d^{\text{in}}(W)) \leq k$$

for each nonempty subset U of V and each partition $\mathcal{P}$ of $V \setminus U$ into nonempty sets.

Proof. Necessity being easy, we show sufficiency. Choose $x \in \mathbb{Z}_+^V$ such that $l \leq x \leq u$ and such that (53.51) holds, with $x(V)$ as small as possible. (Such an x exists since $u(U) \geq k - d^{\text{in}}(U)$ for each nonempty subset U of V.)

We show that $x(V) = k$. Since $x(V) \geq k$ by (53.51), it suffices to show $x(V) \leq k$. Let $\mathcal{P}$ be the collection of inclusionwise maximal sets having equality in (53.51). Then $\mathcal{P}$ is a subpartition, for suppose that $U, W \in \mathcal{P}$ with $U \cap W \neq \emptyset$. Then

$$\text{(53.53)} \qquad \begin{aligned} x(U \cup W) &= x(U) + x(W) - x(U \cap W) \\ &\leq (k - d^{\text{in}}(U)) + (k - d^{\text{in}}(W)) - (k - d^{\text{in}}(U \cap W)) \\ &\leq (k - d^{\text{in}}(U \cup W)), \end{aligned}$$

and hence $U \cup W \in \mathcal{P}$. So $U = W$.

Now for each $v \in V$ with $x_v > l(v)$ there exists a set W in $\mathcal{P}$ containing v, since otherwise we could decrease x_v. Hence

$$(53.54) \qquad x(V) - l(V) = \sum_{W \in \mathcal{P}} (x(W) - l(W)) = \sum_{W \in \mathcal{P}} (k - d^{\text{in}}(W) - l(W)) \\ \leq k - l(V),$$

by (53.52).

So $x(V) = k$. Now let $r_1, \ldots, r_k$ be vertices such that each vertex v occurs x_v times among the r_i. Then by Corollary 53.1a there exist disjoint arborescences $B_1, \ldots, B_k$, where B_i has root r_i. This shows the theorem. ∎

(In this proof we did not use the box-total dual integrality of (53.51), but we applied a similar argument.)

This has as special case the following result of Frank [1981c]:

Corollary 53.11a. *Let $D = (V, A)$ be a digraph such that D has k arc-disjoint arborescences. Let $u \in \mathbb{Z}_+^V$. Then D has k arc-disjoint arborescences such that, for each $v \in V$, at most $u(v)$ of these arborescences have their root in v if and only if*

$$(53.55) \qquad u(U) + d^{\text{in}}(U) \geq k$$

for each nonempty subset U of V.

Proof. Directly from Theorem 53.11. ∎

A related theorem is:

Theorem 53.12. *Let $D = (V, A)$ be a digraph and let $R_1, \ldots, R_k$ be subsets of V. Then there exist disjoint arborescences $B_1, \ldots, B_k$, where B_i has its root in R_i (for $i = 1, \ldots, k$) if and only if*

$$(53.56) \qquad \sum_{U \in \mathcal{P}} (k - d^{\text{in}}(U)) \leq |\{i \mid R_i \cap \bigcup \mathcal{P} \neq \emptyset\}|$$

for each subpartition $\mathcal{P}$ of V with nonempty classes.

Proof. Necessity is easy, since if the B_i exist, with roots $r_i \in R_i$, then for each $U \in \mathcal{P}$ one has that $r_i \in U$ or B_i contains at least one arc entering U. That is,

$$(53.57) \qquad |\{r_i\} \cap U| + d^{\text{in}}_{B_i}(U) \geq 1.$$

Summing this inequality over $U \in \mathcal{P}$ and over $i = 1, \ldots, k$ we obtain (53.56), with R_i replaced by $\{r_i\}$. This implies (53.56) for the original R_i.

To see sufficiency, first observe that the condition implies that the R_i are nonempty (by taking $\mathcal{P} := \{V\}$). If the R_i are singletons, the theorem is equivalent to Corollary 53.1a. So we can assume that $|R_1| \geq 2$. Choose distinct vertices $u, w \in R_1$.

If the condition is maintained after replacing R_1 by $R_1 \setminus \{u\}$, the theorem follows by induction. So we can assume that this violates the condition. That is, there exists a subpartition $\mathcal{P}$ of V into nonempty classes such that (setting $X := \bigcup \mathcal{P}$):

(53.58) $$\sum_{U\in\mathcal{P}}(k - d^{\text{in}}(U)) = |\{i \mid R_i \cap X \neq \emptyset\}|$$

and such that $X \cap R_1 = \{u\}$ (for the original R_1). Similarly we can assume that there exists a subpartition $\mathcal{Q}$ of V into nonempty classes such that (setting $Y := \bigcup\mathcal{Q}$):

(53.59) $$\sum_{U\in\mathcal{Q}}(k - d^{\text{in}}(U)) = |\{i \mid R_i \cap Y \neq \emptyset\}|$$

and such that $Y \cap R_1 = \{w\}$.

Let $\mathcal{F}$ be the union of $\mathcal{P}$ and $\mathcal{Q}$ (any set occurring both in $\mathcal{P}$ and in $\mathcal{Q}$ occurs twice in $\mathcal{F}$). Now iteratively replace any $T, U \in \mathcal{F}$ with $T \cap U \neq \emptyset$ and $T \not\subseteq U \not\subseteq T$ by $T \cap U$ and $T \cup U$. Then the final family $\mathcal{F}$ is laminar. Let $\mathcal{R}$ be the collection of inclusionwise minimal sets in $\mathcal{F}$ and let $\mathcal{S}$ be the collection of inclusionwise maximal sets in $\mathcal{F}$. Then $\mathcal{R}$ and $\mathcal{S}$ are subpartitions of V into nonempty classes, and $\bigcup\mathcal{R} = X \cap Y$ and $\bigcup\mathcal{S} = X \cup Y$. Moreover

(53.60) $$\begin{aligned}&\sum_{U\in\mathcal{R}}(k - d^{\text{in}}(U)) + \sum_{U\in\mathcal{S}}(k - d^{\text{in}}(U)) = \sum_{U\in\mathcal{F}}(k - d^{\text{in}}(U))\\ &\geq \sum_{U\in\mathcal{P}}(k - d^{\text{in}}(U)) + \sum_{U\in\mathcal{Q}}(k - d^{\text{in}}(U))\\ &= |\{i \mid R_i \cap X \neq \emptyset\}| + |\{i \mid R_i \cap Y \neq \emptyset\}|\\ &> |\{i \mid R_i \cap (X \cap Y) \neq \emptyset\}| + |\{i \mid R_i \cap (X \cup Y) \neq \emptyset\}|.\end{aligned}$$

The first inequality follows from the submodularity of $d^{\text{in}}(U)$. The last inequality holds as (i) if R_i intersects $X \cup Y$, then it intersects X or Y, (ii) if R_i intersects $X \cap Y$, then it intersects X and Y, and (iii) R_1 intersects X and Y but not $X \cap Y$, since $R_1 \cap X = \{u\}$ and $R_1 \cap Y = \{w\}$.

However, (53.60) contradicts (53.56). ∎

53.10c. Disclaimers

The equivalence of (i) and (iii) in Corollary 53.1c suggests the following question, raised by A. Frank (cf. Schrijver [1979b], Frank [1995]; it generalizes a similar question for the undirected case, described in Section 51.5b):

(53.61) (?) Let $D = (V, A)$ be a k-arc-connected digraph and let $r \in V$. Suppose that for each $s \in V$ there exist k internally vertex-disjoint $r - s$ paths in D. Then there exist k r-arborescences such that, for any vertex s, the k $r - s$ paths determined by the respective r-arborescences are internally vertex-disjoint. (?)

For $k = 2$ this was proved by Whitty [1987]. However, for $k = 3$, a counterexample was found by Huck [1995].

Two potential generalizations of Edmonds' disjoint arborescences theorem have been raised, neither of which holds however. For vertices s, t, let $\lambda(s, t)$ denote the maximum number of arc-disjoint $s - t$ paths. It is not true that for any digraph $D = (V, A)$, $r \in V$, and $T \subseteq V \setminus \{r\}$, there exist k disjoint subsets $A_1, \ldots, A_k$ of A such that each A_i contains an $r - t$ path for each $t \in T$ if and only if $\lambda(r, t) \geq k$ for each $t \in T$ (see Figure 53.2, for $k = 2$).

N. Robertson raised the question if it is true that in any digraph $D = (V, A)$ and any $r \in V$, there exist partial r-arborescences $B_1, B_2, \ldots$ such that each vertex $v \in V \setminus \{r\}$ is in exactly $\lambda(r, v)$ of them. Lovász [1973b] showed that Figure 53.3 is a counterexample. (Related work is reported by Bang-Jensen, Frank, and Jackson [1995] and Gabow [1996].)

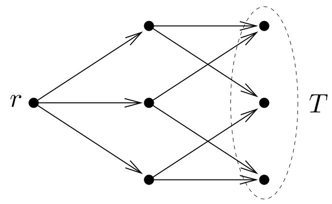

Figure 53.2

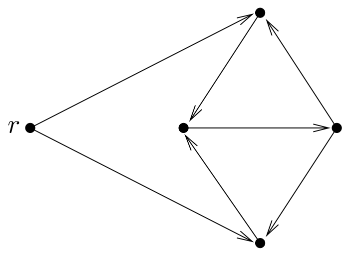

Figure 53.3

53.10d. Further notes

Frank [1981c] gave the following results for mixed graphs. Let $G = (V, E, A)$ be a mixed graph (that is, (V, E) is an undirected graph and (V, A) is a directed graph). A *mixed branching* is a subset B of $E \cup A$ such that the undirected edges in B can be oriented such that B becomes a branching. Then E can be covered by k mixed branchings if and only if

(53.62) (i) $d_A^{\text{in}}(U) + |E[U]| \leq k|U|$ for each $U \subseteq V$,
(ii) $|A[U]| + |E[U]| \leq k(|U| - 1)$ for each $\emptyset \neq U \subseteq V$.

Similarly, a *mixed r-arborescence* is a subset B of $E \cup A$ such that the undirected edges in B can be oriented such that B becomes an r-arborescence. Then for any $r \in V$, G has k disjoint mixed r-arborescences if and only if for each subpartition $\mathcal{P}$ of $V \setminus \{r\}$ with nonempty classes, the number of edges (directed or not) entering any class of $\mathcal{P}$, is at least $k|\mathcal{P}|$.

Cai [1989] characterized when, for given digraphs $D_1 = (V, A_1)$ and $D_2 = (V, A_2)$, $a, b \in \mathbb{Z}_+^V$, and $k \in \mathbb{Z}_+$, there exists an $r \in \mathbb{Z}_+^V$ with $a \leq r \leq b$ and there exist, for $i = 1, 2$, k disjoint arborescences in D_i such that for each $v \in V$, $r(v)$ of these arborescences have root v. This can be proved using polymatroid intersection theory, in particular the box-total dual integrality of

(53.63) $x(U) \geq k - d_{A_i}^{\text{in}}(U)$ for $i = 1, 2$ and nonempty $U \subseteq V$

(Theorem 48.5). (For a generalization, see Cai [1990a,1993].)

Cai [1990b] showed, for given digraph $D = (V, A)$, $r \in V$, $d, c \in \mathbb{Z}_+^A$, and $k \in \mathbb{Z}_+$: there exist k r-arborescences such that each arc is covered at least $d(a)$ times and at most $c(a)$ times, if and only if $d(\delta^{\text{in}}(v)) \leq k$, for each $v \in V \setminus \{r\}$, and

$$\sum_{v \in U} \min\{k - d(\delta^{\text{in}}(v) \setminus \delta^{\text{in}}(U)), c(\delta^{\text{in}}(v) \cap \delta^{\text{in}}(U))\} \geq k \tag{53.64}$$

for each nonempty subset U of $V \setminus \{r\}$.

Chapter 54

Biconnectors and bibranchings

The concept of biconnector is a generalization of that of a connector. Let $G = (V, E)$ be an undirected graph and let V be partitioned into classes R and S. An $R-S$ *biconnector* is a subset F of E such that each component of (V, F) intersects both R and S. So contracting R or S gives a connector. If R is a singleton, $R-S$ biconnectors are precisely the connectors.
For biconnectors, min-max relations, polyhedral characterizations, and complexity results similar to those for connectors hold.
In this chapter we also consider the forest analogue of biconnector, the biforest. An $R-S$ *biforest* is a forest F such that each component of (V, F) has at most one edge in the cut $\delta(R)$. So contracting R or S gives a forest. Also biforests show good polyhedral and algorithmical behaviour.
Similar results hold for the directed analogues of biconnectors and biforests, the bibranchings and the bifurcations. An $R-S$ *bibranching* is a set B of arcs such that for each $s \in S$, B contains an $R-s$ path and for each $r \in R$, B contains an $r-S$ path. Bibranchings form a generalization of arborescences, and give rise to similar min-max relations and polyhedral characterizations. An $R-S$ *bifurcation* is a set B of arcs containing no undirected circuit, such that each vertex in R is left by at most one arc in B, each vertex in S is entered by at most one arc in B, and B contains no arcs from S to R.
Theorem 54.11 on disjoint bibranchings will be the only result of this chapter that will be used later in this book, namely in Chapter 56 to obtain a dual form of the Lucchesi-Younger theorem, on packing directed cut covers in a source-sink connected digraph. The proof of Theorem 54.11 uses no other results from this chapter.

54.1. Shortest $R-S$ biconnectors

Let $G = (V, E)$ be a graph and let V be partitioned into two sets R and S. A subset F of E is called an $R-S$ *biconnector* if each component of the graph (V, F) intersects both R and S. So F is an $R-S$ biconnector if and only if each component of (V, F) has at least one edge in $\delta(R)$.

A min-max relation for the minimum size of an $R-S$ biconnector can be derived easily from the Kőnig-Rado edge cover theorem:

Theorem 54.1. *Let $G = (V, E)$ be a graph and let V be partitioned into sets R and S such that each component of G intersects both R and S. Then the minimum size of an $R - S$ biconnector is equal to the maximum size of a subset of V spanning no edge connecting R and S.*

Proof. To see that the minimum is not less than the maximum, let F be a minimum $R - S$ biconnector and let U attain the maximum. Then F is a forest. For each $r \in U \cap R$, let $\phi(r)$ be the first edge in any $r - S$ path in F; and for each $s \in U \cap S$ let $\phi(s)$ be the first edge in any $s - R$ path in F. Then ϕ is injective from U to F (as U spans no edge in $\delta(R)$). Hence $|U| \leq |F|$.

To see equality, let $H := (N(R) \cup N(S), \delta(R))$, where $N(R)$ and $N(S)$ are the sets of neighbours of R and of S respectively. (So $N(R) \subseteq S$ and $N(S) \subseteq R$, and H is bipartite.) Let U' be a maximum-size stable set in H. Let F' be a minimum-size edge cover in H. By the Kőnig-Rado edge cover theorem (Theorem 19.4) we know $|F'| = |U'|$. Let $U := U' \cup (V \setminus (N(R) \cup N(S)))$. Then U spans no edge connecting R and S. By adding $|V \setminus (N(R) \cup N(S))|$ edges to F' we obtain an $R - S$ biconnector F with

$$|F| = |U'| + |V \setminus (N(R) \cup N(S))| = |U|. \tag{54.1}$$

This shows the required equality. ∎

To obtain a min-max relation for the minimum length of an $R - S$ biconnector (given a length function on the edges), consider the system

(54.2)
(i) $0 \leq x_e \leq 1$ for each $e \in E$,
(ii) $x(\delta(\mathcal{P})) \geq |\mathcal{P}|$ for each subpartition $\mathcal{P}$ of R or S with nonempty classes.

Here a *subpartition* of a set X is a partition of a subset of X (that is, a collection of disjoint subsets of X). $\delta(\mathcal{P})$ denotes the set of edges incident with but not spanned by any set in $\mathcal{P}$. Then system (54.2) determines the *$R - S$ biconnector polytope* — the convex hull of the incidence vectors of $R - S$ biconnectors:

Theorem 54.2. *System* (54.2) *is box-totally dual integral and determines the $R - S$ biconnector polytope.*

Proof. This follows from matroid intersection theory, applied to the matroids M_1 and M_2 on E, where M_1 is obtained from the cycle matroid $M(G)$ of G by contracting R to one vertex, making all edges spanned by R to a loop, and where M_2 is obtained similarly from $M(G)$ by contracting S to one vertex, making all edges spanned by S to a loop.

So the spanning sets of M_1 are the subsets F of E such that each component of (V, F) intersects R. Similarly, the spanning sets of M_2 are the subsets F of E such that each component of (V, F) intersects S. Hence the common spanning sets are precisely the $R - S$ biconnectors. Therefore, by Corollaries

41.12f and 50.8a, system (54.2) determines the convex hull of the incidence vectors of $R-S$ biconnectors. To see that the system is box-TDI, we use Corollary 41.12g and the fact that for each $F \subseteq E$, the inequality

(54.3) $$x(F) \geq r_{M_i}(E) - r_{M_i}(E \setminus F)$$

is a nonnegative integer combination of the inequalities (54.2). Indeed (for $i=1$), if $\mathcal{P}$ denotes the collection of the components of $(V, E \setminus F)$ contained in S, then

(54.4) $$x(F) \geq x(\delta(\mathcal{P})) \geq |\mathcal{P}| \geq r_{M_1}(E) - r_{M_1}(E \setminus F),$$

as $r_{M_1}(E) \leq |S|$ and $r_{M_1}(E \setminus F) = |S| - |\mathcal{P}|$. ∎

This implies a min-max relation for the minimum length of an $R-S$ biconnector. The reduction to matroid intersection also immediately implies that one can find a shortest $R-S$ biconnector in strongly polynomial time.

54.2. Longest $R-S$ biforests

Again, let $G=(V,E)$ be a graph and let V be partitioned into two sets R and S. Call a subset F of E an $R-S$ *biforest* if F is a forest and each component of F contains at most one edge in $\delta(R)$.

A min-max relation for the maximum size of an $R-S$ biforest can be derived easily from Kőnig's matching theorem:

Theorem 54.3. *Let $G=(V,E)$ be a graph and let V be partitioned into sets R and S. Then the maximum size of an $R-S$ biforest is equal to the minimum value of $|V|-|\mathcal{U}|$, where $\mathcal{U}$ is a collection of components of $G-R$ and $G-S$ such that no edge connects any two sets in $\mathcal{U}$.*

Proof. We may assume that G has no loops. To see that the maximum is not more than the minimum, consider any $R-S$ biforest F and any collection $\mathcal{U}$ as in the theorem. Then F contains no path connecting two distinct sets in $\mathcal{U}$. Hence $|F| \leq |V| - |\mathcal{U}|$.

The reverse inequality is proved by induction on the number of edges not in $\delta(R)$.

If $E=\delta(R)$, then G is bipartite, and $R-S$ biforests coincide with matchings. Then the theorem is equivalent to Kőnig's matching theorem (Theorem 16.2).

If $E \neq \delta(R)$, choose an edge $e=uv$ in $E \setminus \delta(R)$. If we contract e, the minimum value in the theorem reduces by precisely 1. Moreover, the maximum reduces by at least 1, since any $R-S$ forest in the contracted graph gives with e an $R-S$ forest in the original graph. So we are done by induction. ∎

To obtain a min-max relation for the maximum length of an $R-S$ biforest (given a length function on the edges), consider the following system:

(54.5)
$$\begin{array}{lll} \text{(i)} & 0 \leq x_e \leq 1 & \text{for each } e \in E, \\ \text{(ii)} & x(E[U]) \leq |U|-1 & \text{for each nonempty subset } U \text{ of } R \text{ or } S, \\ \text{(iii)} & x(E[U] \cup (\delta(U) \cap \delta(R))) \leq |U| & \\ & & \text{for each subset } U \text{ of } R \text{ or } S. \end{array}$$

This determines the $R-S$ *biforest polytope* — the convex hull of the incidence vectors of $R-S$ biforests:

Theorem 54.4. *System* (54.5) *is box-totally dual integral and determines the $R-S$ biforest polytope.*

Proof. This can be reduced to matroid intersection theory, similar to the proof of Theorem 54.3. ∎

Again, this theorem implies a min-max relation for the maximum length of an $R-S$ biforest, and the reduction to matroid intersection also implies that a longest $R-S$ biforest can be found in strongly polynomial time.

54.3. Disjoint $R-S$ biconnectors

We give a min-max relation for the maximum number of disjoint $R-S$ biconnectors (Keijsper and Schrijver [1998]). It generalizes the Tutte-Nash-Williams disjoint trees theorem (Corollary 51.1a) — which theorem however is used in the proof — and the disjoint edge covers theorem for bipartite graphs (Theorem 20.5).

We follow the (algorithmic) proof method of Keijsper [1998a], based on the following lemma:

Lemma 54.5α. *Let $T_1 = (V, E_1)$ and $T_2 = (V, E_2)$ be edge-disjoint spanning trees and let $r \in V$. For each $e = rv \in \delta_{T_1}(r)$, let $\phi(e)$ be the first edge of the $v-r$ path in T_2 that leaves the component of $T_1 - e$ containing v. Let $B \subseteq \delta_{T_1}(r)$ be such that $\phi(B)$ contains at most one edge not in $\delta_{T_2}(r)$. Then $(E_1 \setminus B) \cup \phi(B)$ and $(E_2 \setminus \phi(B)) \cup B$ are spanning trees again.*

Proof. By induction on $|B|$, the case $|B| \leq 1$ being easy. Let $|B| \geq 2$. Then there exists an edge $f = rw \in B$ with $\phi(f) \in \delta_{T_2}(r)$ (by the condition given in the theorem). Define

(54.6) $\quad T_1' := (T_1 \setminus \{f\}) \cup \{\phi(f)\}$ and $T_2' := (T_2 \setminus \{\phi(f)\}) \cup \{f\}$.

Let $B' := B \setminus \{f\}$. Then for each $e = rv \in B'$,

(54.7) $\quad \phi(e)$ is equal to the first edge of the $v-r$ path in T_2' that leaves the component K of $T_1' - e$ containing v.

To see this, let L be the component of $T_1 - f$ containing w. Since $\phi(f)$ connects L and r, K is equal to the component of $T_1 - e$ containing v. Moreover, the

$v - r$ path P in T_2' does not differ from the $v - r$ path in T_2 before entering L, and hence the first edge of P leaving K equals $\phi(e)$. This shows (54.7).

Now $(T_1 \setminus B) \cup \phi(B) = (T_1' \setminus B') \cup \phi(B')$ and $(T_2 \setminus \phi(B)) \cup B = (T_2' \setminus \phi(B')) \cup B'$, and by induction, they are spanning trees. ∎

Notice that the function $\phi : E_1 \cap \delta(r) \to E_2$ defined in the lemma is injective.

In the following lemma, we consider forests as edge sets. We recall that G/R denotes the graph obtained from G by contracting all vertices in R to one new vertex, denoted by R. The edges in the contracted graph are named after the edges in the original graph.

Lemma 54.5β. *Let $G = (V, E)$ be a graph and let V be partitioned into sets R and S. Let X_1 and X_2 be disjoint forests in G/R and let Y_1 and Y_2 be disjoint forests in G/S. Then there exist disjoint forests X_1' and X_2' in G/R and disjoint forests Y_1' and Y_2' in G/S with $X_1' \cup X_2' = X_1 \cup X_2$, $Y_1' \cup Y_2' = Y_1 \cup Y_2$, $X_1' \cap Y_2' = \emptyset$, and $X_2' \cap Y_1' = \emptyset$.*

Proof. By adding new edges spanned by S, we can assume that the X_i are spanning trees in G/R. Similarly, we can assume that the Y_i are spanning trees in G/S. (At the conclusion, we delete the new edges from the X_i' and Y_i'.)

If $X_1 \cap Y_2 = \emptyset$ and $X_2 \cap Y_1 = \emptyset$, we are done. So, by symmetry, we can assume that $X_1 \cap Y_2 \neq \emptyset$.

For each $e = rs \in X_1 \cap \delta(R)$, with $r \in R, s \in S$, let $\phi(e)$ be the first edge on the $s - R$ path in X_2 that leaves the component of $X_1 - e$ containing s. For each $e = rs \in Y_2 \cap \delta(R)$, with $r \in R, s \in S$, let $\psi(e)$ be the first edge on the $r - S$ path in Y_1 that leaves the component of $Y_2 - e$ containing r.

This gives injective functions

(54.8) $$\phi : X_1 \cap \delta(R) \to X_2 \text{ and } \psi : Y_2 \cap \delta(R) \to Y_1.$$

Observe that $X_1 \cap (Y_1 \cup Y_2) \subseteq \delta(R)$ and $Y_2 \cap (X_1 \cup X_2) \subseteq \delta(R)$. Consider the directed graph with vertex set E and arc set

(54.9) $$A := \{(e, \phi(e)) \mid e \in X_1 \cap \delta(R)\} \cup \{(e, \psi(e)) \mid e \in Y_2 \cap \delta(R)\}.$$

Choose $e_0 \in X_1 \cap Y_2$ and set $e_1 := \phi(e_0)$. Then D contains a unique directed path $e_0, e_1, \ldots, e_h$ such that $e_0, \ldots, e_{h-1} \in X_1 \cup Y_2$ and $e_h \notin X_1 \cup Y_2$. (This because each vertex in $X_1 \cap Y_2$ has outdegree 2 and indegree 0 in D, and each vertex in $(X_1 \cup Y_2) \setminus (X_1 \cap Y_2)$ has outdegree 1 and indegree at most 1.)

It follows that for each $j < h$ one has $e_{j+1} = \phi(e_j)$ if j is even and $e_{j+1} = \psi(e_j)$ if j is odd. Define

(54.10) $$B := \{e_j \mid 0 \leq j < h, \ j \text{ even}\} \text{ and } C := \{e_j \mid 1 \leq j < h, \ j \text{ odd}\}.$$

Then by Lemma 54.5α,

(54.11) $X'_1 := (X_1 \setminus B) \cup \phi(B)$, $X'_2 := (X_2 \setminus \phi(B)) \cup B$,
$Y'_1 := (Y_1 \setminus \psi(C)) \cup C$, $Y'_2 := (Y_2 \setminus C) \cup \psi(C)$,

are again spanning tree of G/R and G/S respectively. Note that $X'_1 \cap X'_2 = \emptyset$, $Y'_1 \cap Y'_2 = \emptyset$, $X'_1 \cup X'_2 = X_1 \cup X_2$ and $Y'_1 \cup Y'_2 = Y_1 \cup Y_2$.

Now $\phi(B) \cap \psi(C) = \emptyset$, $\phi(B) \cap (Y_2 \setminus C) = \emptyset$ (since $\phi(B) \cap Y_2 \subseteq C$, as $e_h \not\in Y_2$), and $\psi(C) \cap (X_1 \setminus B) = \emptyset$ (since $\psi(C) \cap X_1 \subseteq B$, as $e_h \not\in X_1$). So $X'_1 \cap Y'_2 \subseteq (X_1 \cap Y_2) \setminus \{e_0\}$ (since $e_0 \not\in X'_1$).

Moreover, $B \cap C = \emptyset$, $B \cap (Y_1 \setminus \psi(C)) = \emptyset$ (since $B \cap Y_1 \subseteq \psi(C)$, as $e_0 \not\in Y_1$), and $C \cap (X_2 \setminus \phi(B)) = \emptyset$ (since $C \cap X_2 \subseteq \phi(B)$, as $e_0 \not\in X_2$). So $X'_2 \cap Y'_1 \subseteq X_2 \cap Y_1$.

Concluding, $|X'_1 \cap Y'_2| + |X'_2 \cap Y'_1| < |X_1 \cap Y_2| + |X_2 \cap Y_1|$. Therefore, iterating this, we obtain trees as required. ∎

Now a min-max relation for disjoint $R-S$ biconnectors can be deduced:

Theorem 54.5. *Let $G = (V, E)$ be a graph, let V be partitioned into sets R and S, and let $k \in \mathbb{Z}_+$. Then there exist k disjoint $R-S$ biconnectors if and only if $|\delta(\mathcal{P})| \geq k|\mathcal{P}|$ for each subpartition $\mathcal{P}$ of R or S with nonempty classes.*

Proof. Necessity being easy, we show sufficiency. By Corollary 51.1a, the graph G/R (obtained from G by contracting R) has k disjoint spanning trees $X_1, \ldots, X_k$. Similarly, the graph G/S has k disjoint spanning trees $Y_1, \ldots, Y_k$. Then $X_i \cap Y_j$ is a subset of $\delta(R)$, for all i, j. Choose the X_i and Y_i in such a way that

$$(54.12) \qquad \sum_{i=1}^{k} |X_i \cap Y_i|$$

is as large as possible.

Then $X_i \cap Y_j = \emptyset$ for all distinct i, j, for if, say, $X_1 \cap Y_2 \neq \emptyset$, we can replace X_1, X_2, Y_1, Y_2 by X'_1, X'_2, Y'_1, Y'_2 as in Lemma 54.5β. Then we have

$$(54.13) \qquad |X'_1 \cap Y'_1| + |X'_2 \cap Y'_2| = |(X'_1 \cup X'_2) \cap (Y'_1 \cup Y'_2)|$$
$$= |(X_1 \cup X_2) \cap (Y_1 \cup Y_2)| > |X_1 \cap Y_1| + |X_2 \cap Y_2|.$$

This contradicts the maximality of sum (54.12).

Hence $X_1 \cup Y_1, \ldots, X_k \cup Y_k$ form k disjoint $R-S$ biconnectors as required. ∎

This proof gives a polynomial-time algorithm to find a maximum number of disjoint $R-S$ biconnectors. Keijsper [1998a] gave an $O(\mathrm{DT}(n, m) + nm)$-time algorithm for this problem, where $\mathrm{DT}(n, m)$ denotes the time needed to find a maximum number of disjoint spanning trees in an undirected graph with n vertices and m edges.

By replacing edges by parallel edges, one obtains a capacitated version of Theorem 54.5. The corresponding optimization problem can be solved in polynomial time, by a straightforward adaptation of the methods described in the proofs of Theorems 51.8 and 51.10 and Corollary 51.8a. However, the capacitated problem cannot be solved in strongly polynomial time if we do not allow rounding (cf. the argument given in Section 51.4).

A generalization of Theorem 54.5 is given by Keijsper [1998a].

54.4. Covering by $R - S$ biforests

With the foregoing two lemmas, one can also derive a min-max relation for the minimum number of $R - S$ biforests that cover all edges (Keijsper [1998b]). It generalizes the Nash-Williams' covering forests theorem (Corollary 51.1c) — which theorem however is used in the proof — and Kőnig's edge-colouring theorem for bipartite graphs (Theorem 20.1).

Theorem 54.6. *Let $G = (V, E)$ be a graph, let V be partitioned into sets R and S, and let $k \in \mathbb{Z}_+$. Then E can be covered by k $R - S$ biforests if and only if*

(54.14) $\quad |E[U]| \leq k(|U| - 1)$ *and* $|E[U]| + |\delta(U) \cap \delta(R)| \leq k|U|$

for each nonempty subset U of R or S.

Proof. Necessity being easy, we show sufficiency. We can assume that G is connected, as otherwise we can consider any component of G separately.

By Corollary 51.1c, the edges of the graph G/R can be partitioned into k forests $X_1, \ldots, X_k$. Similarly, the edges of the graph G/S can be partitioned into k forests $Y_1, \ldots, Y_k$. So $X_i \cap Y_j \subseteq \delta(R)$, for all i, j. Choose the X_i and Y_i in such a way that sum (54.12) is as large as possible. Then, as in the proof of Theorem 54.5, $X_i \cap Y_j = \emptyset$ for distinct i, j. Hence each $e \in \delta(R)$ belongs to $X_i \cap Y_i$ for some $i = 1, \ldots, k$. Concluding, $X_1 \cup Y_1, \ldots, X_k \cup Y_k$ form $R - S$ biforests as required. ∎

This proof gives a polynomial-time algorithm for finding a minimum covering by $R - S$ biforests. The methods of Section 51.4 can be extended to imply the polynomial-time solvability of the corresponding capacitated version, while strong polynomial-time solvability is again impossible.

54.5. Minimum-size bibranchings

We now turn to the directed analogues of biconnectors and biforests. Let $D = (V, A)$ be a digraph and let V be partitioned into two sets R and S. Call a subset B of A an $R - S$ *bibranching* if in the graph (V, B), each vertex in S is reachable from R, and each vertex in R reaches S.

Similarly to minimum $R - S$ biconnectors, a min-max relation for the minimum size of an $R - S$ bibranching follows easily from the Kőnig-Rado edge cover theorem.

Theorem 54.7. *Let $D = (V, A)$ be a graph and let V be partitioned into sets R and S such that each vertex in R can reach S and such that each vertex in S is reachable from R. Then the minimum size of an $R - S$ bibranching is equal to the maximum size of a subset of V spanning no arc in $\delta^{\text{out}}(R)$.*

Proof. To see that the minimum is not less than the maximum, let B be a minimum-size $R - S$ bibranching and let U attain the maximum. For each $r \in U \cap R$, let $\phi(r)$ be any arc in B leaving r, and for each $s \in U \cap S$ let $\phi(s)$ be any arc in B entering s. Then ϕ is injective from U to B, and hence $|U| \leq |B|$.

To see equality, let U' be a maximum stable set in the bipartite graph H with colour classes $N^{\text{in}}(S) \subseteq R$ and $N^{\text{out}}(R) \subseteq S$, with $r \in N^{\text{in}}(S)$ and $s \in N^{\text{out}}(R)$ adjacent if and only if D has an arc from r to s. (Here $N^{\text{out}}(X)$ and $N^{\text{in}}(X)$ are the sets of outneighbours and of inneighbours of X, respectively.)

Let B' be a minimum-size edge cover in H. By the Kőnig-Rado edge cover theorem (Theorem 19.4) we know $|B'| = |U'|$. Now by adding $|V \setminus (N^{\text{out}}(R) \cup N^{\text{in}}(S))|$ arcs to B' we obtain an $R - S$ bibranching B with

$$|B| = |U'| + |V \setminus (N^{\text{out}}(R) \cup N^{\text{in}}(S))| = |U|, \tag{54.15}$$

where $U := U' \cup (V \setminus (N^{\text{out}}(R) \cup N^{\text{in}}(S)))$. This shows the required equality. ∎

If each arc of D belongs to $\delta^{\text{out}}(R)$, then Theorem 54.7 reduces to the Kőnig-Rado edge cover theorem (Theorem 19.4).

The proof gives a polynomial-time algorithm to find a minimum-size $R-S$ bibranching (as we can find a minimum-size edge cover in a bipartite graph in polynomial time (Corollary 19.3a)).

54.6. Shortest bibranchings

To obtain a min-max relation for the minimum length of an $R-S$ bibranching (given a length function on the arcs), define a set of arcs C to be an $R - S$ *bicut* if $C = \delta^{\text{in}}(U)$ for some nonempty proper subset U of V satisfying $U \subseteq S$ or $S \subseteq U$.

Consider the system:

$$\text{(54.16)} \quad \begin{array}{lll} \text{(i)} & x_a \geq 0 & \text{for each } a \in A, \\ \text{(ii)} & x(C) \geq 1 & \text{for each } R - S \text{ bicut } C. \end{array}$$

Then the following implies a min-max relation for the minimum length of an $R-S$ bibranching.

Theorem 54.8. *System* (54.16) *is box-TDI.*

Proof. Let $w : A \to \mathbb{R}_+$. Let $\mathcal{U}$ be the collection of nonempty proper subsets U of V satisfying $U \subseteq S$ or $S \subseteq U$. Consider the maximum value of

(54.17) $$\sum_{U \in \mathcal{U}} y_U$$

where $y : \mathcal{U} \to \mathbb{R}_+$ satisfies

(54.18) $$\sum_{U \in \mathcal{U}} y_U \chi^{\delta^{\text{in}}(U)} \leq w.$$

Choose $y : \mathcal{U} \to \mathbb{R}_+$ attaining the maximum, such that

(54.19) $$\sum_{U \in \mathcal{U}} y_U |U||V \setminus U|$$

is minimized. We show that the collection $\mathcal{F} := \{U \in \mathcal{U} \mid y_U > 0\}$ is cross-free; that is, for all $T, U \in \mathcal{F}$ one has

(54.20) $T \subseteq U$ or $U \subseteq T$ or $T \cap U = \emptyset$ or $T \cup U = V$.

Suppose that this is not true. Let $\alpha := \min\{y_T, y_U\}$. Decrease y_T and y_U by α, and increase $y_{T\cap U}$ and $y_{T \cup U}$ by α. Now (54.18) is maintained, and (54.17) did not change. However, (54.19) decreases (by Theorem 2.1), contradicting our minimality assumption.

So $\mathcal{F}$ is cross-free. Now the $\mathcal{F} \times A$ matrix M with

(54.21) $$M_{U,a} := \begin{cases} 1 & \text{if } a \in \delta^{\text{in}}(U), \\ 0 & \text{otherwise,} \end{cases}$$

is totally unimodular. To see this, let $T = (W, B)$ and $\pi : V \to W$ form a tree-representation of $\mathcal{F}$ (see Section 13.4). That is, T is a directed tree and $\mathcal{F} = \{V_b \mid b \in B\}$, where

(54.22) $V_b := \{v \in V \mid \pi(v)$ belongs to the same component of $T - b$ as the head of $b\}$.

Then for any arc $a = (u, v)$ of D, the set of forward arcs in the undirected $\pi(u) - \pi(v)$ path in T is contiguous, that is, forms a directed path, say from u' to v'. This follows from the fact that there exist no arcs b, c, d in this order on the path with b and d forward and c backward.

Define $a' := (u', v')$, and let $D' = (W, A')$ be the digraph with $A' := \{a' \mid a \in A\}$. Then M is equal to the network matrix generated by T and D' (identifying $b \in B$ with the set V_b in $\mathcal{F}$ determined by b). Hence by Theorem 13.20, M is totally unimodular.

This implies with Theorem 5.35 that (54.16) is box-TDI. ∎

This implies that the $R-S$ *bibranching polytope* — the convex hull of the incidence vectors of $R-S$ bibranchings — can be described as follows:

Corollary 54.8a. *The $R-S$ bibranching polytope is determined by*

(54.23) (i) $0 \leq x_a \leq 1$ for each $a \in A$,
(ii) $x(C) \geq 1$ for each $R-S$ bicut C.

Proof. By Theorem 54.8, (54.23) determines an integer polytope. Necessarily, each vertex of it is the incidence vector of an $R-S$ branching. ∎

The box-total dual integrality of (54.16) has as special case the total dual integrality of (54.23), which is equivalent to:

Corollary 54.8b (optimum bibranching theorem). *Let $D = (V, A)$ be a digraph, let V be partitioned into sets R and S, and let $l : A \to \mathbb{Z}_+$ be a length function. Then the minimum length of an $R-S$ bibranching is equal to the maximum size of a family of $R-S$ dicuts, such that each arc a is in at most $l(a)$ of them.*

Proof. This is a reformulation of the total dual integrality of (54.23), which follows from Theorem 54.8. ∎

We also note that Theorem 54.8 implies that for each $k \in \mathbb{Z}_+$ the system

(54.24) (i) $x_a \geq 0$ for each $a \in A$,
(ii) $x(C) \geq k$ for each $R-S$ bicut C,

is box-TDI (since if $Ax \leq b$ is box-TDI, then for each $k > 0$, $Ax \leq k \cdot b$ is box-TDI).

Keijsper and Pendavingh [1998] gave an $O(n'(m + n \log n))$ algorithm to find a shortest bibranching, where $n' := \min\{|R|, |S|\}$. The strong polynomial-time solvability follows also from the strong polynomial-time solvability of finding a minimum-length strong connector for a source-sink connected digraph, which by the method of Theorem 57.3 can be reduced to finding a minimum-length directed cut cover, which is a special case of weighted matroid intersection (Section 55.5).

54.6a. Longest bifurcations

Let $D = (V, A)$ be a digraph and let V be partitioned into two sets R and S. Call a subset B of A an $R-S$ *bifurcation* if B contains no undirected circuits, each vertex in R is left by at most one arc in B, each vertex in S is entered by at most one arc in B, and B contains no arcs from S to R. So B is an $R-S$ bifurcation if and only if contracting R gives a branching and contracting S gives a cobranching. (A *cobranching* is a set B of arcs whose reversal B^{-1} is a branching.)

Similarly to maximum $R-S$ biforests, a min-max relation for the maximum size of an $R-S$ bifurcation follows from Kőnig's matching theorem:

Theorem 54.9. *Let $D = (V, A)$ be a graph and let V be partitioned into sets R and S, with $\delta^{\text{in}}(R) = \emptyset$. Then the maximum size of an $R - S$ bifurcation is equal to the minimum size of $|V| - |\mathcal{L}|$, where $\mathcal{L}$ is a collection of strong components K of D with either $K \subseteq R$ and $\delta^{\text{out}}(K) \subseteq \delta^{\text{out}}(R)$, or $K \subseteq S$ and $\delta^{\text{in}}(K) \subseteq \delta^{\text{in}}(S)$, such that no arc connects two components in $\mathcal{L}$.*

Proof. To see that the minimum is not less than the maximum, let B be a maximum-size $R - S$ bifurcation and let $\mathcal{L}$ attain the minimum. Let U be the set of vertices v with $v \in R$ and $\delta_B^{\text{out}}(v) = \emptyset$, or $v \in S$ and $\delta_B^{\text{in}}(v) = \emptyset$. Then

(54.25) $$|B| = |V| - |U| - |B \cap \delta^{\text{out}}(R)| \leq |V| - |\mathcal{L}|,$$

since each $K \in \mathcal{L}$ contains a vertex in U or is entered or left by an arc in $B \cap \delta^{\text{out}}(R)$.

To see equality, consider the following bipartite graph H. H has vertex set the set $\mathcal{K}$ of strong components K of D with either $K \subseteq R$ and $\delta^{\text{out}}(K) \subseteq \delta^{\text{out}}(R)$, or $K \subseteq S$ and $\delta^{\text{in}}(K) \subseteq \delta^{\text{in}}(S)$. Two sets $K, L \in \mathcal{K}$ are adjacent if and only if there is an arc connecting K and L. (This implies that one of K, L is contained in R, the other in S.) Let $\mathcal{L}$ be a maximum-size stable set in H and let B' be a maximum-size matching in H. By Kőnig's matching theorem (Theorem 16.2), $|B'| + |\mathcal{L}| = |\mathcal{K}|$. Now by adding $|V| - |\mathcal{K}|$ arcs to the arc set in D corresponding to B', we can obtain an $R - S$ bifurcation of size $|B'| + |V| - |\mathcal{K}| = |V| - |\mathcal{L}|$. ∎

If each arc of D belongs to $\delta^{\text{out}}(R)$, then Theorem 54.9 reduces to Kőnig's matching theorem (Theorem 16.2).

We next give a min-max relation for the maximum length of an $R - S$ bifurcation, by reduction to Theorem 54.8 on minimum-length bibranching:

Theorem 54.10. *Let $D = (V, A)$ be a digraph and let V be partitioned into R and S such that there are no arcs from S to R. Let $l \in \mathbb{Z}_+^A$ be a length function. Then the maximum length of an $R - S$ bifurcation is equal to the minimum value of*

(54.26) $$\sum_{v \in V} y_v + \sum_{U \in \mathcal{U}} z_U(|U| - 1)$$

where $y \in \mathbb{Z}_+^V$ and $z \in \mathbb{Z}_+^{\mathcal{U}}$, with $\mathcal{U} := \{U \mid U \neq \emptyset, U \subseteq R \text{ or } U \subseteq S\}$, such that

(54.27) $$\sum_{v \in R} y_v \chi^{\delta^{\text{out}}(v)} + \sum_{v \in S} y_v \chi^{\delta^{\text{in}}(v)} + \sum_{U \in \mathcal{U}} z_U \chi^{A[U]} \geq l.$$

Proof. To see that the maximum is not more than the minimum, let B be any $R - S$ bifurcation and let y_v, z_U satisfy (54.27). Then

(54.28) $$\begin{aligned} l(B) = \sum_{a \in B} l(a) &\leq \sum_{a \in B} \Big(\sum_{\substack{v \in R \\ a \in \delta^{\text{out}}(v)}} y_v + \sum_{\substack{v \in S \\ a \in \delta^{\text{in}}(v)}} y_v + \sum_{\substack{U \in \mathcal{U} \\ a \in A[U]}} z_U \Big) \\ &= \sum_{v \in R} y_v |B \cap \delta^{\text{out}}(v)| + \sum_{v \in S} y_v |B \cap \delta^{\text{in}}(v)| + \sum_{U \in \mathcal{U}} z_U |B \cap A[U]| \\ &\leq \sum_{v \in V} y_v + \sum_{U \in \mathcal{U}} z_U(|U| - 1). \end{aligned}$$

To see equality, extend D by two new vertices, r and s, and by arcs (r, v) for each $v \in S \cup \{s\}$ and (v, s) for each $v \in R$. This makes the digraph $D' = (V', A')$. Define $R' := R \cup \{r\}$ and $S' := S \cup \{s\}$. Let $L := \max\{l(a) \mid a \in A\} + 1$. Define $l' \in \mathbb{Z}_+^{A'}$ by:

$$l'(a) := \begin{cases} L - l(a) & \text{for each } a \in A[R] \cup A[S], \\ 2L - l(a) & \text{for each } a \in \delta^{\text{out}}(R), \\ L & \text{for each } a = (r,v) \text{ with } v \in S \text{ and } a = (v,s) \\ & \text{with } v \in R, \\ 0 & \text{for } a = (r,s). \end{cases} \tag{54.29}$$

Let $\mathcal{U}'$ be the collection of nonempty subsets U of R' or S'. By Theorem 54.8, applied to D', there exists an $R' - S'$ bibranching B' in D' and a $z : \mathcal{U}' \to \mathbb{Z}_+$ such that

$$l'(B') = \sum_{U \in \mathcal{U}'} z_U, \tag{54.30}$$

and

$$\sum_{\substack{U \in \mathcal{U}' \\ U \subseteq R'}} z_U \chi^{\delta^{\text{out}}(U)} + \sum_{\substack{U \in \mathcal{U}' \\ U \subseteq S'}} z_U \chi^{\delta^{\text{in}}(U)} \leq l'. \tag{54.31}$$

Since $l'(r,s) = 0$ we know that $z_U = 0$ if r or s belongs to U. That is, $z_U = 0$ if $U \in \mathcal{U}' \setminus \mathcal{U}$.

For each $v \in V$, define

$$y_v := L - \sum_{\substack{U \in \mathcal{U} \\ v \in U}} z_U. \tag{54.32}$$

Then $y_v \geq 0$ for each $v \in V$, as

$$y_v = L - \sum_{\substack{U \in \mathcal{U} \\ v \in U}} z_U \geq L - l'(r,v) = 0 \tag{54.33}$$

if $v \in S$, and similarly $y_v \geq 0$ if $v \in R$.

Also, y and z satisfy (54.27), since for any arc $a = (u,v)$ one has, if $u, v \in R$:

$$\begin{aligned} y_u + \sum_{\substack{U \in \mathcal{U} \\ a \in A[U]}} z_U &= L - \sum_{\substack{U \in \mathcal{U} \\ u \in U}} z_U + \sum_{\substack{U \in \mathcal{U} \\ a \in A[U]}} z_U = L - \sum_{\substack{U \in \mathcal{U} \\ a \in \delta^{\text{out}}(U)}} z_U \\ &\geq L - l'(a) = l(a). \end{aligned} \tag{54.34}$$

Similarly, if $u, v \in S$, then

$$y_v + \sum_{\substack{U \in \mathcal{U} \\ a \in A[U]}} z_U \geq l(a). \tag{54.35}$$

Finally, if $u \in R$ and $v \in S$, then:

$$\begin{aligned} y_u + y_v &= 2L - \sum_{\substack{U \in \mathcal{U} \\ u \in U}} z_U - \sum_{\substack{U \in \mathcal{U} \\ v \in U}} z_U \\ &= 2L - \sum_{\substack{U \in \mathcal{U} \\ a \in \delta^{\text{out}}(U)}} z_U - \sum_{\substack{U \in \mathcal{U} \\ a \in \delta^{\text{in}}(U)}} z_U \geq 2L - l'(a) = l(a). \end{aligned} \tag{54.36}$$

So y and z satisfy (54.27).

Note that each $u \in R$ is left by a unique arc in B', since if there is more than one, all arcs leaving u should have their heads in S' (since if (u,v), $(u',v') \in B'$, $v \neq v'$, $v \notin S'$, then $B' \setminus \{(u,v)\}$ is again an $R' - S'$ bibranching). Then replacing one outgoing arc $(u,v) \in B'$ by the arc (r,v) keeps B' an $R - S$ bibranching, however of smaller length. This contradicts our assumption. So each vertex in R is left by exactly one arc in B', and similarly, each vertex in S is entered by exactly one arc in B'. This implies that $B := B' \cap A$ is an $R - S$ bifurcation.

We finally show that equality holds throughout in (54.28). Indeed, if $a \in B$, then $a \in B'$, and hence we have equality in (54.34), implying that the first inequality in (54.28) is satisfied with equality. Moreover, if $y_v > 0$ and $v \in S$, then we have strict inequality in (54.33), and hence $(r,v) \notin B'$. Therefore $|B \cap \delta^{\text{in}}(v)| = 1$. Similarly, $y_v > 0$ and $v \in R$ implies $|B \cap \delta^{\text{out}}(v)| = 1$. Finally, if $z_U > 0$ and (say) $U \subseteq R$, then $|B' \cap \delta^{\text{out}}(U)| = 1$, and hence $|B' \cap A[U]| = |U| - 1$ (since each $v \in R$ is left by precisely one arc in B'), implying $|B \cap A[U]| = |U| - 1$. This shows that also the second inequality in (54.28) is satisfied with equality. ∎

Theorem 54.10 is equivalent to the total dual integrality of the following system:

(54.37)

(i)	$x_a \geq 0$	for each $a \in A$,
(ii)	$x(\delta^{\text{out}}(v)) \leq 1$	for each $v \in R$,
(iii)	$x(\delta^{\text{in}}(v)) \leq 1$	for each $v \in S$,
(iv)	$x(A[U]) \leq \lvert U \rvert - 1$	for each nonempty U with $U \subseteq R$ or $U \subseteq S$.

It yields a description of the $R - S$ *bifurcation polytope* — the convex hull of the incidence vectors of the $R - S$ bifurcations in D.

Corollary 54.10a. *System* (54.37) *is TDI and determines the $R - S$ bifurcation polytope.*

Proof. This is equivalent to Theorem 54.10. ∎

As for the *complexity*, the reduction given in Theorem 54.10 also implies that a maximum-length $R - S$ bifurcation can be found in strongly polynomial time (since a minimum-length $R - S$ bibranching can be found in strongly polynomial time).

54.7. Disjoint bibranchings

Consider the system

(54.38)

(i)	$0 \leq x_a \leq 1$	for each $a \in A$,
(ii)	$x(B) \geq 1$	for each $R - S$ bibranching B.

By the theory of blocking polyhedra, Corollary 54.8a implies:

Corollary 54.10b. *System* (54.38) *determines the convex hull of the incidence vectors of arc sets containing an $R - S$ bicut.*

Proof. Directly from Corollary 54.8a with the theory of blocking polyhedra. ∎

System (54.38) in fact is TDI, which is equivalent to the following statement:

Theorem 54.11 (disjoint bibranchings theorem)**.** *Let $D = (V, A)$ be a digraph and let V be partitioned into sets R and S. Then the maximum number of disjoint $R-S$ bibranchings is equal to the minimum size of an $R-S$ bicut.*

Proof. Let k be the minimum size of an $R - S$ bicut. Clearly, there are at most k disjoint $R - S$ bibranchings. We show equality. For any digraph $D = (V, A)$ and $r \in V$, call a subset B of A an *r-coarborescence* if the set B^{-1} of reverse arcs of B is an r-arborescence.

By Edmonds' disjoint arborescences theorem (Corollary 53.1b), the graph D/R (obtained from D by contracting R to one vertex) has k disjoint R-arborescences $B_1, \ldots, B_k$. Similarly, the graph D/S has k disjoint S-coarborescences $B'_1, \ldots, B'_k$. Choose the B_i and B'_i such that the sum

(54.39) $$\sum_{i=1}^{k} \sum_{\substack{j=1 \\ j \neq i}}^{k} |B_i \cap B'_j|$$

is as small as possible. If the sum is 0, then

(54.40) $$B_1 \cup B'_1, \ldots, B_k \cup B'_k$$

are k disjoint $R - S$ bibranchings in D as required. So we can assume that the sum is positive. Without loss of generality, $B_1 \cap B'_2 \neq \emptyset$.

Define

(54.41) $$X := (B_1 \cup B_2) \cap A[S],\ X' := (B'_1 \cup B'_2) \cap A[R],$$
$$Y := (B_1 \cup B_2) \cap \delta^{\text{out}}(R),\ Y' := (B'_1 \cup B'_2) \cap \delta^{\text{out}}(R).$$

Let $\mathcal{K}$ be the collection of strong components K of the digraph (S, X) with $\delta^{\text{in}}_X(K) = \emptyset$. Similarly, let $\mathcal{K}'$ be the collection of strong components K of the digraph (R, X') with $\delta^{\text{out}}_{X'}(K) = \emptyset$.

Now $d^{\text{in}}_Y(K) = d^{\text{in}}_{B_1 \cup B_2}(K) \geq 2$ for each $K \in \mathcal{K}$, and similarly $d^{\text{out}}_{Y'}(K) \geq 2$ for each $K \in \mathcal{K}'$. Then we can split Y into Y_1 and Y_2 and Y' into Y'_1 and Y'_2 such that

(54.42) $$d^{\text{in}}_{Y_i}(K) \geq 1 \text{ for each } K \in \mathcal{K} \text{ and } i = 1, 2,$$
$$d^{\text{out}}_{Y'_i}(K) \geq 1 \text{ for each } K \in \mathcal{K}' \text{ and } i = 1, 2,$$
$$\text{and } Y_1 \cap Y'_2 = \emptyset \text{ and } Y_2 \cap Y'_1 = \emptyset.$$

This can be seen as follows. Select for each $U \in \mathcal{K}$ a pair e_U from $\delta^{\text{in}}_Y(U)$. Similarly, select for each $U \in \mathcal{K}'$ a pair e_U from $\delta^{\text{out}}_{Y'}(U)$. So the e_U for $U \in \mathcal{K}$ are disjoint, and the e_U for $U \in \mathcal{K}'$ are disjoint. Hence the e_U for $U \in \mathcal{K} \cup \mathcal{K}'$ form a bipartite graph on $Y \cup Y'$ (in fact, a set of vertex-disjoint paths and even circuits). The two colour classes of this bipartite graph give the partitions of Y and Y' as required.

Then by Lemma 53.2α, X can be split into two branchings X_1 and X_2 such that the set of roots of X_i is equal to the set of heads of Y_i ($i = 1, 2$). Similarly, X' can be split into two cobranchings X'_1 and X'_2 such that the set of coroots of X'_i is equal to the set of tails of Y'_i ($i = 1, 2$). (A *cobranching* is a set B of arcs whose reversal B^{-1} is a branching. A *coroot* of B is a root of B^{-1}.)

Define

(54.43) $\qquad \widetilde{B}_i := X_i \cup Y_i$ and $\widetilde{B}'_i := X'_i \cup Y'_i$

for $i = 1, 2$. Since $\widetilde{B}_1 \cap \widetilde{B}'_2 = \emptyset$ and $\widetilde{B}_2 \cap \widetilde{B}'_1 = \emptyset$, replacing B_1, B_2, B'_1, B'_2 by $\widetilde{B}_1, \widetilde{B}_2, \widetilde{B}'_1, \widetilde{B}'_2$ decreases sum (54.39), contradicting the minimality assumption. ∎

The capacitated case can be derived as a consequence:

Corollary 54.11a. *Let $D = (V, A)$ be a digraph, let V be partitioned into sets R and S, and let $c \in \mathbb{Z}_+^A$ be a capacity function. Then the maximum number of $R - S$ bibranchings such that no arc a is in more than $c(a)$ of these bibranchings is equal to the minimum capacity of an $R - S$ bicut.*

Proof. This follows from Theorem 54.11 by replacing any arc a by $c(a)$ parallel arcs. ∎

Equivalently, in TDI terms:

Corollary 54.11b. *System* (54.38) *is totally dual integral.*

Proof. This is a reformulation of Corollary 54.11a. ∎

Another consequence is:

(54.44) For any digraph $D = (V, A)$ and any partition of V into R and S, the $R - S$ bibranching polytope has the integer decomposition property.

As for the *complexity*, the proof of Theorem 54.11 gives a polynomial-time algorithm for finding a maximum number of disjoint $R - S$ bibranchings. For the capacitated case there is a semi-strongly polynomial-time algorithm (that is, where rounding takes one arithmetic step): first find a fractional dual solution, then round (Grötschel, Lovász, and Schrijver [1988]). A combinatorial semi-strongly polynomial-time algorithm follows from the results in Section 57.5.

54.7a. Proof using supermodular colourings

We show how to derive Theorem 54.11 on disjoint bibranchings from Edmonds' disjoint branchings theorem (Theorem 53.1) and Theorem 49.14 on supermodular colourings.

Let $D = (V, A)$ be a digraph and let V be partitioned into R and S. Let $k \in \mathbb{Z}_+$. Define $H := \delta^{\text{out}}(R)$, and define the following collections of subsets of H:

(54.45) $\mathcal{C}_1 := \{\delta^{\text{in}}_H(U) \mid \emptyset \neq U \subseteq S\}$ and $\mathcal{C}_2 := \{\delta^{\text{out}}_H(U) \mid \emptyset \neq U \subseteq R\}$.

Then $\mathcal{C}_1$ and $\mathcal{C}_2$ are intersecting families on H. Define $g_j : \mathcal{C}_j \to \mathbb{Z}$ for $j = 1, 2$ by:

(54.46) $g_1(B) := \max\{k - d^{\text{in}}_{A[S]}(U) \mid \emptyset \neq U \subseteq S, B = \delta^{\text{in}}_H(U)\}$ for $B \in \mathcal{C}_1$,
$g_2(B) := \max\{k - d^{\text{out}}_{A[R]}(U) \mid \emptyset \neq U \subseteq R, B = \delta^{\text{out}}_H(U)\}$ for $B \in \mathcal{C}_2$.

Then g_1 and g_2 are intersecting supermodular. Moreover, if U attains the maximum in (54.46), then

(54.47) $g_1(B) = k - d^{\text{in}}_{A[S]}(U) \leq d^{\text{in}}_A(U) - d^{\text{in}}_{A[S]}(U) = d^{\text{in}}_H(U) = |B|$ if $U \subseteq S$ and
$g_2(B) = k - d^{\text{out}}_{A[R]}(U) \leq d^{\text{out}}_A(U) - d^{\text{out}}_{A[R]}(U) = d^{\text{out}}_H(U) = |B|$ if $U \subseteq R$.

Since $g_j(B) \leq k$ for $j = 1, 2$ and $B \in \mathcal{C}_j$, by Theorem 49.14 we can partition H into classes $H_1, \ldots, H_k$ such that:

(54.48)
(i) if $\emptyset \neq U \subseteq S$, then U is entered by at least $k - d^{\text{in}}_{A[S]}(U)$ of the classes H_i, and
(ii) if $\emptyset \neq U \subseteq R$, then U is left by at least $k - d^{\text{out}}_{A[R]}(U)$ of the classes H_i.

By Edmonds' disjoint branchings theorem, (i) implies that $A[S]$ contains disjoint branchings $B_1, \ldots, B_k$ such that, for each $i = 1, \ldots, k$, the root set of B_i is equal to the set of heads of the arcs in H_i; that is, each vertex in S is entered by at least one arc in $B_i \cup H_i$. Similarly, $A[R]$ contains disjoint cobranchings (= branchings if all orientations are reversed) $B'_1, \ldots, B'_k$ such that, for each $i = 1, \ldots, k$, each vertex in R is left by at least one arc in $B'_i \cup H_i$. Then the $B_i \cup H_i \cup B'_i$ form disjoint $R-S$ bibranchings.

54.7b. Covering by bifurcations

Theorem 54.11 also implies the following characterization of the minimum number of $R - S$ bifurcations needed to cover all arcs (Keijsper [1998b]):

Corollary 54.11c. *Let $D = (V, A)$ be a digraph and let V be partitioned into sets R and S, with no arc from S to R. Then A can be covered by k $R - S$ bifurcations if and only if*

(54.49)
(i) $\deg^{\text{out}}(v) \leq k$ *for each* $v \in R$;
(ii) $\deg^{\text{in}}(v) \leq k$ *for each* $v \in S$;
(iii) $|A[U]| \leq k(|U| - 1)$ *for each nonempty subset* U *of* R *or* S.

Proof. Necessity being easy, we show sufficiency. Extend D by two new vertices r and s, for each $v \in S$ by $k - \deg^{\text{in}}(v)$ parallel arcs from r to v, for each $v \in R$ by $k - \deg^{\text{out}}(v)$ parallel arcs from v to s, and by k parallel arcs from r to s. Let D' be the graph arising in this way. So in D', each $v \in R$ has outdegree k, and each $v \in S$ has indegree k. Define $R' := R \cup \{r\}$ and $S' := S \cup \{s\}$.

Then by Theorem 54.11, D' has k disjoint $R' - S'$ bibranchings. Indeed, any nonempty subset U of R' is left by $k|U| - |A[U]| \geq k$ arcs of D' if $r \notin U$ (since each vertex in R has outdegree k in D'), and by at least k arcs of D' if $r \in U$. Similarly, any nonempty subset of S' is entered by at least k arcs of D'.

Now each of these bibranchings leaves any $v \in R$ exactly once (as v has outdegree k in D'), and (similarly) enters any $v \in S$ exactly once. Moreover, these bibranchings cover A. Hence restricted to A we obtain a covering of A by k $R - S$ bifurcations. ∎

An equivalent way of saying this is (using Corollary 54.10a):

(54.50) For any digraph $D = (V, A)$ and any partition of V into R and S, the $R - S$ bifurcation polytope has the integer decomposition property.

As for the *complexity*, the reduction given in the proof of Corollary 54.11c implies a polynomial-time algorithm to find a minimum number of $R - S$ bifurcations covering the arc set (by reduction to finding a maximum number of disjoint bibranchings). The capacitated version can be solved in semi-strongly polynomial time, with the help of the ellipsoid method, by first finding a fractional packing, and next round (like in Section 51.4).

54.7c. Disjoint $R - S$ biconnectors and $R - S$ bibranchings

As in Keijsper and Schrijver [1998], one can derive Theorem 54.5 on disjoint $R-S$ biconnectors (in an undirected graph) from Theorem 54.11 on disjoint $R-S$ bibranchings (in a directed graph), with the help of the Tutte-Nash-Williams disjoint trees theorem (Corollary 51.1a).

Indeed, the condition in Theorem 54.5 gives, with the Tutte-Nash-Williams disjoint trees theorem, that the graph G/R obtained from G by contracting R to one vertex, has k edge-disjoint spanning trees.

By orienting the edges in these trees appropriately, we see that G/R has an orientation such that any nonempty $U \subseteq S$ is entered by at least k arcs, and such that each edge incident with R is oriented away from R. Similarly, G/S has an orientation such that any nonempty $U \subseteq R$ is left by at least k arcs, and such that each edge incident with S is oriented towards S.

Combining the two orientations, we obtain an orientation $D = (V, A)$ of G such that each $R-S$ bicut has size at least k. Hence, by Theorem 54.11, D has k disjoint $R - S$ bibranchings, and hence, G has k disjoint $R - S$ biconnectors.

54.7d. Covering by $R - S$ biforests and by $R - S$ bifurcations

Similarly, one can derive Theorem 54.6 on covering $R - S$ biforests from Corollary 54.11c on covering $R - S$ bifurcations, with the help of Nash-Williams' covering forests theorem (Corollary 51.1c). Indeed, the condition in Theorem 54.6 gives, with Nash-Williams' covering forests theorem, that the edges of the graph G/R obtained from G by contracting R to one vertex, can be covered by k forests. Hence G/R has an orientation such that any vertex in S is entered by at most k arcs, and such that R is only left by arcs. Similarly, G/S has an orientation such that any vertex in R is left by at most k arcs, and such that S is only entered by arcs.

Combining the two orientations, we obtain an orientation $D = (V, A)$ of G satisfying the condition in Corollary 54.11c. Hence the arcs of D can be covered by k $R - S$ bifurcations, and hence the edges of G can be covered by k $R - S$ biforests.

Chapter 55

Minimum directed cut covers and packing directed cuts

A *directed cut* in a directed graph $D = (V, A)$ is a set of arcs $\delta^{\text{in}}(U)$ for some nonempty proper subset U of V with $\delta^{\text{out}}(U) = \emptyset$. A *directed cut cover* is a set of arcs intersecting each directed cut — equivalent, it is a set of arcs such that their contraction makes the graph strongly connected. For planar digraphs, a directed cut cover corresponds to a *feedback arc set* in the dual digraph — a set of arcs whose removal makes the digraph acyclic. Lucchesi and Younger showed that the minimum size of a directed cut cover is equal to the maximum number of disjoint directed cuts. This min-max relation is the basis for several other results on *shortest* directed cut covers, which we survey in this chapter. In the next chapter we consider the, less tractable, *disjoint* directed cut covers.

55.1. Minimum directed cut covers and packing directed cuts

Let $D = (V, A)$ be a digraph. A subset C of A is called a *directed cut* if there exists a nonempty proper subset U of V with $\delta^{\text{in}}(U) = C$ and $\delta^{\text{out}}(U) = \emptyset$. A *directed cut cover* is a set of arcs intersecting each directed cut.

It is easy to show that for any subset B of A the following are equivalent:

(55.1)
- (i) B is a directed cut cover;
- (ii) adding to D all arcs (u, v) with $(v, u) \in B$ makes the digraph strongly connected;
- (iii) contracting all arcs in B makes the digraph strongly connected.

So a minimum directed cut cover gives a minimum number of arcs in D such that making them two-way we obtain a strongly connected digraph.

Moreover, A. Frank (cf. Lovász [1979a] p. 271) showed:

Theorem 55.1. *Let $D = (V, A)$ be a weakly connected digraph without cut arcs and let $B \subseteq A$. Then B is an inclusionwise minimal directed cut cover if and only if B is an inclusionwise minimal set such that if we invert the orientations of all arcs in B, the digraph becomes strongly connected.*

Proof. Define $\widetilde{A} := (A \setminus B) \cup B^{-1}$, where $B^{-1} := \{a^{-1} \mid a \in B\}$, and where a^{-1} is the arc arising from a by inverting its orientation.

Trivially, if $(V, \widetilde{A})$ is strongly connected, then B is a directed cut cover. Hence it suffices to show that if B is an inclusionwise minimal directed cut cover, then $\widetilde{D} = (V, \widetilde{A})$ is strongly connected.

Suppose that $\widetilde{D}$ is not strongly connected. Let K be a strong component of $\widetilde{D}$ with $\delta^{\text{in}}_{\widetilde{A}}(K) = \emptyset$. Let $\delta^{\text{in}}_{A}(K) = \{a_1, \ldots, a_t\}$. So $a_1, \ldots, a_t$ belong to B. Hence, as B is an inclusionwise minimal directed cut cover, for each $i = 1, \ldots, t$ there exists a subset U_i of V with $\delta^{\text{in}}_{A}(U_i) = \emptyset$ and $\delta^{\text{out}}_{B}(U_i) = \{a_i\}$.

Then for each i, $U_i \cap K = \emptyset$. For suppose that $U_i \cap K \neq \emptyset$. As the head of a_i does not belong to U_i, U_i splits K. Hence some arc $a \in \widetilde{A}$ enters U_i, with a spanned by K. As $\delta^{\text{in}}_{A}(U_i) = \emptyset$, we know $a \in B^{-1}$, and therefore $a^{-1} \in \delta^{\text{out}}_{B}(U_i)$ while $a \neq a_i$, a contradiction.

Also, $U_i \cap U_j = \emptyset$ for $i \neq j$, as $\delta^{\text{in}}_{A}(U_i \cap U_j) = \emptyset$ and

$$d^{\text{out}}_{B}(U_i \cap U_j) \leq d^{\text{out}}_{B}(U_i) + d^{\text{out}}_{B}(U_j) - d^{\text{out}}_{B}(U_i \cup U_j) = 0, \tag{55.2}$$

since both a_i and a_j leave $U_i \cup U_j$.

So $U_1, \ldots, U_t$ are disjoint subsets of $V \setminus K$. As D has no cut arcs, $d^{\text{out}}_{A}(U_i) \geq 2$ for each i. Hence, as no arc in A enters any U_i, and only one arc (namely a_i) leaves U_i to enter K, the set $W := V \setminus (K \cup U_1 \cup \cdots \cup U_t)$ is nonempty. Also, $\delta^{\text{out}}_{A}(W) = \emptyset$, and so $\delta^{\text{in}}_{B}(W) \neq \emptyset$, that is $\delta^{\text{out}}_{B}(K \cup U_1 \cup \cdots \cup U_t) \neq \emptyset$. However, $\delta^{\text{out}}_{B}(K) = \emptyset$ (since $\delta^{\text{in}}_{\widetilde{A}}(K) = \emptyset$) and $\delta^{\text{out}}_{B}(U_i) = \{a_i\}$, implying $\delta^{\text{out}}_{B}(K \cup U_i) = \emptyset$ for each i, a contradiction. ∎

55.2. The Lucchesi-Younger theorem

Lucchesi and Younger [1978] proved the following min-max relation for the minimum size of a directed cut cover, which was conjectured by N. Robertson and by Younger [1965,1969] (for planar graphs by Younger [1963a], inspired by a question suggested by J.P. Runyan to Seshu and Reed [1961]).

The proof below is a variant of the proof of Lovász [1976c] (cf. Lovász [1979b]).

Theorem 55.2 (Lucchesi-Younger theorem)**.** *Let $D = (V, A)$ be a weakly connected digraph. Then the minimum size of a directed cut cover is equal to the maximum number of disjoint directed cuts.*

Proof. For any digraph D, let $\nu(D)$ be the maximum number of disjoint directed cuts in D and let $\tau(D)$ be the minimum size of a directed cut cover. Choose a counterexample $D = (V, A)$ with a minimum number of arcs.

For any $B \subseteq A$, let D_B be the graph obtained from D by replacing each arc (u, v) in B by a directed $u-v$ path of length 2 (the intermediate vertex being new). Choose an inclusionwise maximal subset B of A with $\nu(D_B) = \nu(D)$. Then $B \neq A$, as $\nu(D_A) \geq 2\nu(D) > \nu(D)$.

Choose $b \in A \setminus B$. So $\nu(D_{B\cup\{b\}}) > \nu(D)$. Moreover, as D is a smallest counterexample, the graph D' obtained from D by contracting b satisfies $\nu(D') = \tau(D') \geq \tau(D) - 1 \geq \nu(D)$. Combining a maximum-size packing of directed cuts in D' and one in $D_{B\cup\{b\}}$, we obtain a family $\mathcal{F}$ of nonempty proper subsets of the vertex set V' of D_B with the property that

(55.3) $|\mathcal{F}| = 2\nu(D) + 1$, and the $\delta^{\text{in}}(U)$ for $U \in \mathcal{F}$ are directed cuts in D_B covering any arc of D_B at most twice.

Now we choose $\mathcal{F}$ satisfying (55.3) such that

(55.4) $$\sum_{U\in\mathcal{F}} |U||V' \setminus U|$$

is minimized. Then $\mathcal{F}$ is a cross-free family. Indeed, if $X, Y \in \mathcal{F}$ with $X \not\subseteq Y \not\subseteq X$, $X \cap Y \neq \emptyset$ and $X \cup Y \neq V'$, we can replace X and Y by $X \cap Y$ and $X \cup Y$, while not violating (55.3) but decreasing sum (55.4) (by Theorem 2.1), contradicting its minimality.

So $\mathcal{F}$ is cross-free. For each $X \in \mathcal{F}$, define

(55.5) $$\beta(X) := \{U \in \mathcal{F} \mid U \subseteq X \text{ or } U \cap X = \emptyset\}.$$

Let $\mathcal{F}_2$ be the collection of sets occurring twice in $\mathcal{F}$ and let $\mathcal{F}_1$ be the collection of sets occurring precisely once in $\mathcal{F}$. Then

(55.6) if X and Y are distinct sets in $\mathcal{F}_1$ with $|\beta(X)| \equiv |\beta(Y)| \pmod 2$, then no arc of D enters both X and Y.

Suppose that to the contrary arc a enters both X and Y. As $\mathcal{F}$ is cross-free, we can assume that $X \subset Y$.

If $|\beta(Y)| \leq |\beta(X)|$, then (as $Y \in \beta(Y) \setminus \beta(X)$) there exists a Z in $\beta(X) \setminus \beta(Y)$. So $Z \not\subseteq Y$ and $Z \cap Y \neq \emptyset$. Hence $Z \not\subseteq X$, and so $Z \cap X = \emptyset$. So $Y \not\subseteq Z$, and hence (as $\mathcal{F}$ is cross-free) $Z \cup Y = V'$. So a leaves Z, a contradiction (since no arc leaves any set in $\mathcal{F}$).

If $|\beta(Y)| \geq |\beta(X)|+2$, then there exists a $Z \neq Y$ with $Z \in \beta(Y) \setminus \beta(X)$. So $Z \not\subseteq X$ and $Z \cap X \neq \emptyset$. Hence $Z \cap Y \neq \emptyset$, and so $Z \subseteq Y$. So $Z \cup X \neq V'$, and hence (as $\mathcal{F}$ is cross-free) $X \subset Z$. So a enters X, Y, and Z, a contradiction. This proves (55.6).

It follows that for some $j \in \{0, 1\}$, the collection

(55.7) $$\mathcal{F}_2 \cup \{X \in \mathcal{F}_1 \bigm| |\beta(X)| \equiv j \pmod 2\}$$

has size at least $\nu(D) + 1$. By (55.6), it gives $\nu(D) + 1$ disjoint directed cuts in D_B, contradicting our assumption. ∎

Equivalent to the Lucchesi-Younger theorem is the following weighted version of it:

Corollary 55.2a. *Let $D = (V, A)$ be a digraph and let $l : A \to \mathbb{Z}_+$ be a length function. Then the minimum length of a directed cut cover is equal to* *the maximum number of directed cuts such that each arc a is in at most $l(a)$ of them.*

Proof. Replace any arc a by a path of length $l(a)$ (contracting a if $l(a) = 0$). Then the Lucchesi-Younger theorem applied to the new graph gives the present corollary. ∎

This can be formulated in terms of the total dual integrality of the following system:

(55.8) (i) $0 \leq x_a \leq 1$ for each $a \in A$,
(ii) $x(C) \geq 1$ for each directed cut C.

Define the *directed cut cover polytope* of D as the convex hull of the incidence vectors of directed cut covers. Then:

Corollary 55.2b. *System* (55.8) *is TDI and determines the directed cut cover polytope of D.*

Proof. The total dual integrality is a reformulation of Corollary 55.2a. The total dual integrality of (55.8) implies that it determines an integer polytope. Hence the second part of the corollary follows. ∎

55.3. Directed cut k-covers

In fact, system (55.8) is box-TDI, and more generally, the following system is box-TDI, as was shown by Edmonds and Giles [1977]:

(55.9) $x(C) \geq 1$ for each directed cut C.

Edmonds and Giles' proof gives the following alternative way of proving the Lucchesi-Younger theorem.

Theorem 55.3. *System* (55.9) *is box-TDI.*

Proof. Let $\mathcal{U}$ be the collection of nonempty proper subsets U of V with $\delta^{\text{out}}(U) = \emptyset$. So $\{\delta^{\text{in}}(U) \mid U \in \mathcal{U}\}$ is the collection of all directed cuts.

Choose $w \in \mathbb{R}^A$, and let y achieve the maximum in the dual of minimizing $w^\mathsf{T} x$ over (55.9), that is, in:

(55.10) $$\max\{\sum_{U\in\mathcal{U}} y_U \mid y \in \mathbb{R}^{\mathcal{U}}_+, \sum_{U\in\mathcal{U}} y_U \chi^{\delta^{\text{in}}(U)} = w\},$$

such that

(55.11) $$\sum_{U\in\mathcal{U}} y_U |U||V \setminus U|$$

is as small as possible. Let $\mathcal{F} := \{U \in \mathcal{U} \mid y_U > 0\}$. Then $\mathcal{F}$ is cross-free. Suppose to the contrary that $T, U \in \mathcal{F}$ with $T \not\subseteq U \not\subseteq T$, $T \cap U \neq \emptyset$, $T \cup U \neq V$. Let $\alpha := \min\{y_T, y_U\} > 0$. Then decreasing y_T and y_U by α, and increasing $y_{T\cap U}$ and $y_{T\cup U}$ by α, maintains feasibility of y, while its value is not changed; so it remains an optimum solution. However, sum (55.11) decreases (by Theorem 2.1). This contradicts the minimality of (55.11).

So $\mathcal{F}$ is cross-free, and hence the constraints corresponding to $\mathcal{F}$ form a totally unimodular matrix (Corollary 13.21a). Hence, by Theorem 5.35, system (55.9) is box-TDI. ∎

This implies the box-total dual integrality of (for $k \geq 0$):

(55.12) $\quad x(C) \geq k \quad$ for each directed cut C.

Corollary 55.3a. *For each $k \in \mathbb{R}_+$, system* (55.12) *is box-TDI.*

Proof. Directly from Theorem 55.3, since if a system $Ax \leq b$ is box-TDI, then also $Ax \leq k \cdot b$ is box-TDI. ∎

This has the following consequences. Call a subset C of the arc set A of a digraph $D = (V, A)$ a *directed cut k-cover* if C intersects each directed cut in at least k arcs. Consider the system:

(55.13)
$$\begin{array}{ll} 0 \leq x_a \leq 1 & \text{for } a \in A, \\ x(C) \geq k & \text{for each directed cut } C. \end{array}$$

Then:

Corollary 55.3b. *System* (55.13) *is TDI and determines the convex hull of the incidence vectors of the directed cut k-covers.*

Proof. Directly from Corollary 55.3a. ∎

From this, a min-max relation for the minimum size of a directed cut k-cover can be derived:

Corollary 55.3c. *Let $D = (V, A)$ be a digraph and let $k \in \mathbb{Z}_+$, such that each directed cut has size at least k. Then the minimum size of a directed cut k-cover is equal to the maximum value of*

(55.14)
$$\left|\bigcup \mathcal{C}\right| + k|\mathcal{C}| - \sum_{C \in \mathcal{C}} |C|$$

taken over all collections $\mathcal{C}$ of directed cuts.

Proof. By Corollary 55.3b, the minimum size of a directed cut k-cover is equal to the minimum value of $\mathbf{1}^{\mathsf{T}}x$ over (55.13). Hence, as (55.12) is TDI, the minimum size of a directed cut k-cover is equal to the maximum value of

(55.15) $$k\sum_C y_C - z(A),$$

where $y_C \in \mathbb{Z}_+$ for each directed cut C and where $z \in \mathbb{Z}_+^A$ such that

(55.16) $$\sum_C y_C\chi^C - z \leq \mathbf{1}.$$

Now we can assume that $y_C \in \{0,1\}$ for each C, since if $y_C \geq 2$, then $z_a \geq 1$ for each $a \in C$ (by (55.16)). Hence decreasing y_C by 1 and decreasing z_a by 1 for each $a \in C$, maintains (55.16) while (55.15) is not decreased (as $|C| \geq k$ by assumption).

Let $\mathcal{C} := \{C \mid y_C = 1\}$. As $z(A)$ is minimized, we have

(55.17) $$z = \sum_{C\in\mathcal{C}} \chi^C - \chi^{\bigcup\mathcal{C}},$$

and hence that $z(A)$ is equal to $\sum_{C\in\mathcal{C}} |C| - \left|\bigcup\mathcal{C}\right|$. This proves the corollary. ∎

55.4. Feedback arc sets

The Lucchesi-Younger theorem implies a min-max relation for the minimum size of a feedback arc set in a planar digraph. A *feedback arc set* in a digraph $D = (V, A)$ is a set of arcs intersecting every directed circuit.

In fact, if D has no loops, then a set A' is an inclusionwise minimal feedback arc set if and only if A' is an inclusionwise minimal set of arcs such that inverting all arcs in A' makes the digraph acyclic (Grinberg and Dambit [1966], Gallai [1968a]).

E.L. Lawler and R.M. Karp (see Karp [1972b]) showed that finding a minimum-size feedback arc set in a digraph, is NP-complete. For planar digraphs one has however:

Theorem 55.4. *Let $D = (V, A)$ be a planar digraph. Then the minimum size of a feedback arc set is equal to the maximum number of arc-disjoint directed circuits.*

Proof. Consider the dual digraph D^* of D. Then a set of arcs of D forms a directed circuit if and only if the set of dual arcs forms a directed cut in D^*. Hence the corollary follows immediately from the Lucchesi-Younger theorem (Theorem 55.2). ∎

Notes. Figure 55.1 (from Younger [1965]) shows that we cannot drop the planarity condition. This is a counterexample with a smallest number of vertices, since Barahona, Fonlupt, and Mahjoub [1994] showed that in a digraph with no $K_{3,3}$ minor, the minimum size of a feedback arc set is equal to the maximum number of disjoint

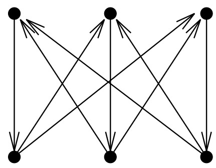

Figure 55.1
The minimum size of a feedback arc set equals 2, while there are no two disjoint directed cuts.

directed circuits. The proof is based on a theorem of Wagner [1937b] on decomposing graphs without $K_{3,3}$ minor into planar graphs and copies of K_5. (Nutov and Penn [1995] gave a similar proof. Related work is done is reported in Nutov and Penn [2000].)

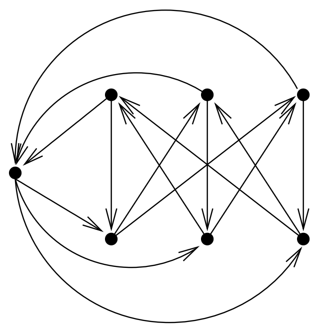

Figure 55.2
An Eulerian digraph where the minimum size of a feedback arc set equals 5, while there are no 5 disjoint directed cuts.

Moreover, Borobia, Nutov, and Penn [1996] showed that in an Eulerian digraph with at most 6 vertices, the minimum size of a feedback arc set is equal to the maximum value of a *fractional* packing of directed circuits. This is not the case if there are more than 6 vertices, as is shown by the graph in Figure 55.2.

Guenin and Thomas [2001] characterized the digraphs D that have the property that for every subhypergraph D' of D, the maximum number of disjoint circuits in D' is equal to the minimum size of a feedback arc set in D'.

More on the polytope determined by the feedback arc sets, equivalently on the *acyclic subgraph polytope* (the convex hull of the incidence vectors of arc sets containing no directed circuit) is presented in Young [1978], Grötschel, Jünger, and Reinelt [1984,1985a,1985b], Reinelt [1993], Leung and Lee [1994], Goemans and Hall [1996], and Bolotashvili, Kovalev, and Girlich [1999]. (Bowman [1972] wrongly claimed to give a system determining the acyclic subgraph polytope.)

The problem of finding a minimum-weight feedback arc set is equivalent to the *linear ordering problem*: given a matrix M, find a permutation matrix P such that the sum of the elements below the main diagonal of $P^{\mathsf{T}}MP$ is minimized. More on this can be found in Younger [1963b], Jünger [1985], Reinelt [1985], Berger and Shor [1990,1997], Arora, Frieze, and Kaplan [1996,2002], Fernandez de la Vega [1996], Frieze and Kannan [1996,1999], and Newman and Vempala [2001].

For feedback arc sets in linklessly embeddable graphs, see Section 55.6b. For feedback vertex sets, see Section 55.6c.

55.5. Complexity

It was shown by Lucchesi [1976], Karzanov [1979c,1981], and Frank [1981b] that a minimum-size directed cut cover and a maximum packing of directed cuts can be found in polynomial time. Lucchesi [1976] also gave a weakly polynomial-time algorithm for the weighted versions of these problems, and Frank [1981b] gave a strongly polynomial-time algorithm for these problems.

Frank and Tardos [1984b] showed that finding a minimum-length directed cut k-cover in fact can be reduced to a weighted matroid intersection problem. Thus all ingredients for a strongly polynomial-time algorithm are ready at hand.

We describe the reduction. Let $D = (V, A)$ be a digraph, let $l : A \to \mathbb{Q}_+$ be a length function, and let $k \in \mathbb{Z}_+$. We want to find a directed cut k-cover of minimum length.

Let $D^{-1} = (V, A^{-1})$ be the reverse digraph of D, where $A^{-1} := \{a^{-1} \mid a \in A\}$ and $a^{-1} = (v, u)$ if $a = (u, v)$. We will define matroids M_1 and M_2 on $A \cup A^{-1}$.

M_1 is easy: it is the partition matroid induced by the sets $\{a, a^{-1}\}$ for $a \in A$. To define M_2, let $\mathcal{U}$ be the collection of nonempty proper subsets U of V with $\delta^{\text{in}}_A(U) = \emptyset$. Define

(55.18) $P := \{x \in \mathbb{Z}^V_+ \mid x(V) = |A| \text{ and } x(U) \geq |A[U]| + k \text{ for each } U \in \mathcal{U}\}$.

Then:

(55.19) for $x, y \in P$ and $u \in V$ with $x_u < y_u$, there exists a $v \in V$ with $x_v > y_v$ and $x + \chi^u - \chi^v \in P$.

Indeed, let $\mathcal{K}$ be the collection of inclusionwise maximal subsets U of $V \setminus \{u\}$ with $U \in \mathcal{U}$ and $x(U) = |A[U]| + k$. As sets with this property are closed under unions of intersecting sets, $\mathcal{K}$ consists of disjoint sets, and no two of them are connected by an arc. Hence for $W := V \setminus \bigcup \mathcal{K}$, we have

(55.20) $$y(W) = y(V) - \sum_{U \in \mathcal{K}} y(U) \leq |A| - \sum_{U \in \mathcal{K}} (|A[U]| + k) = x(W).$$

As $x_u < y_u$ and $u \in W$, we know that $x_v > y_v$ for some $v \in W$. Also, $x + \chi^u - \chi^v \in P$, since there is no subset U of $V \setminus \{u\}$ with $\delta^{\text{in}}(U) = \emptyset$, $x(U) = |A[U]| + k$, and $v \in U$.

This shows (55.19), which implies that

(55.21) $$\mathcal{B} := \{B \subseteq A \cup A^{-1} \mid \deg_B^{\text{in}} \in P\}$$

forms the collection of bases of a matroid M_2 on $A \cup A^{-1}$, provided that $\mathcal{B}$ is nonempty; equivalently, provided that each directed cut in D has size at least k. (That M_2 is a matroid can also be derived from Corollary 49.7a.)

To test independence in M_2, it suffices to have one base of M_2 (which we have: A^{-1}), and to have a test of being a base. Equivalently, we should be able to test membership of P. Let $x \in \mathbb{Z}_+^V$ with $x(V) = |A|$. By Theorem 51.3, we can find a nonempty proper subset U of V minimizing

(55.22) $$\begin{aligned} &x(U) - |A[U]| + (k + |A|)d^{\text{in}}(U) \\ &= x(U) - \sum_{v \in U} \deg^{\text{out}}(v) + d^{\text{out}}(U) + (k + |A|)d^{\text{in}}(U), \end{aligned}$$

in strongly polynomial time. If this minimum is at least k, then x belongs to P. If this minimum is less than k, then $d^{\text{in}}(U) = 0$, and hence $x(U) < |A[U]| + k$, implying that x does not belong to P.

Now a subset C of A is a directed cut k-cover if and only if $B := (A \setminus C) \cup C^{-1}$ is a common base of M_1 and M_2. Hence:

Theorem 55.5. *Given a digraph $D = (V, A)$, a length function $l : A \to \mathbb{Q}_+$, and $k \in \mathbb{Z}_+$, a minimum-length directed cut k-cover can be found in strongly polynomial time.*

Proof. Directly from the above, with Theorem 41.8. We apply the weighted matroid intersection algorithm to find a maximum-length common base B in the matroids M_1 and M_2 on $A \cup A^{-1}$, defining $l(a^{-1}) := 0$ for $a \in A$. Then $A \setminus B$ is a minimum-length directed cut cover. ∎

55.5a. Finding a dual solution

Also a maximum packing of directed cuts can be found in polynomial time. Let B be the maximum-length base found and let C be the directed cut k-cover with $B = (A \setminus C) \cup C^{-1}$.

The weighted matroid intersection algorithm also yields a dual solution. Indeed, if l is integer-valued, it gives length functions $l_1, l_2 : A \cup A^{-1} \to \mathbb{Z}$ such that $l = l_1 + l_2$ and such that B is an l_i-maximal base of M_i, for $i = 1, 2$ (cf. Section 41.3a).

Define

(55.23) $$\mathcal{F} := \{U \subseteq V \mid d_A^{\text{in}}(U) = 0, d_C^{\text{out}}(U) = k\},$$

and define a pre-order $\preceq$ on V by:

(55.24) $$u \preceq v \iff \text{each } U \in \mathcal{F} \text{ containing } u \text{ also contains } v,$$

for $u, v \in V$. It can be checked in polynomial time whether $u \preceq v$ holds, since it is equivalent to: $\deg_B^{\text{in}} - \chi^u + \chi^v \in P$. Indeed, $\deg_B^{\text{in}} - \chi^u + \chi^v$ belongs to P if and only if $v \in U$ for each $U \in \mathcal{U}$ satisfying $u \in U$ and $\sum_{s \in U} \deg_B^{\text{in}}(s) = |A[U]| + k$. Now $\sum_{s \in U} \deg_B^{\text{in}}(s) = |A[U]| + d_C^{\text{out}}(U)$. So it is equivalent to: $u \preceq v$.

Next define for each $u \in V$:

$$p(u) := \max\{l_2(a) \mid \exists v \succeq u : a^{-1} \in \delta_B^{\text{out}}(v)\}. \tag{55.25}$$

Let $h_0 < \cdots < h_t$ be the elements of the set $\{p(u) \mid u \in V\}$. For $j = 1, \ldots, t$, define $V_j := \{u \mid p(u) \geq h_j\}$. Let $\mathcal{K}$ be the collection of all weak components of $D - V_j$, over all $j = 1, \ldots, t$, and for each $K \in \mathcal{K}$, let

$$y_K := \sum (h_j - h_{j-1} \mid j = 1, \ldots, t;\ K \text{ is a weak component of } D - V_j). \tag{55.26}$$

So

$$P = h_0 \chi^V + \sum_{K \in \mathcal{K}} y_K \chi^K. \tag{55.27}$$

Then:

Theorem 55.6. *Each $K \in \mathcal{K}$ belongs to $\mathcal{F}$. Moreover, for each $a = (u, v) \in A$:*

$$\begin{array}{lll} \text{(i)} & \sum(y_K \mid K \in \mathcal{K}, a \in \delta^{\text{out}}(K)) \leq l(a) & \text{if } a \in A \setminus C, \\ \text{(ii)} & \sum(y_K \mid K \in \mathcal{K}, a \in \delta^{\text{out}}(K)) \geq l(a) & \text{if } a \in C. \end{array} \tag{55.28}$$

Proof. Consider any $j = 1, \ldots, t$. By definition of $p(u)$, we know that V_j is a lower ideal with respect to $\preceq$. That is, if $v \in V_j$ and $u \preceq v$, then $u \in V_j$. (Indeed, if $v \in V_j$, then $p(v) \geq h_j$, hence $l_2(a) \geq h_j$ for some a with $a^{-1} \in \delta_B^{\text{out}}(w)$ for some $w \succeq v$. Since $w \succeq u$ we have $p(u) \geq l_2(a) \geq h_j$.)

Hence, for each $v \in V_j$ and $u \notin V_j$ we have $u \not\preceq v$. Therefore, there is a $U \in \mathcal{F}$ with $u \in U$ and $v \notin U$. This implies, as $\mathcal{F}$ is a crossing family, that there is a partition of $V \setminus V_j$ into sets in $\mathcal{F}$. As $d_A^{\text{in}}(U) = 0$ and $d_C^{\text{out}}(U) = k$, it follows that this partition is equal to the collection of weak components of the digraph $D - V_j$. So each weak component K of $D - V_j$ satisfies $d_A^{\text{in}}(K) = 0$ and $d_C^{\text{out}}(K) = k$; that is, K belongs to $\mathcal{F}$.

Consider any arc $a = (v, u) \in B$. As B is an l_1-maximal base of M_1, we have $l_1(a^{-1}) \leq l_1(a)$. Let $p(u) = l_2(b)$ for some $b^{-1} \in \delta_B^{\text{out}}(w)$ and some $w \succeq u$. Since $u \preceq w$, we know that $\deg_B^{\text{in}} - \chi^u + \chi^w \in P$. So $(B \cup \{b\}) \setminus \{a\}$ is again a base of M_2. Hence we have (as B is an l_2-maximal base of M_2) $l_2(b) \leq l_2(a)$. So $l_2(a) \geq l_2(b) \geq p(u)$. Also $p(v) \geq l_2(a^{-1})$, by definition of $p(v)$. Hence

$$\begin{aligned} l(a) - l(a^{-1}) &= l_1(a) + l_2(a) - l_1(a^{-1}) - l_2(a^{-1}) \geq l_2(a) - l_2(a^{-1}) \\ &\geq p(u) - p(v). \end{aligned} \tag{55.29}$$

If $a \in A \setminus C$, we have $l(a^{-1}) = 0$, and obtain (55.28)(i), since a enters no $K \in \mathcal{K}$ and so $p(u) \geq p(v)$. Hence

$$l(a) \geq p(u) - p(v) = \sum (y_K \mid K \in \mathcal{K}, a \in \delta^{\text{out}}(K)). \tag{55.30}$$

If $a \in C^{-1}$, we have $l(a) = 0$ and obtain (55.28)(ii), since a^{-1} enters no $K \in \mathcal{K}$, and so $p(u) \leq p(v)$. Hence

$$l(a^{-1}) \leq p(v) - p(u) = \sum (y_K \mid K \in \mathcal{K}, a^{-1} \in \delta^{\text{out}}(K)). \tag{55.31}$$

∎

For each $a \in C$, let $s(a)$ be the difference of the two terms in (55.28)(ii), and for $a \in A \setminus C$ let $s(a) := 0$. Then

$$\sum_{K \in \mathcal{K}} y_K \chi^{\delta_A^{\text{out}}(K)} - s \leq l \tag{55.32}$$

and

$$\begin{aligned} k \sum_{K \in \mathcal{K}} y_K - s(A) &= \sum_{K \in \mathcal{K}} y_K |\delta_A^{\text{out}}(K) \cap C| - s(A) \\ &= \sum_{a \in C} \sum (y_K \mid K \in \mathcal{K}, a \in \delta^{\text{out}}(K)) - s(A) = l(C). \end{aligned} \tag{55.33}$$

Thus we have an integer optimum dual solution to maximizing $l^{\mathsf{T}} x$ over the system $\mathbf{0} \leq x \leq \mathbf{1}$, $x(Y) \geq k$ (Y directed cut). If $k = 1$, we can do with $s = \mathbf{0}$, and obtain an integer optimum packing of directed cuts subject to l.

So we have:

Theorem 55.7. *Given a digraph $D = (V, A)$ and a length function $l : A \to \mathbb{Z}_+$, an optimum packing of directed cuts subject to l can be found in strongly polynomial time.*

Proof. See above. ∎

55.6. Further results and notes

55.6a. Complexity survey for minimum-size directed cut cover

	$O(n^5 \log n)$	Lucchesi [1976]
	$O(n^3 m)$	Frank [1981b]
	$O(n^2 M(n))$	Frank [1981b]
*	$O(n^2 m)$	Gabow [1993b,1995c]
*	$O(n M(n))$	Gabow [1993b,1995c]

As before, * indicates an asymptotically best bound in the table. $M(n)$ denotes the time to multiply $n \times n$ matrices. Also Karzanov [1979c,1981] gave a polynomial-time algorithm to find a minimum-size directed cut cover. Lucchesi [1976] gave also a polynomial-time algorithm to find a minimum-weight directed cut cover, and Frank [1981b] and Gabow [1993a,1993b,1995c] gave strongly polynomial-time algorithms for this.

55.6b. Feedback arc sets in linklessly embeddable digraphs

An undirected graph is called *linklessly embeddable* if it can be embedded in $\mathbb{R}^3$ such that any two vertex-disjoint circuits C_1 and C_2 are unlinked (that is, there is a topological sphere S such that C_1 is in the interior of S and C_2 is in the exterior of S). A digraph is called linklessly embeddable if its underlying undirected graph is linklessly embeddable. (Linklessly embeddable graphs are characterized by Robertson, Seymour, and Thomas [1995] in terms of 'forbidden minors'.)

Seymour [1996] showed that in an Eulerian linklessly embeddable directed graph, the minimum-size of a feedback arc set is equal to the maximum number of arc-disjoint directed circuits. We sketch the proof.

The basic combinatorial-topological part of the proof consists of showing:

Theorem 55.8. *Let D be an Eulerian linklessly embeddable digraph. Suppose that there exist $2k+1$ directed circuits such that any arc is in at most two of them. Then there exist $k+1$ arc-disjoint directed circuits.*

Sketch of proof. By a theorem of Robertson, Seymour, and Thomas [1995], D can be embedded in $\mathbb{R}^3$ such that for each undirected circuit C in D there exists an open disk B in $\mathbb{R}^3$ with boundary C and disjoint from D. (We identify D with its embedding.)

Let $C_1, \ldots, C_t$ be a maximum number of directed circuits in D such that any arc of D is in at most two of them. So $t \geq 2k+1$. Moreover, each arc of D is contained in *exactly* two of the C_i. Otherwise, the arcs not covered twice contain a directed circuit (as D is Eulerian). This contradicts the maximality of t.

For each $i = 1, \ldots, t$, let B_i by an open disk with boundary C_i and disjoint from D. We can assume that the B_i are pairwise disjoint, as can be seen as follows. First, we can assume that the B_i are tame and in general position. In particular, no point is in four of the B_i. Any point p in three of the B_i is the intersection point of three of the B_i, pairwise crossing at p. Any point p in two of the B_i is the intersection point of two of the B_i, crossing at p. Moreover, any two distinct B_i and B_j intersect each other in a finite number of closed and open curves, each representing crossings of B_i and B_j. Let $c(B_i, B_j)$ denote the number of such components.

We choose the C_i and B_i such that the sum of the $c(B_i, B_j)$ for $i \neq j$ is minimized.

Now it is elementary combinatorial topology to prove that there exist for any distinct i, j, with $B_i \cap B_j \neq \emptyset$, directed circuits C'_i and C'_j in D with

(55.34) $$\chi^{AC'_i} + \chi^{AC'_j} = \chi^{AC_i} + \chi^{AC_j}$$

and open disks B'_i and B'_j with boundaries C'_i and C'_j respectively, such that $B'_i \cap B'_j = \emptyset$ and $c(B'_i, B_h) + c(B'_j, B_h) \leq c(B_i, B_h) + c(B_j, B_h)$ for all $h \neq i, j$.

Hence, by the minimality of the sum of the $c(B_i, B_j)$, it follows that the B_i are disjoint. So D, together with the B_i forms the union of a number of compact surfaces, certain points of which are identified. As these surfaces are orientable (since they are embedded in $\mathbb{R}^3$), the B_i fall apart into two classes: those with boundary oriented clockwise, and those with boundary oriented counter-clockwise. Each of these classes have arc-disjoint boundaries, and at least one of these classes has size at least $k+1$. This proves the theorem. ∎

Seymour [1996] next continues by deriving (for linklessly embeddable graphs) the total dual integrality of the following system in $x \in \mathbb{R}^A$:

(55.35) $x(C) \geq 1$ for each directed circuit C in D.

Note that nonnegativity of x is not required here.

Corollary 55.8a. *For any linklessly embeddable digraph $D = (V, A)$, system* (55.35) *is totally dual integral.*

Proof. Let $w \in \mathbb{Z}^A$ be such that the minimum of $w^\mathsf{T} x$ over (55.35) is finite. Let $\mathcal{C}$ be the collection of directed circuits in D. By Theorem 5.29, it suffices to show that the maximum value μ of $\sum_{C\in\mathcal{C}} y(C)$ taken over $y : \mathcal{C} \to \frac{1}{2}\mathbb{Z}_+$ satisfying

$$\sum_{C\in\mathcal{C}} y_C \chi^{AC} \leq w \tag{55.36}$$

is attained by an integer-valued y.

Now (as the minimum is finite) w belongs to the cone generated by the incidence vectors of directed circuits, and hence w is a nonnegative circulation. Replace any arc $a = (u, v)$ by $w(a)$ parallel arcs from u to v, giving the Eulerian digraph $D' = (V, A')$. Then μ is equal to half of the maximum number μ' of directed circuits in D' such that any arc of D' is in at most two of these circuits. By Theorem 55.8, D' contains at least $\lceil \frac{1}{2}\mu' \rceil$ arc-disjoint directed circuits. Since $\lceil \frac{1}{2}\mu' \rceil \geq \mu$, this gives in D an integer vector $y : \mathcal{C} \to \mathbb{Z}_+$ as required. ∎

This finally gives:

Theorem 55.9. *The minimum size of a feedback arc set in an Eulerian linklessly embeddable digraph $D = (V, A)$ is equal to the maximum number of arc-disjoint directed circuits.*

Proof. Consider the LP-duality relation for maximizing $x(U)$ over (55.35):

$$\begin{aligned} &\min\{x(A) \mid x(C) \geq 1 \text{ for each directed circuit } C\} \\ &= \max\{\sum_C y_C \mid y_C \geq 0, \sum_C y_C \chi^{AC} = \mathbf{1}\}, \end{aligned} \tag{55.37}$$

where C ranges over all directed circuits. By Corollary 55.8a and the theory of total dual integrality (Theorem 5.22), both optima have an integer optimum solution. So the maximum is equal to the maximum number of arc-disjoint directed circuits. Let x attain the minimum. By Theorem 8.2, there exists a ('potential') $p : V \to \mathbb{Z}$ with $x_a \geq p(v) - p(u)$ for each arc $a = (u, v)$ of D. Define $x'(a) := x_a - p(v) + p(u)$ for each arc $a = (u, v)$. Then $x' \in \mathbb{Z}^A_+$, $x'(C) = x(C) \geq 1$ for each directed circuit C, and $x'(A) = x(A)$ (since D is Eulerian). Hence the set of arcs a with $x'(a) \geq 1$ forms a feedback arc set of size at most $x(A)$, proving the theorem. ∎

System (55.35) can be tested in polynomial time, for any digraph (with the Bellman-Ford method). It implies that in an Eulerian linklessly embeddable digraph, a minimum-size feedback arc set can be found in polynomial time (with the ellipsoid method).

55.6c. Feedback vertex sets

A *feedback vertex set* in a digraph $D = (V, A)$ is a subset U of V with $D - U$ acyclic — that is, U intersects every directed circuit. Reed, Robertson, Seymour, and Thomas [1996] proved:

(55.38) for each integer $k \geq 0$ there exists an integer $n_k \geq 0$ such that each digraph $D = (V, A)$ having no k vertex-disjoint directed circuits, has a feedback vertex set of size at most n_k.

(For $k = 2$ this answers a question of Gallai [1968b] and Younger [1973].)

Reed, Robertson, Seymour, and Thomas also showed that for each fixed integer k, there is a polynomial-time algorithm to find k vertex-disjoint directed circuits in a digraph if they exist.

Earlier, progress on (55.38) was made by McCuaig [1993], who proved it for $k = 2$, where $n_2 = 3$, by Seymour [1995b], who proved a fractional version of it (if there is no fractional packing of k directed circuits, then there is a feedback vertex set of size at most n_k), and by Reed and Shepherd [1996] for planar graphs.

According to Reed, Robertson, Seymour, and Thomas [1996], N. Alon proved that n_k is at least $Ck \log k$ for some constant C.

Cai, Deng, and Zang [1999,2002] characterized for which orientations $D = (V, A)$ of a complete bipartite graph the system

(55.39)
$$\begin{aligned} x_v &\geq 0 && \text{for } v \in V,\\ x(VC) &\geq 1 && \text{for each directed circuit } C, \end{aligned}$$

is totally dual integral. (Related results can be found in Cai, Deng, and Zang [1998].)

Guenin [2001b] gave a characterization of digraphs $D = (V, A)$ for which the linear system in $\mathbb{R}^{V \cup A}$ for feedback arc and vertex sets:

(55.40)
$$\begin{aligned} x_v &\geq 0 && \text{for } v \in V,\\ x_a &\geq 0 && \text{for } a \in A,\\ x(VC) + x(AC) &\geq 1 && \text{for each directed circuit } C, \end{aligned}$$

is totally dual integral.

The undirected analogue of (55.38) was proved for $k = 2$ by Bollobás [1963], and for general k by Erdős and Pósa [1965]. Ding and Zang [1999] characterized the undirected graphs $G = (V, E)$ for which the system

(55.41)
$$\begin{aligned} x_v &\geq 0 && \text{for } v \in V,\\ x(VC) &\geq 1 && \text{for each circuit } C, \end{aligned}$$

is totally dual integral. Their characterization implies that (55.41) is totally dual integral if and only if it defines an integer polyhedron.

A polyhedral approach to the feedback vertex set problem was investigated by Funke and Reinelt [1996]. Approximation algorithms for feedback problems were given by Monien and Schulz [1982], Eades, Lin, and Smyth [1993], Bar-Yehuda, Geiger, Naor, and Roth [1994,1998], Becker and Geiger [1994,1996], Bafna, Berman, and Fujito [1995,1999], Even, Naor, Schieber, and Sudan [1995,1998], Even, Naor, and Zosin [1996,2000], Goemans and Williamson [1996,1998], Chudak, Goemans, Hochbaum, and Williamson [1998], Bar-Yehuda [2000], and Cai, Deng, and Zang [2001]. More on the feedback vertex set problem was presented by Smith and Walford [1975], Kevorkian [1980], Rosen [1982], Speckenmeyer [1988], Stamm [1991], Hackbusch [1997], and Pardalos, Qian, and Resende [1999].

55.6d. The bipartite case

McWhirter and Younger [1971] (cf. Younger [1970], Vidyasankar [1978b]) proved the Lucchesi-Younger theorem in case the arcs of D form a directed cut; that is, in case the underlying undirected graph is bipartite, while all arcs are oriented from one colour class to the other. It amounts to the following:

Theorem 55.10. *Let $G = (V, E)$ be a connected bipartite graph and let $\mathcal{F}$ be the collection of subsets $E[U]$ of E for which U is a vertex cover with $E[U]$ nonempty. Then the minimum size of a set of edges intersecting each set in $\mathcal{F}$ is equal to the maximum number of disjoint sets in $\mathcal{F}$.*

Proof. Let U and W be the colour classes of G and let digraph D be obtained from G by orienting each edge from U to W. Then a set of edges belongs to $\mathcal{F}$ if and only if it is a directed cut of D. Hence the theorem follows from the Lucchesi-Younger theorem. ∎

D.H. Younger (cf. Frank [1993b]) showed that the maximum number of disjoint nonempty cuts in a bipartite graph G is equal to the maximum number of disjoint directed cuts in the directed graph obtained from G by orienting all edges from one colour class to the other (cf. Corollary 29.13b). (Vidyasankar [1978b] showed that a set of edges J intersecting each set in $\mathcal{F}$ attains the minimum in Theorem 55.10 if and only if J intersects any circuit C of G in at most $\frac{1}{2}|EC|$ edges; that is, if and only if J is a join — cf. Section 29.11d.)

As noted by Younger [1979], the Lucchesi-Younger theorem, in the form of Corollary 55.2b, implies the Kőnig-Rado edge cover theorem (Theorem 19.4): the minimum size of an edge cover in a bipartite graph $G = (V, E)$ is equal to the maximum size of a stable set in G. To obtain this as a consequence, let U and W be the colour classes of G and let $D = (V, A)$ be the directed graph with vertex set V and arcs all pairs (u, v) with $u \in U$ and $v \in W$. Define a weight function $w : A \to \mathbb{Z}$ by $w(u, v) := 1$ if $uv \in E$, and ∞ otherwise. Then the minimum weight of a directed cut cover in D is equal to the minimum size of an edge cover in G. With this correspondence, Corollary 55.2b gives the Kőnig-Rado edge cover theorem.

55.6e. Further notes

Frank, Tardos, and Sebő [1984] showed that the Lucchesi-Younger theorem implies that in a digraph $D = (V, A)$, the minimum size of a directed cut cover is equal to the maximum value of

$$(55.42) \qquad \sum_{i=1}^{k} \text{number of weak components of } D - V_i,$$

where $\emptyset \neq V_1 \subset V_2 \subset \cdots \subset V_k \subset V$ are such that no arc leaves any V_i and enters at most one of the V_i.

Frank and Tardos [1989] showed that a weakly connected digraph $D = (V, A)$ has a branching that intersects all directed cuts if and only if for each nonempty $U \subseteq V$, the number of weak components K of $D - U$ with $d^{\text{in}}(K) = 0$, is at most $|U|$.

Younger [1965] proved the Lucchesi-Younger theorem for digraphs having an arborescence, and, more generally, Younger [1979] proved it for source-sink connected digraphs (that is, each strong component not left by any arc is reachable by a directed path from each strong component not entered by any arc).

Tuza [1994] showed that for any planar directed graph $D = (V, A)$ and any collection $\mathcal{T}$ of directed triangles in D, the minimum number of arcs intersecting each triangle in $\mathcal{T}$ is equal to the maximum number of arc-disjoint triangles in $\mathcal{T}$.

Chapter 56

Minimum directed cuts and packing directed cut covers

A minimum-capacity directed cut can be found in strongly polynomial time, by applying the minimum-capacity $s-t$ cut algorithm, for all s, t, in some modified digraph.
As for packing directed cut covers it is unknown if it is polynomial-time tractable. Also it is unknown if the maximum number of disjoint directed cut covers is equal to the minimum size of a directed cut — this is Woodall's conjecture. But the capacitated version of it does not hold.
In this chapter we consider a few cases where Woodall's conjecture has been proved, in particular for the *source-sink connected* digraphs.

56.1. Minimum directed cuts and packing directed cut covers

The Lucchesi-Younger theorem states that in a digraph $D = (V, A)$, the minimum size of a directed cut cover is equal to the maximum number of disjoint directed cuts. Woodall [1978a,1978b] ventured the conjecture that this min-max relation would be maintained after interchanging the terms directed cut and directed cut cover:

Conjecture (Woodall's conjecture)**.** In a digraph, the minimum size of a directed cut is equal to the maximum number of disjoint directed cut covers.

This conjecture is open.

A capacitated version of Woodall's conjecture (conjectured by Edmonds and Giles [1977] and D.H. Younger) is however not true. Note that the Lucchesi-Younger theorem is equivalent to its weighted version, by replacing arcs by directed paths of length $l(a)$ if $l(a) \geq 1$, and contracting an arc a if $l(a) = 0$. We could attempt this approach to obtain an equivalent capacitated version from Woodall's conjecture, by replacing any arc a by $c(a)$ parallel arcs, but there is a problem here: if $c(a) = 0$, we delete a and can create new directed cuts.

A capacitated version with capacities 0 and 1 amounts to the statement that each directed cut k-cover can be partitioned into k directed cut covers.

Figure 56.1 gives a counterexample to this for the case $k = 2$ (Schrijver [1980a]). Note that the counterexample is planar, and that therefore the 'planar dual' assertion (on packing feedback arc sets) also does not hold.

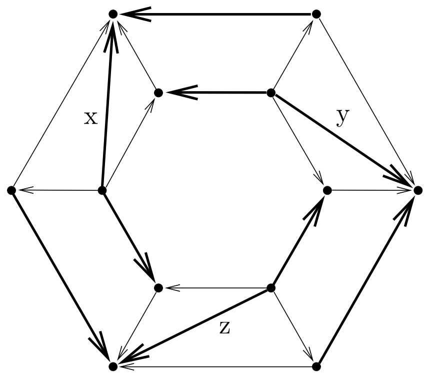

Figure 56.1

A directed cut 2-cover that cannot be split into two directed cut covers. Let C be the set of heavy arcs. Then C is a directed cut 2-cover, since for each arc $c \in C$, the set $C \setminus \{c\}$ is a directed cut cover, which is easy to check since up to symmetry there are only two types of arcs in C.

However, C cannot be split into directed cut covers C_1 and C_2. To see this, observe that each of these C_i must contain exactly one of the two arcs in C meeting any source or sink. Moreover, each C_i must contain at least one of the arcs labeled x, y, z, since the set of arcs from the inner hexagon to the outer hexagon forms a directed cut. Hence we may assume without loss of generality that C_1 contains the arcs x and y, but not z. But then C_1 does not intersect the directed cut of those arcs going from the right half of the figure to the left half.

To interpret this polyhedrally, consider the system:

(56.1) (i) $0 \leq x_a \leq 1$ for each $a \in A$,
(ii) $x(B) \geq 1$ for each directed cut cover B.

With Corollary 55.2b, the theory of blocking polyhedra gives that system (56.1) determines the convex hull of the incidence vectors of arc sets containing a directed cut. However, by the example in Figure 56.1, system (56.1) generally is not TDI, as total dual integrality amounts to the capacitated version of Woodall's conjecture.

In a number of special cases, Woodall's conjecture, and its capacitated extension, have been proved. In the remainder of this chapter we will consider such cases.

Two more counterexamples to the conjecture of Edmonds and Giles were given by Cornuéjols and Guenin [2002c], and they asked if, together with the example of Figure 56.1, these form all minimal counterexamples to the Edmonds-Giles conjecture.

56.2. Source-sink connected digraphs

Feofiloff and Younger [1987] and Schrijver [1982] showed that for source-sink connected digraphs, the min-max relation for packing directed cut covers does hold. Here a digraph is called *source-sink connected* if each strong component not left by any arc is reachable by a directed path from each strong component not entered by any arc. So an acyclic digraph is source-sink connected if each sink is reachable by a directed path from each source. We follow the proof of Schrijver [1982].

Theorem 56.1. *Let $D = (V, A)$ be a source-sink connected digraph and let $k \in \mathbb{Z}_+$. Then any directed cut k-cover C can be partitioned into k directed cut covers.*

Proof. Choose a counterexample with $|V| + |C|$ as small as possible. Then D is acyclic, since any strong component can be contracted to one vertex.

We may assume that if v is reachable in D from u and $v \neq u$, then $(u, v) \in A$. We first show:

(56.2) for any nonempty proper subset U of V with $\delta^{\text{out}}(U) = \emptyset$ and $|\delta^{\text{in}}_C(U)| = k$, one has $|U| = 1$ or $|U| = |V| - 1$.

Suppose not. Let $D' := D/U$ and $D'' := D/\overline{U}$ be the digraphs obtained from D by contracting U and $\overline{U} := V \setminus U$, respectively. Note that D' and D'' are source-sink connected again. Let C' be the set of arcs in C with tail in $\overline{U}$, and let C'' be the set of arcs in C with head in U.

Now each directed cut in D' intersects C' in at least k arcs, as it is a directed cut in D and hence intersects C in at least k arcs. So by the minimality of $|V| + |C|$, C' can be split into k directed cut covers $B'_1, \ldots, B'_k$ for D'. As $|\delta^{\text{in}}_C(U)| = k$, each B'_i has exactly one arc entering U. Similarly, C'' can be split into k directed cut covers $B''_1, \ldots, B''_k$ for D'', such that each B''_i has exactly one arc entering U. By choosing indices appropriately, we can assume that B'_i and B''_i have an arc entering U in common, for each $i = 1, \ldots, k$ (as $|\delta^{\text{in}}_C(U)| = k$).

Then each $B'_i \cup B''_i$ is a directed cut cover for D. For suppose that there is a nonempty proper subset W of V with $\delta^{\text{out}}(W) = \emptyset$ and $\delta^{\text{in}}(W)$ disjoint from $B'_i \cup B'_i$. Then $U \cap W \neq \emptyset$ and $U \not\subseteq W$, since otherwise $\delta^{\text{in}}(W)$ is a directed cut of D', and hence some arc in B'_i enters W. So some arc in B''_i enters $U \cap W$. Similarly, some arc in B'_i enters $U \cup W$. Since exactly one arc in $B_i' \cup B_i''$ enters U, it follows that at least one arc in $B_i' \cup B_i''$ enters W, contradicting our assumption.

So each $B_i' \cup B_i''$ is a directed cut cover for D. As they are disjoint, this contradicts our assumption, thus proving (56.2).

We next show the following. Let X be the set of sources of D and let Y be the set of sinks of D. Then:

(56.3) for each $a = (u, v) \in C$ we have $u \in X$ or $v \in Y$.

For suppose not. Then by (56.2), each directed cut of D intersects $C \setminus \{a\}$ in at least k arcs (as any directed cut intersecting C in exactly k arcs and containing a is equal to $\delta^{\text{in}}(\{v\})$ or $\delta^{\text{in}}(V \setminus \{u\})$, implying that v is a sink or u is a source). So by the minimality of $|V| + |C|$, we can split $C \setminus \{a\}$ into k directed cut covers. This implies that also C can be split into k directed cut covers, contradicting our assumption. This proves (56.3).

Next:

(56.4) if $a = (u, v) \in C$, $a' = (u', v') \in C$, and v is reachable from u', then $u' \in X$ or $v \in Y$.

For suppose not. By (56.3), $u \in X$ and $v' \in Y$, and hence (since D is source-sink connected), $a'' = (u, v') \in A$. Now $a \neq a'$, as $u \in X$ and $u' \notin X$. So $C' := (C \setminus \{a, a'\}) \cup \{a''\}$ is smaller than C. Moreover, C' is a directed cut k-cover. Indeed, let U be a nonempty proper subset of V with $\delta^{\text{out}}(U) = \emptyset$. If $|U| = 1$ or $|U| = |V| - 1$, then $\delta^{\text{in}}_{C'}(U) = \delta^{\text{in}}_{C}(U)$, since then $U = \{r\}$ for some sink r or $U = V \setminus \{s\}$ for some source s. If $1 < |U| < |V| - 1$, then

(56.5) $|\delta^{\text{in}}_{C'}(U)| \geq |\delta^{\text{in}}_{C}(U)| - 1 \geq k,$

since if both a and a' enter U, then $u \notin U$ and $v' \in U$, and hence a'' enters U.

So C' is a directed cut k-cover, and hence, by the minimality of $|V| + |C|$, C' can be split into k directed cut covers. Let B be the directed cut cover containing a''. Then $B' := (B \setminus \{a''\}) \cup \{a, a'\}$ is a directed cut cover, since any directed cut $\delta^{\text{in}}(W)$ containing a'', contains at least one of a, a'. Indeed, otherwise $u, v \notin W$, $u', v' \in W$, but then $u' \neq v$ and arc (u', v) leaves W, contradicting the fact that W determines a directed cut.

So by replacing B by B' we obtain a splitting of C into k directed cut covers, contradicting our assumption. This proves (56.4).

This implies:

(56.6) V can be partitioned into sets R and S such that $\delta^{\text{in}}(R) = \emptyset$, $X \subseteq R$, $Y \subseteq S$, and if any $(u, v) \in C$ leaves R, then $u \in X$ and $v \in Y$.

For define

(56.7)
$$C' := \{(u, v) \in C \mid u \notin X \text{ or } v \notin Y\},$$
$$R := \{v \in V \mid D' = (V, A \cup C^{-1}) \text{ has a directed } v - X \text{ path}\},$$
$$S := V \setminus R.$$

Then $X \subseteq R$, $\delta^{\text{in}}_A(R) = \emptyset$, and any $(u,v) \in C$ leaving R satisfies $u \in X$ and $v \in Y$. To see that $Y \subseteq S$, suppose to the contrary that D' has a directed $Y - X$ path P. Choose P shortest. Then by (56.3), P has of at most three arcs. Let (v', u') and (v, u) be the first and last arc of P. So $v' \in Y$ and $u \in X$. These arcs belong to C'^{-1}, and v is reachable from u' in D. So by (56.4), $u' \in X$ or $v \in Y$. This contradicts the definition of C'. This shows (56.6).

Fix R, S as in (56.6). Let $D' = (V, A')$ be the digraph arising from D by replacing any arc (u,v) of D by k parallel arcs from v to u. Then

(56.8) $$|\delta^{\text{in}}_{A' \cup C}(U)| \geq k$$

for each nonempty proper subset U of V. So by Theorem 54.11, $A' \cup C$ can be split into k $R - S$ bibranchings. Let $B_1, \ldots, B_k$ be the intersections of these bibranchings with C. We show that each B_i is a directed cut cover, contradicting our assumption, and therefore finishing the proof.

Suppose that say B_1 is not a directed cut cover. Let U be a nonempty proper subset of V with $\delta^{\text{out}}(U) = \emptyset$, and suppose that no arc in B_1 enters U. Note that if U contains any source, it contains all sinks, since no arc of D leaves U. So U contains no sources or contains all sinks.

First assume that U contains no sources of D. As U contains at least one sink (as $U \neq \emptyset$ and $\delta^{\text{out}}(U) = \emptyset$), we know $U \not\subseteq R$. As $A' \cup B_1$ is an $R - S$ bibranching, we know that

(56.9) $$\delta^{\text{in}}_{A' \cup B_1}(U \cap S) \neq \emptyset.$$

As $\delta^{\text{out}}_A(U \cap S) = \emptyset$ (since $\delta^{\text{out}}_A(U) = \emptyset$ and $\delta^{\text{in}}_A(R) = \emptyset$), we have $\delta^{\text{in}}_{A'}(U \cap S) = \emptyset$. Hence some arc (u,v) in B_1 enters $U \cap S$. As by assumption (u,v) does not enter U, (u,v) enters S, and $u \in U$. However, by (56.6), u belongs to X. This contradicts our assumption that U contains no sources of D.

The case that U contains all sinks is symmetric, and leads again to a contradiction. ∎

A special case of Theorem 56.1 is Woodall's conjecture for source-sink connected digraphs:

Corollary 56.1a. *Let $D = (V, A)$ be a source-sink connected digraph. Then the minimum size of a directed cut is equal to the maximum number of disjoint directed cut covers.*

Proof. This is the case $C = A$ of Theorem 56.1. ∎

Also, a capacitated version can be derived from the theorem:

Corollary 56.1b. *Let $D = (V, A)$ be a source-sink connected digraph and let $c : A \to \mathbb{Z}_+$ be a capacity function. Then the minimum capacity of a directed cut is equal to the maximum number of directed cut covers such that no arc a is in more than $c(a)$ of these directed cut covers.*

Proof. Directly from Theorem 56.1, by adding, for any arc a of D, $c(a)$ arcs parallel to a, and by taking for C the set of newly added arcs. ∎

Equivalently, in TDI terms:

Corollary 56.1c. *If $D = (V, A)$ is a source-sink connected digraph, then system* (56.1) *is totally dual integral.*

Proof. This is a reformulation of Corollary 56.1b. ∎

Feofiloff [1983] gave a polynomial-time algorithm to find a maximum number of disjoint directed cut covers in a source-sink connected digraph. Also the proof above implies a polynomial-time algorithm.

A polynomial-time algorithm for the capacitated case can be derived from the ellipsoid method (cf. Grötschel, Lovász, and Schrijver [1988]). A semi-strongly polynomial-time algorithm also follows from Section 57.5 below.

Notes. Frank [1979b] showed the special case of Woodall's conjecture for digraphs having an arborescence. (Such digraphs are source-sink connected.) J. Edmonds observed that this can be derived from Edmonds' disjoint arborescences theorem (Corollary 53.1b): Let $D = (V, A)$ have an r-arborescence. Let k be the minimum size of a directed cut in D. Add to D, for each arc (u, v) of D, k parallel arcs from v to u. This makes the digraph $D' = (V, A')$ with $|\delta^{\text{in}}_{A'}(U)| \geq k$ for each nonempty $U \subseteq V \setminus \{r\}$. Hence D' has k disjoint r-arborescences (by Edmonds' disjoint arborescences theorem). Now for any r-arborescence B in D', the set $B \cap A$ is a directed cut cover in D, since if U is a nonempty proper subset of V with $\delta^{\text{out}}_{A}(U) = \emptyset$, then $\delta^{\text{in}}_{A'}(U) = \delta^{\text{in}}_{A}(U)$, and hence $\delta^{\text{in}}_{B \cap A}(U) = \delta^{\text{in}}_{B}(U) \neq \emptyset$. So we obtain k disjoint directed cut covers in D.

56.3. Other cases where Woodall's conjecture is true

Another case where Woodall's conjecture holds is given in:

Theorem 56.2. *Let $D = (V, A)$ be a digraph arising from a directed tree $T = (V, A')$ such that $(u, v) \in A$ if and only if v is reachable in T from u. Let $c : A \to \mathbb{Z}_+$ be a capacity function. Then the minimum capacity of a directed cut is equal to the maximum number of directed cut covers such that each arc a is in at most $c(a)$ of them.*

Proof. The proof is by induction on the minimum capacity k of a directed cut. Then it suffices to show that there exists a directed cut cover B with $\chi^B \leq c$ and with $(c - \chi^B)(C) \geq k - 1$ for each directed cut C.

Let M be the $A' \times A$ network matrix generated by T and D (cf. Section 13.3). Then the rows of M are precisely the incidence vectors of inclusionwise minimal directed cuts. So it suffices to show that there exists an integer solution x of

(56.10) $\mathbf{0} \leq x \leq \mathbf{c}, Mx \geq \mathbf{1}, M(c-x) \geq (k-1)\mathbf{1}$,

since for any such x there is a directed cut cover B satisfying $\chi^B \leq x$.

Since M is totally unimodular, it suffices to show that (56.10) has any solution. Define $x := \frac{1}{k}c$. Then x satisfies (56.10), since $Mc \geq k\mathbf{1}$ and hence $Mx \geq \mathbf{1}$ and $M(c-x) = \frac{k-1}{k}Mc \geq (k-1)\mathbf{1}$. ∎

The theorem can also be formulated in terms of partitioning directed cut k-covers:

Corollary 56.2a. *Let $D = (V, A)$ be a digraph such that A contains a directed spanning tree T with the property that for each arc (u, v) in A there exists a directed $u-v$ path in T. Then any directed cut k-cover in D can be partitioned into k directed cut covers.*

Proof. This follows from Theorem 56.2 by taking $c(u, v)$ equal to the number of times there is an arc from u to v in the directed cut k-cover. ∎

A. Frank also noted that Woodall's conjecture is true if the minimum size of a directed cut is at most 2:

Theorem 56.3. *Let $D = (V, A)$ be a digraph such that each directed cut has size at least two. Then there are two disjoint directed cut covers.*

Proof. As the underlying undirected graph is 2-edge-connected, it has a strongly connected orientation $D' = (V, A')$ (see Corollary 61.3a). Let B_1 be the set of arcs of D that have the same orientation in D' and let $B_2 := A \setminus B_1$. Then B_1 and B_2 are disjoint directed cut covers. ∎

Figure 56.1 shows that this cannot be generalized to each directed cut 2-cover being partitionable into two directed cut covers.

56.3a. Further notes

Karzanov [1985c] gave a strongly polynomial-time algorithm to find a minimum-*mean* capacity directed cut (cf. McCormick and Ervolina [1994]).

Chapter 57

Strong connectors

A *strong connector* is a set of new arcs whose addition to a given digraph D makes it strongly connected. If each potential new arc has been given a length, then finding a shortest strong connector is NP-complete, even if D has no arcs at all: finding a directed Hamiltonian circuit is a special case. (So even if each length is 0 or 1, the problem is NP-complete.)
However, there are a few cases where finding a shortest strong connector is tractable and where min-max relations and polyhedral characterizations hold — for instance, if D is source-sink connected. For these digraphs, packing strong connectors is similarly tractable. These results follow by reduction to directed cut covers discussed in the previous two chapters.

57.1. Making a directed graph strongly connected

Let (V, A) and (V, B) be digraphs. The set B is called a *strong connector for* D if the digraph $(V, A \cup B)$ is strongly connected.

Consider the following *strong connectivity augmentation problem*:

(57.1) Given a digraph $D = (V, A)$ and a cost function $k : V \times V \to \mathbb{Q}$, find a minimum-cost strong connector for D.

Theorem 57.1. *The strong connectivity augmentation problem is* NP*-complete, even if* $A = \emptyset$*.*

Proof. The problem of finding a Hamiltonian circuit in a digraph $D' = (V, A')$ is equivalent to the existence of a strong connector for $(V, \emptyset)$ of cost $|V|$, where $k(u, v) := 1$ if $(u, v) \in A'$, and $k(u, v) := 2$ otherwise. ∎

Eswaran and Tarjan [1976] showed that if the cost of each new arc equals 1, then there is an easy solution:

Theorem 57.2. *If* $D = (V, A)$ *is an acyclic digraph with at least* 2 *vertices, and with* ρ *sources and* σ *sinks, then the minimum size of a strong connector for* D *equals* $\max\{\rho, \sigma\}$*.*

Proof. To see that the minimum is at least $\max\{\rho, \sigma\}$, note that for each source r one should add at least one arc entering r; similarly, for each sink s one should add at least one arc leaving s.

That the bound can be attained is shown by induction on $\max\{\rho, \sigma\}$. If there is a pair of a source r and a sink s such that s is not reachable from r, add an arc (s, r). This reduces both ρ and σ by 1, while maintaining acyclicity, and induction gives the result.

So we can assume that each sink is reachable from each source. We can also assume that $\rho \geq \sigma$ (otherwise, reverse all orientations). Let $r_1, \ldots, r_\rho$ be the sources and let $s_1, \ldots, s_\sigma$ be the sinks. Then adding arcs (s_i, r_i) for $i = 1, \ldots, \sigma$, and arcs (s_i, r_1) for $i = \sigma+1, \ldots, \rho$ makes D strongly connected, proving the theorem. ∎

This implies for not necessarily acyclic digraphs:

Corollary 57.2a. *Let $D = (V, A)$ be a digraph which is not strongly connected, let ρ be the number of strong components K of D with $d^{\mathrm{in}}(K) = 0$ and let σ be the number of strong components K of D with $d^{\mathrm{out}}(K) = 0$. Then the minimum size of a strong connector for D equals* $\max\{\rho, \sigma\}$.

Proof. Apply Theorem 57.2 to the digraph obtained from D by contracting each strong component of D to one vertex. ∎

These proofs also give a polynomial-time algorithm to find a minimum-size strong connector. Eswaran and Tarjan [1976] describe a linear-time implementation.

57.2. Shortest strong connectors

Let $D_0 = (V, A_0)$ and $D = (V, A)$ be digraphs. Call a subset A' of A a D_0-*cut* (*in* D) if $A' = \delta^{\mathrm{in}}_A(U)$ for some nonempty proper subset U of V with $\delta^{\mathrm{in}}_{A_0}(U) = \emptyset$.

It is easy to see that a set B of arcs of D is a strong connector for D_0 if and only if B intersects each D_0-cut in D. The following consequence of the Lucchesi-Younger theorem was given in Schrijver [1982]. It gives a min-max relation for the minimum length of a strong connector, if the following condition holds for digraphs $D_0 = (V, A_0)$ and $D = (V, A)$:

(57.2) for each $(u, v) \in A$ there exist $u', v' \in V$ such that in D_0, u' is reachable from u and from v', and v from v'.

We mention two special cases where this condition is satisfied:

- A is a subset of A_0^{-1},
- D_0 is source-sink connected.

We derive from the Lucchesi-Younger theorem (Schrijver [1982]):

Theorem 57.3. *Let $D_0 = (V, A_0)$ and $D = (V, A)$ be digraphs satisfying (57.2) and let $l : A \to \mathbb{Z}_+$ be a length function. Then the minimum length of a strong connector in D for D_0 is equal to the maximum number of D_0-cuts in D such that no arc a of D is in more than $l(a)$ of these cuts.*

Proof. We can assume that D_0 is acyclic, and that for any $u, v \in V$, if v is reachable in D_0 from u, then $(u, v) \in A_0$. (So $(v, v) \in A_0$ for each $v \in V$.)

We show the theorem by induction on the number τ of arcs $a = (u, v)$ of D for which $(v, u) \notin A_0$. If $\tau = 0$, the theorem is equivalent to Corollary 55.2a.

If $\tau > 0$, choose $(u, v) \in A$ with $(v, u) \notin A_0$. By assumption, there exist $u', v' \in V$ with $(u, u'), (v', u'), (v', v) \in A_0$. Introduce two new vertices, u'' and v'', and add arcs

(57.3) $\qquad (u, u''), (u'', u'), (v'', u''), (v'', v), (v', v'')$

to A_0. Moreover, replace arc (u, v) of A by (u'', v''), with length equal to that of the original arc (u, v). Let $\widetilde{D}_0 = (\widetilde{V}, \widetilde{A}_0)$ and $\widetilde{D} = (\widetilde{V}, \widetilde{A})$ denote the modified graphs.

This transformation decreases the number τ by 1. Moreover,

(57.4) $\qquad$ any subset C of A is a strong connector for D_0 if and only if the set $\widetilde{C} \subseteq \widetilde{A}$ is a strong connector for $\widetilde{D}_0$.

Here $\widetilde{C}$ arises from C by replacing (u, v) by (u'', v'') if $(u, v) \in C$. Proving (57.4) suffices, since it implies that the two numbers in the theorem are invariant under the transformation.

(57.4) can be seen as follows. Choose $C \subseteq A$. First let C be a strong connector for D_0. If $(u, v) \notin C$, then $\widetilde{C} = C$ is also strong connector for $\widetilde{D}_0$ (since in $\widetilde{D}_0$ the new vertex u'' is on a $u - u'$ path, and the new vertex v'' is on a $v' - v$ path). If $(u, v) \in C$, then $\widetilde{C} = (C \setminus \{(u, v)\}) \cup \{(u'', v'')\}$ is a strong connector for $\widetilde{D}_0$, since $A_0 \cup C$ contains the $u-v$ path (u, u''), (u'', v''), (v'', v).

Conversely, let $\widetilde{C}$ be a strong connector for $\widetilde{D}_0$. If $(u'', v'') \notin \widetilde{C}$, then $C = \widetilde{C}$ is also a strong connector for D_0, since any directed path in $\widetilde{A}_0 \cup \widetilde{C}$ connecting two vertices in V and traversing any of the new vertices u'', v'', can be shortcut to a path avoiding u'' and v''.

If $(u'', v'') \in \widetilde{C}$, then $C = (\widetilde{C} \setminus \{(u'', v'')\}) \cup \{(u, v)\}$ is a strong connector for D_0, since any directed path in $\widetilde{A}_0 \cup \widetilde{C}$ connecting two vertices in V and traversing arc (u'', v''), must traverse (u, u''), (u'', v''), and (v'', v), and hence gives a path in $A_0 \cup C$, by replacing this by (u, v). ∎

The proof gives also an algorithmic reduction to the problem of finding a minimum-length directed cut cover, and hence (by Theorem 55.5) a minimum-length strong connector for D_0 can be found in strongly polynomial time.

Theorem 57.3 includes several theorems considered earlier:

- $s, t \in V$ and $A_0 := \{(u, v) \mid u = t \text{ or } v = s\}$: max-potential min-work theorem (Theorem 8.3);
- V is the disjoint union of U and W, $A_0 := \{(u, w) \mid u \in U, w \in W\}$ and $A \subseteq \{(w, u) \mid w \in W, u \in U\}$: weighted version of the Kőnig-Rado edge cover theorem(Theorem 19.4);
- $A_0 = \{(v, r) \mid v \in V\}$ for some $r \in V$: optimum arborescence theorem(Theorem 52.3);
- V is the disjoint union of U and W and $A_0 := \{(u, w) \mid u \in U, w \in W\}$:optimum bibranching theorem(Corollary 54.8b);
- $A \subseteq \{(u, v) \mid (v, u) \in A_0\}$: Lucchesi-Younger theorem (Theorem 55.2).

A cardinality version of the previous theorem is:

Corollary 57.3a. *Let $D_0 = (V, A_0)$ and $D = (V, A)$ be digraphs satisfying* (57.2). *Then the minimum size of a strong connector in D for D_0 is equal to the maximum number of disjoint D_0-cuts in D.*

Proof. This is the case $l = \mathbf{1}$ of Theorem 57.3. ∎

We formulate this for the special case of source-sink connected digraphs. Recall that a digraph $D = (V, A)$ is called *source-sink connected* if each strong component not left by any arc is reachable by a directed path from each strong component not entered by any arc.

Corollary 57.3b. *Let $D_0 = (V, A_0)$ and $D = (V, A)$ be digraphs, with D_0 source-sink connected. Let $l : A \to \mathbb{Z}_+$ be a length function. Then the minimum length of a strong connector in D for D_0 is equal to the maximum number of D_0-cuts in D such that no arc a of D is in more than $l(a)$ of these cuts.*

Proof. Directly from Theorem 57.3, since condition (57.2) is implied by the fact that D_0 is source-sink connected. ∎

The cardinality version is:

Corollary 57.3c. *Let $D_0 = (V, A_0)$ and $D = (V, A)$ be digraphs, with D_0 source-sink connected. Then the minimum size of a strong connector in D for D_0 is equal to the maximum number of disjoint D_0-cuts in D.*

Proof. This is the case $l = \mathbf{1}$ in Corollary 57.3b. ∎

57.3. Polyhedrally

Theorem 57.3 can be equivalently formulated in TDI terms:

Corollary 57.3d. *Let $D_0 = (V, A_0)$ and $D = (V, A)$ be digraphs satisfying* (57.2)*. Then the system*

$$\text{(57.5)} \quad \begin{array}{lll} \text{(i)} & x_a \geq 0 & a \in A, \\ \text{(ii)} & x(\delta^{\text{in}}_A(U)) \geq 1 & U \subset V, U \neq \emptyset, \delta^{\text{in}}_{A_0}(U) = \emptyset, \end{array}$$

is TDI and determines the convex hull of the strong connectors of D_0.

Proof. This is a reformulation of Theorem 57.3. ∎

In fact, system (57.5) is box-TDI, as will follow from Theorem 60.3.

By the theory of blocking polyhedra, Corollary 57.3d implies:

Corollary 57.3e. *Let $D_0 = (V, A_0)$ and $D = (V, A)$ satisfy* (57.2)*. Then the system*

$$\text{(57.6)} \quad \begin{array}{lll} \text{(i)} & 0 \leq x_a \leq 1 & a \in A, \\ \text{(ii)} & x(B) \geq 1 & B \text{ strong connector for } D_0 \end{array}$$

determines the convex hull of subsets of A containing a D_0-cut.

Proof. System (57.6) determines the blocking polyhedron of the polyhedron determined by (57.5), and hence its vertices are the incidence vectors of subsets of A that intersect all strong connectors for D_0. These are precisely the sets of arcs in A containing a D_0-cut. ∎

System (57.6) generally is not TDI, by Figure 56.1. But if D_0 is source-sink connected, system (57.6) is totally dual integral, as is shown in the following section.

57.4. Disjoint strong connectors

Similarly to the derivation of Theorem 57.3 from the Lucchesi-Younger theorem, the following generalization can be derived as a consequence of Theorem 56.1 (Schrijver [1982]):

Theorem 57.4. *Let $D_0 = (V, A_0)$ and $D = (V, A)$ be digraphs, with D_0 source-sink connected. Then the minimum size of a D_0-cut in D is equal to the maximum number of disjoint strong connectors in D for D_0.*

Proof. The proof is similar to the derivation of Theorem 57.3 from the Lucchesi-Younger theorem. We can assume that for any $u, v \in V$, if v is reachable in D_0 from u, then $(u, v) \in A_0$.

We show the theorem by induction on the number τ of arcs (u,v) of D for which $(v,u) \notin A_0$. If $\tau = 0$, the theorem is equivalent to Theorem 56.1 (by taking $C := \{(u,v) \mid (v,u) \in A\}$).

If $\tau > 0$, choose $(u,v) \in A$ with $(v,u) \notin A_0$. Let u' be a sink of D_0 with $(u,u') \in A_0$ and let v' be a source of D_0 with $(v',v) \in A_0$. As D_0 is source-sink connected we know that $(v',u') \in A_0$. Now introduce two new vertices, u'' and v'', and add arcs

$$(u,u''),(u'',u'),(v'',u''),(v'',v),(v',v'') \tag{57.7}$$

to A_0. Moreover, replace arc (u,v) of A by (u'',v''). Let $\widetilde{D}_0 = (\widetilde{V},\widetilde{A}_0)$ and $\widetilde{D} = (\widetilde{V},\widetilde{A})$ denote the modified graphs.

This transformation decreases τ by 1. Again (57.4) holds. This implies that the two numbers in the theorem are invariant under the transformation. Hence the theorem follows by induction. ∎

The condition given in this theorem cannot be relaxed to condition (57.2), as Figure 56.1 shows.

Theorem 57.4 has the following special cases:

- $s,t \in V$ and $A_0 := \{(u,v) \mid u = t$ or $v = s\}$: Menger's theorem (Corollary 9.1b);
- V is the disjoint union of U and W, $A_0 = \{(u,w) \mid u \in U, w \in W\}$ and $A \subseteq \{(w,u) \mid w \in W, u \in U\}$: Gupta's edge-colouring theorem (Theorem 20.5);
- $r \in V$ and $A_0 = \{(v,r) \mid v \in V\}$: Edmonds' disjoint arborescences theorem (Corollary 53.1b);
- V is the disjoint union of U and W and $A_0 = \{(u,w) \mid u \in U, w \in W\}$: disjoint bibranchings theorem(Theorem 54.11);
- $D_0 = (V,A_0)$ is source-sink connected and $A \subseteq \{(u,v) \mid (v,u) \in A_0\}$: Corollary 56.1b.

An equivalent capacitated version of Theorem 57.4 reads:

Corollary 57.4a. *Let $D_0 = (V,A_0)$ and $D = (V,A)$ be digraphs, with D_0 source-sink connected, and let $c \in \mathbb{Z}_+^A$ be a capacity function. Then the minimum capacity of a D_0-cut in D is equal to the maximum number of strong connectors in D for D_0 such that any arc a of D is in at most $c(a)$ of them.*

Proof. Directly from Theorem 57.4 by replacing any arc a of D by $c(a)$ parallel arcs. ∎

Equivalently, in TDI terms:

Corollary 57.4b. *Let $D_0 = (V,A_0)$ and $D = (V,A)$ be digraphs, with D_0 source-sink connected. Then system* (57.6) *is totally dual integral.*

Proof. This is a reformulation of Corollary 57.4a. ∎

57.5. Complexity

As for the complexity of finding disjoint strong connectors for a source-sink connected digraph, the proof of Theorem 57.4 gives a polynomial-time reduction to finding a maximum number of disjoint directed cut covers in a subset of the arcs of a source-sink connected graph. The latter problem is solvable in polynomial time by the methods of Section 56.2.

The capacitated case can be solved in semi-strongly polynomial time (that is, where rounding is taken as one arithmetic operation) with the ellipsoid method (cf. Grötschel, Lovász, and Schrijver [1988]). A combinatorial semi-strongly polynomial-time algorithm is as follows.

Let be given a source-sink connected digraph $D_0 = (V, A_0)$, a digraph $D = (V, A)$, and a capacity function $c : A \to \mathbb{Z}_+$. We show that an optimum *fractional* packing of strong connectors subject to c can be found in strongly polynomial time. Then an *integer* packing can be found by rounding (like in Section 51.4), thus yielding a semi-strongly polynomial-time algorithm.

Define $\mathcal{C} := \{U \mid \emptyset \neq U \subset V, d^{\text{in}}_{A_0}(U) = 0\}$. To find an optimum fractional packing, let κ be the minimum of $c(\delta^{\text{in}}_A(U))$ taken over sets $U \in \mathcal{C}$. (κ can be computed with a maximum flow algorithm.) We keep a subcollection $\mathcal{U}$ of $\mathcal{C}$ with $c(\delta^{\text{in}}_A(U)) = \kappa$ for each $U \in \mathcal{U}$.

Choose a strong connector $B \subseteq A$ for A_0 with $d^{\text{in}}_B(U) = 1$ for each $U \in \mathcal{U}$. (This can be found in strongly polynomial time, by finding a minimum-length strong connector for length function $l := \sum_{U \in \mathcal{U}} \chi^{\delta^{\text{in}}_B(U)}$. It exists by Theorem 57.4.)

If $c = \mathbf{0}$, we are done. If $c \neq \mathbf{0}$, determine a rational λ as follows. First set $\lambda := \min\{c(a) \mid a \in B\}$. Next, iteratively, find a $U \in \mathcal{C}$ minimizing

$$(57.8) \qquad (c - \lambda \cdot \chi^B)(\delta^{\text{in}}_A(U)).$$

If this minimum value is less than $\kappa - \lambda$, reset

$$(57.9) \qquad \lambda := \frac{c(\delta^{\text{in}}_A(U)) - \kappa}{d^{\text{in}}_B(U) - 1},$$

and iterate. If the minimum is equal to $\kappa - \lambda$, this ends the inner iterations. Then we reset $c := c - \lambda \cdot \chi^B$, $\kappa := \kappa - \lambda$, and $\mathcal{U} := \mathcal{U} \cup \{U\}$, and (outer) iterate.

In each outer iteration, the number of arcs a with $c(a) > 0$ decreases or the intersecting family generated by $\mathcal{U}$ increases (since for the U added we have $d^{\text{in}}_B(U) > 1$). Hence the number of outer iterations is bounded by $|A| + |V|^3$ (see the argument given in the proof of Theorem 53.9).

In each outer iteration, the number of inner iterations is at most $|B|$. To see this, consider any inner iteration, and denote by λ' and U' the objects λ and U in the next inner iteration. As U minimizes (57.8), we know

(57.10) $$(c - \lambda \cdot \chi^B)(\delta_A^{\text{in}}(U)) \leq (c - \lambda \cdot \chi^B)(\delta_A^{\text{in}}(U')).$$

Moreover, if the next iteration is not the last iteration, then

(57.11) $$(c - \lambda' \cdot \chi^B)(\delta_A^{\text{in}}(U')) < \kappa - \lambda' = (c - \lambda' \cdot \chi^B)(\delta_A^{\text{in}}(U))$$

(the equality follows from definition (57.9), replacing λ by λ'). Now (57.10) and (57.11) imply

(57.12) $$\lambda'(d_B^{\text{in}}(U) - d_B^{\text{in}}(U')) < c(\delta_A^{\text{in}}(U)) - c(\delta_A^{\text{in}}(U')) \\ \leq \lambda(d_B^{\text{in}}(U) - d_B^{\text{in}}(U')).$$

Since $\lambda' < \lambda$ (as (57.8) is less than $\kappa - \lambda$), we have $d_B^{\text{in}}(U') < d_B^{\text{in}}(U)$. Hence the number of inner iterations is at most $|B|$.

57.5a. Crossing families

Theorem 57.4 and part of Theorem 57.3 were generalized by Schrijver [1983b]. Let $\mathcal{C}$ be a *crossing family* of subsets of a set V; that is:

(57.13) if $U, W \in \mathcal{C}$ and $U \cap W \neq \emptyset$ and $U \cup W \neq V$, then $U \cap W \in \mathcal{C}$ and $U \cup W \in \mathcal{C}$.

Let $D = (V, A)$ be a digraph. Call $B \subseteq A$ a *$\mathcal{C}$-cut* if $B = \delta^{\text{in}}(U)$ for some $U \in \mathcal{C}$. Call $B \subseteq A$ a *$\mathcal{C}$-cover* if B intersects each $\mathcal{C}$-cut.

Let $\mathcal{C}$ be a crossing family of nonempty proper subsets of a set V. In Schrijver [1983b] it is shown that the following are equivalent:

(57.14)
(i) for each digraph $D = (V, A)$, the minimum size of a $\mathcal{C}$-cut is equal to the maximum number of disjoint $\mathcal{C}$-covers;
(ii) for each digraph $D = (V, A)$ and each length function $l : A \to \mathbb{Z}_+$, the minimum length of a $\mathcal{C}$-cover is equal to the maximum number of $\mathcal{C}$-cuts such that no arc a is in more than $l(a)$ of these cuts;
(iii) there are no V_1, V_2, V_3, V_4, V_5 in $\mathcal{C}$ with $V_1 \subseteq V_3 \subseteq V_5$, $V_1 \subseteq V_2$, $V_2 \cup V_3 = V$, $V_3 \cap V_4 = \emptyset$, and $V_4 \subseteq V_5$.

The configuration described in (iii) is depicted in Figure 57.1. As directed graphs may have parallel arcs, property (57.14)(i) is equivalent to its capacitated version. So condition (57.14)(i) is equivalent to the total dual integrality of

(57.15)
$x_a \geq 0$ for $a \in A$,
$x(B) \geq 1$ for each $\mathcal{C}$-cover $B \subseteq A$,

for each digraph (V, A). Similarly, condition (57.14)(ii) is equivalent to the total dual integrality of

(57.16)
$x_a \geq 0$ for $a \in A$,
$x(B) \geq 1$ for each $\mathcal{C}$-cut $B \subseteq A$,

for each digraph (V, A).

Frank [1979b] showed that (57.14)(i) holds if $\mathcal{C}$ is an intersecting family. (For any intersecting family $\mathcal{C}$, (iii) holds if $V \notin \mathcal{C}$, which we may assume without loss of generality.)

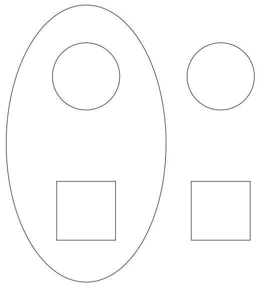

Figure 57.1
The configuration excluded in (57.14)(iii)**.** In this Venn-diagram, the collection is represented by the *interiors* of the ellipses and by the *exteriors* of the rectangles.

In (57.14)(i) and (ii) we require the min-max relation for cuts and covers to hold for *all* directed graphs on V. It is a more general problem to characterize pairs $(\mathcal{C}, D)$ of a crossing family $\mathcal{C}$ on V and a directed graph $D = (V, A)$ having the properties described in (57.14)(i) and (ii), respectively. For example, the Lucchesi-Younger theorem (Theorem 55.2), and its extension by Edmonds and Giles [1977], assert that if $\mathcal{C}$ is a crossing family on V and no arc of D leaves any set $U \in \mathcal{C}$, then $(\mathcal{C}, D)$ has the properties described in (ii). However, the example in Figure 56.1 shows that it generally does not have the property described in (57.14)(i). So for *fixed* graphs D, (57.14)(i) and (ii) are not equivalent.

Theorem 60.3 implies that a pair $(\mathcal{C}, D)$ has property (ii) if $\mathcal{C}$ is a crossing family and D a directed graph such that if $U_1, U_2, U_3 \in \mathcal{C}$ with $U_1 \subseteq V \setminus U_2 \subseteq U_3$, then no arc enters both U_1 and U_3. This generalizes the Lucchesi-Younger theorem.

We show the equivalence of (57.14)(ii) and (iii), for which we show a lemma indicating that condition (57.14)(iii) has a natural characterization in terms of total unimodularity.

For any collection $\mathcal{C}$ of subsets of a set V, let A be the collection of all ordered pairs of elements of V (making the complete directed graph $D = (V, A)$), and let $M_{\mathcal{C}}$ be the $\mathcal{C} \times A$ matrix with

(57.17) $$(M_{\mathcal{C}})_{U,a} := \begin{cases} 1 & \text{if } a \text{ enters } U, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 57.5α. *Let $\mathcal{C}$ be a cross-free collection of nonempty proper subsets of a set V. Then $M_{\mathcal{C}}$ is totally unimodular if and only if $\mathcal{C}$ satisfies* (57.14)(iii)*.*

Proof. To see necessity, let $M_{\mathcal{C}}$ be totally unimodular. Suppose that condition (57.14)(iii) is violated. So there exist V_1, V_2, V_3, V_4, V_5 in $\mathcal{C}$ with $V_1 \subseteq V_3 \subseteq V_5$, $V_1 \subseteq V_2$, $V_2 \cup V_3 = V$, $V_3 \cap V_4 = \emptyset$, and $V_4 \subseteq V_5$. Choose $v_1 \in V_1$, $v_2 \in V \setminus V_2$, $v_4 \in V_4$, and $v_5 \in V \setminus V_5$. Define

(57.18) $$A_0 := \{(v_2, v_1), (v_4, v_1), (v_2, v_4), (v_5, v_4), (v_5, v_2)\}$$

(cf. Figure 57.2). Consider the submatrix of $M_{\mathcal{C}}$ with rows indexed by $V_1, \ldots, V_5$, and columns indexed by the arcs in A_0. Then, as one easily checks:

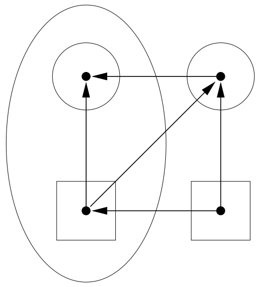

Figure 57.2

(57.19) each set in $\mathcal{C}_0$ is entered by exactly two arcs from A_0, and each arc in A_0 enters exactly two sets in $\mathcal{C}_0$.

So this submatrix has exactly two 1's in each row and each column, and hence is not totally unimodular.

To see sufficiency, let $\mathcal{C}$ satisfy (57.14)(iii). To prove that $M_{\mathcal{C}}$ is totally unimodular, we use the following characterization of Ghouila-Houri [1962b] (cf. Theorem 19.3 in Schrijver [1986b]): a matrix M is totally unimodular if and only if each collection R of rows of M can be partitioned into classes R_1 and R_2 such that the sum of the rows in R_1, minus the sum of the rows in R_2, is a vector with entries $0, \pm 1$ only.

To check this condition, we can assume that we have chosen all rows of $M_{\mathcal{C}}$ (as any subset of the rows gives a matrix of the same type as $M_{\mathcal{C}}$). Make a digraph $D = (\mathcal{C}, A')$, where A' consists of all pairs (T, U) from $\mathcal{C}$ such that

(57.20) $T \subset U$, and there is no $W \in \mathcal{C}$ with $T \subset W \subset U$.

We show that the undirected graph underlying D' is bipartite, which will verify Ghouila-Houri's criterion: let $\mathcal{C}_1$ and $\mathcal{C}_2$ be the two colour classes; then any arc $a = (u, v)$ of D enters a chain of subsets in $\mathcal{C}$ (as $\mathcal{C}$ is cross-free), which subsets are alternatingly in $\mathcal{C}_1$ and $\mathcal{C}_2$. Hence the sum of the rows with index in $\mathcal{C}_1$ minus the sum of the rows with index in $\mathcal{C}_2$, has an entry 0 or ± 1 in position a.

To show that D' is bipartite, suppose that it has an (undirected) circuit of odd length. Since this circuit is odd, and since D' is acyclic, it follows that there are distinct $U_0, U_1, \ldots, U_k, U_{k+1}$ in $\mathcal{C}$, with $k \geq 3$, such that

(57.21) $(U_1, U_0), (U_1, U_2), (U_2, U_3), \ldots, (U_{k-1}, U_k), (U_{k+1}, U_k)$

belong to A'. So U_0 and U_2 are distinct minimal sets in $\mathcal{C}$ containing U_1 as a subset. As $\mathcal{C}$ is cross-free, $U_0 \cup U_2 = V$. Similarly, U_{k-1} and U_{k+1} are distinct maximal subsets of U_k, and hence $U_{k-1} \cap U_{k+1} = \emptyset$. As $U_2 \subseteq U_{k-1}$, it follows that $U_1 \subseteq U_0 \cap U_2$, $U_0 \cup U_2 = V$, $U_2 \cup U_{k+1} \subseteq U_k$, and $U_2 \cap U_{k+1} = \emptyset$. This contradicts (57.14)(iii). ∎

This gives the box-TDI result:

Theorem 57.5. *Let $\mathcal{C}$ be a crossing family of nonempty proper subsets of a set V satisfying* (57.14)(iii) *and let $D = (V, A)$ be a digraph. Then the system*

(57.22) $\quad x_a \geq 0 \quad$ for $a \in A$,
$\quad x(\delta^{\text{in}}(U)) \geq 1 \quad$ for $U \in \mathcal{C}$,

is box-TDI.

Proof. Let $w : A \to \mathbb{R}_+$. Consider the maximum value of

(57.23) $$\sum_{U \in \mathcal{C}} y_U$$

where $y : \mathcal{C} \to \mathbb{R}_+$ satisfies

(57.24) $$\sum_{U \in \mathcal{C}} y_U \chi^{\delta^{\text{in}}(U)} \leq w.$$

Choose $y : \mathcal{C} \to \mathbb{R}_+$ attaining the maximum, such that

(57.25) $$\sum_{U \in \mathcal{C}} y_U |U||V \setminus U|$$

is minimized. We show that the collection $\mathcal{F} := \{U \in \mathcal{C} \mid y_U > 0\}$ is cross-free; that is, for all $T, U \in \mathcal{F}$ one has

(57.26) $\quad T \subseteq U$ or $U \subseteq T$ or $T \cap U = \emptyset$ or $T \cup U = V$.

Suppose that this is not true. Let $\alpha := \min\{y_T, y_U\}$. Decrease y_T and y_U by α, and increase $y_{T\cap U}$ and $y_{T \cup U}$ by α. Now (57.24) is maintained, and (57.23) did not change. However, (57.25) decreases (Theorem 2.1), contradicting our minimality assumption.

So $\mathcal{F}$ is cross-free. As $M_{\mathcal{F}}$ is totally unimodular by Lemma 57.5α, this gives the box-total dual integrality of (57.22) by Theorem 5.35. ∎

Condition (57.14)(iii) is necessary and sufficient for integrality of the polyhedron:

Corollary 57.5a. *For any crossing family $\mathcal{C}$ of nonempty proper subsets of a set V, (57.22) defines an integer polyhedron for each digraph $D = (V, A)$ if and only if condition (57.14)(iii) holds.*

Proof. Sufficiency follows from Theorem 57.5. To see necessity, suppose that (57.22)(iii) does not hold. Let $V_1, \ldots, V_5$ in $\mathcal{C}$ with $V_1 \subseteq V_3 \subseteq V_5$, $V_1 \subseteq V_2$, $V_2 \cup V_3 = V$, $V_3 \cap V_4 = \emptyset$, and $V_4 \subseteq V_5$. Let $\mathcal{C}_0 := \{V_1, \ldots, V_5\}$ and $\mathcal{C}_1 := \mathcal{C} \setminus \mathcal{C}_0$. Choose $v_1 \in V_1$, $v_2 \in V \setminus V_2$, $v_4 \in V_4$, $v_5 \in V \setminus V_5$. Let $D = (V, A)$ be a digraph, with $A = A_0 \cup A_1$, where A_0 is as defined in (57.18) and where

(57.27) $\quad A_1 := \{(u, v) \mid u, v \in V$ such that (u, v) enters no V_i $(i = 1, \ldots, 5)\}$.

Then

(57.28) each set in $\mathcal{C}_1$ is either entered by at least one arc in A_1 or by at least two arcs in A_0.

To see this, by definition of A_1, a subset U of V is entered by no arc in A_1 if and only if U belongs to the lattice generated by $\mathcal{C}_0$ (with respect to inclusion). This lattice consists of the sets

(57.29) $\emptyset, V, V_1, \ldots, V_5, V_1 \cup V_4, V_2 \cap V_3, (V_2 \cap V_3) \cup V_4, V_3 \cup V_4, V_2 \cap V_5,$

as (57.29) is closed under taking unions and intersections, and as each set in (57.29) is generated by taking unions and intersections from $\mathcal{C}_0$. Since each of the sets in (57.29), except $\emptyset$ and V, is entered by at least two arcs in A_0, we have (57.28).

(57.19) and (57.28) give:

(57.30) any $\mathcal{C}$-cover in A contains at least three arcs in A_0, and any $\mathcal{C}$-cut contains at least one arc in A_1 or at least two arcs in A_0.

Define $x : A \to \mathbb{Q}$ be $x := \chi^{A_1} + \frac{1}{2}\chi^{A_0}$ and a length function $l : A \to \mathbb{Z}$ by $l := \chi^{A_0}$. Then x satisfies (57.22) and $l^{\mathsf{T}}x = \frac{5}{2}$. However, $l(C) \geq 3$ for each $\mathcal{C}$-cover C. So (57.22) determines no integer polyhedron. ∎

Theorem 57.5 and Corollary 57.5a imply the equivalence of (57.14)(ii) and (iii). For the proof of the equivalence of (57.14)(i) and (iii), we refer to Schrijver [1983b].

Chapter 58

The traveling salesman problem

The traveling salesman problem (TSP) asks for a shortest Hamiltonian circuit in a graph. It belongs to the most seductive problems in combinatorial optimization, thanks to a blend of complexity, applicability, and appeal to imagination.

The problem shows up in practice not only in routing but also in various other applications like machine scheduling (ordering jobs), clustering, computer wiring, and curve reconstruction.

The traveling salesman problem is an NP-complete problem, and no polynomial-time algorithm is known. As such, the problem would not fit in the scope of the present book. However, the TSP is closely related to several of the problem areas discussed before, like 2-matching, spanning tree, and cutting planes, which areas actually were stimulated by questions prompted by the TSP, and often provide subroutines in solving the TSP.

Being NP-complete, the TSP has served as prototype for the development and improvement of advanced computational methods, to a large extent utilizing polyhedral techniques. The basis of the solution techniques for the TSP is branch-and-bound, for which good bounding techniques are essential. Here 'good' is determined by two, often conflicting, criteria: the bound should be *tight* and *fast* to compute. Polyhedral bounds turn out to be good candidates for such bounds.

58.1. The traveling salesman problem

Given a graph $G = (V, E)$, a *Hamiltonian circuit* in G is a circuit C with $VC = V$. The *symmetric traveling salesman problem* (TSP) is: given a graph $G = (V, E)$ and a length function $l : E \to \mathbb{R}$, find a Hamiltonian circuit C of minimum length.

The directed version is as follows. Given a digraph $D = (V, A)$, a *directed Hamiltonian circuit*, or just a *Hamiltonian circuit*, in D is a directed circuit C with $VC = V$. The *asymmetric traveling salesman problem* (TSP or ATSP) is: given a digraph $D = (V, A)$ and a length function $l : A \to \mathbb{R}$, find a Hamiltonian circuit C of minimum length.

In the context of the traveling salesman problem, vertices are sometimes called *cities*, and a Hamiltonian circuit a *traveling salesman tour*. If the vertices are represented by points in the plane and each pair of vertices is connected by an edge of length equal to the Euclidean distance between the two points, one speaks of the *Euclidean traveling salesman problem*.

58.2. NP-completeness of the TSP

The problem of finding a Hamiltonian circuit and (hence) the traveling salesman problem are NP-complete. Indeed, in Theorem 8.11 and Corollary 8.11b we proved the NP-completeness of the directed and undirected Hamiltonian circuit problem. This implies the NP-completeness of the TSP, both in the undirected and the directed case:

Theorem 58.1. *The symmetric TSP and the asymmetric TSP are* NP-*complete.*

Proof. Given an undirected graph $G = (V, E)$, define $l(e) := 0$ for each edge e. Then G has a Hamiltonian circuit if and only if G has a Hamiltonian circuit of length ≤ 0. This reduces the undirected Hamiltonian circuit problem to the symmetric TSP.

One similarly shows the NP-completeness of the asymmetric TSP. ∎

This method also gives that the symmetric TSP remains NP-complete if the graph is complete and the length function satisfies the *triangle inequality*:

(58.1) $$l(uw) \leq l(uv) + l(vw) \text{ for all } u, v, w \in V.$$

Indeed, to test if a graph $G = (V, E)$ has a Hamiltonian circuit, define $l(uv) := 1$ if u and v are adjacent and $l(uv) := 2$ otherwise (for $u \neq v$). Then G has a Hamiltonian circuit if and only if there exists a traveling salesman tour of length $\leq |V|$.

Garey, Graham, and Johnson [1976] and Papadimitriou [1977a] showed that even the Euclidean traveling salesman problem is NP-complete. (Similarly for several other metrics, like l_1.) More on complexity can be found in Section 58.8b below.

58.3. Branch-and-bound techniques

The traveling salesman problem is NP-complete, and no polynomial-time algorithm is known. Most exact methods known are essentially enumerative, aiming at minimizing the enumeration. A general framework is that of *branch-and-bound*. The idea of branch-and-bound applied to the traveling salesman problem roots in papers of Tompkins [1956], Rossman and Twery [1958], and Eastman [1959]. The term ‘branch and bound’ was introduced by Little, Murty, Sweeney, and Karel [1963].

A rough, elementary description is as follows. Let $G = (V, E)$ be a graph and let $l : E \to \mathbb{R}$ be a length function. For any set $\mathcal{C}$ of Hamiltonian circuits, let $\mu(\mathcal{C})$ denote the minimum length of the Hamiltonian circuits in $\mathcal{C}$.

Keep a collection Γ of sets of Hamiltonian circuits and a function $\lambda : \Gamma \to \mathbb{R}$ satisfying:

(58.2) (i) $\bigcup \Gamma$ contains a shortest Hamiltonian circuit;
(ii) $\lambda(\mathcal{C}) \leq \mu(\mathcal{C})$ for each $\mathcal{C} \in \Gamma$.

A typical iteration is:

(58.3) Select a collection $\mathcal{C} \in \Gamma$ with $\lambda(\mathcal{C})$ minimal. Either find a circuit $C \in \mathcal{C}$ with $l(C) = \lambda(\mathcal{C})$ or replace $\mathcal{C}$ by (zero or more) smaller sets such that (58.2) is maintained.

Obviously, if we find $C \in \mathcal{C}$ with $l(C) = \lambda(\mathcal{C})$, then C is a shortest Hamiltonian circuit.

This method always terminates, but the method and its efficiency heavily depend on how the details in this framework are filled in: how to bound (that is, how to define and calculate $\lambda(\mathcal{C})$), how to branch (that is, which smaller sets replace $\mathcal{C}$), and how to find the circuit C.

As for branching, the classes $\mathcal{C}$ in Γ can be stored implicitly: for example, by prescribing sets B and F of edges such that $\mathcal{C}$ consists of all Hamiltonian circuits whose edge set contains B and is disjoint from F. Then we can split $\mathcal{C}$ by selecting an edge $e \in E \setminus (B \cup F)$ and replacing $\mathcal{C}$ by the classes determined by $B \cup \{e\}, F$ and by $B, F \cup \{e\}$ respectively.

As for bounding, one should choose $\lambda(\mathcal{C})$ that is fast to compute and close to $\mu(\mathcal{C})$. For this, polyhedral bounds seem good candidates, and in the coming sections we consider a number of them.

For finding the circuit $C \in \mathcal{C}$, a heuristic or exact method can be used. If it returns a circuit C with $l(C) > \lambda(\mathcal{C})$, we can delete all sets $\mathcal{C}'$ from Γ with $\lambda(\mathcal{C}') \geq l(C)$, thus saving computer space.

58.4. The symmetric traveling salesman polytope

The *(symmetric) traveling salesman polytope* of an undirected graph $G = (V, E)$ is the convex hull of the incidence vectors (in $\mathbb{R}^E$) of the Hamiltonian circuits. The TSP is equivalent to minimizing a function $l^{\mathsf{T}}x$ over the traveling salesman polytope. Hence this is NP-complete.

The NP-completeness of the TSP also implies that, unless NP=co-NP, no description in terms of inequalities of the traveling salesman polytope may be expected (Corollary 5.16a). In fact, as deciding if a Hamiltonian circuit exists is NP-complete, it is NP-complete to decide if the traveling salesman polytope is nonempty. Hence, if NP$\neq$co-NP, there exist no inequalities satisfied by the traveling salesman polytope such that their validity can be certified in polynomial time and such that they have no common solution.

58.5. The subtour elimination constraints

Polynomial-time computable lower bounds on the minimum length of a Hamiltonian circuit can be obtained by including the traveling salesman polytope in a larger polytope (a *relaxation*) over which $l^{\mathsf{T}}x$ can be minimized in polynomial time.

Dantzig, Fulkerson, and Johnson [1954a,1954b] proposed the following relaxation:

$$\text{(58.4)} \qquad \begin{array}{rll} \text{(i)} & 0 \leq x_e \leq 1 & \text{for each edge } e, \\ \text{(ii)} & x(\delta(v)) = 2 & \text{for each vertex } v, \\ \text{(iii)} & x(\delta(U)) \geq 2 & \text{for each } U \subseteq V \text{ with } \emptyset \neq U \neq V. \end{array}$$

The integer solutions of (58.4) are precisely the incidence vectors of the Hamiltonian circuits. If (ii) holds, then (iii) is equivalent to:

$$\text{(58.5)} \qquad \text{(iii')} \; x(E[U]) \leq |U| - 1 \text{ for each } U \subseteq V \text{ with } \emptyset \neq U \neq V.$$

These conditions are called the *subtour elimination constraints*.

It can be shown with the ellipsoid method that the minimum of $l^{\mathsf{T}}x$ over (58.4) can be found in strongly polynomial time (cf. Theorem 5.10). For this it suffices to show that the conditions (58.4) can be tested in polynomial time. This is easy for (i) and (ii). If (i) and (ii) are satisfied, we can test (iii) by taking x as capacity function, and test if there is a cut $\delta(U)$ of capacity less than 2, with $\emptyset \neq U \neq V$.

No combinatorial polynomial-time algorithm is known to minimize $l^{\mathsf{T}}x$ over (58.4). In practice, one can apply the simplex method to minimize $l^{\mathsf{T}}x$ over the constraints (i) and (ii), test if the solution satisfies (iii) by finding a cut $\delta(U)$ minimizing $x(\delta(U))$. If this cut has capacity at least 2, then x minimizes $l^{\mathsf{T}}x$ over (58.4). Otherwise, we can add the constraint $x(\delta(U)) \geq 2$ to the simplex tableau (a *cutting plane*), and iterate. (This method is implicit in Dantzig, Fulkerson, and Johnson [1954b].)

Branch-and-bound methods that incorporate such a cutting plane method to obtain bounds and that extend the cutting plane found to all other nodes of the branching tree to improve their bounds, are called *branch-and-cut*.

System (58.4) generally is not enough to determine the traveling salesman polytope: for the Petersen graph $G = (V, E)$, the vector x with $x_e = \frac{2}{3}$ for each $e \in E$ satisfies (58.4) but is not in the traveling salesman polytope of G (as it is empty).

Wolsey [1980] (also Shmoys and Williamson [1990]) showed that if G is complete and the length function l satisfies the triangle inequality, then the minimum of $l^{\mathsf{T}}x$ over (58.4) is at least $\frac{2}{3}$ times the minimum length of a Hamiltonian circuit. It is conjectured (cf. Carr and Vempala [2000]) that for any length function, a lower bound of $\frac{3}{4}$ holds (which is best possible). Related results are given by Papadimitriou and Vempala [2000] and Boyd and Labonté [2002] (who verified the conjecture for $n \leq 10$).

Maurras [1975] and Grötschel and Padberg [1979b] showed that, if G is the complete graph on V and $2 \leq |U| \leq |V|-2$, then the subtour elimination constraint (58.4)(iii) determines a facet of the traveling salesman polytope.

Chvátal [1989] showed the NP-completeness of recognizing if the bound given by the subtour elimination constraints is equal to the length of a shortest tour. He also showed that there is no nontrivial upper bound on the relative error of this bound.

58.6. 1-trees and Lagrangean relaxation

Held and Karp [1971] gave a method to find the minimum value of $l^{\mathsf{T}}x$ over (58.4), with the help of 1-*trees* and *Lagrangean relaxation*.

Let $G = (V, E)$ be a graph and fix a vertex, say 1, of G. A 1-*tree* is a subset F of E such that $|F \cap \delta(1)| = 2$ and such that $F \setminus \delta(1)$ forms a spanning tree on $V \setminus \{1\}$. So each Hamiltonian circuit is a 1-tree with all degrees equal to 2.

It is easy to find a shortest 1-tree F, as it consists of a shortest spanning tree of the graph $G - 1$, joined with the two shortest edges incident with vertex 1. Corollary 50.7c implies that the convex hull of the incidence vectors of 1-trees is given by:

(58.6)
$$
\begin{array}{rll}
\text{(i)} & 0 \leq x_e \leq 1 & \text{for each } e \in E,\\
\text{(ii)} & x(\delta(1)) = 2, & \\
\text{(iii)} & x(E[U]) \leq |U| - 1 & \text{for each nonempty } U \subseteq V \setminus \{1\},\\
\text{(iv)} & x(E) = |V|. &
\end{array}
$$

Then (58.4) is equivalent to (58.6) added with (58.4)(ii).

The Lagrangean relaxation approach to find the minimum of $l^{\mathsf{T}}x$ over (58.4) is based on the following result. For any $y \in \mathbb{R}^V$ define

(58.7) $$l_y(e) := l(e) - y_u - y_v$$

for $e = uv \in E$, and define

(58.8) $$f(y) := 2y(V) + \min_F l_y(F),$$

where F ranges over all 1-trees. Christofides [1970] and Held and Karp [1970] observed that for each $y \in \mathbb{R}^V$:

(58.9) $$f(y) \leq \text{the minimum length of a Hamiltonian circuit,}$$

since if C is a shortest Hamiltonian circuit, then $f(y) \leq 2y(V)+l_y(C) = l(C)$.

The function f is concave. Since a shortest 1-tree can be found fast, also $f(y)$ can be computed fast. Held and Karp [1970] showed:

Theorem 58.2. *The minimum value of* $l^{\mathsf{T}}x$ *over* (58.4) *is equal to the maximum value of* $f(y)$ *over* $y \in \mathbb{R}^V$.

Proof. This follows from general linear programming theory. Let $Ax = b$ be system (58.4)(ii) and let $Cx \geq d$ be system (58.6). As (58.4) is equivalent to $Ax = b$, $Cx \geq d$, we have, using LP-duality:

(58.10)
$$\begin{aligned}\min_{\substack{Ax=b\\ Cx\geq d}} l^{\mathsf{T}}x &= \max_{\substack{y,z\\ z\geq \mathbf{0}\\ y^{\mathsf{T}}A+z^{\mathsf{T}}C=l^{\mathsf{T}}}} y^{\mathsf{T}}b + z^{\mathsf{T}}d\\ &= \max_y (y^{\mathsf{T}}b + \max_{\substack{z\geq\mathbf{0}\\ z^{\mathsf{T}}C=l^{\mathsf{T}}-y^{\mathsf{T}}A}} z^{\mathsf{T}}d) = \max_y \big(y^{\mathsf{T}}b + \min_{Cx\geq d}(l^{\mathsf{T}} - y^{\mathsf{T}}A)x\big)\\ &= \max_y f(y).\end{aligned}$$

The last inequality holds as $Cx \geq d$ determines the convex hull of the incidence vectors of 1-trees. ∎

This translates the problem of minimizing $l^{\mathsf{T}}x$ over (58.4) to finding the maximum of the concave function f. We can find this maximum with a subgradient method (cf. Chapter 24.3 of Schrijver [1986b]). The vector y (the *Lagrangean multipliers*) can be used as a correction mechanism to urge the 1-tree to have degree 2 at each vertex. That is, if we calculate $f(y)$, and see that the 1-tree F minimizing $l_y(F)$ has degree more than 2 at a vertex v, we can increase l_y on $\delta(v)$ by decreasing y_v. Similarly, if the degree is less than 2, we can increase y_v. This method was proposed by Held and Karp [1970,1971].

The advantage of this approach is that one need not implement a linear programming algorithm with a constraint generation technique, but that instead it suffices to apply the more elementary tools of finding a shortest 1-tree and updating y. More can be found in Jünger, Reinelt, and Rinaldi [1995].

58.7. The 2-factor constraints

A strengthening of relaxation (58.4) is obtained by using the facts that each Hamiltonian circuit is a 2-factor and that the convex hull of the incidence vectors of 2-factors is known (Corollary 30.8a) (this idea goes back to Robinson [1949] for the asymmetric TSP and Bellmore and Malone [1971] for the symmetric TSP, and was used for the symmetric TSP by Grötschel [1977a] and Pulleyblank [1979b]):

(58.11)
$$\begin{aligned}&\text{(i)}\ 0 \leq x_e \leq 1 && \text{for each edge } e,\\ &\text{(ii)}\ x(\delta(v)) = 2 && \text{for each vertex } v,\\ &\text{(iii)}\ x(\delta(U)) \geq 2 && \text{for each } U \subseteq V \text{ with } \emptyset \neq U \neq V,\\ &\text{(iv)}\ x(\delta(U)\setminus F) - x(F) \geq 1 - |F| &&\\ &&& \text{for } U \subseteq V,\ F \subseteq \delta(U),\ F \text{ matching},\ |F| \text{ odd}.\end{aligned}$$

Since a minimum-length 2-factor can be found in polynomial time, the inequalities (i), (ii), and (iv) can be tested in polynomial time (cf. Theorem 32.5). Hence the minimum of $l^{\mathsf{T}}x$ over (58.11) can be found in strongly polynomial time.

System (58.11) generally is not enough to determine the traveling salesman polytope, as can be seen, by taking the Petersen graph $G = (V, E)$ and $x_e := \frac{2}{3}$ for each edge e.

Grötschel and Padberg [1979b] showed that, for complete graphs, each of the inequalities (58.11)(iv) determines a facet of the traveling salesman polytope (if $|F| \geq 3$). Boyd and Pulleyblank [1991] studied optimization over (58.11).

58.8. The clique tree inequalities

Grötschel and Pulleyblank [1986] found a large class of facet-inducing inequalities, the 'clique tree inequalities', that generalize the 'comb inequalities' (see below), which generalize both the subtour elimination constraints (58.4)(iii) and the 2-factor constraints (58.11)(iv). However, no polynomial-time test of clique tree inequalities is known.

A *clique tree inequality* is given by:

(58.12)
$$\sum_{i=1}^{r} x(\delta(H_i)) + \sum_{j=1}^{s} x(\delta(T_j)) \geq 2r + 3s - 1,$$

where $H_1, \ldots, H_r$ are pairwise disjoint subsets of V and $T_1, \ldots, T_s$ are pairwise disjoint proper subsets of V such that

(58.13)
(i) no T_j is contained in $H_1 \cup \cdots \cup H_r$,
(ii) each H_i intersects an odd number of the T_j,
(iii) the intersection graph of $H_1, \ldots, H_r, T_1, \ldots, T_s$ is a tree.

(Here, the *intersection graph* is the graph with vertices $H_1, \ldots, H_r, T_1, \ldots, T_s$, two of them being adjacent if and only if they intersect. Each H_i is called a *handle* and each T_j a *tooth*.)

Theorem 58.3. *The clique tree inequality* (58.12) *is valid for the traveling salesman polytope.*

Proof. It suffices to show that each Hamiltonian circuit C satisfies:

(58.14)
$$\sum_{i=1}^{r} d_C(H_i) + \sum_{j=1}^{s} d_C(T_j) \geq 2r + 3s - 1.$$

We apply induction on r, the case $r = 0$ being easy (as it implies $s = 1$). For each $i = 1, \ldots, r$, let β_i be the number of T_j intersecting H_i.

If there is an i with $d_C(H_i) \geq \beta_i$, say $i = 1$, then, by parity, $d_C(H_1) \geq \beta_1 + 1$. The sets $H_2, \ldots, H_r, T_1, \ldots, T_s$ fall apart into β_1 collections of type (58.13), to which we can apply induction. Adding up the inequalities obtained, we get:

$$\text{(58.15)} \qquad \sum_{i=2}^{r} d_C(H_i) + \sum_{j=1}^{s} d_C(T_j) \geq 2(r-1) + 3s - \beta_1.$$

Then (58.14) follows, as $d_C(H_1) \geq \beta_1 + 1$.

So we can assume that $d_C(H_i) \leq \beta_i - 1$ for each i. For all i, j, let $\alpha_{i,j} := 1$ if $T_j \cap H_i \neq \emptyset$ and C has no edge connecting $T_j \cap H_i$ and $T_j \setminus H_i$, and let $\alpha_{i,j} := 0$ otherwise. Then

$$\text{(58.16)} \qquad d_C(T_j) \geq 2 + 2\sum_{i=1}^{r} \alpha_{i,j},$$

since C restricted to T_j falls apart into at least $1 + \sum_{i=1}^{r} \alpha_{i,j}$ components (using (58.13)(i)).

Moreover, for each $i = 1, \ldots, r$, there exist at least $\beta_i - d_C(H_i)$ indices j with $\alpha_{i,j} = 1$. Hence

$$\begin{aligned}\text{(58.17)} \qquad \sum_{j=1}^{s} d_C(T_j) &\geq 2s + 2\sum_{i=1}^{r}\sum_{j=1}^{s} \alpha_{i,j} \geq 2s + 2\sum_{i=1}^{r}(\beta_i - d_C(H_i)) \\ &\geq 2s + r + \sum_{i=1}^{r}(\beta_i - d_C(H_i)) = 2r + 3s - 1 - \sum_{i=1}^{r} d_C(H_i),\end{aligned}$$

since $\sum_{i=1}^{r} \beta_i = r + s - 1$, as the intersection graph of the H_i and the T_j is a tree with $r + s$ vertices, and hence with $r + s - 1$ edges.

(58.17) implies (58.14). ∎

Notes. Grötschel and Pulleyblank [1986] also showed that, if G is a complete graph, then any clique tree inequality determines a facet if and only if each H_i intersects at least three of the T_j.

The clique tree inequalities are not enough to determine the traveling salesman polytope, as is shown again by taking the Petersen graph $G = (V, E)$ and $x_e := \frac{2}{3}$ for all $e \in E$.

The special case $r = 1$ of the clique tree inequality is called a *comb inequality*, and was introduced by Grötschel and Padberg [1979a] and proved to be facet-inducing (if G is complete and $s \geq 3$) by Grötschel and Padberg [1979b].

The special case of the comb inequality with $|H_1 \cap T_j| = 1$ for all $j = 1, \ldots, s$ is called a *Chvátal comb inequality*, introduced by Chvátal [1973b]. The special case of the Chvátal comb inequalities with $|T_j| = 2$ for each $j = 1, \ldots, s$ gives the 2-factor constraints (58.11)(iv) (since $2x(F) + \sum_{f \in F} x(\delta(f)) = 4|F|$).

No polynomial-time algorithm is know to test the clique tree inequalities, or the comb inequalities, or the Chvátal comb inequalities. Carr [1995,1997] showed that for each constant K, there is a polynomial-time algorithm to test the clique tree inequalities with at most K teeth and handles. (This can be done by first fixing intersection points of the $H_i \cap T_j$ (if nonempty) and points in $T_j \setminus (H_1 \cup \cdots \cup H_r)$, and next finding minimum-capacity cuts separating the appropriate sets of these points (taking x as capacity function). We can make them disjoint where necessary by the usual uncrossing techniques. As K is fixed, the number of vertices to be chosen is also bounded by a polynomial in $|V|$.)

Letchford [2000] gave a polynomial-time algorithm for testing a superclass of the comb inequalities in planar graphs. Related results are given in Carr [1996], Fleischer and Tardos [1996,1999], Letchford and Lodi [2002], and Naddef and Thienel [2002a,2002b].

58.8a. Christofides' heuristic for the TSP

Christofides [1976] designed the following algorithm to find a short Hamiltonian circuit in a complete graph $G = (V, E)$ (generally not the shortest however). It assumes a nonnegative length function l satisfying the following *triangle inequality*:

(58.18) $\qquad l(uw) \leq l(uv) + l(vw)$

for all $u, v, w \in V$.

First determine a shortest spanning tree T (with the greedy algorithm). Next, let U be the set of vertices that have odd degree in T. Find a shortest perfect matching M on U. Now $ET \cup M$ forms a set of edges such that each vertex has even degree. (If an edge occurs both in ET and in M, we take it as two parallel edges.) So we can make a closed path C such that each edge in $ET \cup M$ is traversed exactly once. Then C traverses each vertex at least once. By shortcutting we obtain a Hamiltonian circuit C' with $l(C') \leq l(C)$.

How far away is the length of C' from the minimum length μ of a Hamiltonian circuit?

Theorem 58.4. $l(C') \leq \frac{3}{2}\mu$.

Proof. Let C'' be a shortest Hamiltonian circuit. Then $l(T) \leq l(C'') = \mu$, since C'' contains a spanning tree. Also, $l(M) \leq \frac{1}{2}l(C'') = \frac{1}{2}\mu$, since we can split C'' into two collections of paths, each having U as set of end vertices. They give two perfect matchings on U, of total length at most $l(C'')$ (by the triangle inequality (58.18)). Hence one of these matchings has length at most $\frac{1}{2}l(C'')$. So $l(M) \leq \frac{1}{2}l(C'') = \frac{1}{2}\mu$.

Combining the two inequalities, we obtain

(58.19) $\qquad l(C') \leq l(C) = l(T) + l(M) \leq \frac{3}{2}\mu,$

which proves the theorem. ∎

The factor $\frac{3}{2}$ seems quite large, but it is the smallest factor for which a polynomial-time method is known. Don't forget moreover that it is a *worst-case* bound, and that in practice (or on average) the algorithm might have a much better performance.

Wolsey [1980] showed more strongly that (if l satisfies the triangle inequality) the length of the tour found by Christofides' algorithm, is at most $\frac{3}{2}$ times the lower bound based on the subtour elimination constraints (58.4). If all distances are 1 or 2, Papadimitriou and Yannakakis [1993] gave a polynomial-time algorithm with worst-case factor $\frac{7}{6}$. Hoogeveen [1991] analyzed the behaviour of Christofides' heuristic when applied to finding shortest Hamiltonian paths.

58.8b. Further notes on the symmetric traveling salesman problem

Adjacency of vertices of the symmetric traveling salesman polytope of a graph $G = (V, E)$ is co-NP-complete, as was shown by Papadimitriou [1978].

Norman [1955] remarked that the symmetric traveling salesman polytope of the complete graph K_n has dimension $\frac{1}{2}n(n-3) = \binom{n}{2} - n$ (if $n \geq 3$). Proofs were given by Maurras [1975] and Grötschel and Padberg [1979a].

The symmetric traveling salesman polytopes of K_n for small n were studied by Norman [1955], Boyd and Cunningham [1991], Christof, Jünger, and Reinelt [1991] ($n = 8$), and Naddef and Rinaldi [1992,1993]. Weinberger [1974a] showed that the up hull of the symmetric traveling salesman polytope of K_6 is not determined by inequalities with $0, 1$ coefficients only.

Rispoli and Cosares [1998] showed that the diameter of the symmetric traveling salesman polytope of a complete graph is at most 4. Grötschel and Padberg [1985] conjecture that it is at most 2. (See Sierksma and Tijssen [1992] and Sierksma, Teunter, and Tijssen [1995] for supporting results.) Further work on the symmetric traveling salesman polytope includes Naddef and Rinaldi [1993], Queyranne and Wang [1993], Carr [2000], Cook and Dash [2001], and Naddef and Pochet [2001].

Rispoli [1998] showed that the monotonic diameter of the symmetric traveling salesman polytope of K_n is $\lfloor n/2 \rfloor - 1$ if $n \geq 6$. (The *monotonic diameter* of a polytope is the minimum λ such that for each linear function $l^{\mathsf{T}}x$ and each pair of vertices y, z such that $l^{\mathsf{T}}x$ is maximized over P at z, there is a $y - z$ path along vertices and edges of the polytope such that the function $l^{\mathsf{T}}x$ is monotonically nondecreasing and such that the number of edges in the path is at most λ.)

Sahni and Gonzalez [1976] showed that for any constant c, unless P=NP, there is no polynomial-time algorithm finding a Hamiltonian circuit of length at most c times the minimum length of a Hamiltonian circuit. Johnson and Papadimitriou [1985a] showed that unless P=NP there is no *fully polynomial approximation scheme* for the Euclidean traveling salesman problem (that is, there is no algorithm that gives for any $\varepsilon > 0$, a Hamiltonian circuit of length at most $1+\varepsilon$ times the minimum length of a Hamiltonian circuit, with running time bounded by a polynomial in the size of the problem and in $1/\varepsilon$).

However, Arora [1996,1997,1998] showed that for the Euclidean TSP there is a polynomial approximation scheme: there is an algorithm that gives, for any n vertices in the plane and any $\varepsilon > 0$, a Hamiltonian circuit of length at most $1 + \varepsilon$ times the minimum length of a Hamiltonian circuit, in $n^{O(1/\varepsilon)}$ time. The method also applies to several other metrics. Mitchell [1999] noticed that the methods of Mitchell [1996] imply similar results. Related work is reported in Trevisan [1997, 2000], Rao and Smith [1998], and Dumitrescu and Mitchell [2001]. Earlier work on plane TSP includes Karp [1977], Steele [1981], Moran [1984], Karloff [1989], and Clarkson [1991].

A polynomial-time approximation scheme for the traveling salesman problem where the length is determined by the shortest path metric in a weighted planar graph was given by Arora, Grigni, Karger, Klein, and Woloszyn [1998] (extending the unweighted case proved by Grigni, Koutsoupias, and Papadimitriou [1995]).

Yannakakis [1988,1991] showed that the traveling salesman problem on K_n cannot be expressed by a linear program of polynomial size that is invariant under the symmetric group on K_n. (A similar negative result was proved by Yannakakis for the perfect matching polytope.)

More valid inequalities for the symmetric traveling salesman polytope were given by Grötschel [1980a], Papadimitriou and Yannakakis [1984], Fleischmann [1988], Boyd and Cunningham [1991], Naddef [1992], Naddef and Rinaldi [1992], and Boyd, Cunningham, Queyranne, and Wang [1995].

Jünger, Reinelt, and Rinaldi [1995] gave a comparison of the values of various relaxations for several instances of the symmetric traveling salesman problem. Johnson, McGeoch, and Rothberg [1996] report on an 'asymptotic experimental analysis' of the Held-Karp bound. A probabilistic analysis of the Held-Karp bound for the Euclidean TSP was presented by Goemans and Bertsimas [1991].

A worst-case comparison of several classes of valid inequalities for the traveling salesman polytope was given by Goemans [1995]. Several integer programming formulations for the TSP were compared by Langevin, Soumis, and Desrosiers [1990]. Althaus and Mehlhorn [2000,2001] showed that the subtour elimination constraints solve traveling salesman problems coming from curve reconstruction, under appropriate sampling conditions.

Semidefinite programming was applied to the symmetric TSP by Cvetković, Čangalović, and Kovačević-Vujčić [1999a,1999b] and Iyengar and Çezik [2001].

Let $G = (V, E)$ be an undirected graph. The symmetric traveling salesman polytope of G is a face of the convex hull of all integer solutions of

(58.20) (i) $x_e \geq 0$ for each $e \in E$,
(ii) $x(\delta(U)) \geq 2$ for each $U \subseteq V$ with $\emptyset \neq U \neq V$.

Fonlupt and Naddef [1992] characterized for which graphs G each vertex x of (58.20) is integer and has $x(\delta(v)) \equiv 0 \pmod 2$ for each vertex v of G.

Grötschel [1980a] studied the *monotone traveling salesman polytope* of a graph, which is the convex hull of the incidence vectors of subsets of Hamiltonian circuits.

Cornuéjols, Fonlupt, and Naddef [1985] considered the related problem of finding a shortest tour in a graph such that each vertex is traversed *at least* once, and the related polytope (cf. Naddef and Rinaldi [1991]). Further and related studies (also on shortest k-connected spanning subgraphs, on the 'Steiner network problem', and on the (equivalent) 'survivable network design problem') include Bienstock, Brickell, and Monma [1990], Grötschel and Monma [1990], Monma, Munson, and Pulleyblank [1990], Kelsen and Ramachandran [1991,1995], Barahona and Mahjoub [1992,1995], Chopra [1992,1994], Goemans and Williamson [1992,1995a], Grötschel, Monma, and Stoer [1992], Han, Kelsen, Ramachandran, and Tarjan [1992,1995], Khuller and Vishkin [1992,1994], Nagamochi and Ibaraki [1992a], Cheriyan, Kao, and Thurimella [1993], Gabow, Goemans, and Williamson [1993,1998], Garg, Santosh, and Singla [1993], Naddef and Rinaldi [1993], Queyranne and Wang [1993], Williamson, Goemans, Mihail, and Vazirani [1993,1995], Aggarwal and Garg [1994], Goemans, Goldberg, Plotkin, Shmoys, Tardos, and Williamson [1994], Khuller, Raghavachari, and Young [1994,1995a,1996], Mahjoub [1994,1997], Agrawal, Klein, and Ravi [1995], Khuller and Raghavachari [1995], Ravi and Williamson [1995, 1997], Cheriyan and Thurimella [1996a,2000], Didi Biha and Mahjoub [1996], Fernandes [1997,1998], Carr and Ravi [1998], Cheriyan, Sebő, and Szigeti [1998,2001], Auletta, Dinitz, Nutov, and Parente [1999], Czumaj and Lingas [1998,1999], Jain [1998,2001], Fonlupt and Mahjoub [1999], Fleischer, Jain, and Williamson [2001], Cheriyan, Vempala, and Vetta [2002], and Gabow [2002]. This problem relates to connectivity augmentation — see Chapter 63.

58.9. The asymmetric traveling salesman problem

We next consider the *asymmetric* traveling salesman problem. Let $D = (V, A)$ be a directed graph. The *(asymmetric) traveling salesman polytope* of D is the convex hull of the incidence vectors (in $\mathbb{R}^A$) of Hamiltonian circuits in D. Again, since the asymmetric traveling salesman problem is NP-complete, we know that unless NP=co-NP there is no system of linear inequalities that describes the traveling salesman polytope of a digraph such that their validity can be certified in polynomial time.

Again, we can obtain lower bounds on the minimum length of a Hamiltonian circuit in D by including the traveling salesman polytope in a larger polytope (a *relaxation*) over which $l^{\mathsf{T}}x$ can be minimized in polynomial time. The analogue of relaxation (58.4) for the directed case is:

(58.21)
$$
\begin{array}{rll}
\text{(i)} & 0 \leq x_a \leq 1 & \text{for } a \in A,\\
\text{(ii)} & x(\delta^{\text{in}}(v)) = 1 & \text{for } v \in V,\\
\text{(iii)} & x(\delta^{\text{out}}(v)) = 1 & \text{for } v \in V,\\
\text{(iv)} & x(\delta^{\text{in}}(U)) \geq 1 & \text{for } U \subseteq V \text{ with } \emptyset \neq U \neq V.
\end{array}
$$

With the ellipsoid method, the minimum of $l^{\mathsf{T}}x$ over (58.21) can be found in strongly polynomial time. However, no combinatorial polynomial-time algorithm is known. (The relaxation (i), (ii), (iii) is due to Robinson [1949].)

Grötschel and Padberg [1977] showed that each inequality (58.21)(iv) determines a facet of the traveling salesman polytope of the complete directed graph, if $2 \leq |U| \leq |U| - 2$. (This result was announced in Grötschel and Padberg [1975].)

(58.21) is not enough to determine the traveling salesman polytope, even not for digraphs on 4 vertices only. This is shown by Figure 58.1. Another example is obtained from the Petersen graph, by replacing each edge by two oppositely oriented edges and putting value $\frac{1}{3}$ on each arc.

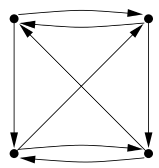

Figure 58.1

Setting $x_a := \frac{1}{2}$ for each arc a, we have a vector x satisfying (58.21) but not belonging to the traveling salesman polytope.

58.10. Directed 1-trees

As in the undirected case, Held and Karp [1970] showed that the minimum of $l^{\mathsf{T}}x$ over (58.21) can be obtained as follows.

Let $D=(V,A)$ be a digraph and fix a vertex 1 of D. Call a subset F of A a *directed* 1*-tree* if F contains exactly one arc, a say, entering 1 and if $F\setminus\{a\}$ is a directed 1-tree such that exactly one arc leaves 1.[7] Each Hamiltonian circuit is a directed 1-tree, and a minimum-length directed 1-tree can be found in strongly polynomial time (by adapting Theorem 52.1).

From Corollary 52.3b one may derive that the convex hull of the incidence vectors of directed 1-trees is determined by:

$$\text{(58.22)}\qquad \begin{array}{rll} \text{(i)} & 0\le x_a\le 1 & \text{for } a\in A,\\ \text{(ii)} & x(\delta^{\text{in}}(v))=1 & \text{for each } v\in V,\\ \text{(iii)} & x(\delta^{\text{out}}(1))=1, & \\ \text{(iv)} & x(\delta^{\text{in}}(U))\ge 1 & \text{for each nonempty } U\subseteq V\setminus\{1\}. \end{array}$$

Again, a *Lagrangean relaxation* approach can find the minimum of $l^{\mathsf{T}}x$ over (58.21), for $l\in\mathbb{R}^A$. For any $y\in\mathbb{R}^V$ define

$$\text{(58.23)}\qquad l_y(a):=l(a)-y(u)$$

for any arc $a=(u,v)\in A$, and define

$$\text{(58.24)}\qquad f(y):=\min_F l_y(F)+y(V),$$

where F ranges over directed 1-trees.

Then the minimum of $l^{\mathsf{T}}x$ over (58.21) is equal to the maximum of $f(y)$ over $y\in\mathbb{R}^V$. The proof is similar to that of Theorem 58.2.

58.10a. An integer programming formulation

The integer solutions of (58.21) are precisely the incidence vectors of Hamiltonian circuits, so it gives an integer programming formulation of the asymmetric traveling salesman problem. The system has exponentially many constraints. A.W. Tucker showed in 1960 (cf. Miller, Tucker, and Zemlin [1960]) that the asymmetric TSP can be formulated as the following integer programming problem, of polynomial size only. Set $n:=|V|$, fix a vertex v_0 of D, and minimize $l^{\mathsf{T}}x$ where $x\in\mathbb{Z}^A$ and $z\in\mathbb{R}^V$ are such that

$$\text{(58.25)}\qquad \begin{array}{rll} \text{(i)} & x_a\ge 0 & \text{for } a\in A,\\ \text{(ii)} & x(\delta^{\text{in}}(v))=1 & \text{for } v\in V,\\ \text{(ii)} & x(\delta^{\text{out}}(v))=1 & \text{for } v\in V,\\ \text{(iv)} & z_u-z_v+nx_a\le n-1 & \text{for } a=(u,v)\in A \text{ with } u,v\ne v_0. \end{array}$$

[7] Held and Karp used the term 1-*arborescence* for a directed 1-tree. To avoid confusion with r-arborescence (a slightly different notion), we have chosen for directed 1-tree.

The conditions (i), (ii), and (iii) and the integrality of x guarantee that x is the incidence vector of a set C of arcs forming directed circuits partitioning V. Then condition (iv) says the following. For any arc $a = (u, v)$ not incident with v_0, one has: if a belongs to C, then $z_u \leq z_v - 1$; if a does not belong to C, then $z_u - z_v \leq n - 1$. This implies that C contains no directed circuit disjoint from v_0. Hence C is a Hamiltonian circuit.

Conversely, for any incidence vector x of a Hamiltonian circuit, one can find $z \in \mathbb{R}^V$ satisfying (58.25).

Unfortunately, the linear programming bound one may derive from (58.25) is generally much worse than that obtained from (58.21).

58.10b. Further notes on the asymmetric traveling salesman problem

Bartels and Bartels [1989] gave a system of inequalities determining the traveling salesman polytope of the complete directed graph on 5 vertices (correcting Heller [1953a] and Kuhn [1955a]).

Padberg and Rao [1974] showed that the diameter of the asymmetric traveling salesman polytope of the complete directed graph on n vertices is equal to 1 if $3 \leq n \leq 5$, and to 2 if $n \geq 6$. Rispoli [1998] showed that the monotonic diameter of the asymmetric traveling salesman polytope of the complete directed graph on n vertices equals $\lfloor n/3 \rfloor$ if $n \geq 3$. (For the definition of monotonic diameter, see Section 58.8b.)

Adjacency of vertices of the asymmetric traveling salesman polytope of a graph $G = (V, E)$ is co-NP-complete, as was shown by Papadimitriou [1978][8]. The number of edges of the asymmetric traveling salesman polytope was estimated by Sarangarajan [1997].

H.W. Kuhn (cf. Heller [1953a], Kuhn [1955a]) claimed that the dimension of the asymmetric traveling salesman polytope of the complete directed graph on n vertices is equal to $n^2 - 3n + 1$ (if $n \geq 3$). A proof of this was supplied by Grötschel and Padberg [1977]. Further work on this polytope is reported in Kuhn [1991].

More valid inequalities for the asymmetric traveling salesman polytope were given by Grötschel and Padberg [1977], Grötschel and Wakabayashi [1981a,1981b], Balas [1989], Fischetti [1991,1992,1995], Balas and Fischetti [1993,1999], and Queyranne and Wang [1995].

A polytope generalizing the directed 1-tree polytope and the asymmetric traveling salesman polytope, the 'fixed-outdegree 1-arborescence polytope', was studied by Balas and Fischetti [1992]. Another polyhedron related to the asymmetric traveling salesman polytope was studied by Chopra and Rinaldi [1996].

Billera and Sarangarajan [1996] showed that each 0,1 polytope is affinely equivalent to the traveling salesman polytope of some directed graph.

Frieze, Karp, and Reed [1992,1995] investigated the tightness of the assignment bound (determined by (58.21)(i)-(iii)). Williamson [1992] compared the Held-Karp lower bound for the asymmetric TSP with the assignment bound.

Carr and Vempala [2000] related the relative error of the asymmetric TSP bound obtained from (58.21) to that of the symmetric TSP bound obtained from (58.4).

[8] Murty [1969] gave a characterization of adjacency that was shown to be false by Rao [1976].

Padberg and Sung [1991] compared different formulations of the asymmetric traveling salesman problem.

An analogue of Christofides' algorithm (Section 58.8a) for the asymmetric case is not known: no factor c and polynomial-time algorithm are known that give a Hamiltonian circuit in a digraph of length at most c times the length of a shortest Hamiltonian circuit, even not if the lengths satisfy the triangle inequality.

58.11. Further notes on the traveling salesman problem

58.11a. Further notes

There is an abundance of papers presenting algorithms, heuristics, and computational results for the traveling salesman problem. We give a short selection of it.

Milestones in solving large-scale symmetric traveling salesman problems were achieved by Dantzig, Fulkerson, and Johnson [1954b] (42 cities), Held and Karp [1962] (48 cities), Karg and Thompson [1964] (57 cities), Held and Karp [1971] (64 cities), Helbig Hansen and Krarup [1974] (80 cities), Camerini, Fratta, and Maffioli [1975] (100 cities), Grötschel [1980b] (120 cities), Crowder and Padberg [1980] and Padberg and Hong [1980] (318 cities), Padberg and Rinaldi [1987] (532 cities), Grötschel and Holland [1991] (666 cities), Padberg and Rinaldi [1990b,1991] (2392 cities), Applegate, Bixby, Chvátal, and Cook [1995] (7397 cities), and Applegate, Bixby, Chvátal, and Cook [1998] (13,509 cities). Although the complexity of a TSP instance is not simply a function of the number of cities, these papers represent substantial steps forward in developing computational techniques for the traveling salesman problem.

Dynamic programming approaches were proposed by Bellman [1962] and Held and Karp [1962]. Several methods were compared by computer experiments by Lin [1965]. The Lagrangean relaxation technique was introduced by Christofides [1970] and Held and Karp [1970,1971]. The Held-Karp method was implemented and extended by Helbig Hansen and Krarup [1974], Smith and Thompson [1977], and Volgenant and Jonker [1982,1983]. Related work includes Bazaraa and Goode [1977].

Miliotis [1976,1978] described a constraint generation approach, mixing subtour elimination constraints with Gomory cuts or with branching. Focusing on the asymmetric TSP are Little, Murty, Sweeney, and Karel [1963] (first reports on a branch-and-bound method), Bellmore and Malone [1971] (on the effect of the subtour elimination constraints), (cf. Garfinkel [1973], Smith, Srinivasan, and Thompson [1977], Lenstra and Rinnooy Kan [1978], Carpaneto and Toth [1980b], Zhang [1997a]), Balas and Christofides [1981] (a Lagrangean approach based on the assignment problem, solving randomly generated asymmetric TSP's with up to 325 cities), Miller and Pekny [1989,1991], Pekny and Miller [1992], and Carpaneto, Dell'Amico, and Toth [1995].

Further bounds for the symmetric and asymmetric TSP were given by Christofides [1972], Carpaneto, Fischetti, and Toth [1989] and Fischetti and Toth [1992].

Important heuristics (algorithms that yield a tour that is expected to be short, but not necessarily shortest) and local search techniques include the *nearest neighbour heuristic*: always go to the closest city not yet visited (Menger [1932a], Gavett

[1965], Bellmore and Nemhauser [1968]), the *Lin-Kernighan heuristic*: start with a Hamiltonian circuit and iteratively replace a limited number of edges by other edges as long as it makes the circuit shorter (Lin and Kernighan [1973]), and Christofides' heuristic discussed in Section 58.8a. From the further work on, and analyses of, heuristics and local search techniques we mention Christofides and Eilon [1972], Rosenkrantz, Stearns, and Lewis [1977], Cornuéjols and Nemhauser [1978], Frieze [1979], d'Atri [1980], Bentley and Saxe [1980], Ong and Moore [1984], Golden and Stewart [1985] (survey), Johnson and Papadimitriou [1985b] (survey), Karp and Steele [1985] (survey), Johnson, Papadimitriou, and Yannakakis [1988], Kern [1989], Bentley [1990,1992], Papadimitriou [1992] (showing that unless P=NP, any local search method taking polynomial time per iteration, can lead to a locally optimum tour that is arbitrarily far from the optimum), Fredman, Johnson, McGeoch, and Ostheimer [1993,1995], Chandra, Karloff, and Tovey [1994,1999], Tassiulas [1997], and Frieze and Sorkin [2001]. A survey and comparison of heuristics and local search techniques for the traveling salesman problem was given by Johnson and McGeoch [1997].

Polynomial-time solvable special cases of the traveling salesman problem were given by Gilmore and Gomory [1964a,1964b], Gilmore [1966], Lawler [1971a], Sysło [1973], Cornuéjols, Naddef, and Pulleyblank [1983], and Hartvigsen and Pulleyblank [1994]. Surveys of such problems were given by Gilmore, Lawler, and Shmoys [1985] and Burkard, Deĭneko, van Dal, van der Veen, and Woeginger [1998].

The standard reference book on the traveling salesman problem, covering a wide variety of aspects, was edited by Lawler, Lenstra, Rinnooy Kan, and Shmoys [1985]. In this book, Grötschel and Padberg [1985] considered the traveling salesman polytope, Padberg and Grötschel [1985] computation with the help of polyhedra, Johnson and Papadimitriou [1985a] the computational complexity of the TSP, and Balas and Toth [1985] branch-and-bound method methods. Computational methods and results are surveyed in the book by Reinelt [1994].

Survey articles on the traveling salesman problem were given by Gomory [1966], Bellmore and Nemhauser [1968], Gupta [1968], Tyagi [1968], Burkard [1979], Christofides [1979], Grötschel [1982] (also on other NP-complete problems), and Johnson and McGeoch [1997] (local search techniques). Introductions are given in the books by Minieka [1978], Sysło, Deo, and Kowalik [1983], Cook, Cunningham, Pulleyblank, and Schrijver [1998], and Korte and Vygen [2000]. An insightful survey of the computational methods for the symmetric TSP was given by Jünger, Reinelt, and Rinaldi [1995]. A framework for guaranteeing quality of TSP solutions was presented by Jünger, Thienel, and Reinelt [1994]. An early survey on branch-and-bound method techniques was given by Lawler and Wood [1966].

Barvinok, Johnson, Woeginger, and Woodroofe [1998] showed that there is a polynomial-time algorithm to find a *longest* Hamiltonian circuit in a complete graph with length determined by a polyhedral norm. Related work was done by Barvinok [1996]. More on the longest Hamiltonian circuit can be found in Fisher, Nemhauser, and Wolsey [1979], Serdyukov [1984], Kostochka and Serdyukov [1985], Kosaraju, Park, and Stein [1994], Hassin and Rubinstein [2000,2001], and Bläser [2002].

58.11b. Historical notes on the traveling salesman problem

Mathematically, the traveling salesman problem is related to, in fact generalizes, the question for a Hamiltonian circuit in a graph. This question goes back to Kirkman

[1856] and Hamilton [1856,1858] and was also studied by Kowalewski [1917b,1917a] — see Biggs, Lloyd, and Wilson [1976]. We restrict our survey to the traveling salesman problem in its general form.

The mathematical roots of the traveling salesman problem are obscure. Dantzig, Fulkerson, and Johnson [1954a] say:

> It appears to have been discussed informally among mathematicians at mathematics meetings for many years.

A 1832 manual

The traveling salesman problem has a natural interpretation, and Müller-Merbach [1983] detected that the problem was formulated in a 1832 manual for the successful traveling salesman, *Der Handlungsreisende — wie er sein soll und was er zu thun hat, um Aufträge zu erhalten und eines glücklichen Erfolgs in seinen Geschäften gewiß zu sein — Von einem alten Commis-Voyageur*[9] ('ein alter Commis-Voyageur' [1832]). (Whereas the politically correct nowadays prefer to speak of the traveling sales*person* problem, the manual presumes that the 'Handlungsreisende' is male, and it warns about the risks of women in or out of business.)

The booklet contains no mathematics, and formulates the problem as follows:

> Die Geschäfte führen die Handlungsreisenden bald hier, bald dort hin, und es lassen sich nicht füglich Reisetouren angeben, die für alle vorkommende Fälle passend sind; aber es kann durch eine zweckmäßige Wahl und Eintheilung der Tour, manchmal so viel Zeit gewonnen werden, daß wir es nicht glauben umgehen zu dürfen, auch hierüber einige Vorschriften zu geben. Ein Jeder möge so viel davon benutzen, als er es seinem Zwecke für dienlich hält; so viel glauben wir aber davon versichern zu dürfen, daß es nicht wohl thunlich sein wird, die Touren durch Deutschland in Absicht der Entfernungen und, worauf der Reisende hauptsächlich zu sehen hat, des Hin- und Herreisens, mit mehr Oekonomie einzurichten. Die Hauptsache besteht immer darin: so viele Orte wie möglich mitzunehmen, ohne den nämlichen Ort zweimal berühren zu müssen.[10]

The manual suggests five tours through Germany (one of them partly through Switzerland). In Figure 58.2 we compare one of the tours with a shortest tour, found with 'modern' methods. (Most other tours given in the manual do not qualify for 'die Hauptsache' as they contain subtours, so that some places are visited twice.)

Menger's Botenproblem 1930

K. Menger seems to be the first mathematician to have written about the traveling salesman problem. The root of his interest is given in his paper Menger [1928c]. In

[9] 'The traveling salesman — how he should be and what he has to do, to obtain orders and to be sure of a happy success in his business — by an old traveling salesman'

[10] Business brings the traveling salesman now here, then there, and no travel routes can be properly indicated that are suitable for all cases occurring; but sometimes, by an appropriate choice and arrangement of the tour, so much time can be gained, that we don't think we may avoid giving some rules also on this. Everybody may use that much of it, as he takes it for useful for his goal; so much of it however we think we may assure, that it will not be well feasible to arrange the tours through Germany with more economy in view of the distances and, which the traveler mainly has to consider, of the trip back and forth. The main point always consists of visiting as many places as possible, without having to touch the same place twice.

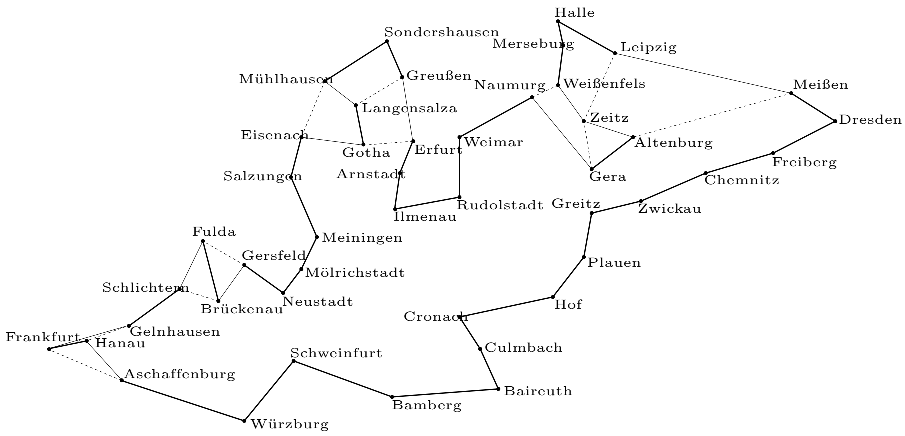

Figure 58.2

A tour along 45 German cities, as described in the 1832 traveling salesman manual, is given by the unbroken (bold and thin) lines (1285 km). A shortest tour is given by the unbroken bold and by the dashed lines (1248 km). We have taken geodesic distances — taking local conditions into account, the 1832 tour might be optimum.

this, he studies the *length* $l(C)$ of a simple curve C in a metric space S, which is, by definition,

$$l(C) := \sup \sum_{i=1}^{n-1} \operatorname{dist}(x_i, x_{i+1}), \tag{58.26}$$

where the supremum ranges over all choices of $x_1, \ldots, x_n$ on C *in the order determined by* C. What Menger showed is that we may relax this to finite subsets X of C and minimize over all possible orderings of X. To this end he defined, for any finite subset X of a metric space, $\lambda(X)$ to be the shortest length of a path through X (in graph terminology: a *Hamiltonian path*), and he showed that

$$l(C) = \sup_X \lambda(X), \tag{58.27}$$

where the supremum ranges over all finite subsets X of C. It amounts to showing that for each $\varepsilon > 0$ there is a finite subset X of C such that $\lambda(X) \geq l(C) - \varepsilon$.

Menger [1929a] sharpened this to:

$$l(C) = \sup_X \kappa(X), \tag{58.28}$$

where again the supremum ranges over all finite subsets X of C, and where $\kappa(X)$ denotes the minimum length of a *spanning tree* on X.

These results were reported also in Menger [1930]. In a number of other papers, Menger [1928b,1929b,1929a] gave related results on these new characterizations of the length function.

The parameter $\lambda(X)$ clearly is close to the practical interpretation of the traveling salesman problem. This relation was made explicit by Menger in the session of 5 February 1930 of his *mathematisches Kolloquium* in Vienna. Menger [1931a, 1932a] reported that he first asked if a further relaxation is possible by replacing $\kappa(X)$ by the minimum length of an (in current terminology) *Steiner tree* connecting X — a spanning tree on a superset of X in S. (So Menger toured along some basic combinatorial optimization problems.) This problem was solved for Euclidean spaces by Mimura [1933].

Next Menger posed the traveling salesman problem, as follows:

> Wir bezeichnen als *Botenproblem* (weil diese Frage in der Praxis von jedem Postboten, übrigens auch von vielen Reisenden zu lösen ist) die Aufgabe, für endlichviele Punkte, deren paarweise Abstände bekannt sind, den kürzesten die Punkte verbindenden Weg zu finden. Dieses Problem ist natürlich stets durch endlichviele Versuche lösbar. Regeln, welche die Anzahl der Versuche unter die Anzahl der Permutationen der gegebenen Punkte herunterdrücken würden, sind nicht bekannt. Die Regel, man solle vom Ausgangspunkt erst zum nächstgelegenen Punkt, dann zu dem diesem nächstgelegenen Punkt gehen usw., liefert im allgemeinen nicht den kürzesten Weg.[11]

So Menger asked for a shortest Hamiltonian path through the given points. He was aware of the complexity issue in the traveling salesman problem, and he realized that the now well-known nearest neighbour heuristic might not give an optimum solution.

Harvard, Princeton 1930-1934

Menger spent the period September 1930-February 1931 as visiting lecturer at Harvard University. In one of his seminar talks at Harvard, Menger presented his results (quoted above) on lengths of arcs and shortest paths through finite sets of points. According to Menger [1931b], a suggestion related to this was given by Hassler Whitney, who at that time did his Ph.D. research in graph theory at Harvard. This paper of Menger however does not mention if the practical interpretation was given in the seminar talk.

The year after, 1931-1932, Whitney was a National Research Council Fellow at Princeton University, where he gave a number of seminar talks. In a seminar talk, he mentioned the problem of finding the shortest route along the 48 States of America.

There are some uncertainties in this story. It is not sure if Whitney spoke about the 48 States problem during his 1931-1932 seminar talks (which talks he did give), or later, in 1934, as is said by Flood [1956] in his article on the traveling salesman problem:

> This problem was posed, in 1934, by Hassler Whitney in a seminar talk at Princeton University.

That memory can be shaky might be indicated by the following two quotes. Dantzig, Fulkerson, and Johnson [1954a] remark:

---

[11] We denote by *messenger problem* (since in practice this question should be solved by each postman, anyway also by many travelers) the task to find, for finitely many points whose pairwise distances are known, the shortest route connecting the points. Of course, this problem is solvable by finitely many trials. Rules which would push the number of trials below the number of permutations of the given points, are not known. The rule that one first should go from the starting point to the closest point, then to the point closest to this, etc., in general does not yield the shortest route.

> Both Flood and A.W. Tucker (Princeton University) recall that they heard about the problem first in a seminar talk by Hassler Whitney at Princeton in 1934 (although Whitney, recently queried, does not seem to recall the problem).

However, when asked by David Shmoys, Tucker replied in a letter of 17 February 1983 (see Hoffman and Wolfe [1985]):

> I cannot confirm or deny the story that I heard of the TSP from Hassler Whitney. If I did (as Flood says), it would have occurred in 1931-32, the first year of the old Fine Hall (now Jones Hall). That year Whitney was a postdoctoral fellow at Fine Hall working on Graph Theory, especially planarity and other offshoots of the 4-color problem. ... I was finishing my thesis with Lefschetz on n-manifolds and Merrill Flood was a first year graduate student. The Fine Hall Common Room was a very lively place — 24 hours a day.

(Whitney finished his Ph.D. at Harvard University in 1932.)

Another uncertainty is in which form Whitney has posed the problem. That he might have focused on finding a shortest route along the 48 states in the U.S.A., is suggested by the reference by Flood, in an interview on 14 May 1984 with Tucker [1984a], to the problem as the '48 States Problem of Hassler Whitney'. In this respect Flood also remarked:

> I don't know who coined the peppier name 'Traveling Salesman Problem' for Whitney's problem, but that name certainly has caught on, and the problem has turned out to be of very fundamental importance.

TSP, Hamiltonian paths, and school bus routing

Flood [1956] remembered that in 1937, A.W. Tucker pointed out to him the connections of the TSP with Hamiltonian games and Hamiltonian paths in graphs:

> I am indebted to A.W. Tucker for calling these connections to my attention, in 1937, when I was struggling with the problem in connection with a schoolbus routing study in New Jersey.

In the following quote from the interview by Tucker [1984a], Flood referred to school bus routing in a different state (West Virginia), and he mentioned the involvement in the TSP of Koopmans, who spent 1940-1941 at the Local Government Surveys Section of Princeton University ('the Princeton Surveys'):

> Koopmans first became interested in the "48 States Problem" of Hassler Whitney when he was with me in the Princeton Surveys, as I tried to solve the problem in connection with the work by Bob Singleton and me on school bus routing for the State of West Virginia.

1940

In 1940, some papers appeared that study the traveling salesman problem, in a different context. They seem to be the first containing mathematical results on the problem.

In the American continuation of Menger's *mathematisches Kolloquium*, Menger [1940] returned to the question of the shortest path through a given set of points in a metric space, followed by investigations of Milgram [1940] on the shortest Jordan curve that covers a given, not necessarily finite, set of points in a metric space. As the set may be infinite, a shortest curve need not exist.

Fejes [1940] investigated the problem of a shortest curve through n points in the unit square. In consequence of this, Verblunsky [1951] showed that its length is less than $2 + \sqrt{2.8n}$. Later work in this direction includes Few [1955], Beardwood, Halton, and Hammersley [1959], Steele [1981], Moran [1984], Karloff [1989], and Goddyn [1990].

Lower bounds on the expected value of a shortest path through n random points in the plane were studied by Mahalanobis [1940] in order to estimate the cost of a sample survey of the acreage under jute in Bengal. This survey took place in 1938 and one of the major costs in carrying out the survey was the transportation of men and equipment from one survey point to the next. He estimated (without proof) the minimum length of a tour along n random points in the plane, for Euclidean distance:

> It is also easy to see in a general way how the journey time is likely to behave. Let us suppose that n sampling units are scattered at random within any given area ; and let us assume that we may treat each such sample unit as a geometrical point. We may also assume that arrangements will usually be made to move from one sample point to another in such a way as to keep the total distance travelled as small as possible ; that is, we may assume that the path traversed in going from one sample point to another will follow a straight line. In this case it is easy to see that the mathematical expectation of the total length of the path travelled in moving from one sample point to another will be $(\sqrt{n} - 1/\sqrt{n})$. The cost of the journey from sample to sample will therefore be roughly proportional to $(\sqrt{n} - 1/\sqrt{n})$. When n is large, that is, when we consider a sufficiently large area, we may expect that the time required for moving from sample to sample will be roughly proportional to $\sqrt{n}$, where n is the total number of samples in the given area. If we consider the journey time per sq. mile, it will be roughly proportional to $\sqrt{y}$, where y is the density of number of sample units per sq. mile.

This research was continued by Jessen [1942], who estimated empirically a similar result for l_1-distance (Manhattan distance), in a statistical investigation of a sample survey for obtaining farm facts in Iowa:

> If a route connecting y points located at random in a fixed area is minimized, the total distance, D, of that route is[12]
>
> $$D = d\left(\frac{y-1}{\sqrt{y}}\right)$$
>
> where d is a constant.
> This relationship is based upon the assumption that points are connected by direct routes. In Iowa the road system is a quite regular network of mile square mesh. There are very few diagonal roads, therefore, routes between points resemble those taken on a checkerboard. A test wherein several sets of different members of points were located at random on an Iowa county road map, and the minimum distance of travel from a given point on the border of the county through all the points and to an end point (the county border nearest the last point on route), revealed that
>
> $$D = d\sqrt{y}$$
>
> works well. Here y is the number of randomized points (border points not included). This is of great aid in setting up a cost function.

[12] at this point, Jessen referred in a footnote to Mahalanobis [1940].

Marks [1948] gave a proof of Mahalanobis' bound. In fact he showed that $\sqrt{\frac{1}{2}A}(\sqrt{n}-1/\sqrt{n})$ is a lower bound, where A is the area of the region. Ghosh [1949] showed that this bound asymptotically is close to the expected value, by giving a heuristic for finding a tour, yielding an upper bound of $1.27\sqrt{An}$. He also observed the complexity of the problem:

> After locating the n random points in a map of the region, it is very difficult to find out *actually* the shortest path connecting the points, unless the number n is very small, which is seldom the case for a large-scale survey.

TSP, transportation, and assignment

As is the case for several other combinatorial optimization problems, the RAND Corporation in Santa Monica, California, played an important role in the research on the TSP. Hoffman and Wolfe [1985] write that

> John Williams urged Flood in 1948 to popularize the TSP at the RAND Corporation, at least partly motivated by the purpose of creating intellectual challenges for models outside the theory of games. In fact, a prize was offered for a significant theorem bearing on the TSP. There is no doubt that the reputation and authority of RAND, which quickly became the intellectual center of much of operations research theory, amplified Flood's advertizing.

(John D. Williams was head of the Mathematics Division of RAND at that time.)

At RAND, researchers considered the idea of transferring the successful methods for the transportation problem to the traveling salesman problem. Flood [1956] mentioned that this idea was brought to his attention by Koopmans in 1948. In the interview with Tucker [1984a], Flood remembered:

> George Dantzig and Tjallings Koopmans met with me in 1948 in Washington, D.C., at the meeting of the International Statistical Institute, to tell me excitedly of their work on what is now known as the linear programming problem and with Tjallings speculating that there was a significant connection with the Traveling Salesman Problem.

The issue was taken up in a RAND Report by Julia Robinson [1949], who, in an 'unsuccessful attempt' to solve the traveling salesman problem, considered, as a relaxation, the assignment problem, for which she found a cycle reduction method. The relation is that the assignment problem asks for an optimum permutation, and the TSP for an optimum *cyclic* permutation.

Robinson's RAND report might be the earliest mathematical reference using the term 'traveling salesman problem':

> The purpose of this note is to give a method for solving a problem related to the traveling salesman problem. One formulation is to find the shortest route for a salesman starting from Washington, visiting all the state capitals and then returning to Washington. More generally, to find the shortest closed curve containing n given points in the plane.

Flood wrote (in a letter of 17 May 1983 to E.L. Lawler) that Robinson's report stimulated several discussions on the TSP of him with his research assistant at RAND, D.R. Fulkerson, during 1950-1952[13].

It was noted by Beckmann and Koopmans [1952] that the TSP can be formulated as a quadratic assignment problem, for which however no fast methods are known.

[13] Fulkerson started at RAND only in March 1951.

Dantzig, Fulkerson, Johnson 1954

Fundamental progress on the traveling salesman was made in a seminal paper by the RAND researchers Dantzig, Fulkerson, and Johnson [1954a] — according to Hoffman and Wolfe [1985] 'one of the principal events in the history of combinatorial optimization'. The paper introduced several new methods for solving the traveling salesman problem that are now basic in combinatorial optimization. In particular, it shows the importance of *cutting planes* for combinatorial optimization.

While the subtour elimination constraints (58.4)(iii) are enough to cut off the noncyclic permutation matrices from the polytope of doubly stochastic matrices (determined by (58.4)(i) and (ii)), they generally do not yield all facets of the traveling salesman polytope, as was observed by Heller [1953a]: there exist doubly stochastic matrices, of any order $n \geq 5$, that satisfy (58.4) but are not a convex combination of cyclic permutation matrices.

The subtour elimination constraints can nevertheless be useful for the TSP, since it gives a lower bound for the optimum tour length if we minimize over the constraints (58.4). This lower bound can be calculated with the simplex method, taking the (exponentially many) constraints (58.4)(iii) as *cutting planes* that can be added during the process when needed. In this way, Dantzig, Fulkerson, and Johnson were able to find the shortest tour along cities chosen in the 48 U.S. states and Washington, D.C. Incidentally, this is close to the problem mentioned by Julia Robinson in 1949 (and maybe also by Whitney in the 1930s).

The Dantzig-Fulkerson-Johnson paper gives no algorithm, but rather gives a tour and proves its optimality with the help of the subtour elimination constraints. This work forms the basis for most of the later work on large-scale traveling salesman problems.

Early studies of the traveling salesman polytope were reported by Heller [1953a,1953b,1955a,1955b,1956a,1956b], Kuhn [1955a], Norman [1955], and Robacker [1955b], who also made computational studies of the probability that a random instance of the traveling salesman problem needs the subtour elimination constraints (58.4)(iii) (cf. Kuhn [1991]). This made Flood [1956] remark on the intrinsic complexity of the traveling salesman problem:

> Very recent mathematical work on the traveling-salesman problem by I. Heller, H.W. Kuhn, and others indicates that the problem is fundamentally complex. It seems very likely that quite a different approach from any yet used may be required for succesful treatment of the problem. In fact, there may well be no general method for treating the problem and impossibility results would also be valuable.

Flood mentioned a number of other applications of the traveling salesman problem, in particular in machine scheduling, brought to his attention in a seminar talk at Columbia University in 1954 by George Feeney.

Other work on the traveling salesman problem in the 1950s was done by Morton and Land [1955] (a linear programming approach with a 3-exchange heuristic), Barachet [1957] (a graphic solution method), Bock [1958], Croes [1958] (a heuristic), and Rossman and Twery [1958]. In a reaction to Barachet's paper, Dantzig, Fulkerson, and Johnson [1959] showed that their method yields the optimality of Barachet's (heuristically found) solution.

In 1962, the soap company Proctor and Gamble run a contest, requiring to solve a traveling salesman problem along 33 U.S. cities. Little, Murty, Sweeney, and Karel [1963] report:

> The traveling salesman problem recently achieved national prominence when a soap company used it as the basis of a promotional contest. Prizes up to $10,000 were offered for identifying the most correct links in a particular 33-city problem. Quite a few people found the best tour. (The tie-breaking contest for these successful mathematicians was to complete a statement of 25 words or less on "I like...because...".) A number of people, perhaps a little over-educated, wrote the company that the problem was impossible—an interesting misinterpretation of the state of the art.

Chapter 59

Matching forests

Giles [1982a,1982b,1982c] introduced the concept of a *matching forest* in a mixed graph (V, E, A), which is a subset F of $E \cup A$ such that $F \cap A$ is a branching and $F \cap E$ is a matching only covering roots of the branching $F \cap A$. Equivalently, F contains no circuit (in the underlying undirected graph) and each $v \in V$ is head of at most one $e \in F$. (Here, for an undirected edge e, both ends of e are called head of e.)
Matching forests generalize both matchings in undirected graphs and branchings in directed graphs. Giles gave a polynomial-time algorithm to find a maximum-weight matching forest, yielding as a by-product a characterization of the matching forest polytope (the convex hull of the incidence vectors of matching forests).
Giles' results generalize the polynomial-time solvability and the polyhedral characterizations for matchings (Chapters 24–26) and for branchings (Chapter 52).

59.1. Introduction

A *mixed graph* is a triple (V, E, A), where (V, E) is an undirected graph and (V, A) is a directed graph. In this chapter, a graph can have multiple edges, but no loops. The *underlying* undirected graph of a mixed graph is the undirected graph obtained from the mixed graph by forgetting the orientations of the directed edges.

As usual, if an edge e is directed from u to v, then u is called the *tail* and v the *head* of e. In this chapter, if e is undirected and connects u and v, then both u and v will be called *head* of e.

A subset F of $E \cup A$ is called a *matching forest* if F contains no circuits (in the underlying undirected graph) and any vertex v is head of at most one edge in F. We call a vertex v a *root* of F if v is head of no edge in F. We denote the set of roots of F by $R(F)$.

It is convenient to consider the relations of matching forests with matchings in undirected graphs and branchings in directed graphs: M is a matching in an undirected graph (V, E) if and only if M is a matching forest in the mixed graph $(V, E, \emptyset)$. In this case, the roots of M are the vertices not covered by M. Similarly, B is a branching in a directed graph (V, A) if and only if B

is a matching forest in the mixed graph $(V, \emptyset, A)$. In this case, the concept of root of a branching and root of a matching forest coincide.

In turn, we can characterize matching forests in terms of matchings and branchings: for any mixed graph (V, E, A), a subset F of $E \cup A$ is a matching forest if and only if $F \cap A$ is a branching in (V, A) and $F \cap E$ is a matching in (V, E) such that $F \cap E$ only covers roots of $F \cap A$.

It will be useful to observe the following formulas, for any matching forest F in a mixed graph (V, E, A), setting $M := F \cap E$ and $B := F \cap A$:

(59.1) $\qquad R(F) = R(M) \cap R(B)$ and $V = R(M) \cup R(B)$.

In fact, for any matching M in (V, E) and any branching B in (V, A), the set $M \cup B$ is a matching forest if and only if $R(M) \cup R(B) = V$.

59.2. The maximum size of a matching forest

Giles [1982a] described a min-max formula for the maximum size of a matching forest. It can be derived from the Tutte-Berge formula with the following direct formula:

Theorem 59.1. *Let (V, E, A) be a mixed graph and let $\mathcal{K}$ be the collection of those strong components K of the directed graph (V, A) that satisfy $d_A^{\text{in}}(K) = 0$. Consider the undirected graph H with vertex set $\mathcal{K}$, where two distinct $K, L \in \mathcal{K}$ are adjacent if and only if there is an edge in E connecting K and L. Then the maximum size of a matching forest in (V, E, A) is equal to*

(59.2) $\qquad \nu(H) + |V| - |\mathcal{K}|.$

Here $\nu(H)$ denotes the maximum size of a matching in H.

Proof. Let M' be a matching in H of size $\nu(H)$. Then M' yields a matching M of size $\nu(H)$ in (V, E), where each edge in M connects two components in $\mathcal{K}$. Now there exists a branching B in (V, A) such that B has exactly $|\mathcal{K}|$ roots, such that each $K \in \mathcal{K}$ contains exactly one root, and such that each vertex covered by M is a root of B. (To see that such a branching B exists, choose, for any $K \in \mathcal{K}$ not intersecting M, an arbitrary vertex in K. Let X be the set of chosen vertices together with the vertices covered by M. As X intersects each $K \in \mathcal{K}$, each vertex in V is reachable in (V, A) by a directed path from X. Hence there exists a branching B with root set X. This B has the required properties.)

Then $M \cup B$ is a matching forest, of size $\nu(H) + |V| - |\mathcal{K}|$ (as B has size $|V| - |\mathcal{K}|$).

To see that there is no larger matching forest, let F be any matching forest. Let $U := \bigcup \mathcal{K}$. Then F has at most $|V \setminus U|$ edges with at least one head in $V \setminus U$. Since no directed edge enters U, all other edges are contained in

U. So it suffices to show that F has at most $\nu(H)+|U|-|\mathcal{K}|$ edges contained in U.

Let N be the set of (necessarily undirected) edges in F connecting two different components in $\mathcal{K}$. For each $K \in \mathcal{K}$, let α_K be the number of edges in N incident with K. Then

(59.3) $$|N| - \sum_{K\in\mathcal{K}} \max\{0, \alpha_K - 1\} \leq \nu(H),$$

since by deleting, for each $K \in \mathcal{K}$, at most $\max\{0, \alpha_K - 1\}$ edges from N incident with K, we obtain a matching in the graph H defined above.

We have moreover that any $K \in \mathcal{K}$ spans at most $|K| - \max\{1, \alpha_K\}$ edges of F. With (59.3) this implies that the number of edges in F contained in U is at most

(59.4) $$|N| + \sum_{K\in\mathcal{K}} (|K| - \max\{1, \alpha_K\}) \leq \nu(H) + \sum_{K\in\mathcal{K}} (|K| - 1) = \nu(H) + |U| - |\mathcal{K}|,$$

as required. ∎

The method described in this proof also directly implies that a maximum-size matching forest can be found in polynomial time (Giles [1982a]).

59.3. Perfect matching forests

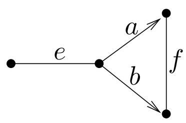

Figure 59.1
$\{e, f\}$ and $\{e, a, b\}$ are perfect matching forests.

A matching forest F is called *perfect* if each vertex is head of exactly one edge in F. (So a perfect matching forest need not be a maximum-size matching forest — cf. Figure 59.1.) The following is easy to see:

(59.5) A mixed graph (V, E, A) contains a perfect matching forest F if and only if the graph (V, E) contains a matching M such that each strong component K of (V, A) with $d^{\text{in}}(K) = 0$ is intersected by at least one edge in M.

Indeed, if a perfect matching forest F exists, then $M := F \cap E$ is such a matching. Conversely, if such a matching M exists, any vertex is reachable by a directed path from at least one vertex covered by M; hence M can be augmented with directed arcs to a perfect matching forest.

This shows (59.5), which implies the following characterization for perfect matching forests of Giles [1982b]:

Theorem 59.2. *Let (V, E, A) be a mixed graph and let $\mathcal{K}$ be the collection of strong components K of (V, A) with $d_A^{\text{in}}(K) = 0$. Then (V, E, A) has a perfect matching forest if and only if for each $U \subseteq V$ and $\mathcal{L} \subseteq \mathcal{K}$ the graph $(V, E) - U$ has at most $|U| + \left|\bigcup \mathcal{L}\right| - |\mathcal{L}|$ odd components that are contained in $\bigcup \mathcal{L}$.*

Proof. Extend $G = (V, E)$ by, for each $K \in \mathcal{K}$, a clique C_K of size $|K| - 1$, such that each vertex in C_K is adjacent to each vertex in K. This makes the undirected graph H. Then (V, E, A) has a perfect matching forest if and only if graph H has a matching covering $\bigcup \mathcal{K}$. So we can apply Corollary 24.6a. ∎

This method also gives a polynomial-time algorithm to find a perfect matching forest.

59.4. An exchange property of matching forests

As a preparation for characterizing the matching forest polytope, we show an exchange property of matching forests. It generalizes the well-known and trivial exchange property of matchings in an undirected graph, based on considering the union of two matchings.

Lemma 59.3α. *Let F_1 and F_2 be matching forests in a mixed graph (V, E, A). Let $s \in R(F_2) \setminus R(F_1)$. Then there exist matching forests F_1' and F_2' such that $F_1' \cap F_2' = F_1 \cap F_2$, $F_1' \cup F_2' = F_1 \cup F_2$, $s \in R(F_1')$, and*

(59.6) (i) $|F_1'| < |F_1|$,
or (ii) $|F_1'| = |F_1|$ and $|R(F_1')| > |R(F_1)|$,
or (iii) $|F_1'| = |F_1|$, $R(F_1') = (R(F_1) \setminus \{t\}) \cup \{s\}$ for some $t \in R(F_1)$, and $|R(F_1' \cap A) \cap K| = |R(F_1 \cap A) \cap K|$ for each strong component K of the directed graph (V, A).

Proof. We may assume that F_1 and F_2 partition $E \cup A$, as we can delete edges that are not in $F_1 \cup F_2$, and add parallel edges to those in $F_1 \cap F_2$.

Define $M_i := F_i \cap E$ and $B_i := F_i \cap A$ for $i = 1, 2$. Let $\mathcal{K}$ be the collection of strong components K of the directed graph (V, A) with $\delta_A^{\text{in}}(K) = \emptyset$. Then each set in $\mathcal{K}$ intersects both $R(B_1)$ and $R(B_2)$, and $\{v\} \in \mathcal{K}$ for each $v \in R(B_1) \cap R(B_2)$.

So each $K \in \mathcal{K}$ with $|K| \geq 2$ intersects $R(B_1)$ and $R(B_2)$ in disjoint subsets. Hence we can choose for each such K

(59.7) a pair $e_K \subseteq K$ consisting of a vertex in $R(B_1) \setminus R(B_2)$ and a vertex in $R(B_2) \setminus R(B_1)$.

Let N be the set of pairs e_K for $K \in \mathcal{K}$ with $|K| \geq 2$. So N consists of disjoint pairs.

Then the undirected graph H on V with edge set

(59.8) $M_1 \cup M_2 \cup N$

consists of a number of vertex-disjoint paths and circuits, since no vertex in $R(B_1) \setminus R(B_2)$ is covered by M_2, and no vertex in $R(B_2) \setminus R(B_1)$ is covered by M_1.

Moreover, s has degree at most one in H. Indeed, s is not covered by M_2, as $s \in R(F_2) = R(M_2) \cap R(B_2)$. If s is covered by M_1, then $s \in R(B_1)$, and so $s \in R(B_1) \cap R(B_2)$, implying that s is not covered by N.

So s is the starting vertex of a path component P of H (possibly only consisting of s). Let Y be the set of edges in $M_1 \cup M_2$ occurring in P, and set

(59.9) $M_1' := M_1 \triangle Y$ and $M_2' := M_2 \triangle Y$

(where $\triangle$ denotes symmetric difference). Since Y is the union of the edge sets of some path components of the graph $(V, M_1 \cup M_2)$, we know that M_1' and M_2' are matchings again.

Then, obviously, $R(M_1')$ and $R(M_2')$ arise from $R(M_1)$ and $R(M_2)$ by exchanging these sets on VP; that is:

(59.10) $R(M_1') = (R(M_1) \setminus VP) \cup (R(M_2) \cap VP)$ and
$R(M_2') = (R(M_2) \setminus VP) \cup (R(M_1) \cap VP)$.

We show that a similar operation can be performed with respect to B_1 and B_2; that is, we show that there exist disjoint branchings B_1' and B_2' in (V, A) satisfying

(59.11) $R(B_1') = (R(B_1) \setminus VP) \cup (R(B_2) \cap VP)$ and
$R(B_2') = (R(B_2) \setminus VP) \cup (R(B_1) \cap VP)$.

By Lemma 53.2α, it suffices to show that each $K \in \mathcal{K}$ intersects both sets in (59.11). If $|K| = 1$, then K is contained in both $R(B_1)$ and $R(B_2)$, and hence in both sets in (59.11). If $|K| \geq 2$, then e_K intersects both $R(B_1)$ and $R(B_2)$. Since e_K is either contained in VP or disjoint from VP, e_K intersects both sets in (59.11). Hence, as $e_K \subseteq K$, also K intersects both sets in (59.11). Therefore, branchings B_1' and B_2' satisfying (59.11) exist.

(59.10) and (59.11) imply:

(59.12) $F_1' := M_1' \cup B_1'$ and $F_2' := M_2' \cup B_2'$ are matching forests.

To see this, we must show that $R(M_1') \cup R(B_1') = V$ and $R(M_2') \cup R(B_2') = V$. Since $R(M_1) \cup R(B_1) = V$ and $R(M_2) \cup R(B_2) = V$, this follows directly from (59.10) and (59.11). This shows (59.12).

Since $R(F) = R(M) \cap R(B)$ for any matching forest F (with $M := F \cap E$ and $B := F \cap A$), (59.10) and (59.11) imply that also $R(F'_1)$ and $R(F'_2)$ arise from $R(F_1)$ and $R(F_2)$ by swapping on P; that is:

(59.13) $\quad R(F'_1) = (R(F_1) \setminus VP) \cup (R(F_2) \cap VP)$ and
$R(F'_2) = (R(F_2) \setminus VP) \cup (R(F_1) \cap VP)$.

This implies:

(59.14) $\quad s \in R(F'_1) \setminus R(F'_2)$,

since $s \in VP$ and $s \in R(F_2) \setminus R(F_1)$.

We study the effects of the exchanges (59.10) and (59.11), to show that one of the alternatives (59.6) holds. It is based on the following observations on the sizes of M'_1 and B'_1. Let t be the last vertex of P (possible $t = s$).

Suppose that none of the alternatives (59.6) hold. If $s = t$, then s is not covered by M_1, and so $M'_1 = M_1$ and $R(B'_1) = R(B_1) \cup \{s\}$, implying $|F'_1| < |F_1|$, which is alternative (59.6)(i). So $s \neq t$.

By the exchanges made, $|M_1| - |M'_1| = |M_1 \cap EP| - |M_2 \cap EP|$ and $|R(F_1)| - |R(F'_1)| = |R(F_1) \cap VP| - |R(F_2) \cap VP|$. This gives, as $|F'_1| \geq |F_1|$, since alternative (59.6)(i) does not hold:

(59.15) $\quad |M_1 \cap EP| - |M_2 \cap EP| + |R(F_1) \cap VP| - |R(F_2) \cap VP|$
$= |M_1| + |R(F_1)| - |M'_1| - |R(F'_1)| = |F'_1| - |F_1| \geq 0.$

(The last equality holds as $|F'_i| = |V| - |M'_i| - |R(F'_i)|$ for $i = 1, 2$, since $|F'_i| + |M'_i|$ is the number of heads of edges in F'_i.)

We next note:

(59.16) $\quad$ no intermediate vertex v of P belongs to $R(F_1) \cup R(F_2)$.

For suppose that $v \in R(F_1)$. Then (as v is an intermediate vertex of P) v is covered by M_2 and some $e_K \in N$. Hence $v \in R(B_2)$, and therefore $v \notin R(B_1)$ (by (59.7)), contradicting the fact that $v \in R(F_1)$. One similarly shows that $v \notin R(F_2)$, proving (59.16).

As $s \in R(F_2) \setminus R(F_1)$, (59.16) implies that

(59.17) $\quad |R(F_1) \cap VP| \leq |R(F_2) \cap VP|$, with equality if and only if $t \in R(F_1) \setminus R(F_2)$.

With (59.15) this gives that $|M_1 \cap EP| \geq |M_2 \cap EP|$.

Let k be the number of edges in $M_1 \cup M_2$ on P. Note that the edges in $M_1 \cup M_2$ occur along P alternatingly in M_1 and M_2, as any intermediate $e_K \in N$ on P connects an edge in M_1 and an edge in M_2 (as by (59.7), $e_K \in N$ consists of a vertex not in $R(B_2)$ and a vertex not in $R(B_1)$).

Suppose that k is odd. Then $|M_1 \cap EP| = |M_2 \cap EP| + 1$. So the last edge in $M_1 \cup M_2$ along P (seen from s) belongs to M_1. Moreover, one has that $t \notin R(F_1)$. For if $t \in R(F_1)$, then t is not covered by M_1, and hence t belongs to some $e_K = \{v, t\} \in N$ with v covered by M_1. Hence $v \in R(B_1)$, and hence $t \notin R(B_1)$ (by (59.7)), contradicting the fact that $t \in R(F_1)$. So $t \notin R(F_1)$.

Then (59.17) implies that $|R(F_2) \cap VP| > |R(F_1) \cap VP|$. This implies with (59.15) that $|F_1'| = |F_1|$ (as $|M_1 \cap EP| = |M_2 \cap EP| + 1$), and with (59.13) that $|R(F_1')| > |R(F_1)|$. So (59.6)(ii) holds, a contradiction.

So k is even, and hence $|M_1 \cap EP| = |M_2 \cap EP|$, which implies with (59.13), (59.15), and (59.17) that $|R(F_1)| = |R(F_2)|$ and $t \in R(F_1) \setminus R(F_2)$. Therefore, $|F_1'| = |F_1|$ (by (59.16)) and $R(F_1') = (R(F_1) \setminus \{t\}) \cup \{s\}$.

Finally, $|R(B_1') \cap K| = |R(B_1) \cap K|$ for each strong component K of D. This follows directly (with (59.11)) from the fact that for any $v \in K \cap VP$ one has either $K = \{v\}$ (if $|K| = 1$) or $v \in e_K$ (if $|K| \geq 2$). For suppose that $v \in VP$ is incident with no $e_K \in N$. We show that $v \in R(B_1) \cap R(B_2)$, implying $\{v\} \in \mathcal{K}$. If v is an intermediate vertex of P, then v is covered by M_1 and M_2 and hence v belongs to $R(B_1)$ and $R(B_2)$. If $v = s$, then $v \in R(F_2)$ (so $v \in R(B_2)$) and v is covered by M_1, so $v \in R(B_1)$. If $v = t$, then $v \in R(F_1)$ (so $v \in R(B_1)$) and v is covered by M_2, so $v \in R(B_2)$. ∎

59.5. The matching forest polytope

The *matching forest polytope* of a mixed graph (V, E, A) is the convex hull of the incidence vectors of the matching forests. So the matching forest polytope is a polytope in $\mathbb{R}^{E \cup A}$.

Giles [1982b] showed that the matching forest polytope is determined by the following inequalities:

(59.18)
(i)	$x_e \geq 0$	for each $e \in E \cup A$,
(ii)	$x(\delta^{\text{head}}(v)) \leq 1$	for each $v \in V$,
(iii)	$x(\gamma(\mathcal{L})) \leq \lfloor \lvert \bigcup \mathcal{L} \rvert - \frac{1}{2}\lvert\mathcal{L}\rvert \rfloor$	for each subpartition $\mathcal{L}$ of V with $\lvert\mathcal{L}\rvert$ odd and all classes nonempty.

Here we use the following notation and terminology. $\delta^{\text{head}}(v)$ denotes the set of edges with head v. A *subpartition* of V is a collection of disjoint subsets of V. As usual, $\bigcup \mathcal{L}$ denotes the union of the sets in $\mathcal{L}$. For each subpartition $\mathcal{L}$, we define:

(59.19) $\gamma(\mathcal{L}) :=$ the set of undirected edges spanned by $\bigcup \mathcal{L}$ and directed edges spanned by any set in $\mathcal{L}$.

The inequalities (i) and (ii) in (59.18) are trivially valid for the incidence vector of any matching forest F. To see that (iii) is valid, we can assume that $F \subseteq \gamma(\mathcal{L})$ and that $V = \bigcup \mathcal{L}$. Then $|R(F \cap A)| \geq |\mathcal{L}|$, since each set in $\mathcal{L}$ contains at least one root of $F \cap A$ (since no directed edge enters any set in $\mathcal{L}$). Moreover, $|F \cap E| \leq \lfloor \frac{1}{2}|R(F \cap A)| \rfloor$, since $F \cap E$ is a matching on a subset of $R(F \cap A)$. As $|F \cap A| = |V| - |R(F \cap A)|$, this gives:

(59.20)
$$|F| = |F \cap E| + |F \cap A| \leq \lfloor \tfrac{1}{2}|R(F \cap A)| \rfloor + (|V| - |R(F \cap A)|) = \lfloor |V| - \tfrac{1}{2}|R(F \cap A)| \rfloor \leq \lfloor \left|\bigcup \mathcal{L}\right| - \tfrac{1}{2}|\mathcal{L}| \rfloor,$$

as required.

Each integer solution x of (59.18) is the incidence vector of a matching forest. Indeed, as x is a 0,1 vector by (i) and (ii), we know that $x = \chi^F$ for some $F \subseteq E \cup A$. By (ii), each vertex is head of at most one edge in F. Hence, if F would contain a circuit (in the underlying undirected graph), it is a directed circuit C. But then for $\mathcal{L} := \{VC\}$, condition (iii) is violated. So F is a matching forest.

We show that system (59.18) is totally dual integral. This implies that it determines an integer polytope, which therefore is the matching forest polytope.

The proof method is a generalization of the method in Section 25.3a for proving the Cunningham-Marsh formula, stating that the matching constraints are totally dual integral.

The total dual integrality of (59.18) is equivalent to the following. For any weight function $w : E \cup A \to \mathbb{Z}$, let ν_w denote the maximum weight of a matching forest. Call a matching forest F *w-maximal* if $w(F) = \nu_w$. Let Λ be the set of subpartitions $\mathcal{L}$ of V with $|\mathcal{L}|$ odd and with all classes nonempty.

Then the total dual integrality of (59.18) is equivalent to: for each weight function $w : E \cup A \to \mathbb{Z}$, there exist $y : V \to \mathbb{Z}_+$ and $z : \Lambda \to \mathbb{Z}_+$ satisfying

(59.21) $$\sum_{v \in V} y_v + \sum_{\mathcal{L} \in \Lambda} z(\mathcal{L}) \lfloor |\bigcup \mathcal{L}| - \tfrac{1}{2}|\mathcal{L}| \rfloor \leq \nu_w$$

and

(59.22) $$\sum_{v \in V} y_v \chi^{\delta^{\text{head}}(v)} + \sum_{\mathcal{L} \in \Lambda} z(\mathcal{L}) \chi^{\gamma(\mathcal{L})} \geq w.$$

Now we can derive (Schrijver [2000b]):

Theorem 59.3. *For each mixed graph* (V, E, A)*, system* (59.18) *is totally dual integral.*

Proof. We must prove that for each mixed graph (V, E, A) and each function $w : E \cup A \to \mathbb{Z}$, there exist y, z satisfying (59.21) and (59.22).

In proving this, we can assume that w is nonnegative. For suppose that w has negative entries, and let w' be obtained from w by setting all negative entries to 0. As $\nu_{w'} = \nu_w$ and $w' \geq w$, any y, z satisfying (59.21) and (59.22) with respect to w', also satisfy (59.21) and (59.22) with respect to w.

Suppose that the theorem is not true. Choose a counterexample (V, E, A) and $w : E \cup A \to \mathbb{Z}_+$ with $|V| + |E \cup A| + \sum_{e \in E \cup A} w(e)$ as small as possible.

Then the underlying undirected graph of (V, E, A) is connected, since otherwise one of the components will form a smaller counterexample. Moreover, $w(e) \geq 1$ for each edge e, since otherwise we can delete e to obtain a smaller counterexample.

Next:

(59.23) for each $v \in V$, there exists a w-maximal matching forest F with $v \in R(F)$.

For suppose that no such matching forest exists. For any edge e, let $w'(e) := w(e) - 1$ if v is head of e and $w'(e) := w(e)$ otherwise. Then $\nu_{w'} = \nu_w - 1$. By the minimality of w, there exist y, z satisfying (59.21) and (59.22) with respect to w'. Replacing y_v by $y_v + 1$ we obtain y, z satisfying (59.21) and (59.22) with respect to w, contradicting our assumption. This proves (59.23).

This implies:

(59.24) each weak component of the directed graph (V, A) is strongly connected.

To see this, it suffices to show that each directed edge $e = (u, v)$ is contained in some directed circuit. By (59.23) there exists a w-maximal matching forest F with $v \in R(F)$. Then the weak component of F containing v is an arborescence rooted at v. As F has maximum weight, $F \cup \{e\}$ is not a matching forest, and hence $F \cap A$ contains a directed $v - u$ path. This makes a directed circuit containing e, and proves (59.24).

Let $\mathcal{K}$ denote the collection of strong components of (V, A). Define $w'(e) := w(e) - 1$ for each edge e. The remainder of this proof consists of showing that $|\mathcal{K}|$ is odd (so $\mathcal{K} \in \Lambda$), and that

(59.25) $\nu_w \geq \nu_{w'} + \lfloor |V| - \frac{1}{2}|\mathcal{K}| \rfloor$.

This is enough, since, by the minimality of w, there exist y, z satisfying (59.21) and (59.22) with respect to w'. Replacing $z(\mathcal{K})$ by $z(\mathcal{K}) + 1$ we obtain y, z satisfying (59.21) and (59.22) with respect to w (note that $\gamma(\mathcal{K}) = E \cup A$), contradicting our assumption.

To show (59.25), choose a w'-maximal matching forest F of maximum size $|F|$. Under this condition, choose F such that it maximizes $|R(F)|$.

We show that for each $s \in V$ the following holds, where r is the root of the arborescence[14] in $F \cap A$ containing s:

(59.26) there exist a $t \in R(F)$ and a w'-maximal matching forest F' satisfying $|F'| = |F|$, $R(F') = (R(F) \setminus \{t\}) \cup \{s\}$, and $|R(F' \cap A) \cap K| = R(F \cap A) \cap K|$ for each strong component K of (V, A); if $r \in R(F)$, then moreover $t = r$ and $R(F' \cap A) = (R(F \cap A) \setminus \{r\}) \cup \{s\}$.

Let $F_1 := F$ and let F_2 be a w-maximal forest with $s \in R(F_2)$ (which exists by (59.23)). We first find F_1' and F_2' as follows.

If $r \notin R(F)$, then $s \notin R(F) = R(F_1)$ (since otherwise s is a root of $F \cap A$, and hence $r = s \in R(F)$). Applying Lemma 59.3α to F_1 and F_2 yields the matching forests F_1' and F_2'.

If $r \in R(F)$, then $s \notin R(F \cap A)$. Apply Theorem 53.2 to $B_1 := F_1 \cap A$ and $B_2 := F_2 \cap A$. It yields branchings B_1' and B_2' in (V, A) satisfying $B_1' \cap B_2' =$

[14] An *arborescence in* a branching B is a weak component of (V, B), or just the arc set of it.

$B_1 \cap B_2$, $B_1' \cup B_2' = B_1 \cup B_2$, and $R(B_1') = R(B_1) \cup \{s\}$ or $R(B_1') = (R(B_1) \setminus \{r\}) \cup \{s\}$. This implies $R(B_2') = R(B_2) \setminus \{s\}$ or $R(B_2') = (R(B_2) \setminus \{s\}) \cup \{r\}$. Now define $F_i' := (F_i \cap E) \cup B_i'$ for $i = 1, 2$. Then the F_i' are matching forests, since $r \in R(F_1 \cap E)$ and $s \in R(F_2 \cap E)$.

In both constructions, $|F_1'| \leq |F_1|$, and if $|F_1'| = |F_1|$, then $|R(F_1')| \geq |R(F_1)|$. Moreover,

(59.27) $$\chi^{F_1'} + \chi^{F_2'} = \chi^{F_1} + \chi^{F_2},$$

which implies that $w(F_1') + w(F_2') = w(F_1) + w(F_2)$. Hence

(59.28) $$\begin{aligned} w'(F_1') + w(F_2') &= w(F_1') + w(F_2') - |F_1'| \geq w(F_1) + w(F_2) - |F_1| \\ &= w'(F_1) + w(F_2). \end{aligned}$$

Therefore, since F_1 is a w'-maximal matching forest and F_2 is a w-maximal matching forest, we have equality throughout in (59.28). So F_1' is w'-maximal and $|F_1'| = |F_1|$. Hence $|R(F_1')| \geq |R(F_1)|$. Then, by the maximality of $|R(F)|$, we know that $|R(F_1')| = |R(F_1)|$.

Set $F' := F_1'$. If $r \notin R(F)$, we know that (59.6)(iii) holds, which gives (59.26). If $r \in R(F)$, then (59.26) holds for $t := r$, and $R(B_1') = (R(B_1) \setminus \{t\}) \cup \{s\}$ or $R(B_2') = (R(B_2) \setminus \{s\}) \cup \{t\}$ (since $|F_1'| = |F_1|$, $F_1' \cap E = F_1 \cap E$, $|F_2'| = |F_2|$, $F_2' \cap E = F_2 \cap E$). Moreover, s and t belong to the same strong component of (V, A): as $r = t$ is the root of the arborescence in $F_1 \cap A$ containing s, there exists a $t - s$ path in (V, A); since each weak component of (V, A) is a strong component (by (59.24)), there is a directed $s - t$ path in (V, A). This implies (59.26).

Note that (59.26) implies in particular that $R(F) \neq \emptyset$. Suppose $|R(F)| \geq 2$. Choose F under the additional condition that the minimum distance in (V, E, A) between distinct vertices $u, v \in R(F)$ is as small as possible. Here, the distance in (V, E, A) is the length of a shortest $u - v$ path in the underlying undirected graph.

Necessarily, this distance is at least two, since otherwise we can extend F by an edge connecting u and v, thereby maintaining w'-maximality but increasing the size. This contradicts the maximality of $|F|$.

So we can choose an intermediate vertex s on a shortest $u - v$ path. Let F' be the matching forest described in (59.26), with $t \in R(F)$. By symmetry of u and v we can assume that $t \neq u$. So $u, s \in R(F')$, contradicting the choice of F, as the distance of u and s is smaller than that of u and v.

This implies that $|R(F)| = 1$. Let $R(F) = \{r\}$ and let K be the strong component of (V, A) containing r. We choose F (and r) under the additional constraint that $|R(F \cap A) \cap K|$ is as large as possible.

Suppose $|R(F \cap A) \cap K| \geq 2$. Choose F under the additional constraint that r has minimal distance in (V, A) from some root u of $F \cap A$ in $K \setminus \{r\}$. In this case, the distance in (V, A) from u to r is the length of a shortest directed $u - r$ path. (Such a path exists, since K is strongly connected.)

Let T be the arborescence in $F \cap A$ containing r. Let s be the first vertex on a shortest directed $u - r$ path Q in (V, A) that belongs to T. Necessarily

$s \neq r$, since otherwise we can extend F by the last edge of Q, contradicting the maximality of $|F|$.

Let F' be the matching forest described in (59.26). Then $s \in R(F')$ and $R(F' \cap A) = (R(F \cap A) \setminus \{r\}) \cup \{s\}$. Hence u remains a root of $F' \cap A$, while the distance in (V, A) from u to s is shorter than that from u to r. This contradicts our choice of F (replacing K, r by L, s).

So $|R(F \cap A) \cap K| = 1$. Suppose that there exists a component L of (V, A) with $|R(F \cap A) \cap L| \geq 2$. Choose s in L arbitrarily. Let F' be the matching forest described in (59.26). Then $s \in R(F')$ while $|R(F' \cap A) \cap L| \geq 2$, contradicting the choice of F.

So no such component L exists; that is, each $L \in \mathcal{K}$ contains exactly one root of $F \cap A$. So $|F \cap A| = |V| - |\mathcal{K}|$. Moreover, as $|R(F)| = 1$, $|\mathcal{K}|$ is odd and $|F \cap E| = \lfloor \frac{1}{2}|\mathcal{K}| \rfloor$. So $|F| = |F \cap A| + |F \cap E| = \lfloor |V| - \frac{1}{2}|\mathcal{K}| \rfloor$. Hence

(59.29) $$\nu_w \geq w(F) = w'(F) + |F| = \nu_{w'} + |F| = \nu_{w'} + \lfloor |V| - \tfrac{1}{2}|\mathcal{K}| \rfloor,$$

thus proving (59.25). ∎

We remark that the optimum dual solution y, z constructed in this proof has the following additional property: if $\mathcal{K}, \mathcal{L} \in \Lambda$ and $z(\mathcal{K}), z(\mathcal{L}) > 0$, then $\mathcal{K}$ and $\mathcal{L}$ are 'laminar' in the following sense:

(59.30) $$\begin{aligned} &\forall K \in \mathcal{K} \;\; \exists L \in \mathcal{L} : K \subseteq L, \\ &\text{or } \forall L \in \mathcal{L} \;\; \exists K \in \mathcal{K} : L \subseteq K, \\ &\text{or } \forall K \in \mathcal{K} \;\; \forall L \in \mathcal{L} : K \cap L = \emptyset. \end{aligned}$$

Theorem 59.3 implies the characterization of the matching forest polytope of Giles [1982b]:

Corollary 59.3a. *For each mixed graph* (V, E, A)*, the matching forest polytope is determined by* (59.18).

Proof. By Theorems 59.3 and 5.22, the vertices of the polytope determined by (59.18) are integer. Since the integer solutions of (59.18) are the incidence vectors of matching forests, this proves the corollary. ∎

59.6. Further results and notes

59.6a. Matching forests in partitionable mixed graphs

Call a mixed graph $G = (V, E, A)$ *partitionable (into R and S)* if V can be partitioned into classes R and S such that each undirected edge connects R and S, while each directed arc is spanned by R or by S.

Trivially, a mixed graph is partitionable if and only if each circuit has an even number of undirected edges. That is, by contracting all directed arcs we obtain a bipartite graph. (Another characterization is: the incidence matrix is totally unimodular.)

In a different form, we have studied matching forests in partitionable mixed graphs before. Let $G = (V, E, A)$ be a mixed graph partitionable into R and S. Orient the edges in E from R to S, and turn the orientation of any arc in A spanned by R. We obtain a directed graph $D' = (V, A')$. Then it is easy to see that:

(59.31) a set of edges and arcs of G is a matching forest $\iff$ the corresponding arcs in D' form an $R - S$ bifurcation.

This implies that a number of theorems on matching forests in a partitionable mixed graph can be obtained from those on $R - S$ bifurcations. First we have:

Theorem 59.4. *Let $G = (V, E, A)$ be a partitionable mixed graph. Then the maximum size of a matching forest in G is equal to the minimum size of $|V| - |\mathcal{L}|$, where $\mathcal{L}$ is a collection of strong components K of the directed graph $D = (V, A)$ with $d_D^{\text{in}}(K) = 0$ such that no edge in E connects two components in $\mathcal{L}$.*

Proof. This is equivalent to Theorem 54.9. ∎

We similarly obtain a min-max relation for the maximum weight of a matching forest in a partitionable mixed graph, by the total dual integrality of the following system:

(59.32)
(i) $x_e \geq 0$ for each $e \in E \cup A$,
(ii) $x(\delta^{\text{head}}(v)) \leq 1$ for each $v \in V$,
(iii) $x(A[U]) \leq |U| - 1$ for each nonempty U with $U \subseteq R$ or $U \subseteq S$.

Here $\delta^{\text{head}}(v)$ is the set of edges and arcs having v as head.

Theorem 59.5. *If G is a mixed graph partitionable into R and S, then* (59.32) *is TDI and determines the matching forest polytope.*

Proof. This is equivalent to Corollary 54.10a. ∎

For covering by matching forests in partitionable mixed graphs we have:

Theorem 59.6. *Let $G = (V, E, A)$ be a mixed graph partitionable into R and S. Then $E \cup A$ can be covered by k matching forests if and only if*

(59.33)
(i) $|\delta^{\text{head}}(v)| \leq k$ *for each* $v \in V$;
(ii) $|A[U]| \leq k(|U| - 1)$ *for each nonempty subset* U *of* R *or* S.

Proof. This is equivalent to Corollary 54.11c. ∎

The case $A = \emptyset$ is Kőnig's edge-colouring theorem (Theorem 20.1).

An equivalent, polyhedral way of formulating Theorem 59.6 is:

Corollary 59.6a. *If G is a partitionable mixed graph, then the matching forest polytope has the integer decomposition property.*

Proof. Directly from Theorem 59.6. ∎

59.6b. Further notes

The facets of the matching forest polytope are characterized in Giles [1982c].

Matching forests form a special case of matroid matching. Let $G = (V, E, A)$ be a mixed graph. Consider the space $\mathbb{R}^V \times \mathbb{R}^V$. Associate with any undirected edge $e = uv \in E$, the pair $(\chi^u, 0)$, $(\chi^v, 0)$ of vectors in $\mathbb{R}^V \times \mathbb{R}^V$. Associate with any directed arc $a = (u, v) \in A$, the pair $(\chi^v, 0)$, $(0, \chi^u - \chi^v)$ of vectors in $\mathbb{R}^V \times \mathbb{R}^V$. One easily checks that $M \subseteq E \cup A$ is a matching forest if and only if its associated pairs form a matroid matching. Thus matroid matching theory implies a min-max relation and a polynomial-time algorithm for the maximum size of a matching forest. However, as we saw in Section 59.2, there is an easy direct method for this.

Chapter 60

Submodular functions on directed graphs

At two structures we came across the proof technique of making a collection of subsets cross-free: at submodular functions (like in polymatroid intersection) and at directed graphs (like in the proof of the Lucchesi-Younger theorem).

Edmonds and Giles [1977] combined the two structures into one general framework, consisting of a submodular function defined on the vertex set of a directed graph. Johnson [1975a] and Frank [1979b] designed a variant of Edmonds and Giles' framework, containing the polymatroid intersection theorem and the optimum arborescence theorem as special cases.

We first describe the results of Edmonds and Giles, and after that we present a variant, from which the results of Frank can be derived. At the base is the method of Edmonds and Giles to represent any cross-free family by a directed tree (the *tree-representation*) and to derive a network matrix if the family consists of subsets of the vertex set of a directed graph — see Section 13.4.

60.1. The Edmonds-Giles theorem

Let $D = (V, A)$ be a digraph and let $\mathcal{C}$ be a crossing family of subsets of V (that is, if $T, U \in \mathcal{C}$ with $T \cap U \neq \emptyset$ and $T \cup U \neq V$, then $T \cap U, T \cup U \in \mathcal{C}$). A function $f : \mathcal{C} \to \mathbb{R}$ is called *submodular on crossing pairs*, or *crossing submodular*, if for all $T, U \in \mathcal{C}$ with $T \cap U \neq \emptyset$ and $T \cup U \neq V$ one has

(60.1) $\qquad f(T) + f(U) \geq f(T \cap U) + f(T \cup U).$

Given such $D, \mathcal{C}, f$, a *submodular flow* is a function $x \in \mathbb{R}^A$ satisfying:

(60.2) $\qquad x(\delta^{\text{in}}(U)) - x(\delta^{\text{out}}(U)) \leq f(U)$ for each $U \in \mathcal{C}$.

The set P of all submodular flows is called the *submodular flow polyhedron*.

Equivalently, P is equal to the set of all vectors x in $\mathbb{R}^A$ with the property that the 'gain' vector of x is in the extended polymatroid EP_f. (The *excess function* of x equals Mx where M is the $V \times A$ incidence vector of D.)

Then Edmonds and Giles [1977] showed:

Theorem 60.1 (Edmonds-Giles theorem)**.** *If f is crossing submodular, then* (60.2) *is box-TDI.*

Proof. Choose $w \in \mathbb{R}^A$, and let y be an optimum solution to the dual of maximizing $w^{\mathsf{T}}x$ over (60.2):

$$\min\{\sum_{U\in\mathcal{C}} y(U)f(U) \mid y \in \mathbb{R}^{\mathcal{C}}_{+}, \sum_{U\in\mathcal{C}} y(U)(\chi^{\delta^{\mathrm{in}}(U)} - \chi^{\delta^{\mathrm{out}}(U)}) = w\}. \tag{60.3}$$

Choose y such that

$$\sum_{U\in\mathcal{C}} y(U)|U||V \setminus U| \tag{60.4}$$

is as small as possible. Let $\mathcal{C}_0 := \{U \in \mathcal{C} \mid y(U) > 0\}$. We first prove that $\mathcal{C}_0$ is cross-free.

Suppose to the contrary that $T, U \in \mathcal{C}_0$ with $T \not\subseteq U \not\subseteq T$, $T \cap U \neq \emptyset$, $T \cup U \neq V$. Let $\alpha := \min\{y(T), y(U)\} > 0$. Then decreasing $y(T)$ and $y(U)$ by α, and increasing $y(T \cap U)$ and $y(T \cup U)$ by α, maintains feasibility of z, u, y, while its value is not increased (hence it remains optimum). However, sum (60.4) decreases (by Theorem 2.1). This contradicts the minimality of (60.4).

As $\mathcal{C}_0$ is cross-free, the submatrix formed by the constraints corresponding to $\mathcal{C}_0$ is totally unimodular (by Corollary 13.21a). Hence, by Theorem 5.35, (60.2) is box-TDI. ∎

Note that the proof also yields that the solution y in (60.3) can be taken such that the collection $\{U \in \mathcal{C} \mid y(U) > 0\}$ is cross free.

Box-TDI implies primal integrality (a polyhedron P is *box-integer* if $P \cap \{x \mid d \leq x \leq c\}$ is integer for all integer vectors d, c):

Corollary 60.1a. *If f is integer, the polyhedron determined by* (60.2) *is box-integer.*

Proof. By Theorem 60.1, $\max\{w^{\mathsf{T}}x \mid x \in P\}$ is achieved by an integer solution x, for each vector w. ∎

Complexity. The algorithmic results on polymatroid intersection of Cunningham and Frank [1985] and Fujishige, Röck, and Zimmermann [1989] imply that the optimization problem associated with the Edmonds-Giles theorem can be solved in strongly polynomial time.

Indeed, let $D = (V, A)$ be a digraph, let $\mathcal{C}$ be a crossing family, let $f : \mathcal{C} \to \mathbb{Q}$ be crossing submodular, and let $c, d, l : A \to \mathbb{Q}$. If we want to find a submodular flow x with $d \leq x \leq c$ minimizing $l^{\mathsf{T}}x$, we can assume that all arcs in A are vertex-disjoint. Moreover, we can assume that for each arc $a = (u, v) \in A$ we have $f(\{v\}) = c(a)$ and $f(\{u\}) = -d(a)$. Hence we can ignore d and c, and assume that we want to find a submodular flow x minimizing $l^{\mathsf{T}}x$.

Now define $\mathcal{C}_2 := \{\{u,v\} \mid (u,v) \in A\}$ and $f_2(\{u,v\}) := 0$, $w(v) := l(u,v)$, and $w(u) := 0$, for each $(u,v) \in A$. Then the problem is equivalent to finding a vector x in $EP_f \cap EP_{f_2}$ with $x(V) = 0$ and minimizing $w^{\mathsf{T}}x$. This can be solved in strongly polynomial time by Theorem 49.9.

(Frank [1982b] gave a strongly polynomial-time algorithm for the special case if f is integer, $c = \mathbf{1}$, and $d = \mathbf{0}$.)

A similar reduction of submodular flows to polymatroid intersection was given by Kovalev and Pisaruk [1984].

60.1a. Applications

Network flows. If we take $\mathcal{C} := \{\{v\} \mid v \in V\}$ and $f = \mathbf{0}$, then (60.2) determines circulations, and Theorem 60.1 passes into a theorem on minimum-cost circulations. It may be specialized easily to several other results on flows in networks, e.g., to the max-flow min-cut theorem (Theorem 10.3; take $d = \mathbf{0}, c \geq \mathbf{0}$, and $w(a) = 0$ for $a \neq (s,r)$ and $w((s,r)) = 1$) and to Hoffman's circulation theorem (Theorem 11.2).

Lucchesi-Younger theorem. Let $D = (V, A)$ be a digraph and define

(60.5) $\qquad \mathcal{C} := \{U \subseteq V \mid \emptyset \neq U \neq V \text{ and } d_A^{\text{out}}(U) = 0\}.$

So $\mathcal{C}$ consists of all sets U such that the collection of arcs entering U forms a directed cut. Taking $f := -\mathbf{1}$, $c := \mathbf{0}$, $d := -\boldsymbol{\infty}$, and $w := \mathbf{1}$, Theorem 60.1 passes into the Lucchesi-Younger theorem (Theorem 55.2, cf. Corollary 55.2b): the minimum size of a directed cut cover is equal to the maximum number of disjoint directed cuts. For arbitrary w we obtain a weighted version.

Polymatroid intersection. Let f_1 and f_2 be nonnegative submodular set function on S. Let S' and S'' be two disjoint copies of S, let $V = S' \cup S''$, and define $\mathcal{C}$ by

(60.6) $\qquad \mathcal{C} := \{U' \mid U \subseteq S\} \cup \{S' \cup U'' \mid U \subseteq S\}$

where U' and U'' denote the sets of copies of elements of U in S' and S''. Define $f : \mathcal{C} \to \mathbb{R}_+$ by

(60.7)
$$\begin{aligned} f(U') &:= f_1(U) && \text{for } U \subseteq S,\\ f(V \setminus U'') &:= f_2(U) && \text{for } U \subseteq S,\\ f(S') &:= \min\{f_1(S), f_2(S)\}. \end{aligned}$$

Then $\mathcal{C}$ and f satisfy (60.1). If we take $d = \mathbf{0}$ and $c = \boldsymbol{\infty}$, Theorem 60.1 passes into the polymatroid intersection theorem (Corollary 46.1a, cf. Theorem 46.1).

Frank and Tardos [1989] showed that also Theorem 44.7 (a generalization of Lovász [1970a] of Kőnig's matching theorem) fits into the Edmonds-Giles model. For applications of the Edmonds-Giles theorem to graph orientation, see Chapter 61.

60.1b. Generalized polymatroids and the Edmonds-Giles theorem

The Edmonds-Giles theorem (Theorem 60.1) also comprises the total dual integrality of the system defining the intersection of two generalized polymatroids (Section 49.11b). Indeed, let S be a finite set, let, for $i = 1, 2$, $\mathcal{C}_i$ and $\mathcal{D}_i$ be collections of subsets of S, and let $f_i : \mathcal{C}_i \to \mathbb{R}$ and $g_i : \mathcal{D}_i \to \mathbb{R}$ form a paramodular pair (f_i, g_i). Then the system

(60.8)
$$\begin{array}{ll} x(U) \leq f_1(U) & \text{for } U \in \mathcal{C}_1, \\ x(U) \geq g_1(U) & \text{for } U \in \mathcal{D}_1, \\ x(U) \leq f_2(U) & \text{for } U \in \mathcal{C}_2, \\ x(U) \geq g_2(U) & \text{for } U \in \mathcal{D}_2, \end{array}$$

is box-totally dual integral, which is Corollary 49.12b.

To see this as a special case of the Edmonds-Giles theorem, let S_1 and S_2 be disjoint copies of S, and let $V := S_1 \cup S_2$. For each $s \in S$, let a_s be the arc (s_2, s_1), where s_1 and s_2 are the copies of s in S_1 and S_2 respectively. Let $A := \{a_s \mid s \in S\}$.

Let

(60.9)
$$\mathcal{C} := \{U_1 \mid U \in \mathcal{C}_1\} \cup \{V \setminus U_1 \mid U \in \mathcal{D}_1\} \cup \{V \setminus U_2 \mid U \in \mathcal{C}_2\} \cup \{U_2 \mid U \in \mathcal{D}_2\},$$

where U_i denotes the set of copies of the elements in U in S_i $(i = 1, 2)$. It is easy to see that $\mathcal{C}$ is a crossing family.

Define $f : \mathcal{C} \to \mathbb{R}$ by:

(60.10)
$$\begin{array}{ll} f(U_1) := f_1(U) & \text{for } U \in \mathcal{C}_1, \\ f(V \setminus U_1) := -g_1(U) & \text{for } U \in \mathcal{D}_1, \\ f(V \setminus U_2) := f_2(U) & \text{for } U \in \mathcal{C}_2, \\ f(U_2) := -g_2(U) & \text{for } U \in \mathcal{D}_2. \end{array}$$

(In case that $f(S_1)$ or $f(S_2)$ would be defined more than once, we take the smallest of the values.) Then f is submodular on crossing pairs. Now the system (in $x \in \mathbb{R}^A$)

(60.11)
$$x(\delta^{\text{in}}(U)) - x(\delta^{\text{out}}(U)) \leq f(U) \text{ for } U \in \mathcal{C}$$

is the same as (60.8) (after renaming each variable $x(s)$ to $x(a_s)$). So the box-total dual integrality of (60.8) follows from the Edmonds-Giles theorem.

Frank [1984b] showed that, conversely, the solution set of the 'Edmonds-Giles' system (60.2) is the projection of the intersection of two generalized polymatroids.

60.2. A variant

We now give a theorem similar to Theorem 60.1, which includes as special cases again the Lucchesi-Younger theorem and the polymatroid intersection theorem, and moreover theorems on optimum arborescences, bibranchings, and strong connectors.

For any digraph $D = (V, A)$ and any family $\mathcal{C}$ of subsets of V, define the $\mathcal{C} \times A$ matrix M by

(60.12)
$$M_{U,a} := \begin{cases} 1 & \text{if } a \text{ enters } U, \\ 0 & \text{otherwise,} \end{cases}$$

for $U \in \mathcal{C}$ and $a \in A$.

This matrix is totally unimodular if $\mathcal{C}$ is cross-free and the following condition holds:

(60.13) if $X, Y, Z \in \mathcal{C}$ with $X \subseteq V \setminus Y \subseteq Z$, then no arc of D enters both X and Z.

Theorem 60.2. *If $\mathcal{C}$ is cross-free and* (60.13) *holds, then M is totally unimodular.*

Proof. Let $T = (W, B)$ and $\pi : V \to W$ form a tree-representation for $\mathcal{C}$. For any arc $a = (u, v)$ of D, the set of forward arcs in the undirected $\pi(u) - \pi(v)$ path in T is contiguous, that is, forms a directed path, say from u' to v'. This follows from the fact that there exist no arcs b, c, d in this order on the path with b and d forward and c backward, by (60.13).

Define $a' := (u', v')$, and let $D' = (W, A')$ be the digraph with $A' := \{a' \mid a \in A\}$. Then M is equal to the network matrix generated by T and D' (identifying $b \in B$ with the set X_b in $\mathcal{C}$ determined by b). Hence by Theorem 13.20, M is totally unimodular. ∎

Recall that a function g on a crossing family $\mathcal{C}$ is called *supermodular on crossing pairs*, or *crossing supermodular*, if for all $T, U \in \mathcal{C}$:

(60.14) if $T \cap U \neq \emptyset$ and $T \cup U \neq V$, then $g(T) + g(U) \leq g(T \cap U) + g(T \cup U)$.

Consider the polyhedron P determined by:

(60.15)
$$\begin{aligned} &x_a \geq 0 && \text{for } a \in A,\\ &x(\delta^{\text{in}}(U)) \geq g(U) && \text{for } U \in \mathcal{C}. \end{aligned}$$

Theorem 60.3. *If g is crossing supermodular and* (60.13) *holds, then system* (60.15) *is box-TDI.*

Proof. Let $w \in \mathbb{R}^A$ and let y achieve the maximum in the dual of minimizing $w^{\mathsf{T}}x$ over (60.15):

(60.16)
$$\max\{\sum_{U \in \mathcal{C}} y(U)g(U) \mid y \in \mathbb{R}_+^{\mathcal{C}}, \sum_{U \in \mathcal{C}} y(U)\chi^{\delta^{\text{in}}(U)} \geq w\},$$

in such a way that

(60.17)
$$\sum_{U \in \mathcal{C}} y(U)|U||V \setminus U|$$

is as small as possible. Define

(60.18)
$$\mathcal{C}_0 := \{U \in \mathcal{C} \mid y(U) > 0\}.$$

We first show that $\mathcal{C}_0$ is cross-free. Suppose to the contrary that there are T, U in $\mathcal{C}$ with $T \not\subseteq U \not\subseteq T$, $T \cap U \neq \emptyset$, and $T \cup U \neq V$. Let $\alpha := \min\{y(T), y(U)\}$. Now decrease $y(T)$ and $y(U)$ by α, and increase $y(T \cap U)$ and $y(T \cup U)$ by α. Then y remains feasible and optimum, while sum (60.17) decreases (Theorem 2.1), a contradiction.

Since $\mathcal{C}_0$ determines a totally unimodular submatrix by Theorem 60.2, by Corollary 5.20b system (60.15) is box-TDI. ∎

Note that the proof yields that (60.16) has a solution y with $\{U \in \mathcal{C} \mid y(U) > 0\}$ cross-free. Condition (60.13) cannot be deleted, as is shown by Figure 60.1.

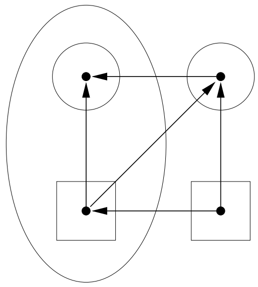

Figure 60.1
A collection and a digraph showing that condition (60.13) **cannot be deleted in Theorem 60.3.** In this Venn-diagram, the collection is represented by the *interiors* of the ellipses and by the *exteriors* of the rectangles.

Again, there is the following standard corollary for primal integrality:

Corollary 60.3a. *If g is integer, the polyhedron determined by* (60.15) *is box-integer.*

Proof. As before. ∎

Notes. Johnson [1975a] proved Theorem 60.3 for the special case that $\mathcal{C}$ is the collection of all nonempty subsets of $V \setminus \{r\}$ (where r is a fixed element of V), and Frank [1979b] extended this result to the case where $\mathcal{C}$ is any intersecting family of subsets of $V \setminus \{r\}$. Note that in this case condition (60.13) is trivially satisfied.

60.2a. Applications

We list some applications of Theorem 60.3, which may be compared with the applications of the Edmonds-Giles theorem (Section 60.1a).

Kőnig-Rado edge cover theorem. Let $G = (V, E)$ be a bipartite graph, with colour classes V_1 and V_2. Let $D = (V, A)$ be the digraph arising from G by orienting all edges from V_2 to V_1. Define $\mathcal{C} := \{\{v\} \mid v \in V_1\} \cup \{V \setminus \{v\} \mid v \in V_2\}$ and let $d := \mathbf{0}$, $c := \boldsymbol{\infty}$, $g := \mathbf{1}$, $w := \mathbf{1}$. Then Theorem 60.3 gives the Kőnig-Rado edge cover theorem (Theorem 19.4): the minimum size of an edge cover in a bipartite graph is equal to the maximum size of a stable set. Taking w arbitrary gives a weighted version.

Optimum arborescence theorem. Let $D = (V, A)$ be a digraph, let $r \in V$, and let $\mathcal{C}$ be the collection of all nonempty subsets of $V \setminus \{r\}$. Let $g := \mathbf{1}$, $d := \mathbf{0}$, $c := \boldsymbol{\infty}$, and let $w : A \to \mathbb{Z}_+$. Theorem 60.3 now gives the optimum arborescence theorem (Theorem 52.3): the minimum weight of an r-arborescence is equal to the maximum number of r-cuts such that no arc a is in more than $w(a)$ of these r-cuts.

Optimum bibranching theorem. Let $D = (V, A)$ be a digraph and let V be split into sets R and S. Define $\mathcal{C} := \{U \subseteq V \mid \emptyset \neq U \subseteq S \text{ or } S \subseteq U \subset V\}$ and $d := \mathbf{0}$, $c := \boldsymbol{\infty}$, $g := \mathbf{1}$, and let $w : A \to \mathbb{Z}_+$. Then Theorem 60.3 gives Corollary 54.8b: the minimum weight of a bibranching is equal to the maximum number of subsets in $\mathcal{C}$ such that no arc a enters more than $w(a)$ of these subsets.

Lucchesi-Younger theorem. Let $D = (V, A)$ be a digraph and let $\mathcal{C}$ be the collection of all nonempty proper subsets U of V with $\delta^{\text{out}}_A(U) = \emptyset$. Let $g := \mathbf{1}$, $d := \mathbf{0}$, $c := \boldsymbol{\infty}$, and $w := \mathbf{1}$. Then Theorem 60.3 gives the Lucchesi-Younger theorem (Theorem 55.2): the minimum size of a directed cut cover is equal to the maximum number of disjoint directed cuts. Taking w arbitrary, gives a weighted version.

Strong connectors. Suppose that $g = \mathbf{1}$, $d = \mathbf{0}$, and $c = \boldsymbol{\infty}$, and that for all $V_1, V_2 \in \mathcal{C}$ we have: if $V_1 \cap V_2 \neq \emptyset$, then $V_1 \cap V_2 \in \mathcal{C}$, and if $V_1 \cup V_2 \neq V$, then $V_1 \cup V_2 \in \mathcal{C}$. Then Theorem 60.3 is equivalent to Theorem 57.3.

Indeed, let $D = (V, A)$ and $D_0 = (V, A_0)$ be digraphs such that for each arc $a = (u, v)$ of D there are vertices u' and v' such that D_0 contains directed paths from u to u', from v' to v, and from v' to u'. Let $w : A \to \mathbb{Z}_+$. Then the minimum weight of a strong connector in D for D_0 is equal to the maximum number of D_0-cuts in D such that no arc a of D is in more than $w(a)$ of these D_0-cuts.

This can be derived from Theorem 60.3 by taking $\mathcal{C} := \{U \subseteq V \mid \emptyset \neq U \neq V, \delta^{\text{in}}_{A_0}(U) = \emptyset\}$. Conversely, if $\mathcal{C}$ satisfies the condition given above, we can take $A_0 := \{(u, v) \mid u, v \in V, (u, v) \text{ enters no } U \in \mathcal{C}\}$.

Polymatroid intersection. Let g_1 and g_2 be integer supermodular nondecreasing set functions on S with $g_1(\emptyset) = g_2(\emptyset) = 0$. Then

$$\begin{aligned}(60.19)\qquad &\min\{x(S) \mid x \in \mathbb{Z}^S_+, x(U) \geq g_i(U) \text{ for } U \subseteq S, i = 1, 2\}\\ &= \max_{U \subseteq S}(g_1(U) + g_2(S \setminus U)).\end{aligned}$$

This follows by taking disjoint copies S' and S'' of S, and setting $V := S' \cup S''$, $\mathcal{C} := \{T \subseteq V \mid T \subseteq S' \text{ or } S' \subseteq T\}$, $A := \{(s'', s') \mid s \in S\}$, $g(U') := g_1(U)$ and $g(V \setminus U'') := g_2(U)$ for $U \subseteq S$ (without loss of generality, $g_1(S) = g_2(S)$), $d := \mathbf{0}$, $c := \boldsymbol{\infty}$, $w := \mathbf{1}$.

By taking d, c, w arbitrary, several other (contra)polymatroid intersection theorems follow.

60.3. Further results and notes

60.3a. Lattice polyhedra

In a series of papers, Hoffman [1976a,1978] and Hoffman and Schwartz [1978] developed a theory of 'lattice polyhedra', which extends results of Johnson [1975a]. This theory has much in common with the theories described above.

Let $(L, \leq)$ be a partially ordered set and let $\wedge : L \times L \to L$ be a function such that

(60.20) for all $a, b \in L$: $a \wedge b \leq a$ and $a \wedge b \leq b$.

Let S be a finite set and let $\phi : L \to \mathcal{P}(S)$ be such that

(60.21) if $a < b < c$, then $\phi(a) \cap \phi(c) \subseteq \phi(b)$

for a, b, c in L. Let $\vee : L \times L \to L$ and let $f : L \to \mathbb{R}_+$ satisfy:

(60.22) $f(a \wedge b) + f(a \vee b) \leq f(a) + f(b)$

for all a, b in L. So f is, in a sense, *submodular*.

Define

(60.23) $S' := \{u \in S \mid \forall a, b \in L : \chi^{\phi(a\wedge b)}(u) + \chi^{\phi(a\vee b)}(u) \leq \chi^{\phi(a)}(u) + \chi^{\phi(b)}(u)\}$ and
$S'' := \{u \in S \mid \forall a, b \in L : \chi^{\phi(a\wedge b)}(u) + \chi^{\phi(a\vee b)}(u) \geq \chi^{\phi(a)}(u) + \chi^{\phi(b)}(u)\}$.

The polyhedron determined by:

(60.24)
$$\begin{array}{ll} x_u \geq 0 & (u \in S \setminus S'), \\ x_u \leq 0 & (u \in S \setminus S''), \\ x(\phi(a)) \leq f(a) & (a \in L). \end{array}$$

is called a *lattice polyhedron*. Hoffman and Schwartz [1978] showed that system (60.24) is box-totally dual integral.

Theorem 60.4. *System* (60.24) *is box-TDI.*

Proof. Choose $w \in \mathbb{R}^S_+$. Consider the dual of maximizing $w^{\mathsf{T}}x$ over (60.24):

(60.25) $\min\{y^{\mathsf{T}}f \mid y \in \mathbb{R}^L_+, \sum_{a\in L} y_a\chi^{\phi(a)} \leq w(u)$ if $u \in S'$ and $\sum_{a\in L} y_a\chi^{\phi(a)} \geq w(u)$ if $u \in S''\}$.

Order the elements of L as $a_1, \ldots, a_n$ such that if $a_i \leq a_j$, then $i \leq j$. Let y attain (60.25), such that $y(L)$ is minimal, and, under this condition, such that

(60.26) $(y(a_1), \ldots, y(a_n))$

is lexicographically maximal.

Then the collection $C := \{a \in L \mid y_a > 0\}$ is a chain in L. For suppose to the contrary that $a, b \in C$ with $a \not\leq b \not\leq a$. Let $\alpha := \min\{y_a, y_b\}$. Reset y by decreasing y_a and y_b by α, and increasing $y(a \wedge b)$ and $y(a \vee b)$ by α. One easily checks, using (60.22) and (60.23), that the new y again attains the minimum (60.25), and moreover that $(y(a_1), \ldots, y(a_n))$ lexicographically increases, contradicting our assumption.

By (60.21) for each u in S, the set of a in C with $u \in \phi(a)$ forms an interval in C. So the linear inequalities corresponding to C make up a totally unimodular matrix (as it is a network matrix generated by a directed path and a directed graph (Theorem 13.20)). Therefore, by Theorem 5.35, system (60.24) is box-TDI. ∎

We give some applications of Theorem 60.4 (more applications are in Hoffman [1976a], Hoffman and Schwartz [1978], and Gröflin [1984,1987]).

Shortest paths (Johnson [1975a]). Let $D = (V, A)$ be a digraph and let $s, t \in V$. Let $L := \{U \subseteq V \mid s \in U, t \notin U\}$ and let $\leq := \subseteq$, $\wedge := \cap$, $\vee := \cup$. Let $S := A$ and let for each $U \in L$, $\phi(U) := \delta^{\text{out}}(U)$. These data satisfy (60.20) and (60.21), where $S' := S$. If $f = -\mathbf{1}$, Theorem 60.4 gives: the minimum length of an $s-t$ path is equal to the maximum number of $s-t$ cuts such that no arc a is in more than $c(a)$ of these $s-t$ cuts — the max-potential min-work theorem (Theorem 7.1).

Matroid intersection (Hoffman [1976a]). Let $(S, \mathcal{I})$ and $(S, \mathcal{I}_2)$ be matroids, with rank functions r_1 and r_2 and assume $r_1(S) = r_2(S)$. Let S' and S'' be two disjoint copies of S and let $V := S' \cup S''$. Let $L := \{U \subseteq V \mid U \subseteq S' \text{ or } S' \subseteq U\}$. Let $\leq := \subseteq$, $\wedge := \cap$, $\vee := \cup$. Define for $T \subseteq S$:

(60.27)
$$\begin{aligned} f(T') &:= r_1(T), & \phi(T') &:= T, \\ f(V \setminus T'') &:= r_2(T), & \phi(V \setminus T'') &:= T. \end{aligned}$$

As these data satisfy (60.20), (60.21), and (60.22), Theorem 60.4 yields the matroid intersection theorem. Polymatroid intersection can be included similarly.

Chains and antichains in partially ordered sets (Hoffman and Schwartz [1978]). Let $(V, \preceq)$ be a partially ordered set and let L be the collection of lower ideals of V (a subset Y of V is a *lower ideal* if $y \preceq x \in V$ implies $y \in V$). Define $\leq := \subseteq$, $\wedge := \cap$, $\vee := \cup$.

First, let $S := V$. For $Y \in L$, let $\phi(Y)$ be the collection of maximal elements of Y. These data satisfy (60.20) and (60.21), and $S' = S'' = S$.

Theorem 60.4 with $f(Y) := k$ for each $Y \in L$ then gives the theorem of Greene [1976] (Corollary 14.10b) that the maximum size of the union of k chains is equal to the minimum value of $|V \setminus Y| + k \cdot c_1(Y)$, where Y ranges over all subsets of V and where $c_1(Y)$ denotes the maximum size of a chain contained in Y.

Indeed, Theorem 60.4 gives the total dual integrality of

(60.28)
$$\begin{aligned} &0 \leq x_v \leq 1 && \text{for } v \in V, \\ &x(A) \leq k && \text{for each antichain } A. \end{aligned}$$

Hence the maximum size of the union of k chains is, by Dilworth's decomposition theorem, equal to (where $\mathcal{A}$ denotes the collection of antichains in V)

(60.29)
$$\begin{aligned} &\max\{\mathbf{1}^{\mathsf{T}} x \mid x \in \{0,1\}^V, x(A) \leq k \text{ for } A \in \mathcal{A}\} \\ &= \min\{k \sum_{A \in \mathcal{A}} y_A + z(V) \mid y \in \mathbb{Z}_+^{\mathcal{A}}, z \in \mathbb{Z}_+^V, \sum_{A \in \mathcal{A}} y_A \chi^A + z \geq \mathbf{1}\} \\ &= \min_{Y \subseteq V}(|V \setminus Y| + k \cdot (\text{minimum number of antichains covering } Y)) \\ &= \min_{Y \subseteq V}(|V \setminus Y| + k \cdot c_1(Y)). \end{aligned}$$

Also the dual result (exchanging 'chain' and 'antichain') due to Greene and Kleitman [1976] (Corollary 14.8b) can be derived. Let $L, \leq, \wedge, \vee$ be as above and let $S := V \cup \{w\}$, where w is some new element. For $Y \in L$, let $\phi(Y)$ be the collection of maximal elements of Y together with w and let $f(Y) := -|\phi(Y)|$. These data again satisfy (60.20) and (60.21), and $S' = S'' = S$.

Then Theorem 60.4 gives the box-total dual integrality of the system

(60.30) $\quad x(A) + \lambda \leq -|A|$ for each antichain A,

and hence of the system

(60.31) $\quad x(A) + \lambda \geq |A|$ for each antichain A.

Then the maximum union of k antichains is equal to

(60.32)
$$\begin{aligned} &\max\{\sum_{A\in\mathcal{A}} y_A|A| \mid y \in \mathbb{Z}_+^{\mathcal{A}}, \sum_{A\in\mathcal{A}} y_A\chi^A \leq \mathbf{1}, \sum_{A\in\mathcal{A}} y_A = k\} \\ &= \min\{x(V) + k\cdot\lambda \mid x \in \mathbb{Z}_+^V, \lambda \in \mathbb{Z}, x(A)+\lambda \geq |A| \text{ for each } A \in \mathcal{A}\} \\ &\geq \min_{Y\subseteq V}(|V \setminus Y| + k\cdot(\text{maximum size of an antichain contained in } Y)). \end{aligned}$$

The equality follows from the box-total dual integrality of (60.31). The inequality follows by taking $Y := \{v \in V \mid x_v = 0\}$. Then λ is at least the maximum size of an antichain contained in Y, since for any antichain $A \subseteq Y$: $\lambda = x(A) + \lambda \geq |A|$.

Common base vectors in two polymatroids (Gröflin and Hoffman [1981]). Let f_1 and f_2 be submodular set functions on S. The polymatroid intersection theorem gives:

(60.33)
$$\begin{aligned} f(T) &:= \max\{x(T) \mid x(U) \leq f_i(U) \text{ for } U \subseteq S \text{ and } i = 1,2\} \\ &= \min_{U\subseteq T}(f_1(U) + f_2(T \setminus U)), \end{aligned}$$

for $T \subseteq S$. Gröflin and Hoffman [1981] showed that Theorem 46.4 follows from Theorem 60.4 above as follows. (The proof of Theorem 46.4 was modelled after the proof of Theorem 60.4.)

Let L be the set of all pairs (T, U) of subsets of S with $T \cap U = \emptyset$, partially ordered by $\leq$ as follows:

(60.34) $\quad (T, U) \leq (T', U')$ and only if $T \subseteq T'$ and $U \supseteq U'$.

Then $(L, \leq)$ is a lattice with lattice operations $\wedge$ and $\vee$ (say). Define $\phi(T, U) := |S \setminus (T \cup U)|$ and $f(T, U) := f_1(T) + f_2(U) - f(S)$. As these data satisfy (60.20), (60.21), and (60.22), Theorem 60.4 applies. We have $S' = S'' = S$. Hence the system

(60.35) $\quad x(S \setminus (T \cup U)) \geq f_1(T) + f_2(U) - f(S)$ for $(T, U) \in L$

is box-TDI. With the definition of f, this implies the box-total dual integrality of

(60.36) $\quad x(T) \leq f(S \setminus T) - f(S)$ for $T \subseteq S$,

and (equivalently) of

(60.37) $\quad x(T) \geq f(S) - f(S \setminus T)$ for $T \subseteq S$.

That is, we have Theorem 46.4.

Convex sets in partially ordered sets (Gröflin [1984]). Let $(S, \leq)$ be a partially ordered set. A subset C of S is called *convex* if $a, b \in C$ and $a \leq x \leq b$ imply $x \in C$. Then the system

(60.38) $\qquad x(C) \leq 1$ for each convex subset C of S,

is box-TDI. Note that this system describes the polar of the convex hull of the incidence vectors of convex sets.

To see the box-total dual integrality of (60.38), define

(60.39) $\qquad L := \{(A, B) \mid A$ lower ideal and B upper ideal in S with $A \cup B = S\}$.

(An *upper ideal* is a subset B such that if $b \in B$ and $x \geq b$, then $x \in B$. Similarly, a *lower ideal* is a subset B such that if $b \in B$ and $x \leq b$, then $x \in B$.) Make L to a lattice by defining a partial order $\preceq$ on L by:

(60.40) $\qquad (A, B) \preceq (A', B') \iff A \subseteq A', B \supseteq B'.$

Define $f : L \to \mathbb{R}$ and $\phi : L \to \mathcal{P}(S)$ by: $f(A, B) := 1$ and $\phi(A, B) := A \cap B$, for $(A, B) \in L$. Applied to this structure, Theorem 60.4 gives the box-total dual integrality of (60.38).

('Greedy' algorithms for some lattice polyhedra problems were investigated by Kornblum [1978].)

An extension of lattice polyhedra, to handle rooted-connectivity augmentation of a digraph, was given by Frank [1999b].

60.3b. Polymatroidal network flows

Hassin [1978,1982] and Lawler and Martel [1982a,1982b] gave the following 'polymatroidal network flow' model equivalent to that of Edmonds and Giles. Let $D = (V, A)$ be a digraph. For each $v \in V$, let $\mathcal{C}_v^{\text{out}}$ and $\mathcal{C}_v^{\text{in}}$ be intersecting families of subsets of $\delta^{\text{out}}(v)$ and $\delta^{\text{in}}(v)$, respectively, and let $f_v^{\text{out}} : \mathcal{C}_v^{\text{out}} \to \mathbb{R}$ and $f_v^{\text{in}} : \mathcal{C}_v^{\text{in}} \to \mathbb{R}$ be submodular on intersecting pairs. Then the system

(60.41)
$$\begin{array}{ll} x(\delta^{\text{out}}(v)) = x(\delta^{\text{in}}(v)) & \text{for } v \in V, \\ x(B) \leq f_v^{\text{in}}(B) & \text{for each } v \in V \text{ and } B \in \mathcal{C}_v^{\text{in}}, \\ x(B) \leq f_v^{\text{out}}(B) & \text{for each } v \in V \text{ and } B \in \mathcal{C}_v^{\text{out}}, \end{array}$$

is box-TDI. Frank [1982b] showed that this can be derived from the Edmonds-Giles theorem (Theorem 60.1) as follows. Make a digraph $D' = (V', A')$, where A' consists of disjoint arcs $a' := (u_a, v_a)$ for each $a \in A$. Let $\mathcal{C}$ consist of all subsets U of V' such that there exists a $v \in V$ satisfying:

(60.42)
$$\begin{array}{l} U = \{v_a \mid a \in \delta^{\text{in}}(v)\} \cup \{u_a \mid a \in \delta^{\text{out}}(v)\}, \\ \text{or } \exists B \in \mathcal{C}_v^{\text{in}} : U = \{v_a \mid a \in B\}, \\ \text{or } \exists B \in \mathcal{C}_v^{\text{out}} : U = V' \setminus \{u_a \mid a \in B\}. \end{array}$$

Define $f(U) := 0$, $f(U) := f_v^{\text{in}}(B)$, and $f(U) := f_v^{\text{in}}(B)$, respectively. Then the box-total dual integrality of (60.41) is equivalent to that of

(60.43) $\qquad x(\delta_{A'}^{\text{in}}(U)) - x(\delta_{A'}^{\text{out}}(U)) \leq f(U)$ for $U \in \mathcal{C}$, which follows from Theorem 60.1.

Lawler [1982] showed that, conversely, the Edmonds-Giles model is a special case of the polymatroidal network flow model. To see this, let $D = (V, A)$ be a digraph, let $\mathcal{C}$ be a crossing family of subsets of V, and let $f : \mathcal{C} \to \mathbb{R}$ be crossing submodular. Let $\hat{\mathcal{C}}$ be the collection of all sets $U = U_1 \cap \cdots \cap U_t$ with $U_1, \ldots, U_t \in \mathcal{C} \setminus \{V\}$ such that $U_i \cup U_j = V$ for all i, j with $1 \leq i < j \leq t$. Define $\hat{f} : \hat{\mathcal{C}} \to \mathbb{R}$ by

(60.44) $$\hat{f}(U) := \min(f(U_1) + \cdots + f(U_t)),$$

where the minimum ranges over sets $U_1, \ldots, U_t$ as above. Then $\hat{\mathcal{C}}$ is an intersecting family and $\hat{f}$ is intersecting submodular (Theorem 49.6).

Now extend D by a new vertex r, and arcs (v, r) for $v \in V$, thus making the digraph $D' = (V \cup \{r\}, A')$. Let $\mathcal{C}_r^{\mathrm{in}}$ consist of all subsets B of $\delta_{A'}^{\mathrm{in}}(r)$ for which there is a $U \in \hat{\mathcal{C}}$ satisfying

(60.45) $$B = \{(v, r) \mid v \in U\}.$$

Define $f_r^{\mathrm{in}}(B) := \hat{f}(U)$. Then

(60.46)
$$\begin{aligned} &x(\delta_{A'}^{\mathrm{out}}(v)) = x(\delta_{A'}^{\mathrm{in}}(v)) && \text{for } v \in V,\\ &x(\delta_{A'}^{\mathrm{in}}(r)) = 0, &&\\ &x(B) \leq f_r^{\mathrm{in}}(B) && \text{for } B \in \mathcal{C}_r^{\mathrm{in}}, \end{aligned}$$

is a special case of (60.41). Moreover, the box-total dual integrality of (60.46) implies the box-total dual integrality of

(60.47) $$x(\delta_A^{\mathrm{in}}(U)) - x(\delta^{\mathrm{out}}(U)) \leq f(U) \text{ for } U \in \mathcal{C},$$

since in (60.46) we can restrict B to those B for which there exists a $U \in \mathcal{C}$ with $B = \{(v, r) \mid v \in U\}$ (since $x(\delta_{A'}^{\mathrm{in}}(r)) = 0$). Then

(60.48) $$x(B) = \sum_{v \in U} \big(x(\delta_A^{\mathrm{in}}(v)) - x(\delta_A^{\mathrm{out}}(v))\big) = x(\delta_A^{\mathrm{in}}(U)) - x(\delta_A^{\mathrm{out}}(U)).$$

So it implies the Edmonds-Giles theorem (Theorem 60.1).

60.3c. A general model

In Schrijver [1984a] the following general framework was given. Let S be a finite set, let $n \in \mathbb{Z}_+$, let $\mathcal{C}$ be a collection of subsets of S, let $b, c \in (\mathbb{R} \cup \{\pm\infty\})^n$, and let $f : \mathcal{C} \to \mathbb{R}$ and $h : \mathcal{C} \to \{0, \pm 1\}^n$ satisfy:

(60.49)
(i) if $\{T_1, T_2, T_3\}$ is a cross-free subcollection of $\mathcal{C}$, then for each $j = 1, \ldots, n$, there exist $u, v \in S$ such that for $i = 1, 2, 3$: $h(T_i)_j = +1$ if and only if (u, v) enters T_i, and $h(T_i)_j = -1$ if and only if (u, v) leaves T_i;

(ii) if T and U are crossing sets in $\mathcal{C}$, then there exist $T', U' \in \mathcal{C}$ such that $T' \subset T$ and
$$f(T) + f(U) - f(T') - f(U') \geq (h(T) + h(U) - h(T') - h(U'))x$$
for each x with $b \leq x \leq c$.

Then the system (in $x \in \mathbb{R}^n$)

(60.50) $$\begin{aligned} & b \leq x \leq c, \\ & h(T)x \leq f(T) \quad \text{for } T \in \mathcal{C}, \end{aligned}$$

is box-TDI. This contains the Edmonds-Giles theorem (Theorem 60.1) and Theorems 60.3 and 60.4 as special cases.

A proof of the box-total dual integrality of (60.50) can be sketched as follows. If we maximize a linear functional $w^{\mathsf{T}}x$ over (60.50), condition (60.49)(ii) implies that there exists an optimum dual solution whose active constraints correspond to a cross-free subfamily of $\mathcal{C}$. Next, condition (60.49)(i) implies that these constraints form a network matrix, hence a totally unimodular matrix, proving the box-total dual integrality of (60.50) with Theorem 5.35.

60.3d. Packing cuts and Győri's theorem

Let $D = (V, A)$ be a digraph and let $g : \mathcal{P}(V) \to \mathbb{Z}_+$ satisfy the supermodular inequality

(60.51) $$g(U) + g(W) \leq g(U \cap W) + g(U \cup W)$$

for all $U, W \subseteq V$ such that $\delta^{\text{in}}(U) \cap \delta^{\text{in}}(W) \neq \emptyset$ and $g(U) > 0$, $g(W) > 0$.

The following was shown by Frank and Jordán [1995b] (in the terminology of bisupermodular functions — see Corollary 60.5a):

Theorem 60.5. *Let $D = (V, A)$ be a digraph satisfying:*

(60.52) *V can be partitioned into two sets S and T such that A consists of all arcs from S to T.*

Let g be as above, with $g(U) = 0$ if $\delta^{\text{in}}(U) = \emptyset$. Then the minimum of $x(A)$ taken over all $x : A \to \mathbb{Z}_+$ satisfying

(60.53) *$x(\delta^{\text{in}}(U)) \geq g(U)$ for each $U \subseteq V$,*

is equal to the maximum value of $\sum_{U \in \mathcal{B}} g(U)$, where $\mathcal{B}$ is a collection of subsets U such that the $\delta^{\text{in}}(U)$ for $U \in \mathcal{B}$ are disjoint.

Proof. Let $\tau(g)$ and $\nu(g)$ denote the minimum and maximum value, respectively.

Then $\nu(g) \leq \tau(g)$, since, if $x : A \to \mathbb{Z}_+$ satisfies (60.53) and $\mathcal{B}$ is as described, then

(60.54) $$x(A) \geq \sum_{U \in \mathcal{B}} x(\delta^{\text{in}}(U)) \geq \sum_{U \in \mathcal{B}} g(U).$$

The reverse inequality $\tau(g) \leq \nu(g)$ is shown by induction on $\nu(g)$. If $\nu(g) = 0$, then $g(U) = 0$ for all $U \subseteq V$, and hence $\tau(g) = 0$. Now let $\nu(g) \geq 1$.

For each $a \in A$, define a function g^a by

(60.55) $$g^a(U) := \begin{cases} g(U) - 1 & \text{if } a \in \delta^{\text{in}}(U) \text{ and } g(U) \geq 1, \\ g(U) & \text{otherwise,} \end{cases}$$

for $U \subseteq V$. In other words:

(60.56) $$g^a(U) = \max\{g(U) - d^{\text{in}}_{\{a\}}(U), 0\}.$$

Then

(60.57) g^a again satisfies (60.51).

Indeed, if $\delta^{\text{in}}(U) \cap \delta^{\text{in}}(W) \neq \emptyset$, and $g^a(U) > 0$ and $g^a(W) > 0$, then $g(U) > 0$ and $g(W) > 0$, and $g^a(U) = g(U) - d^{\text{in}}_{\{a\}}(U)$ and $g^a(W) = g(W) - d^{\text{in}}_{\{a\}}(W)$. Hence

(60.58)
$$\begin{aligned} g^a(U) + g^a(W) &= g(U) + g(W) - d^{\text{in}}_{\{a\}}(U) - d^{\text{in}}_{\{a\}}(W) \\ &\leq g(U \cap W) + g(U \cup W) - d^{\text{in}}_{\{a\}}(U \cap W) - d^{\text{in}}_{\{a\}}(U \cup W) \\ &\leq g^a(U \cap W) + g^a(U \cup W). \end{aligned}$$

So g^a satisfies (60.51).

The following is the key of the proof:

(60.59) there exists an arc a with $\nu(g^a) \leq \nu(g) - 1$.

Suppose to the contrary that $\nu(g^a) = \nu(g)$ for all $a \in A$. As $\nu(g) \geq 1$, there exists a $W \subseteq V$ with $g(W) \geq 1$. For each $a \in \delta^{\text{in}}(W)$, as $\nu(g^a) = \nu(g)$, there exists a collection $\mathcal{B}^a$ such that any arc of D enters at most one $U \in \mathcal{B}^a$, such that $g^a(\mathcal{B}^a) = \nu_g$, and such that $g(U) > 0$ for each $U \in \mathcal{B}^a$. As $g(\mathcal{B}^a) \leq g^a(\mathcal{B}^a)$, a enters no $U \in \mathcal{B}^a$.

Now for each $U \subseteq V$, let $w(U)$ be the number of times U occurs among the $\mathcal{B}^a$ (over all $a \in \delta^{\text{in}}(W)$). Reset $w(W) := w(W) + 1$. Then w has the following properties:

(60.60)
$$\text{(i)} \sum_{U \subseteq V} w(U)\chi^{\delta^{\text{out}}(U)} \leq |\delta^{\text{in}}(W)| \cdot \mathbf{1} \text{ and}$$
$$\text{(ii)} \sum_{U \subseteq V} w(U)g(U) > |\delta^{\text{in}}(W)|\nu(g).$$

Moreover, $g(U) \geq 1$ whenever $w(U) > 0$.

Now as long as there exist $U, U' \subseteq V$ with $w(U) > 0$ and $w(U') > 0$ and not satisfying:

(60.61) $\delta^{\text{in}}(U) \cap \delta^{\text{in}}(U') = \emptyset$ or $U \subseteq U'$ or $U' \subseteq U$,

decrease $w(U)$ and $w(U')$ by 1, and increase $w(U \cap U')$ and $w(U \cup U')$ by 1. This operation maintains (60.60) and decreases

(60.62)
$$\sum_{U \in \mathcal{P}(V)} w(U)|U||V \setminus U|$$

(by Theorem 2.1). So after a finite number of these operations, w satisfies (60.60) and all U, U' with $w(U) > 0$ and $w(U') > 0$ satisfy (60.61).

Let $\mathcal{F}$ be the collection of $U \subseteq V$ with $w(U) > 0$. We apply the length-width inequality for partially ordered sets (Theorem 14.5) to $(\mathcal{F}, \subseteq)$. By (60.60)(i), the maximum of $w(\mathcal{C})$ taken over chains in $\mathcal{F}$ is at most $|\delta^{\text{in}}(W)|$, since by (60.52), there is an arc $a \in A$ entering all $U \in \mathcal{C}$ (as $\delta^{\text{in}}(U) \neq \emptyset$, since $g(U) \geq 1$, for each $U \in \mathcal{F}$). Moreover, the maximum of $g(\mathcal{B})$ taken over antichains $\mathcal{B}$ in $\mathcal{F}$ is at most $\nu(g)$, since the elements in $\mathcal{F}$ satisfy (60.61), and therefore $\mathcal{B}$ gives a collection of disjoint cuts. But then (60.60)(ii) contradicts the length-width inequality. This proves (60.59).

We now can apply induction, since trivially $\tau(g) \leq \tau(g^a) + 1$, as increasing x_a by 1 for any x satisfying (60.53) with respect to g^a, gives an x satisfying (60.53) with respect to g. So $\tau(g) \leq \tau(g^a) + 1 = \nu(g^a) + 1 \leq \nu(g)$. ∎

This theorem can be equivalently formulated as follows. Let S and T be finite sets. Consider functions $h : \mathcal{P}(S) \times \mathcal{P}(T) \to \mathbb{R}$ satisfying

(60.63) $h(X_1 \cap X_2, Y_1 \cup Y_2) + h(X_1 \cup X_2, Y_1 \cap Y_2) \geq h(X_1, Y_1) + h(X_2, Y_2)$ for all $X_1, X_2 \subseteq S$ and $Y_1, Y_2 \subseteq T$ with $X_1 \cap X_2 \neq \emptyset$, $Y_1 \cap Y_2 \neq \emptyset$, $h(X_1, Y_1) > 0$, and $h(X_2, Y_2) > 0$.

Call a collection $\mathcal{F} \subseteq \mathcal{P}(S) \times \mathcal{P}(T)$ *independent* if $X_1 \cap X_2 = \emptyset$ or $Y_1 \cap Y_2 = \emptyset$ for all distinct (X_1, Y_1), (X_2, Y_2) in $\mathcal{F}$. So $\mathcal{F}$ is independent if the sets $X \times Y$ for $(X, Y) \in \mathcal{F}$ are disjoint.

As usual,

(60.64) $$h(\mathcal{F}) := \sum_{(X,Y) \in \mathcal{F}} h(X, Y).$$

Moreover, if $z : S \times T \to \mathbb{R}$, denote

(60.65) $$z(X \times Y) := \sum_{(x,y) \in X \times Y} z(x, y)$$

for $X \subseteq S$ and $Y \subseteq T$.

Corollary 60.5a. *Let $h : \mathcal{P}(S) \times \mathcal{P}(T) \to \mathbb{Z}_+$ satisfy* (60.63)*, such that $h(X, Y) = 0$ if $X = \emptyset$ or $Y = \emptyset$. Then the minimum value of $z(S \times T)$ where $z : S \times T \to \mathbb{Z}_+$ satisfies*

(60.66) $z(X \times Y) \geq h(X, Y)$ *for all* $X \subseteq S$, $Y \subseteq T$,

is equal to the maximum value of $h(\mathcal{F})$ where $\mathcal{F}$ is independent.

Proof. We can assume that S and T are disjoint. Let $V := S \cup T$, and define a set function g on V by:

(60.67) $$g(U) := h(S \setminus U, T \cap U)$$

for $U \subseteq V$. Let $D = (V, A)$ be the digraph with A consisting of all arcs from S to T. Then D and g satisfy the condition of Theorem 60.5, and the min-max equality proved in Theorem 60.5 is equivalent to the min-max equality described in the present corollary. ∎

Frank and Jordán showed that this theorem implies the following 'minimax theorem for intervals' of Győri [1984]. Let $\mathcal{I}$ and $\mathcal{J}$ be collections of sets. Then $\mathcal{J}$ is said to *generate* $\mathcal{I}$ if each set in $\mathcal{I}$ is a union of sets in $\mathcal{J}$. Győri's theorem characterizes the minimum size of a collection of intervals generating a given finite collection $\mathcal{I}$ of intervals. For this, we can take an 'interval' to be a finite, contiguous set of integers.

To describe the min-max equality, consider the undirected graph $G_{\mathcal{I}} = (V_{\mathcal{I}}, E_{\mathcal{I}})$ with

(60.68) $$V_{\mathcal{I}} := \{(s, I) \mid s \in I \in \mathcal{I}\},$$

where two distinct pairs (s, I) and (s', I') are adjacent if and only if $s \in I'$ and $s' \in I$. Call a subset C of $V_{\mathcal{I}}$ *stable* if any two elements of C are nonadjacent (in other words, C is a stable set in $G_{\mathcal{I}}$).

Corollary 60.5b (Győri's theorem)**.** *Let $\mathcal{I}$ be a finite collection of intervals. Then the minimum size of a collection of intervals generating $\mathcal{I}$ is equal to the maximum size of a stable subset of $V_{\mathcal{I}}$.*

Proof. To see that the minimum is not less than the maximum, observe that if $\mathcal{J}$ generates $\mathcal{I}$ and C is a stable subset of $V_{\mathcal{I}}$, then for any $J \in \mathcal{J}$, there is at most one $(s, I) \in C$ with $s \in J \subseteq I$, while for any $(s, I) \in C$ there is at least one such $J \in \mathcal{J}$. So $|\mathcal{J}| \geq |C|$.

Equality is shown with Corollary 60.5a. Let S be the union of the intervals in $\mathcal{I}$. Define a function $h : \mathcal{P}(S) \times \mathcal{P}(S) \to \{0, 1\}$ by

(60.69) $h(X, Y) = 1$ if and only if X and Y are nonempty intervals such that $\max X = \min Y$ and $X \cup Y \in \mathcal{I}$, and such that there is no $I \in \mathcal{I}$ with $X \cap Y \subseteq I \subset X \cup Y$,

for $X, Y \subseteq S$. (Here $\max Z$ and $\min Z$ denote the maximum and minimum element of Z, respectively.)

Then h satisfies (60.63). To see this, note first that each (X, Y) with $h(X, Y) = 1$ is characterized by a point $s \in S$ and an inclusionwise minimal interval $I \in \mathcal{I}$ containing s (inclusionwise minimal among all intervals in $\mathcal{I}$ containing s). The relation is that $\{s\} = X \cap Y$ and $I = X \cup Y$.

To see that h satisfies (60.63), let $h(X_1, Y_1) = 1$ and $h(X_2, Y_2) = 1$ and $X_1 \cap X_2 \neq \emptyset$ and $Y_1 \cap Y_2 \neq \emptyset$. We show $h(X_1 \cap X_2, Y_1 \cup Y_2) = 1$ (then $h(X_1 \cup X_2, Y_1 \cap Y_2) = 1$ follows by symmetry).

In fact, this is straightforward case-checking. Let $X_i = [a_i, b_i]$, $Y_i = [b_i, c_i]$, and $I_i := X_i \cup Y_i$ for $i = 1, 2$. As $X_1 \cap I_2 \neq \emptyset \neq Y_1 \cap I_2$, we know that $b_1 \in [a_2, c_2]$, and similarly $b_2 \in [a_1, c_1]$. By symmetry, we can assume that $a_1 \leq a_2$. Hence, by the minimality of $X_1 \cup Y_1$ as an interval containing b_1, $c_1 \leq c_2$. Now, if $b_2 \leq b_1$, we have $X_1 \cap X_2 = [a_2, b_2] = X_2$ and $Y_1 \cup Y_2 = [b_2, c_2] = Y_2$, and therefore $h(X_1 \cap X_2, Y_1 \cup Y_2) = h(X_2, Y_2) = 1$. If $b_1 < b_2$, then $X_1 \cap X_2 = [a_2, b_1]$ and $Y_1 \cup Y_2 = [b_1, c_2]$. Suppose that there is an $I \in \mathcal{I}$ with $b_1 \in I \subset [a_2, c_2]$. By the minimality of $[a_2, c_2]$ as an interval containing b_2, we know $b_2 \notin I$. Hence $I \subset [a_1, c_1]$, contradicting the minimality of $[a_1, c_1]$ as an interval containing b_1. Therefore, no such I exists, and hence we have $h(X_1 \cap X_2, Y_1 \cup Y_2) = 1$.

So Corollary 60.5a applies (taking $T := S$). Let z and $\mathcal{F}$ attain the minimum and maximum respectively. Let $\mathcal{J}$ be the collection of intervals $[s, t]$ with $z(s, t) \geq 1$ and $s \leq t$. Let C be the collection of pairs (s, I) with $s \in S$ and $I \in \mathcal{I}$ such that there is an $(X, Y) \in \mathcal{F}$ with $h(X, Y) = 1$, $X \cap Y = \{s\}$, and $X \cup Y = I$. Then $\mathcal{J}$ generates $\mathcal{I}$, since $z(X \times Y) \geq h(X, Y)$ for all X, Y. Moreover, C is stable as $\mathcal{F}$ is independent. Finally, $|\mathcal{J}| \leq z(S \times S) = h(\mathcal{F}) = |C|$. ∎

(Frank [1999a] gave an alternative, algorithmic proof.)

Győri's theorem in fact states that the colouring number of the complementary graph of $G_{\mathcal{I}}$ is equal to its clique number. It has the following consequence, proved by Chaiken, Kleitman, Saks, and Shearer [1981] and conjectured by V. Chvátal. Let P be a rectilinear polygon in $\mathbb{R}^2$ (where each side horizontal or vertical), such that the intersection of P with each horizontal or vertical line is convex. Then the minimum number of rectangles contained in P needed to cover P, is equal to the maximum number of points in P no two of which are contained in any rectangle contained in P.

Franzblau and Kleitman [1984] gave an $O(|\mathcal{I}|^2)$-time algorithm to find the optima in Győri's theorem, with a proof of equality as by-product.

Győri's theorem was extended by Lubiw [1991a] to a weighted version. She noted that a fully weighted version of the theorem does not hold; that is, taking integer weights $w(s, I)$ on any pair (s, I), the maximum weight of a stable set need not be equal to the minimum size of a family $\mathcal{J}$ of intervals such that for any (s, I) there are at least $w(s, I)$ intervals J in $\mathcal{J}$ satisfying $s \in J \subseteq I$. (In other words, $G_{\mathcal{I}}$ need not be perfect.)

However, as Lubiw showed, these two optima are equal if $w(s, I)$ only depends on s; that is, if $w(s, I) = w(s)$ for some $w : S \to \mathbb{Z}_+$. Also this can be derived from Frank and Jordán's theorem: instead of defining $h(X, Y) := 1$ in (60.69), define $h(X, Y) := w(s)$ where $\{s\} = X \cap Y$.

As Frank and Jordán observed, their method extends Győri's theorem to the case where we take 'interval' to mean: interval on the circle (instead of just the real line).

Other applications of Theorem 60.5 are to vertex-connectivity augmentation — see Theorem 63.11.

60.3e. Further notes

For another model equivalent to that of Edmonds and Giles, based on distributive lattices, see Gröflin and Hoffman [1982] — cf. Schrijver [1984b]. Grishuhin [1981] gave a lattice model requiring total unimodularity as a condition.

Further algorithms for minimum-cost submodular flow were given by Fujishige [1978a,1987], Zimmermann [1982b,1992], Barahona and Cunningham [1984], Cui and Fujishige [1988], Chung and Tcha [1991], Gabow [1993a], McCormick and Ervolina [1993], Iwata, McCormick, and Shigeno [1998,1999,2000,2002], Wallacher and Zimmermann [1999], Fleischer and Iwata [2000], and Fleischer, Iwata, and McCormick [2002]. A survey on algorithms for submodular flows was presented by Fujishige and Iwata [2000].

Zimmermann [1982a,1982b,1985] considered group-valued extensions of some of the models. Federgruen and Groenevelt [1988] extended some models to more general objective functions. Zimmermann [1986] considered duality for balanced submodular flows. Qi [1988a] and Murota [1999] gave generalizations of submodular flows. Convex-cost submodular flows were considered by Iwata [1996,1997].

An algorithm for a model comprising the Edmonds-Giles and the lattice polyhedron model (Section 60.3a) was given by Karzanov [1983]. For 0,1 problems it is polynomial-time.

The effectivity of uncrossing techniques is studied by Hurkens, Lovász, Schrijver, and Tardos [1988] and Karzanov [1996].

The facets of submodular flow polyhedra were studied by Giles [1975].

For a comparison of models, see Schrijver [1984b], and for a survey, including results on the dimension of faces of submodular flow polyhedra, see Frank and Tardos [1988]. A survey of submodular functions and flows is given by Murota [2002].

Chapter 61

Graph orientation

Orienting an undirected graph so as to obtain a k-arc-connected directed graph is the object of study in this chapter. Recall that a directed graph D is called an *orientation* of an undirected graph G if G is the underlying undirected graph of D.
Central result is a deep theorem of Nash-Williams showing that each undirected graph has an orientation that keeps at least half of the connectivity (rounded down) between any two vertices.
It implies that a $2k$-edge-connected undirected graph has a k-arc-connected orientation. This can be proved alternatively and easier with the help of submodular functions (cf. Section 61.4).

61.1. Orientations with bounds on in- and outdegrees

We first consider orientations obeying bounds on the indegrees and/or outdegrees. The results follow quite directly from bipartite matching or (equivalently) flow theory.

Hakimi [1965] considered lower bounds on the indegrees:

Theorem 61.1. *Let $G = (V, E)$ be an undirected graph and let $l : V \to \mathbb{Z}_+$. Then G has an orientation $D = (V, A)$ with $\deg^{\text{in}}_A(v) \geq l(v)$ for each $v \in V$ if and only if each $U \subseteq V$ is incident with at least $l(U)$ edges.*

Proof. Let $\mathcal{A}$ be the family of subsets of E obtained by taking set $\delta(v)$ with multiplicity $l(v)$, for each $v \in V$. Then the existence of an orientation as required is equivalent to the existence of a transversal of $\mathcal{A}$. By Hall's marriage theorem (Theorem 22.1), this is equivalent to the condition in the theorem. ∎

A direct consequence is:

Corollary 61.1a. *Let $G = (V, E)$ be an undirected graph and let $l : V \to \mathbb{Z}_+$. Then G has an orientation $D = (V, A)$ with $\deg^{\text{in}}_A(v) = l(v)$ for each $v \in V$ if and only if $l(V) = |E|$ and each $U \subseteq V$ is incident with at least $l(U)$ edges.*

Proof. Directly from Theorem 61.1. ∎

Another consequence concerns upper bounds:

Corollary 61.1b. *Let $G = (V, E)$ be an undirected graph and let $u : V \to \mathbb{Z}_+$. Then G has an orientation $D = (V, A)$ with $\deg^{\text{in}}_A(v) \leq u(v)$ for each $v \in V$ if and only if each $U \subseteq V$ spans at most $u(U)$ edges.*

Proof. For each $v \in V$, define $l(v) := \deg(v) - u(v)$. (We may assume that, for each $v \in V$, $u(v) \leq \deg(v)$, since otherwise resetting $u(v) := \deg(v)$ does not change the conditions in the theorem.)

Now G has an orientation with $\deg^{\text{in}}(v) \leq u(v)$ for each v if and only if G has an orientation with $\deg^{\text{in}}(v) \geq l(v)$ for each v (just by reversing the orientation of all edges). By Theorem 61.1, the latter is equivalent to: each $U \subseteq V$ is incident with at least $l(U)$ edges; that is: $E[U]| + |\delta_E(U)| \geq l(U)$. Since

$$l(U) = \sum_{v \in U} (\deg(v) - u(v)) = 2|E[U]| + |\delta_E(U)| - u(U), \tag{61.1}$$

it is equivalent to: $|E[U]| \leq u(U)$, as required. ∎

Frank and Gyárfás [1978] gave a characterization for the case of lower bounds on both indegrees and outdegrees:

Theorem 61.2. *Let $G = (V, E)$ be an undirected graph and let $l, u : V \to \mathbb{Z}_+$ with $l \leq u$. Then G has an orientation $D = (V, A)$ with $l(v) \leq \deg^{\text{in}}_A(v) \leq u(v)$ for each $v \in V$ if and only if each $U \subseteq V$ is incident with at least $l(U)$ edges and spans at most $u(U)$ edges.*

Proof. The condition trivially being necessary, we prove sufficiency. Let $D = (V, A)$ be an arbitrary orientation of G. It suffices to show that there exists a function $x : A \to \{0, 1\}$ such that for each $v \in V$:

$$l(v) \leq \deg^{\text{in}}_A(v) - x(\delta^{\text{in}}_A(v)) + x(\delta^{\text{out}}_A(v)) \leq u(v), \tag{61.2}$$

since reversing the orientation of the arcs a with $x(a) = 1$ then gives an orientation as required. Condition (61.2) is equivalent to:

$$\deg^{\text{in}}_A(v) - u(v) \leq x(\delta^{\text{in}}_A(v)) - x(\delta^{\text{out}}_A(v)) \leq \deg^{\text{in}}_A(v) - l(v). \tag{61.3}$$

By Corollary 11.2i, such an x exists if and only if

$$|\delta^{\text{in}}_A(U)| \geq \max\{\sum_{v \in U} (\deg^{\text{in}}_A(v) - u(v)), \sum_{v \in V \setminus U} (l(v) - \deg^{\text{in}}_A(v))\} \tag{61.4}$$

for each $U \subseteq V$. Since $|\delta^{\text{in}}_A(U)| + \sum_{v \in V \setminus U} \deg^{\text{in}}_A(v)$ is equal to the number of edges incident with $V \setminus U$ and since $\sum_{v \in U} \deg^{\text{in}}_A(v) - |\delta^{\text{in}}_A(U)|$ is equal to the number of edges spanned by U, this is equivalent to the condition given in the theorem. ∎

Ford and Fulkerson [1962] observed that the undirected edges of a mixed graph (V, E, A) can be oriented so as to obtain an Eulerian directed graph if and only if

(61.5) (i) $\deg_E(v) + \deg_A^{\text{in}}(v) + \deg_A^{\text{out}}(v)$ is even for each $v \in V$,

(ii) $d_A^{\text{out}}(U) - d_A^{\text{in}}(U) \leq d_E(U)$ for each $U \subseteq V$.

This can be proved similarly.

61.2. 2-edge-connectivity and strongly connected orientations

Each $2k$-edge-connected undirected graph has a k-arc-connected orientation, which will be seen in Section 61.3. In the present section we consider the special case $k = 2$, which goes back to a theorem of Robbins [1939]. Tarjan [1972] showed that depth-first search is the tool behind. We follow his approach.

Theorem 61.3. *Given an undirected graph $G = (V, E)$ we can find an orientation D of G, in linear time, such that for each $u, v \in V$, if G has two edge-disjoint $u - v$ paths, then D has a directed $u - v$ path.*

Proof. Choose $s \in V$ arbitrarily, and consider a depth-first search tree T starting at s. Orient each edge in T away from s. For each remaining edge $e = uv$, there is a directed path in T that connects u and v. Let the path run from u to v. Then orient e from v to u. This gives the orientation D of G.

Then any edge not in T belongs to a directed circuit in D. Moreover, any edge f in T that is not a cut edge, belongs to a directed circuit in D (since there is an edge $e \notin T$ connecting the two components of $T - f$). This implies that D is as required. ∎

This implies the theorem of Robbins [1939] on strongly connected orientations:

Corollary 61.3a (Robbins' theorem)**.** *An undirected graph G has a strongly connected orientation if and only if G is 2-edge-connected.*

Proof. Necessity is easy, and sufficiency follows from Theorem 61.3. ∎

(The proof by Robbins [1939] uses the fact that each 2-edge-connected graph has an 'ear-decomposition' — cf. Section 15.5a.)

The above proof also shows that a strongly connected orientation can be found in linear time:

Corollary 61.3b. *Given a 2-edge-connected graph G, a strongly connected orientation of G can be found in linear time.*

Proof. Directly from Theorem 61.3. ∎

Robbins' theorem (Corollary 61.3a) extends to the following result of Frank [1976a] and Boesch and Tindell [1980] for mixed graphs.

Theorem 61.4. *Let $G = (V, E)$ be a graph in which part of the edges is oriented. Then the remainder of the edges can be oriented so as to obtain a strongly connected digraph if and only if G is 2-edge-connected and there is no nonempty proper subset U of V such that all edges in $\delta(U)$ are oriented from U to $V \setminus U$.*

Proof. Necessity being easy, we show sufficiency. Let G be a counterexample with a minimum number of undirected edges. Then there is at least one undirected edge, say $e = uv$. By the minimality assumption, orienting e from s to t violates the condition. That is, there exists a $U \subseteq V$ with $u \in U$, $v \in V \setminus U$, such that each edge $\neq e$ in $\delta(U)$ is oriented from U to $V \setminus U$. Similarly, there exists a $T \subseteq V$ with $v \in T$, $u \in V \setminus T$, such that each edge $\neq e$ $\delta(T)$ is oriented from T to $V \setminus T$.

Then each edge in $\delta(U \cap T)$ is oriented from $U \cap T$ to $V \setminus (U \cap T)$, and hence $U \cap T = \emptyset$. Similarly, $U \cup T = V$. Hence $\delta(U) = \{e\}$, a contradiction. ∎

The graph $K_{2,3}$ shows that a 2-edge-connected graph need not have an orientation in which each two vertices belong to a directed circuit; that is an orientation such that for each two vertices s, t there exists an arc-disjoint pair of an $s - t$ and a $t - s$ path.

Chung, Garey, and Tarjan [1985] gave a linear-time algorithm to find an orientation as described in Theorem 61.4.

61.2a. Strongly connected orientations with bounds on degrees

Robbins' theorem (Corollary 61.3a) states that an undirected graph G has a strongly connected orientation if and only if G is 2-edge-connected. Frank and Gyárfás [1978] extended this to the case where upper and lower bounds are prescribed on the indegrees of the orientation.

Let $\kappa(G)$ denote the number of its components of any graph G.

Theorem 61.5. *Let $G = (V, E)$ be a 2-edge-connected undirected graph and let $l, u \in \mathbb{Z}_+^V$ with $l \leq u$. Then G has a strongly connected orientation $D = (V, A)$ satisfying $l(v) \leq \deg_A^{\text{in}}(v) \leq u(v)$ for each $v \in V$ if and only if for each $U \subseteq V$:*

(61.6) (i) $|E[U]| + \kappa(G - U) \leq u(U)$,
(ii) $|E[U]| + |\delta(U)| - \kappa(G - U) \geq l(U)$.

Proof. It is easy to see that condition (61.6) is necessary. To see sufficiency, let (61.6) hold. Let $D = (V, A)$ be a strongly connected orientation of G with

$$\sum_{v\in V} \max\{0, \deg^{\text{in}}_A(v) - u(v), l(v) - \deg^{\text{in}}_A(v)\} \tag{61.7}$$

as small as possible. (A strongly connected orientation exists by Corollary 61.3a.)

If sum (61.7) is 0, we are done, so assume that it is positive. Then there exists a vertex v_0 with $\deg^{\text{in}}_A(v_0) > u(v_0)$ or $l(v_0) > \deg^{\text{in}}_A(v_0)$. Suppose that $\deg^{\text{in}}_A(v_0) > u(v_0)$. Let U be the set of vertices v for which D has two arc-disjoint $v - v_0$ paths. Then $\deg^{\text{in}}_A(v) \geq u(v)$ for each $v \in U$, since otherwise we can reverse the orientation on the arcs of one of the two arc-disjoint $v - v_0$ paths, thereby keeping the orientation strongly connected while decreasing sum (61.7).

We claim that U violates (61.6)(i). To this end, we show that

(61.8) each component K of $G - U$ is left by exactly one arc of D.

This can be seen as follows. For each $v \in K$, there exists a $U_v \subseteq V$ with $d^{\text{out}}_A(U_v) = 1$ and $v \in U_v$, $v_0 \notin U_v$ (as there exist no two arc-disjoint $v_0 - v$ paths). We choose each U_v inclusionwise maximal.

It suffices to show that

(61.9) $U_v = K$ for each $v \in K$.

To see this, note first that, for each $v \in K$, we have $U_v \subseteq K$. Indeed, $U_v \cap U = \emptyset$, since if say $v_1 \in U \cap U_v$, then there exist no two arc-disjoint $v_1 - v_0$ paths in D, contradicting the definition of U. If U_v would intersect another component K' of $G - U$, then $d^{\text{out}}_A(U_v) = d^{\text{out}}_A(U_v \cap K) + d^{\text{out}}_A(U_v \cap K') \geq 2$ — a contradiction.

Moreover, if $v \neq v'$ and $U_v \cap U_{v'} \neq \emptyset$, then $U_v = U_{v'}$. This follows from:

$$1 \leq d^{\text{out}}_A(U_v \cup U_{v'}) \leq d^{\text{out}}_A(U_v) + d^{\text{out}}_A(U_{v'}) - d^{\text{out}}_A(U_v \cap U_{v'}) \leq 1 + 1 - 1 = 1. \tag{61.10}$$

So, if $U_v \neq U_{v'}$, we would increase U_v or $U_{v'}$ by replacing it by $U_v \cup U_{v'}$ — a contradiction.

So the U_v partition K. Now if $U_v \neq K$, there exist v and v' such that $U_v \neq U_{v'}$ and such that G has an edge connecting U_v and $U_{v'}$. We can assume that it is oriented from $U_{v'}$ to U_v. So it is the unique edge leaving $U_{v'}$. Hence $d^{\text{out}}_A(U_v \cup U_{v'}) \leq d^{\text{out}}_A(U_v) = 1$. So replacing U_v by $U_v \cup U_{v'}$ would increase U_v — a contradiction. This shows (61.9), and hence (61.8).

So $d^{\text{out}}_A(K) = 1$ for each component K of $G - U$. Therefore

$$|E[U]| + \kappa(G - U) = \sum_{v \in U} \deg^{\text{in}}_A(v) > u(U). \tag{61.11}$$

Thus U violates (61.6)(i).

One similarly shows that $\deg^{\text{in}}_A(v_0) < l(v_0)$ implies violation of (61.6)(ii). ∎

This implies an alternative characterization:

Corollary 61.5a. *Let $G = (V, E)$ be a 2-edge-connected undirected graph and let $l, u \in \mathbb{Z}^V_+$ with $l \leq u$. Then G has a strongly connected orientation $D = (V, A)$ satisfying $l(v) \leq \deg^{\text{in}}_A(v) \leq u(v)$ for each $v \in V$ if and only if G has strongly connected orientations $D' = (V, A')$ and $D'' = (V, A'')$ with $l(v) \leq \deg^{\text{in}}_{A'}(v)$ and $\deg^{\text{in}}_{A''}(v) \leq u(v)$ for each $v \in V$.*

Proof. Directly from Theorem 61.5, as (61.6)(i) is void if $u = \infty$ and as (61.6)(ii) is void if $l = \mathbf{0}$ (since $\kappa(G - U) \leq d_G(U)$). ∎

For further results, see Theorem 61.7.

61.3. Nash-Williams' orientation theorem

The result of Robbins [1939] was extended deeply by Nash-Williams [1960]. Before stating and proving it, we give a useful lemma of Nash-Williams [1960]. Let $\phi : V \times V \to \mathbb{R}$ be a symmetric function (that is, $\phi(u, v) = \phi(v, u)$ for all $u, v \in V$). Define a set function R on V by:

(61.12) $$R(U) := \max_{u \in U, v \in V \setminus U} \phi(u, v) \text{ if } \emptyset \subset U \subset V, \text{ and}$$
$$R(\emptyset) := R(V) := 0.$$

Lemma 61.6α. *For all $T, U \subseteq V$:*

(61.13) $$R(T) + R(U) \leq R(T \cap U) + R(T \cup U)$$
$$\text{or } R(T) + R(U) \leq R(T \setminus U) + R(U \setminus T).$$

Proof. Suppose not. Then $\emptyset \neq T \neq V$ and $\emptyset \neq U \neq V$. Choose $s \in T$, $t \in V \setminus T$, $u \in U$, $v \in V \setminus U$ such that $R(T) = \phi(s, t)$ and $R(U) = \phi(u, v)$. By symmetry, we can assume that $R(T) \leq R(U)$ and $u \in T$. So $u \in T \cap U$, and hence $T \cap U$ splits[15] $\{u, v\}$. This implies that $T \cup U$ splits neither $\{s, t\}$, nor $\{u, v\}$, as otherwise the first inequality in (61.13) holds (as $\phi(s, t) \leq \phi(u, v)$).

Hence $t, v \in T \cup U$, and so $t \in U \setminus T$ and $v \in T \setminus U$. Then $T \setminus U$ splits $\{u, v\}$, and $U \setminus T$ splits $\{s, t\}$, implying the second inequality in (61.13). ∎

For any undirected graph $G = (V, E)$ and $s, t \in V$, let $\lambda_G(s, t)$ denote the maximum number of edge-disjoint $s - t$ paths in G. Similarly, for any directed graph $D = (V, A)$ and $s, t \in V$, let $\lambda_D(s, t)$ denote the maximum number of arc-disjoint $s - t$ paths in D.

Theorem 61.6 (Nash-Williams' orientation theorem)**.** *Any undirected graph $G = (V, E)$ has an orientation $D = (V, A)$ with*

(61.14) $$\lambda_D(s, t) \geq \lfloor \tfrac{1}{2} \lambda_G(s, t) \rfloor$$

for all $s, t \in V$.

Proof. Call a partition of a set T into pairs, a *pairing* of T. For any number k, define

(61.15) $$k^* := 2\lfloor \tfrac{1}{2} k \rfloor.$$

For any subset U of V, define

[15] Set X *splits* set Y if both $Y \cap X$ and $Y \setminus X$ are nonempty.

(61.16) $$r(U) := \max_{u \in U, v \notin U} \lambda_G(u,v),$$

setting $r(U) := 0$ if $U = \emptyset$ or $U = V$. Let T be the set of vertices of odd degree of G.

I. It suffices to show that T has a pairing P such that

(61.17) $d_G(U) - d_P(U) \geq r(U)^*$ for each $U \subseteq V$.

To see that this is sufficient, let $G' = (V, E \cup P)$. That is, G' is the graph obtained from G by adding all pairs in P as new edges (possibly in parallel). Then all degrees in G' are even, and hence G' has an Eulerian orientation $D' = (V, A')$. So

(61.18) $$\deg^{\text{out}}_{D'}(v) = \deg^{\text{in}}_{D'}(v) = \tfrac{1}{2}\deg_{G'}(v)$$

for each $v \in V$. This implies that, for each $U \subseteq V$,

(61.19) $$d^{\text{out}}_{D'}(U) = \tfrac{1}{2} d_{G'}(U).$$

Let A be the restriction of A' to the original edges of G and let $D = (V, A)$. We claim that D is an orientation of G as required. Indeed, by (61.17), for each $U \subseteq V$,

(61.20) $$\begin{aligned} d^{\text{out}}_{D}(U) &\geq d^{\text{out}}_{D'}(U) - d_P(U) = \tfrac{1}{2} d_{G'}(U) - d_P(U) \\ &= \tfrac{1}{2}(d_G(U) - d_P(U)) \geq \lfloor \tfrac{1}{2} r(U) \rfloor. \end{aligned}$$

Hence, for any $u, v \in V$, if $U \subseteq V$ with $u \in U$, $v \notin U$, and $\lambda_D(u,v) = d^{\text{out}}_D(U)$, then

(61.21) $$\lambda_D(u,v) = d^{\text{out}}_D(U) \geq \lfloor \tfrac{1}{2} r(U) \rfloor \geq \lfloor \tfrac{1}{2} \lambda_G(u,v) \rfloor.$$

II. We now prove the theorem. Define for any $Y \subseteq V$ and $U \subseteq V$,

(61.22) $$r_Y(U) := \max_{u \in Y \cap U, v \in Y \setminus U} \lambda_G(u,v),$$

setting $r_Y(U) := 0$ if $Y \cap U = \emptyset$ or $Y \subseteq U$.

By Lemma 61.6α, for any $U, W \subseteq V$,

(61.23) $$\begin{aligned} &r_Y(U)^* + r_Y(W)^* \leq r_Y(U \cap W)^* + r_Y(U \cup W)^* \\ &\text{or } r_Y(U)^* + r_Y(W)^* \leq r_Y(U \setminus W)^* + r_Y(W \setminus U)^*. \end{aligned}$$

This follows from Lemma 61.6α by taking $\phi(u,v) := \lambda_G(u,v)^*$ if $u, v \in Y$ and $\phi(u,v) := 0$ otherwise.

Suppose that there exist graphs G for which T has no pairing P satisfying (61.17). Choose G with $|V| + |E|$ minimal.

Choose $Y \subseteq V$ such that T has no pairing P satisfying

(61.24) $d_G(U) - d_P(U) \geq r_Y(U)^*$ for each $U \subseteq V$,

with $|Y|$ as small as possible. Then

(61.25) For any subset X of V splitting Y and satisfying $d_G(X)^* = r_Y(X)^*$, one has $|X| = 1$ or $|X| = |V| - 1$.

For suppose $1 < |X| < |V| - 1$. Consider the graph $G_1 = (V_1, E_1)$ obtained from G by contracting $V \setminus X$ to one vertex, v_1 say. Let T_1 be the set of vertices of odd degree of G_1. By the minimality of $|V| + |E|$, T_1 has a pairing P_1 such that for each subset U of X:

(61.26) $$d_{G_1}(U) - d_{P_1}(U) \geq r(U)^*.$$

(Note that $r(U) \leq \max_{u \in U, v \in V \setminus U} \lambda_{G_1}(u, v)$.)

Similarly, consider the graph $G_2 = (V_2, E_2)$ obtained from G by contracting X to one vertex, v_2 say. Let T_2 be the set of vertices of odd degree of G_2. Again by the minimality of $|V| + |E|$, T_2 has a pairing P_2 such that for each subset U of $V \setminus X$ and for each $u \in U$, $v \in V_2 \setminus U$:

(61.27) $$d_{G_2}(U) - d_{P_2}(U) \geq r(U)^*.$$

Now define a pairing P of T as follows. (Observe that $v_1 \in T_1$ if and only if $v_2 \in T_2$.) If $v_1 \notin T_1$ and $v_2 \notin T_2$, let $P := P_1 \cup P_2$. If $v_1 \in T_1$ and $v_2 \in T_2$, let $u_1 \in X$ and $u_2 \in V \setminus X$ be such that $u_1v_1 \in P_1$ and $u_2v_2 \in P_2$. Then define

(61.28) $$P := (P_1 \setminus \{u_1v_1\}) \cup (P_2 \setminus \{u_2v_2\}) \cup \{u_1u_2\}.$$

We claim that P satisfies (61.24). To show this, we may assume by (61.23) that $r_Y(U \cap X)^* + r_Y(U \cup X)^* \geq r_Y(U \setminus X)^* + r_Y(X \setminus U)^*$. (Otherwise, replace U by $V \setminus U$.)

Set $U_1 := U \cap X$ and $U_2 := V \setminus (U \cup X)$. Then

(61.29) $$\begin{aligned} & d_G(U) + d_G(X) \geq d_G(U_1) + d_G(U_2) = d_{G_1}(U_1) + d_{G_2}(U_2) \\ & \geq r(U_1)^* + d_{P_1}(U_1) + r(U_2)^* + d_{P_2}(U_2) \geq r_Y(U)^* + r_Y(X)^* + d_P(U) \\ & \geq r_Y(U)^* + d_G(X) + d_P(U) - 1, \end{aligned}$$

using (61.26) and (61.27). As $d_G(U) + d_P(U)$ is even, (61.24) follows. This contradicts our assumption, showing (61.25).

We next show that

(61.30) each edge of G intersects Y.

For assume that G has an edge $e = st$ disjoint from Y. By (61.25), there is no U splitting Y with $d_G(U)^* = r_Y(U)^*$ and $s \in U$, $t \notin U$. So deleting edge e, changes no $r_Y(U)^*$. Let G' be the graph obtained from G by deleting e. Let T' be the set of vertices of G' of odd degree. (So $T' = T \triangle \{s, t\}$.) Then, by the minimality of $|V| + |E|$, we know that T' has a pairing P' such that, for each $U \subseteq V$,

(61.31) $$d_{G'}(U) - d_{P'}(U) \geq r(U)^* \geq r_Y(U)^*.$$

It is not difficult to transform pairing P' of T' to a pairing P of T with the property that $|P \setminus P'| \leq 1$.[16] Then (61.24) holds. Indeed, $d_G(U) \geq d_{G'}(U)$

[16] If $s, t \in T'$ and $st \in P'$, define $P := P' \setminus \{st\}$. If $s, t \in T'$ and $st \notin P'$, let s' and t' be such that $ss' \in P'$ and $tt' \in P'$, and define $P := (P' \setminus \{ss', tt'\}) \cup \{s't'\}$. If $s \in T'$ and $t \notin T'$, let s' be such that $ss' \in P'$, and define $P := (P' \setminus \{ss'\}) \cup \{ts'\}$. If $s \notin T'$ and $t \in T'$, let t' be such that $tt' \in P'$, and define $P := (P' \setminus \{tt'\}) \cup \{st'\}$. If $s \notin T'$ and $t \notin T'$, define $P := P' \cup \{st\}$.

and $d_P(U) \leq d_{P'}(U)+1$ (as $|P \setminus P'| \leq 1$). Hence (61.24) follows from (61.31), with parity. This contradiction proves (61.30).

Next:

(61.32) $\quad |Y| \geq 2.$

For suppose that $|Y| \leq 1$. In G there exist $\frac{1}{2}|T|$ edge-disjoint paths such that each vertex in T occurs exactly once as an end vertex of one of these paths. (This can be seen by taking an arbitrary pairing Q of T, and considering an Eulerian tour C in the graph $G = (V, E \cup Q)$. Then removing Q from C decomposes C into paths as required.) Let P be the set of pairs of end vertices of these paths. Then $d_G(U) \geq d_P(U)$ for each $U \subseteq V$, and (61.24) follows, contradicting our assumption. So we know (61.32).

Choose a set X splitting Y with $d_G(X)$ minimal. Then $d_G(X) = r_Y(X)$. By (61.25), we may assume that $X = \{x\}$ for some $x \in Y$. So $r_Y(U) = d_G(x)$ for any U splitting Y, since for any $y \in Y \setminus U$ we have

(61.33) $\quad d_G(x) \leq d_G(U) \leq r_Y(U) \leq \lambda_G(x, y) \leq d_G(x).$

.

Define $Y' := Y \setminus \{x\}$. Then, by the minimality of $|Y|$, T has a pairing P such that for each $U \subseteq V$,

(61.34) $\quad d_G(U) - d_P(U) \geq r_{Y'}(U)^*.$

We show that (61.24) holds, which forms a contradiction. To this end, choose $U \subseteq V$.

First assume that U splits Y'. Then $r_{Y'}(U) \geq r_Y(U)$, since $\lambda_G(x, y) = d_G(X) \leq r_{Y'}(U)$ for each $y \in Y'$ (by the minimality of $d_G(X)$, since any splitting of Y' also splits Y). This implies (61.34).

So we can assume that U splits Y but does not split Y'; that is, $U \cap Y = \{x\}$. Consider any $u \in U \setminus \{x\}$. Let α_u denote the number of edges connecting u and x and let β_u denote the number of edges connecting u and $Y \setminus \{x\}$. By (61.30), $\alpha_u + \beta_u = \deg_G(u)$. Since $X = \{x\}$ splits Y with $d_G(X)$ minimum, we have $d_G(\{x, u\}) \geq \deg_G(x)$. Hence $\beta_u \geq \alpha_u$, with strict inequality if $u \in T$ (since then $\alpha_u + \beta_u$ is odd).

Therefore, setting $U' := U \setminus \{x\}$ and $\lambda :=$ number of edges connecting x and $V \setminus U$,

(61.35)
$$\begin{aligned} d_G(U) &= \lambda + \sum_{u \in U'} \beta_u \geq \lambda + \sum_{u \in U'} \alpha_u + |U' \cap T| = \deg_G(x) + |U' \cap T| \\ &= r_Y(U) + |U' \cap T| \geq r_Y(U) + |U \cap T| - 1 \geq r_Y(U) + d_P(U) - 1. \end{aligned}$$

Hence, with parity, we have (61.24). ∎

(This is the original proof of Nash-Williams [1960]. Mader [1978a] and Frank [1993a] gave alternative proofs.)

An orientation satisfying the condition described in Theorem 61.6 is called *well-balanced*. Nash-Williams [1969] (giving an introduction to the proof above) remarks that with methods similar to those used in the proof of Theorem 61.6, one can prove that for any graph G and any subgraph H of G, there is a well-balanced orientation of G such that the restriction to H is well-balanced again.

61.4. k-arc-connected orientations of $2k$-edge-connected graphs

Nash-Williams' orientation theorem directly implies:

Corollary 61.6a. *An undirected graph G has a k-arc-connected orientation if and only if G is $2k$-edge-connected.*

Proof. Directly from Theorem 61.6. ∎

A direct proof of this corollary, based on total dual integrality, was given by Frank [1980b] and Frank and Tardos [1984b], and is as follows.

Orient the edges of G arbitrarily, yielding the directed graph $D = (V, A)$. Consider the system

(61.36) (i) $0 \leq x_a \leq 1$ for each $a \in A$,
(ii) $x(\delta^{\text{in}}(U)) - x(\delta^{\text{out}}(U)) \leq d^{\text{in}}(U) - k$ for each nonempty $U \subset V$.

By the Edmonds-Giles theorem (Theorem 60.1), this system is TDI, and hence determines an integer polytope P. If G is $2k$-edge-connected, then P is nonempty, since the vector $x := \frac{1}{2} \cdot \mathbf{1}$ belongs to P.

As P is nonempty and integer, (61.36) has an integer solution x. Then G has a k-arc-connected orientation D': reversing the orientation of the arcs a of D with $x_a = 1$ gives a k-arc-connected orientation D', since

(61.37) $$d^{\text{in}}_{D'}(U) = d^{\text{in}}_D(U) - x(\delta^{\text{in}}_D(U)) + x(\delta^{\text{out}}_D(U)) \geq k$$

for any nonempty proper subset U of V.

Notes. The total dual integrality of (61.36) implies also the following result of Frank, Tardos, and Sebő [1984] (denoting the number of (weak) components of a (di)graph G by $\kappa(G)$). Let $G = (V, E)$ be a 2-edge-connected undirected graph and let $U \subseteq V$. Then the minimum number of arcs entering U over all strongly connected orientations of G is equal to the maximum of

(61.38) $$\sum_{T \in \mathcal{P}} \kappa(G - T),$$

taken over partitions $\mathcal{P}$ of U into nonempty classes such that no edge connects different classes of $\mathcal{P}$.

This implies another result of Frank, Tardos, and Sebő [1984]: Let $D = (V, A)$ be a digraph and let $C = \delta^{\text{in}}(U)$ be a directed cut. Then the minimum of $|B \cap C|$ where B is a directed cut cover is equal to the maximum of

$$(61.39) \qquad \sum_{T \in \mathcal{P}} \kappa(D - T),$$

taken over partitions $\mathcal{P}$ of U into nonempty classes such that no arc of D connects distinct classes of $\mathcal{P}$.

As A. Frank (personal communication 2002) observed, the proof above yields a stronger result of Nash-Williams [1969]: let $G = (V, E)$ be a $2k$-edge-connected graph and let $F \subseteq E$ have an Eulerian orientation; then the remaining edges have an orientation so as to obtain a k-arc-connected digraph. This follows by taking for A any orientation extending the orientation of F, and by setting $x_a := 0$ for each arcs in F, and $x_a := \frac{1}{2}$ for all other arcs a. Then x satisfies (61.36), and the result follows as above.

61.4a. Complexity

By the results in Section 60.1 on the complexity of the Edmonds-Giles problem, one can find a k-arc-connected orientation of a $2k$-edge-connected undirected graph in polynomial time; more generally, one can find a minimum-length k-arc-connected orientation in strongly polynomial time, if we are given a length for each orientation of each edge.

A direct method of finding a minimum-length k-arc-connected orientation can be based on weighted matroid intersection, similarly to the method described in Section 55.5 to find a minimum-length directed k-cover in a directed graph (such that the k-arc-connected orientations form the common bases of two matroids).

61.4b. *k*-arc-connected orientations with bounds on degrees

Frank [1980b] extended Corollary 61.6a to the case where lower and upper bounds on the indegrees of the vertices are prescribed:

Theorem 61.7. *Let $G = (V, E)$ be a $2k$-connected undirected graph and let $l, u \in \mathbb{Z}_+^V$ with $l \leq u$. Then G has a k-arc-connected orientation D with $l(v) \leq \deg_D^{\text{in}}(v) \leq u(v)$ for each $v \in V$ if and only if*

$$(61.40) \qquad |E[W]| + |\delta(\mathcal{P})| \geq k|\mathcal{P}| + \max\{\sum_{v \in W} l(v), \sum_{v \in W} (\deg_G(v) - u(v))\}$$

for each subpartition $\mathcal{P}$ of V with nonempty classes, where $W := V \setminus \bigcup \mathcal{P}$.

Proof. It is not difficult to see that the condition is necessary. To show sufficiency, by Corollary 61.6a, G has a k-arc-connected orientation D. Choose D such that

$$(61.41) \qquad \sum_{v \in V} \max\{0, \deg_D^{\text{in}}(v) - u(v), l(v) - \deg_D^{\text{in}}(v)\}$$

is as small as possible. If sum (61.41) is 0 we are done, so assume that it is positive. By symmetry, we can assume that there is a vertex r with $\deg_D^{\text{in}}(r) < l(r)$.

Let $\mathcal{P}$ be the collection of inclusionwise maximal nonempty subsets U of $V \setminus \{r\}$ with $d^{\text{in}}(U) = k$, and let $W := V \setminus \bigcup \mathcal{P}$.

Then the sets in $\mathcal{P}$ are disjoint. For let $U, W \in \mathcal{P}$ with $U \cap W \neq \emptyset$. Then

$$(61.42) \qquad 2k \leq d_D^{\text{in}}(U \cap W) + d_D^{\text{in}}(U \cup W) \leq d_D^{\text{in}}(U) + d_D^{\text{in}}(W) = 2k,$$

implying $d_D^{\text{in}}(U \cup W) = k$, and so $U = W = U \cup W$.

Suppose that there exists a vertex $s \in W$ with $\deg_D^{\text{in}}(s) > l(s)$. Then reversing the orientations of the arcs of any $r - s$ path in D gives again a k-arc-connected orientation (since there is no $U \subseteq V$ with $d_D^{\text{in}}(U) = k$ and $s \in U$, $r \notin U$), but decreases sum (61.41). This contradicts our minimality assumption.

So $\deg_D^{\text{in}}(v) \leq l(v)$ for each $v \in W$, with strict inequality for at least one $v \in W$ (namely for r). Now each edge of G that is spanned by no set in $\mathcal{P}$, either enters some $U \in \mathcal{P}$, or has its head in W. So the number of such edges is

(61.43) $$k|\mathcal{P}| + \sum_{v \in W} \deg_D^{\text{in}}(v), \text{ which is less than } k|\mathcal{P}| + \sum_{v \in W} l(v).$$

This contradicts the condition. ∎

This has an alternative characterization as consequence:

Corollary 61.7a. *Let $G = (V, E)$ be an undirected graph, let $k \in \mathbb{Z}_+$, and let $l, u \in \mathbb{Z}_+^E$ with $l \leq u$. Then G has a k-arc-connected orientation D with $l(v) \leq \deg_D^{\text{in}}(v) \leq u(v)$ for each $v \in V$ if and only if G has k-arc-connected orientations D' and D'' with $l(v) \leq \deg_{D'}^{\text{in}}(v)$ and $\deg_{D''}^{\text{in}}(v) \leq u(v)$ for each $v \in V$.*

Proof. This follows from the fact that the condition in Theorem 61.7 can be decomposed into a condition on l and one on u. ∎

61.4c. Orientations of graphs with lower bounds on indegrees of sets

Let $G = (V, E)$ be an undirected graph and let $l : \mathcal{P}(V) \to \mathbb{Z}_+$ be such that

(61.44) $l(T) + l(U) - d(T, U) \leq l(T \cap U) + l(T \cup U)$, for all $T, U \subseteq V$ with $T \cap U \neq \emptyset$ and $T \cup U \neq V$,

where $d(T, U)$ denotes the number of edges connecting $T \setminus U$ and $U \setminus T$.

Frank [1980b] showed with submodularity theory:

Theorem 61.8. *Let $G = (V, E)$ be a graph and let $l : \mathcal{P}(V) \to \mathbb{Z}_+$ satisfy* (61.44). *Then G has an orientation $D = (V, A)$ with*

(61.45) $$d_A^{\text{in}}(U) \geq l(U)$$

for each $U \subseteq V$ if and only if

(61.46) $$|\delta(\mathcal{P})| \geq \max\{\sum_{U \in \mathcal{P}} l(U), \sum_{U \in \mathcal{P}} l(V \setminus U)\}$$

for each partition $\mathcal{P}$ of V into nonempty proper subsets, where $\delta(\mathcal{P})$ denotes the set of edges of G connecting different classes of $\mathcal{P}$.

Proof. The necessity of the condition is obvious. To prove sufficiency, let $D = (V, A)$ be an arbitrary orientation of G. Define for each nonempty proper subset U of V

(61.47) $$f(U) := d_A^{\text{in}}(U) - l(U).$$

One easily checks, using (61.44), that f is crossing submodular. Moreover, if $x : A \to \{0,1\}$ is such that

$$x(\delta_A^{\text{in}}(U)) - x(\delta_A^{\text{out}}(U)) \leq f(U) \tag{61.48}$$

for each nonempty proper subset U of V, then the digraph $D' = (V, A')$ obtained from $D = (V, A)$ by reversing the direction of the arcs a with $x_a = 1$, has indegrees as required by (61.45), since

$$d_{A'}^{\text{in}}(U) = d_A^{\text{in}}(U) - x(\delta_A^{\text{in}}(U)) + x(\delta_A^{\text{out}}(U)) \geq d_A^{\text{in}}(U) - f(U) = l(U). \tag{61.49}$$

Hence it suffices to show that (61.48) has an integer solution x with $\mathbf{0} \leq x \leq \mathbf{1}$.

Consider x as a transshipment. The 'excess function' $\text{excess}_x \in \mathbb{R}^V$ of x is given by:

$$\text{excess}_v := x(\delta_A^{\text{in}}(v)) - x(\delta_A^{\text{out}}(v)) \tag{61.50}$$

for $v \in V$. Then (61.48) is equivalent to

$$y(U) \leq f(U). \tag{61.51}$$

Now y is the excess function of some $x \in \{0,1\}^A$ if and only if x is an integer y-transshipment with $\mathbf{0} \leq x \leq \mathbf{1}$. So, by Corollary 11.2f, y is the excess function of some $x \in \{0,1\}^A$ if and only if y is integer, $y(V) = 0$, and

$$y(U) \leq d_A^{\text{in}}(U) \tag{61.52}$$

for each $U \subseteq V$. Since $l(U) \geq 0$, (61.52) is implied by (61.51).

So it suffices to show that (61.51) has an integer solution y with $y(V) = 0$. By Theorem 49.10, y exists if and only if

$$\sum_{U \in \mathcal{P}} f(U) \geq 0 \tag{61.53}$$

for each partition or copartition $\mathcal{P}$ of V, where each set in $\mathcal{P}$ is a nonempty proper subset of V. (A *copartition* of V is a collection of sets whose complements form a partition of V.) This is equivalent to the condition given in the present theorem. ∎

61.4d. Further notes

Frank [1980b] observed that Edmonds' disjoint arborescences theorem implies:

Corollary 61.8a. *Let $G = (V, E)$ be an undirected graph and $r \in V$. Then G has an orientation such that each nonempty subset U of $V \setminus \{r\}$ is entered by at least k arcs if and only if G contains k edge-disjoint spanning trees.*

Proof. Necessity follows from the fact that if the orientation $D = (V, A)$ as required exists, then by Edmonds' disjoint arborescences theorem (Corollary 53.1b), D has k disjoint r-arborescences. Hence G has k edge-disjoint spanning trees.

Sufficiency follows from the fact that we can orient each spanning tree in G so as to become an r-arborescence. Orienting the remaining edges arbitrarily, we obtain an orientation as required. ∎

Frank [1993c] gave a direct proof of the existence of this orientation from the conditions given in the Tutte-Nash-Williams disjoint trees theorem (Corollary

51.1a), yielding (with Edmonds' disjoint arborescences theorem) a proof of the Tutte-Nash-Williams disjoint trees theorem.

Frank [1982a] showed that each k-arc-connected orientation of an undirected graph can be obtained from any other by reversing iteratively directed paths and circuits, without destroying k-arc-connectivity. This can be derived from a result of L. Lovász that two k-arc-connected orientations are adjacent on the polytope determined by (61.36) if and only if they differ on a directed circuit or on a collection of vertex-disjoint directed paths. Frank [1982b] showed that a minimum-cost k-arc-connected orientation can be found in strongly polynomial time, by reduction to the Edmonds-Giles model. Accelerations were given by Gabow [1993a,1993b,1994, 1995c].

Frank, Jordán, and Szigeti [1999,2001] and Frank and Király [1999,2002] studied graph orientations that satisfy parity and connectivity conditions. Orientations preserving prescribed shortest paths are considered by Hassin and Megiddo [1989].

Chvátal and Thomassen [1978] showed that each 2-edge-connected graph of radius r has a strongly connected orientation of radius at most $r^2 + r$. This was extended to mixed graphs by Chung, Garey, and Tarjan [1985].

For surveys on applying submodularity to orientation problems, see Frank [1993a,1996b].

Chapter 62

Network synthesis

The network synthesis problem asks for a graph having prescribed connectivity properties, with a minimum number of edges. If the edges have costs, a minimum total cost is required.
The problem can be seen as the special case of the connectivity augmentation problem where the input graph is edgeless. Connectivity augmentation in general will be discussed in Chapter 63.

62.1. Minimal k-(edge-)connected graphs

We first consider the easy problem of finding a graph of given connectivity, with a minimal number of edges. First, vertex-connectivity:

Theorem 62.1. *Let k and n be positive integers with $n \geq 2$. The minimum number of edges of a k-vertex-connected graph with n vertices is $n-1$ if $k = 1$, $\lceil \frac{1}{2}kn \rceil$ if $1 < k < n$, and $\frac{1}{2}n(n-1)$ otherwise.*

Proof. Since any k-vertex-connected graph contains a spanning tree, has minimum degree at least k if $k < n$, and is a complete graph if $k \geq n$, the values given are lower bounds. Moreover, if $k = 1$ or $k \geq n$, the bound is tight. So we can assume $1 < k < n$, and it suffices to show that there exists a k-vertex-connected graph $G = (V, E)$ with $|V| = n$ and $|E| = \lceil \frac{1}{2}kn \rceil$.

Let $V := \{1, \ldots, n\}$ and let C be the circuit on V with edge set $\{\{i, i+1\} \mid i \in V\}$, taking addition mod n. Let G be the graph on V with edges all pairs of vertices at distance at most $\frac{1}{2}k$ in C.

First assume that k is even. Then G has $\frac{1}{2}kn$ edges. We show that G is k-vertex-connected. Suppose to the contrary that G has a vertex-cut U of size less than k. There are at least two components K of $C[U]$ such that the two neighbours of K in C belong to different components of $G-U$ (as $G-U$ is disconnected). In particular, the two neighbours have distance more than $\frac{1}{2}k$ in C, and so these components each have size at least $\frac{1}{2}k$. This contradicts the fact that $|U| < k$.

Next, if k is odd, we add to G $\lceil \frac{1}{2}n \rceil$ edges $\{i, j\}$, where i and j have distance $\lfloor \frac{1}{2}n \rfloor$ in C, and such that these edges cover all vertices in V. So G has $\lceil \frac{1}{2}kn \rceil$ edges. We show that G is k-vertex-connected.

Suppose that G has a vertex-cut U of size less than k. By the above, $C[U]$ consists of two components of size $l := \frac{1}{2}(k-1)$ each. We can assume that $U = [1,l] \cup [s+1, s+l]$ for some s with $l < s$ and $s+l < n$. Now n is adjacent to no vertex in $[l+1, s]$, while n is adjacent to at least one of $\lfloor \frac{1}{2}n \rfloor$ and $\lceil \frac{1}{2}n \rceil$. So $\lfloor \frac{1}{2}n \rfloor < l+1$ or $\lceil \frac{1}{2}n \rceil > s$, implying $n > 2s$ (as $k < n$). By symmetry of the two components we similarly have $n > 2(n-s)$, that is $n < 2s$, a contradiction. ∎

For edge-connectivity the answer is almost the same:

Theorem 62.2. *Let k and n be positive integers with $n \geq 2$. The minimum number of edges of a k-edge-connected graph with n vertices is $n-1$ if $k = 1$, and $\lceil \frac{1}{2}kn \rceil$ otherwise. If $k \leq n-1$ the minimum is attained by a simple graph.*

Proof. Again, the values are lower bounds, as a k-edge-connected graph contains a spanning tree and has each degree at least k. Clearly the lower bound can be attained if $k = 1$, so assume $k \geq 2$. Let C be a graph on $V := \{1, \ldots, n\}$ with edges all pairs $\{i, i+1\}$ for $i \in V$ (with addition mod n). Let G be the graph obtained from C by replacing each edge by $\lfloor \frac{1}{2}k \rfloor$ parallel edges.

If k is even, then G is k-edge-connected as required. If k is odd, add $\lceil \frac{1}{2}n \rceil$ edges $\{i, j\}$ to G, where i and j have distance $\lfloor \frac{1}{2}n \rfloor$ in C, and such that these edges cover all vertices in V. So G has $\lceil \frac{1}{2}kn \rceil$ edges. We show that G is k-edge-connected. Suppose that $d_G(U) < k$ for some nonempty proper subset U of V. By symmetry, we can assume that $|U| \geq \frac{1}{2}n$. Now $C[U]$ is connected (as otherwise $d_G[U] \geq 4\lfloor \frac{1}{2}k \rfloor \geq k$, since $k > 1$). So we can assume that $U = [1, s]$, with $s \geq \lceil \frac{1}{2}n \rceil$. However, $n \in V \setminus U$ is adjacent to at least one of $\lfloor \frac{1}{2}n \rfloor$ and $\lceil \frac{1}{2}n \rceil$. As both of these vertices belong to U, we have $d_G(U) \geq 2\lfloor \frac{1}{2}k \rfloor + 1 = k$, a contradiction.

Finally, if $k \leq n-1$, the minimum is attained by a simple graph. Indeed, by Theorem 62.1, there is a k-vertex-connected graph $G = (V, E)$ with n vertices and $\lceil \frac{1}{2}kn \rceil$ edges. Necessarily, G is simple. We show that G is k-edge-connected. Suppose that there is a nonempty $U \subset V$ with $d_G(U) < k$. Then $|U||V \setminus U| \geq n-1 \geq k$, and hence there exist $s \in U$ and $t \in V \setminus U$ that are not adjacent. Hence G has k internally vertex-disjoint $s-t$ paths, and therefore k edge-disjoint $s-t$ paths. This contradicts the fact that $d_G(U) < k$. ∎

The directed case is even simpler. For vertex-connectivity one has:

Theorem 62.3. *Let k and n be positive integers with $n \geq 2$. Then the minimum number of arcs of a k-vertex-connected directed graph with n vertices is kn if $k \leq n-1$, and $n(n-1)$ otherwise.*

Proof. Since each vertex should have at least $\min\{k, n-1\}$ outneighbours, the values are lower bounds. Trivially it is attained if $k \geq n$.

If $k \leq n-1$, let D be the directed graph on $V := \{1, \ldots, n\}$ with arcs all pairs $(i, i+l)$ with $i \in V$ and $1 \leq l \leq k$, taking addition mod n. Then D has kn arcs. To see that D is k-vertex-connected, let U be a vertex-cut. Choose $i, j \in V \setminus U$ with j not reachable from i in $D - U$. We may assume that $1 \leq i < j \leq n$ and that $j - i$ is as small as possible. Then $j - i > k$ and $i+1, \ldots, j-1$ belong to U. So $|U| \geq k$. ∎

Finally, for arc-connectivity (Fulkerson and Shapley [1971]):

Theorem 62.4. *Let k and n be positive integers with $n \geq 2$. Then the minimum number of arcs of a k-arc-connected directed graph with n vertices is kn. If $k \leq n-1$, the minimum is attained by a simple directed graph.*

Proof. Since each vertex should be left by at least k arcs, kn is a lower bound. It is attained by the directed graph obtained from a directed circuit on n vertices, by replacing any arc by k parallel arcs.

If $k \leq n-1$, the minimum is attained by a simple directed graph. Indeed, by Theorem 62.3, there is a k-vertex-connected directed graph $D = (V, A)$ with n vertices and kn arcs. Necessarily, D is simple. We show that D is k-arc-connected. Suppose that there is a nonempty $U \subset V$ with $d_D^{\text{out}}(U) < k$. Then $|U||V \setminus U| \geq n-1 \geq k$, and hence there exist $s \in U$ and $t \in V \setminus U$ with $(s,t) \notin A$. Hence D has k internally vertex-disjoint $s-t$ paths, and therefore k arc-disjoint $s-t$ paths. This contradicts the fact that $d_D^{\text{out}}(U) < k$. ∎

Notes. Edmonds [1964] showed that for each simple graph with all degrees at least k, there exists a k-edge-connected simple graph with the same degree-sequence.

62.2. The network synthesis problem

Let V be a finite set and let $r : V \times V \to \mathbb{R}_+$. A *realization* of r is a pair of a directed graph $D = (V, A)$ and a capacity function $c : A \to \mathbb{R}_+$ such that for all $s, t \in V$, each $s-t$ cut in G has capacity at least $r(s,t)$. The pair D, c is called an *exact realization* if for all $s, t \in V$ with $s \neq t$, the minimum capacity of an $s-t$ cut in D as equal to $r(s,t)$.

Obviously, any function r has a realization. We say that r is *exactly realizable* if it has an exact realization. The *network synthesis problem* is the problem to find an exact or cheapest realization for a given r (or to decide that none exist).

The following theorem due to Gomory and Hu [1961] characterizes the exactly realizable *symmetric*[17] functions. It also shows that if $r : V \times V \to \mathbb{R}$ is exactly realizable and symmetric, then r has an *undirected* exact realization

[17] A function $r : V \times V \to \mathbb{R}$ is called *symmetric* if $r(u,v) = r(v,u)$ for all $u, v \in V$.

(more precisely, an exact realization D, c where for each arc $a = (u, v)$ of D, also (v, u) is an arc, with $c(u, v) = c(v, u)$).

Theorem 62.5. *A symmetric function $r : V \times V \to \mathbb{R}_+$ is exactly realizable if and only if*

(62.1) $$r(u, w) \geq \min\{r(u, v), r(v, w)\}$$

for all distinct $u, v, w \in V$. If r is exactly realizable, there is a tree that gives an exact realization of r.

Proof. Necessity being easy, we show sufficiency. Let $T = (V, E)$ be a tree on V maximizing

(62.2) $$\sum_{uv \in E} r(u, v).$$

Taking $c(uv) := r(u, v)$ for each edge $uv \in E$ gives an exact realization of r. To see this, note that for all s, t, the minimum capacity of an $s - t$ cut is equal to $\min_{uv \in EP} r(u, v)$, where P is the $s - t$ path in T. By (62.1) we know that $r(s, t)$ is not smaller than this minimum. To show equality, suppose to the contrary that $r(u, v) < r(s, t)$ for some $uv \in P$. Then replacing T by $(T - uv) \cup st$ gives a tree with larger sum (62.2). ∎

Notes. Obviously, condition (62.1) remains necessary for exact realizability of non-symmetric functions. Resh [1965] claimed that (62.1) also remains sufficient, but a counterexample is given by the function $r : V \times V \to \mathbb{R}_+$ with $V = \{1, 2, 3, 4\}$, and $r(1, 2) = r(1, 3) = r(1, 4) = r(2, 4) = r(3, 4) = 1$, and $r(s, t) = 0$ for all other s, t (cf. Mayeda [1962]).

62.3. Minimum-capacity network design

Theorem 62.5 yields a tree as an exact realization of a given function $r : V \times V \to \mathbb{R}_+$. A tree is a most economical realization in the sense of having a minimum number of edges with nonzero capacity. It generally gives no exact realization for which the sum of the capacities is minimum. Such an exact realization has been characterized by Chien [1960] (extending Mayeda [1960]), while Gomory and Hu [1961] showed that if r is integer, there is a half-integer optimum exact realization.

As a preparation, we first show the following lemma of Gomory and Hu [1961] ($\lambda_G(s, t)$ denotes the maximum number of edge-disjoint $s - t$ paths in G):

Lemma 62.6α. *Let $r : V \times V \to \mathbb{R}_+$ be symmetric and let T be a spanning tree on V maximizing $r(T)$. Then any graph $G = (V, E)$ satisfies*

(62.3) $$\lambda_G(s, t) \geq r(s, t)$$

for all $s, t \in V$ if and only if (62.3) *is satisfied for each edge st of T.*

Proof. Necessity being trivial, we show sufficiency. Let $s, t \in V$ and let P be the $s - t$ path in T. By the maximality of $r(T)$, we know that $r(s,t) \leq r(e)$ for each edge e on P. Hence

(62.4) $$\lambda_G(s,t) \geq \min_{e=uv \in EP} \lambda_G(u,v) \geq \min_{e \in EP} r(e) \geq r(s,t),$$

as required. ∎

We also use the following lemma:

Lemma 62.6β. *If $r : V \times V \to \mathbb{R}_+$ is symmetric and exactly realizable, then there exists a spanning tree T on V that maximizes $r(T)$ over all spanning trees, and that moreover is a Hamiltonian path.*

Proof. Let T maximize $r(T)$. Choose T and k such that T contains a path $v_1, \ldots, v_k$, with k as large as possible. Choose T, k moreover such that the vector $(\deg_T(v_1), \ldots, \deg_T(v_k))$ is lexicographically minimal. If T is not a path, there is a j with $1 < j < k$ and $\deg_T(v_j) \geq 3$. Let $v_j u$ be an edge of T incident with v_j, with $u \neq v_{j-1}, v_{j+1}$. If $r(v_{j+1}, u) \geq r(v_j, u)$, we can replace edge $v_j u$ of T by $v_{j+1}u$, contradicting the lexicographic minimality. So $r(v_j, u) > r(v_{j+1}, u)$, and so $r(v_j, v_{j+1}) \leq r(v_{j+1}, u)$, since $r(v_{j+1}, u) \geq \min\{r(v_j, v_{j+1}), r(v_j, u)\}$ by (62.1). Hence replacing edge $v_j v_{j+1}$ of T by $v_{j+1}u$ would give a tree with a longer path, contradicting our assumption. ∎

Now we can formulate and prove the theorem. For any $r : V \times V \to \mathbb{R}$ and $u \in V$ define

(62.5) $$R(u) := \max_{v \neq u} r(u,v).$$

Theorem 62.6. *Let $r : V \times V \to \mathbb{R}_+$ be symmetric and exactly realizable. Then the minimum value of $\sum_{e \in E} c(e)$ where $G = (V, E)$ and c form an (undirected) exact realization of r, is equal to*

(62.6) $$\tfrac{1}{2} \sum_{u \in V} R(u).$$

Moreover, if r is integer, the minimum is attained by a half-integer exact realization c.

Proof. We may assume that r is integer. (62.6) indeed is a lower bound, since for each exact realization $G = (V, E), c$ of r one has

(62.7) $$\sum_{e \in E} c(e) = \tfrac{1}{2} \sum_{u \in V} \sum_{e \in \delta(u)} c(e) \geq \tfrac{1}{2} \sum_{u \in V} R(u).$$

To see that the lower bound is attained by a half-integer exact realization, let T be a spanning tree on V maximizing $r(T)$. By Lemma 62.6β, we can assume that T is a path $v_1, \ldots, v_n$.

Let $k := \max_{u,v} r(u, v)$. For $i = 0, \ldots, k$, let E_i be the set of edges e of T with $r(e) \leq i$, and for each nonsingleton component P of $T - E_i$, make a circuit consisting of edges parallel to P and one edge connecting the end vertices of P. Let $G = (V, E)$ arise by taking the edge-disjoint union of these circuits. Let $c(e) := \frac{1}{2}$ for each $e \in E$. Then

(62.8) $$\lambda_G(v_j, v_i) = 2r(v_j, v_i)$$

for all i, j with $1 \leq j < i \leq n$.

Indeed, in proving $\geq$, we can assume that $j = i - 1$ (by Lemma 62.6α). As v_{i-1}, v_i are contained in $r(v_{i-1}, v_i)$ edge-disjoint circuits, we have $\lambda_G(v_{i-1}, v_i) \geq 2r(v_{i-1}, v_i)$.

Conversely, the inequality $\leq$ in (62.8) follows from

(62.9) $$\lambda_G(v_j, v_i) \leq \min_{j<h\leq i} 2r(v_{h-1}, v_h) \leq 2r(v_j, v_i).$$

The first inequality here follows from the fact that for each h with $j < h \leq i$, the number of edges connecting $\{v_1, \ldots, v_{h-1}\}$ and $\{v_h, \ldots, v_n\}$ is equal to $2r(v_{h-1}, v_h)$. The second inequality follows from (62.1). ∎

Notes. Note that also any nonexact realization has size at least (62.6), and therefore, the theorem also characterizes the minimum size of any realization.

Wing and Chien [1961] observed that a minimum-capacity realization can be found by linear programming, and Gomory and Hu [1962,1964] showed that also the weighted case can be solved by linear programming. Indeed, the polyhedron P of all realizations of a given function $r : V \times V \to \mathbb{Q}_+$ can be described as follows. Let E is the collection of all unordered pairs of elements of V. Then P is determined by:

(62.10)
$$\begin{array}{ll} x_e \geq 0 & \text{for all } e \in E, \\ x(\delta(U)) \geq R(U) & \text{for all nonempty } U \subset V, \end{array}$$

where $R(U) := \max_{u \in U, v \in V \setminus U} r(u, v)$.

This formulation was given by Gomory and Hu [1962] and applied to finding a minimum-cost realization with linear programming, by a row-generating implementation of the simplex method (thus avoiding listing the exponential number of constraints). Bland, Goldfarb, and Todd [1981] observed that description (62.10) implies polynomial-time solvability with the ellipsoid method, since the constraints (62.10) can be tested in polynomial time.

A direct, polynomial-size linear programming formulation was given by Gomory and Hu [1964], by extending the number of variables. Indeed, P consists of those $x \in \mathbb{R}^E_+$ such that for all distinct $s, t \in V$, there exists an $s - t$ flow $f_{s,t} : E \to \mathbb{R}^E_+$ with $f \leq x$ and of value $r(s, t)$.

The latter description implies that a minimum-weight realization can be determined in polynomial time, by solving an explicit linear programming problem — in fact, in strongly polynomial time, with the method of Tardos [1986].

Note that the *exact* realizations do not form a convex set; for instance, if $V = \{u, v, w\}$ and $r(s,t) = 1$ for all $s, t \in V$, then $x(uv) = x(vw) = x(uw) = \frac{2}{3}$ is a convex combination of exact realizations, but is not itself an exact realization.

62.4. Integer realizations and r-edge-connected graphs

In Section 62.3, the fractional version of the minimum-capacity network design problem was discussed. We now consider the case where all capacities are required to be integer. It relates to: given $r : V \times V \to \mathbb{Z}_+$, find an r-edge-connected undirected graph $G = (V, E)$ with a minimum number of edges. Here a graph $G = (V, E)$ is called *r-edge-connected* if $\lambda_G(s,t) \geq r(s,t)$ for all $s, t \in V$ with $s \neq t$.

Eswaran and Tarjan [1976] observed that the weighted version of the integer realization problem is NP-complete, as finding a Hamiltonian circuit in an undirected graph can be reduced to it. (So even if $r = \mathbf{2}$ and all weights belong to $\{0, 1\}$, it is NP-complete.)

Chou and Frank [1970] gave a polynomial-time algorithm for finding a minimum-size integer realization, implying the following characterization of the minimum number $\gamma(r)$ of edges of an r-edge-connected graph[18].

To this end we can assume that r is symmetric and exactly realizable, that is, satisfies (62.1) (since resetting $r(s,t)$ to the maximum of $\min_{e=uv \in EP} r(u,v)$ over all $s - t$ paths P, does not change the problem).

Again, define for each $u \in V$,

(62.11) $$R(u) := \max_{v \neq u} r(u, v).$$

Theorem 62.7. *Let $r : V \times V \to \mathbb{Z}_+$ be symmetric and satisfy* (62.1)*.*
(i) *If $R(u) = 1$ for some $u \in V$, then $\gamma(r) = \gamma(r') + 1$, where r' is the restriction of r to $(V \setminus \{u\}) \times (V \setminus \{u\})$.*
(ii) *If $R(u) \neq 1$ for all $u \in V$, then*

(62.12) $$\gamma(r) = \lceil \tfrac{1}{2} \sum_{u \in V} R(u) \rceil.$$

Proof. We first show (i). The inequality $\gamma(r) \leq \gamma(r') + 1$, is easy, since an r-edge-connected graph can be obtained from an r'-edge-connected graph by adding one edge connecting u with some $v \neq u$ with $r(u,v) = 1$.

To see the reverse inequality, let $G = (V, E)$ be an r-edge-connected graph with $|E| = \gamma(r)$. As $R(u) = 1$, we have $\deg_G(u) \geq 1$. Let ut be an edge incident with u. Let H be the graph obtained from G by contracting ut. Then H is r'-edge-connected and has $|E| - 1$ edges, showing $\gamma(r') \leq |E| - 1 = \gamma(r) - 1$.

[18] The construction of Chou and Frank [1970] is lacunary, and does not apply, e.g., to the case where $r(u,v) = 3$ for all u, v and $|V|$ is odd.

We next show (ii). Trivially, for any r-edge-connected graph $G = (V, E)$:

(62.13) $$|E| = \tfrac{1}{2}\sum_{u \in V} \deg_G(u) \geq \tfrac{1}{2}\sum_{u \in V} R(u).$$

This proves $\geq$ in (62.12). To prove $\leq$, order the vertices as $v_1, \ldots, v_n$ such that $R(v_1) \geq R(v_2) \geq \cdots \geq R(v_n)$. Note that $R(v_1) = R(v_2)$.

Let $k := \lfloor \frac{1}{2}R(v_1) \rfloor$. Let W be the set of vertices v with $R(v)$ odd. Let M be a set of $\lceil \frac{1}{2}|W| \rceil$ edges covering W such that if $v_1, v_2 \in W$, then $v_1v_2 \in M$.

For $i = 1, \ldots, k$, let C_i be a circuit on $\{v \in V \mid R(v) \geq 2i\}$. So C_1 is a Hamiltonian circuit. We choose C_1 in such a way that the components of $C_1 - v_1 - v_2$ span no edge in M. Let H be the (edge-disjoint) union of M and C_1. Then for any $U \subseteq V$:

(62.14) if $d_H(U) = 2$ and $U \cap \{v_1, v_2\} = \emptyset$, then $U \cap W = \emptyset$.

Indeed, if $d_H(U) = 2$, then U induces a path on C_1. As $U \cap \{v_1, v_2\} = \emptyset$, U is contained in a component of $C_1 - v_1 - v_2$. Hence each edge in M incident with U belongs to $d_H(U)$. As $d_H(U) = 2$, it follows that no edge in M is incident with U. So $U \cap W = \emptyset$, proving (62.14).

Let G be the (edge-disjoint) union of $M, C_1, \ldots, C_k$. Note that the number of edges of G is equal to

(62.15) $$|M| + \sum_{i=1}^{k} |EC_i| = |M| + \sum_{u \in V} \lfloor \tfrac{1}{2}R(u) \rfloor = \lceil \tfrac{1}{2}|W| \rceil + \sum_{u \in V} \lfloor \tfrac{1}{2}R(u) \rfloor = \lceil \tfrac{1}{2}\sum_{u \in V} R(u) \rceil.$$

We finally show that G is r-connected, for which it suffices to show that for $i = 2, \ldots, n$:

(62.16) $$\lambda_G(v_{i-1}, v_i) \geq R(v_i).$$

To see that this is sufficient, note that for $h < j$ one has

(62.17) $$\lambda_G(v_h, v_j) \geq \min_{h<i\leq j} \lambda_G(v_{i-1}, v_i) \geq \min_{h<i\leq j} R(v_i) = R(v_j) \geq r(v_h, v_j).$$

To prove (62.16), choose the smallest $i \geq 2$ for which it is not true. Then G has a cut $\delta(U)$ with $v_i \in U$, $v_{i-1} \notin U$, and $d_G(U) < R(v_i)$. By the minimality of i, $\delta(U)$ separates no pair among $v_1, \ldots, v_{i-1}$, and hence $v_1, \ldots, v_{i-1} \notin U$. Now, setting $l := \lfloor \frac{1}{2}R(v_i) \rfloor$:

(62.18) $$2l + 1 \geq R(v_i) > d_G(U) \geq d_H(U) + \sum_{j=2}^{l} d_{C_j}(U) \geq 2l$$

(as C_j covers v_{i-1} and v_i for $j = 1, \ldots, l$). Hence $d_H(U) = 2$ and $R(v_i)$ is odd. So $U \cap W \neq \emptyset$. Hence, by (62.14), $i = 2$. Then $v_1v_2 \in M$, and so $d_H(U) \geq 3$, a contradiction. ∎

Notes. In fact, by choosing M in this proof in such a way that $\deg_M(u) = 1$ if $u \in W \setminus \{v_1\}$, and $\deg_M(u) = 0$ if $u \in (V \setminus W) \setminus \{v_1\}$, we obtain a graph G with $\deg_G(u) = R(u)$ for each $u \neq v_1$. Hence $\lambda_G(u, v) = \min\{R(u), R(v)\}$ for all distinct $u, v \in V$.

The construction can be extended to obtain a strongly polynomial-time algorithm that, for given integer function r, finds a minimum-capacity integer realization c (Sridhar and Chandrasekaran [1990,1992]).

(Chou and Frank [1970] claim to give an algorithm to find a minimum-size *exact* realization, but their construction fails when taking $r(1, 2) := r(3, 4) := r(4, 5) := 5$, $r(2, 3) := r(5, 6) := 3$, and $r(i, j) := \min_{i<h\leq j} r(h-1, h)$ for $i < j$. The construction gives 15 edges, while there is an exact realization with 14 edges only.)

Frank and Chou [1970] announced a polynomial-time algorithm for the problem: given a symmetric $r : V \times V \to \mathbb{Z}_+$, find a *simple* r-edge-connected graph $G = (V, E)$ (if any) with $|E|$ minimal.

Wang and Kleitman [1973] characterized the degree-sequences of k-vertex-connected simple undirected graphs.

Chapter 63

Connectivity augmentation

This last chapter of Part V is devoted to the connectivity augmentation problem: given a graph, find the minimum number of edges to be added to make it k-connected. There is an undirected and a directed variant, and a vertex-connectivity and an edge/arc-connectivity variant. Thus we will come across:

- making a directed graph k-arc-connected (Section 63.1),
- making an undirected graph k-edge-connected (Section 63.3),
- making a directed graph k-vertex-connected (Section 63.5),
- making an undirected graph k-vertex-connected (Section 63.6).

For the first three problems, min-max relations and polynomial-time algorithms have been found. The core is formed by fundamental theorems of Frank and Jordán. As for the fourth problem, only for fixed k the polynomial-time solvability has been proved. The complexity for general k is open.

Two special cases of connectivity augmentation have been considered before: making a digraph 1-arc-connected — that is, strongly connected (Chapter 57), and making an edge- or arcless (di)graph k-vertex- or edge/arc-connected — the network synthesis problem (Chapter 62).

63.1. Making a directed graph k-arc-connected

Let (V, A) and (V, B) be directed graphs. The set B is called a *k-arc-connector for D* if the directed graph $(V, A \cup B)$ is k-arc-connected (where in $A \cup B$ arcs are taken parallel if they occur both in A and in B). So 1-arc-connectors are precisely the strong connectors, which we discussed in Chapter 57.

Frank [1990a,1992a] characterized the minimum size of a k-arc-connector for a directed graph, with the help of the following result of Mader [1982] (we follow the proof of Frank [1992a]).

Lemma 63.1α. *Let $D = (V, A)$ be a directed graph, let $k \in \mathbb{Z}_+$, and let $x, y : V \to \mathbb{Z}_+$. Then D has a k-arc-connector B with $\deg_B^{\text{in}}(v) = x_v$ and $\deg_B^{\text{out}}(v) = y_v$ for each $v \in V$ if and only if $x(V) = y(V)$ and*

(63.1) $\qquad x(U) \geq k - d_A^{\text{in}}(U)$ *and* $y(U) \geq k - d_A^{\text{out}}(U)$

for each nonempty proper subset U of V.

Proof. Necessity is easy, since for each nonempty $U \subset V$,

(63.2) $$k \leq d^{\text{in}}_{A\cup B}(U) = d^{\text{in}}_B(U) + d^{\text{in}}_A(U) \leq x(U) + d^{\text{in}}_A(U),$$

and similarly for y.

To see sufficiency, choose a counterexample with $x(V)$ minimal. Trivially, $x(V) \geq 1$. Let $\mathcal{X}$ be the collection of inclusionwise maximal proper subsets U of V satisfying $x(U)+d^{\text{in}}(U) = k$, and let $\mathcal{Y}$ be the collection of inclusionwise maximal proper subsets U of V satisfying $y(U) + d^{\text{out}}(U) = k$. (We set d^{in} and d^{out} for d^{in}_A and d^{out}_A.)

Let $R := \{v \in V \mid x_v \geq 1\}$ and $S := \{v \in V \mid y_v \geq 1\}$. Then

(63.3) for all $r \in R$ and $s \in S$, there exists a $U \in \mathcal{X} \cup \mathcal{Y}$ with $r, s \in U$.

Otherwise, we could augment D by a new arc (s, r) and decrease both x_r and y_s by 1. Then (63.1) is maintained, and we obtain a smaller counterexample, contradicting our assumption. This shows (63.3).

Now note that for each $U \in \mathcal{X}$:

(63.4) $$y(V \setminus U) \geq k - d^{\text{out}}(V \setminus U) = k - d^{\text{in}}(U) = x(U).$$

This implies, for each $U \in \mathcal{X}$:

(63.5) if $S \subseteq U$, then $U \cap R = \emptyset$; if $R \subseteq U$, then $U \cap S = \emptyset$.

Indeed, if $S \subseteq U$, then $y(V \setminus U) = 0$, implying (with (63.4)) that $x(U) = 0$, that is, $U \cap R = \emptyset$. Similarly, if $R \subseteq U$, then $x(U) = x(V)$, implying (with (63.4)) that $y(V \setminus U) = y(V)$, that is, $U \cap S = \emptyset$. This proves (63.5).

Now choose $r \in R$, $s \in S$, and let $U \in \mathcal{X} \cup \mathcal{Y}$ with $r, s \in U$. By symmetry, we may assume that $U \in \mathcal{X}$. By (63.5), $S \not\subseteq U$. Choose $t \in S \setminus U$. Let $T \in \mathcal{X} \cup \mathcal{Y}$ contain r and t.

First assume that $T \in \mathcal{X}$. Then $T \cup U = V$, by the maximality of T and U and the submodularity of the set function $x(W) + d^{\text{in}}(W)$. This implies (using (63.4)):

(63.6) $$\begin{aligned} y(V) &\geq y(V \setminus U) + y(V \setminus T) \geq x(U) + x(T) = x(T \cup U) + x(T \cap U) \\ &> x(V) = y(V) \end{aligned}$$

(since $V \setminus U$ and $V \setminus T$ are disjoint, and since $r \in T \cap U$), a contradiction.

So $T \in \mathcal{Y}$. But then

(63.7) $$\begin{aligned} 2k &= x(T) + d^{\text{in}}(T) + y(U) + d^{\text{out}}(U) \\ &\geq x(T \setminus U) + d^{\text{in}}(T \setminus U) + y(U \setminus T) + d^{\text{out}}(U \setminus T) + x(T \cap U) \\ &\quad + y(T \cap U) \geq 2k, \end{aligned}$$

implying equality throughout. Hence $x(T \cap U) = 0$, contradicting the fact that $r \in T \cap U$. ∎

From this, the min-max relation for minimum-size k-arc-connectors of Frank [1990a,1992a] (generalizing Corollary 57.2a) easily follows:

Theorem 63.1. *Let $D = (V, A)$ be a directed graph and let $k, \gamma \in \mathbb{Z}_+$. Then D has a k-arc-connector of size at most γ if and only if*

(63.8) $$\gamma \geq \sum_{X \in \mathcal{P}} (k - d^{\text{in}}(X)) \text{ and } \gamma \geq \sum_{X \in \mathcal{P}} (k - d^{\text{out}}(X))$$

for each collection $\mathcal{P}$ of disjoint nonempty proper subsets of V.

Proof. Necessity follows since for each nonempty subset X of V, at least $k - d^{\text{in}}(X)$ arcs entering X must be in any k-arc-connector, and at least $k - d^{\text{out}}(X)$ arcs leaving X must be in any k-arc-connector. As any new arc can enter at most one set in $\mathcal{P}$, we have (63.8).

To see sufficiency, choose $x : V \to \mathbb{Z}_+$ satisfying $x(U) \geq k - d^{\text{in}}(U)$ for each nonempty $U \subset V$, with $x(V)$ as small as possible.

We show $x(V) \leq \gamma$. Let $\mathcal{P}$ be the collection of inclusionwise maximal proper subsets U of V satisfying $x(U) = k - d^{\text{in}}(U)$. Any two distinct sets $T, U \in \mathcal{P}$ satisfy $T \cup U \neq V$, since otherwise $V \setminus T$ and $V \setminus U$ are disjoint, and we obtain the contradiction

(63.9) $$\begin{aligned}\gamma &\geq k - d^{\text{out}}(V \setminus T) + k - d^{\text{out}}(V \setminus U) = 2k - d^{\text{in}}(T) - d^{\text{in}}(U) \\ &= x(T) + x(U) \geq x(T \cup U) = x(V) > \gamma.\end{aligned}$$

Moreover, any two distinct $T, U \in \mathcal{P}$ are disjoint, since otherwise we obtain the contradiction

(63.10) $$\begin{aligned}x(T)+x(U) &= 2k - d^{\text{in}}(T) - d^{\text{in}}(U) \leq 2k - d^{\text{in}}(T \cap U) - d^{\text{in}}(T \cup U) \\ &< x(T \cap U) + x(T \cup U) = x(T) + x(U),\end{aligned}$$

by the maximality of T.

Now each $v \in V$ with $x_v \geq 1$ is contained in some $U \in \mathcal{P}$, as otherwise we could decrease x_v. This gives

(63.11) $$x(V) = \sum_{U \in \mathcal{P}} x(U) = \sum_{U \in \mathcal{P}} (k - d^{\text{in}}(U)) \leq \gamma.$$

Hence $x(V) \leq \gamma$. Similarly, there exists a $y : V \to \mathbb{Z}_+$ satisfying $y(U) \geq k - d^{\text{out}}(U)$ for each nonempty proper subset U of V and $y(V) \leq \gamma$. We can assume that $x(V) = y(V) = \gamma$. So we can apply Lemma 63.1α, which gives the theorem. ∎

The proof yields a polynomial-time algorithm, as the proof reduces to a polynomial-time number of tests if a given $x : V \to \mathbb{Z}_+$ satisfies

(63.12) $$x(U) \geq k - d^{\text{in}}(U) \text{ for each nonempty } U \subset V.$$

(Similarly for y.) This can be done by maximum flow calculations: add a new vertex s and for each $v \in V$, add x_v (parallel) arcs from s to v and k parallel arcs from v to s. Then (63.12) is satisfied if and only if in the extended graph there exist k arc-disjoint $u - v$ paths, for all $u, v \in V$.

Thus (Frank [1990a,1992a]):

Theorem 63.2. *Given a directed graph $D = (V, A)$ and $k \in \mathbb{Z}_+$, a minimum-size k-arc-connector can be found in time bounded by a polynomial in the size of D and in k.*

Proof. See above. ∎

63.1a. *k*-arc-connectors with bounds on degrees

Frank [1990a,1992a] derived similarly characterizations of the existence of k-arc-connectors of given size and satisfying given lower and upper bounds on the in- and outdegrees:

Theorem 63.3. *Let $D = (V, A)$ be an undirected graph, let $k, \gamma \in \mathbb{Z}_+$, and let $l^{\text{in}}, l^{\text{out}}, u^{\text{in}}, u^{\text{out}} \in \mathbb{Z}_+^V$ with $l^{\text{in}} \leq u^{\text{in}}$ and $l^{\text{out}} \leq u^{\text{out}}$. Then D has a k-arc-connector B of size at most γ satisfying $l^{\text{in}}(v) \leq \deg_B^{\text{in}}(v) \leq u^{\text{in}}(v)$ and $l^{\text{out}}(v) \leq \deg_B^{\text{out}}(v) \leq u^{\text{out}}(v)$ for each $v \in V$ if and only if $\gamma \leq u^{\text{in}}(V)$, $\gamma \leq u^{\text{out}}(V)$,*

(63.13) $$k - d^{\text{in}}(U) \leq u^{\text{in}}(U) \text{ and } k - d^{\text{out}}(U) \leq u^{\text{out}}(U)$$

for each nonempty proper subset U of V, and

(63.14) $$\gamma \geq l^{\text{in}}(V \setminus \bigcup \mathcal{P}) + \sum_{X \in \mathcal{P}} (k - d^{\text{in}}(X)) \text{ and}$$
$$\gamma \geq l^{\text{out}}(V \setminus \bigcup \mathcal{P}) + \sum_{X \in \mathcal{P}} (k - d^{\text{out}}(X))$$

for each collection $\mathcal{P}$ of disjoint nonempty proper subsets of V.

Proof. Necessity is easy. To see sufficiency, choose $x : V \to \mathbb{Z}_+$ satisfying $l^{\text{in}} \leq x \leq u^{\text{in}}$, $x(V) \geq \gamma$, and $x(U) \geq k - d^{\text{in}}(U)$ for each nonempty $U \subset V$, with $x(V)$ as small as possible.

We show $x(V) \leq \gamma$. Let $\mathcal{P}$ be the collection of inclusionwise maximal proper subsets U of V satisfying $x(U) = k - d^{\text{in}}(U)$. Any two distinct sets $T, U \in \mathcal{P}$ satisfy $T \cup U \neq V$, since otherwise $V \setminus T$ and $V \setminus U$ are disjoint, and we obtain the contradiction

(63.15) $$\begin{aligned}\gamma &\geq k - d^{\text{out}}(V \setminus T) + k - d^{\text{out}}(V \setminus U) = 2k - d^{\text{in}}(T) - d^{\text{in}}(U) \\ &= x(T) + x(U) \geq x(T \cup U) \geq x(V) > \gamma.\end{aligned}$$

Moreover, any two distinct $T, U \in \mathcal{P}$ are disjoint, since otherwise we obtain the contradiction

(63.16) $$\begin{aligned}x(T) + x(U) &= 2k - d^{\text{in}}(T) - d^{\text{in}}(U) \leq 2k - d^{\text{in}}(T \cap U) - d^{\text{in}}(T \cup U) \\ &< x(T \cap U) + x(T \cup U) = x(T) + x(U),\end{aligned}$$

by the maximality of T.

Now each $v \in V$ with $x_v > l^{\text{in}}(v)$ is contained in some $U \in \mathcal{P}$, as otherwise we could decrease x_v. This gives

(63.17) $$x(V) = l^{\text{in}}(V \setminus \bigcup \mathcal{P}) + \sum_{U \in \mathcal{P}} x(U) = l^{\text{in}}(V \setminus \bigcup \mathcal{P}) + \sum_{U \in \mathcal{P}} (k - d^{\text{in}}(U)) \leq \gamma.$$

Hence $x(V) = \gamma$. Similarly, there exists a $y : V \to \mathbb{Z}_+$ satisfying $l^{\text{out}} \leq y \leq u^{\text{out}}$, $y(V) = \gamma$, and $y(U) \geq k - d^{\text{out}}(U)$ for each nonempty $U \subset V$. So we can apply Lemma 63.1α, which gives the theorem. ∎

Again, this proof yields a polynomial-time algorithm to find a minimum-size k-arc-connector satisfying prescribed bounds on the in- and outdegrees.

Notes. The following problem is NP-complete: given a directed graph $D = (V, A)$, a function $r : V \times V \to \mathbb{Z}_+$, and a cost function $k : V \times V \to \mathbb{Q}_+$, find a minimum-cost set of new arcs whose addition to D makes the graph r-arc-connected. Frank [1990a,1992a] showed that if there are functions $k', k'' : V \to \mathbb{Q}_+$ with $k(u, v) = k'(u) + k''(v)$ for all $u, v \in V$, then this problem is solvable in polynomial time.

Gusfield [1987a] gave a linear-time algorithm to find a minimum number of directed arcs to be added to a mixed graph such that it becomes strongly connected (that is, for all vertices u, v there is a $u - v$ path traversing directed edges in the right direction only).

Frank and Jordán [1995b] gave an alternative proof of Theorem 63.1 based on bisubmodular functions, and showed a number of related results.

Frank [1993c] gave some further methods for the problems discussed in these sections.

Kajitani and Ueno [1986] showed that the minimum size of a k-arc-connector for a directed tree $D = (V, A)$ is equal to the maximum of $\sum_{v \in V} \max\{0, k - \deg^{\text{in}}(v)\}$ and $\sum_{v \in V} \max\{0, k - \deg^{\text{out}}(v)\}$.

Frank [1990a,1992a] gave a polynomial-time algorithm for: given directed graph $D = (V, A)$, $r \in V$, and $k \in \mathbb{Z}_+$, find a minimum number of arcs to be added to D such that for each $s \in V$ there exist k arc-disjoint $r - s$ paths in the augmented graph. The complexity was improved by Gabow [1991b].

63.2. Making an undirected graph 2-edge-connected

Let (V, E) and (V, F) be undirected graphs. The set F is called a *k-edge-connector* for G if the graph $(V, E \cup F)$ is k-edge-connected (where in $E \cup F$ edges are taken parallel if they occur both in E and in F).

The minimum size of a 1-edge-connector of a graph G trivially is one less than the number of components of G. Eswaran and Tarjan [1976] and Plesník [1976] characterized the minimum size of a 2-edge-connector, by first showing:

Theorem 63.4. *Let $G = (V, E)$ be a forest with at least two vertices and with p vertices of degree 1 and q isolated vertices. Then the minimum size of a 2-edge-connector for G equals $\lceil \frac{1}{2}p \rceil + q$.*

Proof. Each vertex of degree 1 should be incident with at least one new edge, and each vertex of degree 0 should be incident with at least two new edges. So any 2-edge-connector has size at least $\frac{1}{2}p + q$.

To see that $\lceil \frac{1}{2}p \rceil + q$ can be attained, first assume that G is not connected. Choose vertices u and v in different components, with $\deg(u) \leq 1$ and $\deg(v) \leq 1$. Adding edge uv, reduces $\frac{1}{2}p + q$ by 1, as one easily checks.

So we can assume that G is a tree. If $p \leq 3$, the graph is a path or a subdivision of $K_{1,3}$, and the theorem is easy.

If $p \geq 4$, there is a pair of end vertices u, v such that at least two edges of G leave the $u - v$ path P in G. Let G' be the tree obtained from G by contracting P to one vertex. Then G' has $p - 2$ end vertices. Applying induction shows that G' has a 2-edge-connector F of size $\lceil \frac{1}{2}p \rceil - 1$. By adding edge uv we obtain a 2-edge-connector of G, proving the theorem. ∎

This implies, for not necessarily forests:

Corollary 63.4a. *Let $G = (V, E)$ be a non-2-edge-connected undirected graph. For $i = 0, 1$, let p_i be the number of 2-edge-connected components K with $d_E(K) = i$. Then the minimum size of a 2-edge-connector equals $\lceil \frac{1}{2}p_1 \rceil + p_0$.*

Proof. Directly from Theorem 63.4, by contracting each 2-edge-connected component to one vertex. ∎

These proofs give polynomial-time algorithms to find a minimum-size 2-edge-connector for a given undirected graph. Eswaran and Tarjan [1976] gave a linear-time algorithm.

63.3. Making an undirected graph k-edge-connected

Watanabe and Nakamura [1987] gave a min-max formula and a polynomial-time algorithm for the minimum size of a k-edge-connector for any undirected graph. Cai and Sun [1989] and Frank [1992a] showed that the min-max relation can be derived from the following lemma (given, in a different, 'vertex-splitting' terminology, by Mader [1978a] and Lovász [1979a] (Problem 6.53); the proof below follows Frank [1992a]):

Lemma 63.5α. *Let $G = (V, E)$ be a graph, let $k \in \mathbb{Z}_+$, with $k \geq 2$, and let $x : V \to \mathbb{Z}_+$. Then G has a k-edge-connector F with $\deg_F(v) = x_v$ for each $v \in V$ if and only if $x(V)$ is even and*

(63.18) $$x(U) \geq k - d_E(U)$$

for each nonempty proper subset U of V.

Proof. Necessity is easy, since for each nonempty $U \subset V$,

(63.19) $$k \leq d_{E \cup F}(U) = d_F(U) + d_E(U) \leq x(U) + d_E(U).$$

To see sufficiency, choose a counterexample with $x(V)$ minimal. Trivially, $x(V) \geq 2$.

Let $S := \{v \in V \mid x_v \geq 1\}$, and fix $s \in S$. Let $\mathcal{U}$ be the collection of inclusionwise maximal sets $U \subset V$ containing s and satisfying $x(U)+d_E(U) \leq k+1$. Note that

(63.20) $\quad x(U) \leq \frac{1}{2}x(V)$ for each $U \in \mathcal{U}$,

since otherwise $x(V \setminus U) \leq x(U) - 2$, implying the contradiction $k \leq x(V \setminus U) + d_E(V \setminus U) \leq x(U) - 2 + d_E(U) \leq k - 1$.

Moreover,

(63.21) $\quad$ for all $t \in S \setminus \{s\}$, there exists a $U \in \mathcal{U}$ containing t.

Otherwise, we could augment G by a new edge st and decrease both x_s and x_t by 1. Then (63.18) is maintained, and we obtain a smaller counterexample, contradicting our assumption. This shows (63.21).

Next:

(63.22) $\quad$ for any two distinct $T, U \in \mathcal{U}$, G has an edge leaving $T \cap U$, and no edge connecting $T \cap U$ and $V \setminus (T \cup U)$.

Consider:

(63.23)
$$\begin{aligned} &2(k+1) \geq x(T) + d_E(T) + x(U) + d_E(U) \\ &= x(T \setminus U) + d_E(T \setminus U) + x(U \setminus T) + d_E(U \setminus T) \\ &+2|E[T \cap U, V \setminus (T \cup U)]| + 2x(T \cap U) \geq 2k+2, \end{aligned}$$

implying equality throughout. So $x(T \cap U) = 1$ and $|E[T \cap U, V \setminus (T \cup U)] = \emptyset$. Since $d_E(T \cap U) \geq k - x(T \cap U) = k - 1 \geq 1$, this proves (63.22).

Now by (63.21) and (63.20) we can choose three sets $T, U, W \in \mathcal{U}$. Then

(63.24) $\quad T \cap U = T \cap W = U \cap W.$

Indeed, by symmetry it suffices to prove that $T \cap U \cap W = U \cap W$. Let $M := U \cap W$. Suppose $T \cap M \neq M$; so $M \not\subseteq T$, and hence $T \cup M \neq T$. Defining $\phi(X) := x(X) + d_E(X)$ for $X \subseteq V$, we obtain the contradiction

(63.25)
$$\begin{aligned} &k \leq \phi(T \cap M) \leq \phi(T) + \phi(M) - \phi(T \cup M) \\ &\leq \phi(T) + \phi(U) + \phi(W) - \phi(U \cup W) - \phi(T \cup M) \\ &\leq 3(k+1) - 2(k+2) = k - 1, \end{aligned}$$

since the maximality of T, U, and W gives $\phi(U \cup W) \geq k+2$ and $\phi(T \cup M) \geq k+2$. This shows (63.24).

Now by (63.22), G has an edge leaving $T \cap U$, while the other end should be in each of $T \cup U$, $T \cup W$, and $U \cup W$, and hence in $T \cap U$, a contradiction. ∎

From this, the min-max result for minimum-size k-edge-connectors of Watanabe and Nakamura [1987] follows:

Theorem 63.5. *Let $G = (V, E)$ be an undirected graph and let $k, \gamma \in \mathbb{Z}_+$, with $k \geq 2$. Then G has a k-edge-connector of size at most γ if and only if*

$$2\gamma \geq \sum_{U \in \mathcal{P}} (k - d(U)) \tag{63.26}$$

for each collection $\mathcal{P}$ of disjoint nonempty proper subsets of V.

Proof. Necessity follows since for each nonempty proper subset U of V, at least $k - d(U)$ edges entering U must be in any k-edge-connector. As any new edge can enter at most two sets in $\mathcal{P}$, we have (63.26).

To see sufficiency, choose $x : V \to \mathbb{Z}_+$ satisfying (63.18), with $x(V)$ as small as possible. By Lemma 63.5α it suffices to show that $x(V) \leq 2\gamma$.

Let $\mathcal{P}$ be the collection of inclusionwise maximal subsets U of V satisfying $x(U) = k - d(U)$. Any two distinct sets $T, U \in \mathcal{P}$ satisfy $T \cup U \neq V$, since otherwise we obtain the contradiction

$$\begin{aligned} 2\gamma < x(V) = x(T \cup U) &\leq x(T) + x(U) = k - d(T) + k - d(U) \\ &= k - d(V \setminus T) + k - d(V \setminus U) \leq 2\gamma, \end{aligned} \tag{63.27}$$

using (63.26). Moreover, any two distinct $T, U \in \mathcal{P}$ are disjoint, since otherwise we obtain the contradiction

$$\begin{aligned} x(T) + x(U) &= 2k - d(T) - d(U) \leq 2k - d(T \cap U) - d(T \cup U) \\ &< x(T \cap U) + x(T \cup U) = x(T) + x(U), \end{aligned} \tag{63.28}$$

by the maximality of T. Now each $v \in V$ with $x_v \geq 1$ is contained in some $U \in \mathcal{P}$, as otherwise we could decrease x_v. This gives

$$x(V) = \sum_{U \in \mathcal{P}} x(U) = \sum_{U \in \mathcal{P}} (k - d(U)) \leq 2\gamma, \tag{63.29}$$

which proves the theorem. ∎

Similarly to the directed case, the proof implies that a minimum-size k-edge-connector can be found in polynomial time (Watanabe and Nakamura [1987]):

Theorem 63.6. *Given an undirected graph G and $k \in \mathbb{Z}_+$, a minimum-size k-edge-connector can be found in strongly polynomial time.*

Proof. The proof method reduces to a polynomially bounded number of tests of (63.18), which can be performed in strongly polynomial time by reducing it to maximum flow computations. ∎

Notes. Also Naor, Gusfield, and Martel [1990,1997], and Gabow [1991b] gave polynomial-time algorithms to find a minimum-size k-edge-connector. Frank [1990a, 1992a] and Benczúr [1994,1999] gave strongly polynomial-time algorithms for the (integer) capacitated version of the problem. More and related results can be found in Gabow [1994], Benczúr [1995], Nagamochi and Ibaraki [1995,1996,1997,1999c], Nagamochi, Shiraki, and Ibaraki [1997], Benczúr and Karger [1998,2000], Nagamochi and Eades [1998], Bang-Jensen and Jordán [2000], and Nagamochi, Nakamura, and Ibaraki [2000].

63.3a. k-edge-connectors with bounds on degrees

Frank [1990a,1992a] derived similarly characterizations of the existence of k-edge-connectors of given size and satisfying given lower and upper bounds on the degrees:

Theorem 63.7. *Let $G = (V, E)$ be an undirected graph, let $k, \gamma \in \mathbb{Z}_+$, with $k \geq 2$, and let $l, u \in \mathbb{Z}_+^V$ with $l \leq u$. Then G has a k-edge-connector F of size at most γ satisfying $l(v) \leq \deg_F(v) \leq u(v)$ for each $v \in V$ if and only if $2\gamma \leq u(V)$,*

(63.30) $$k - d_E(U) \leq u(U)$$

for each nonempty proper subset U of V, and

(63.31) $$2\gamma \geq l(V \setminus \bigcup \mathcal{P}) + \sum_{U \in \mathcal{P}} (k - d_E(U)).$$

for each collection $\mathcal{P}$ of disjoint nonempty proper subsets of V.

Proof. The conditions can be easily seen to be necessary.

To see sufficiency, let $x : V \to \mathbb{Z}_+$ satisfy $l \leq x \leq u$, $x(V) \geq 2\gamma$, $x(U) \geq k - d_E(U)$ for each nonempty $U \subset V$, and with $x(V)$ as small as possible. Such an x exists, by (63.30). By Lemma 63.5α, it suffices to show that $x(V) \leq 2\gamma$.

Let $\mathcal{P}$ be the collection of inclusionwise maximal proper subsets U of V satisfying $x(U) = k - d(U)$. Any two distinct sets $T, U \in \mathcal{P}$ satisfy $T \cup U \neq V$, since otherwise we obtain the contradiction

(63.32) $$\begin{aligned} 2\gamma < x(V) &= x(T \cup U) \leq x(T) + x(U) = k - d(T) + k - d(U) \\ &= k - d(V \setminus T) + k - d(V \setminus U) \leq 2\gamma, \end{aligned}$$

by (63.31). Moreover, any two distinct $T, U \in \mathcal{P}$ are disjoint, since otherwise we obtain the contradiction

(63.33) $$\begin{aligned} x(T) + x(U) &= 2k - d(T) - d(U) \leq 2k - d(T \cap U) - d(T \cup U) \\ &< x(T \cap U) + x(T \cup U) = x(T) + x(U), \end{aligned}$$

by the maximality of T. Now each $v \in V$ with $x_v > l(v)$ is contained in some $U \in \mathcal{P}$, as otherwise we could decrease x_v. This gives

(63.34) $$x(V) = l(V \setminus \bigcup \mathcal{P}) + \sum_{U \in \mathcal{P}} x(U) = l(V \setminus \bigcup \mathcal{P}) + \sum_{U \in \mathcal{P}} (k - d(U)) \leq 2\gamma,$$

as required. ∎

Notes. T. Jordán (cf. Bang-Jensen and Jordán [1997]) showed that finding a minimum number of edges that makes a given simple graph k-edge-connected and keeps it simple, is NP-complete. On the other hand, Bang-Jensen and Jordán [1997,1998] gave, for any fixed k, an $O(n^4)$-time algorithm for this problem. Taoka, Watanabe, and Takafuji [1994] gave an $O(m + n \log n)$-time algorithm for $k = 4$ and an $O(n^2 + m)$-time algorithm for $k = 5$ (assuming the input graph is $k-1$-edge-connected). Other fast algorithms for undirected edge-connectivity augmentation were given by Benczúr [1999].

Ueno, Kajitani, and Wada [1988] gave a polynomial-time algorithm for finding a minimum-size k-edge-connector for a tree.

63.4. r-edge-connectivity and r-edge-connectors

Let $G = (V, E)$ be an undirected graph and let $r : V \times V \to \mathbb{Z}_+$. G is called *r-edge-connected* if for all $u, v \in V$ there exist $r(u, v)$ edge-disjoint paths connecting u and v. So, by Menger's theorem, G is r-edge-connected if and only if $d_E(U) \geq r(u, v)$ for all $U \subseteq V$ and $u \in U$, $v \in V \setminus U$.

An *r-edge-connector* for G is a set F of edges on V such that the graph $G' := (V, E \cup F)$ satisfies $\lambda_{G'}(u, v) \geq r(u, v)$ for all $u, v \in V$. (Again, $E \cup F$ is the disjoint union, allowing parallel edges.) Define $\gamma(G, r)$ as the minimum size of an r-edge-connector for G.

Given an undirected graph $G = (V, E)$, a function $r : V \times V \to \mathbb{Z}_+$, and a cost function $k : V \times V \to \mathbb{Q}_+$, it is NP-complete to find a minimum-cost r-edge-connector (since for $E = \emptyset$, $r = \mathbf{2}$, it is the traveling salesman problem).

Frank [1990a,1992a] gave a polynomial-time algorithm and a min-max formula for the cardinality case: given a graph $G = (V, E)$ and $r : V \times V \to \mathbb{Z}_+$, find the minimum number of edges to be added to make G r-edge-connected. We describe the method in this section.

It is based on the following theorem of Mader [1978a] (conjectured by Lovász [1976a]; we follow the proof of Frank [1992b]):

Lemma 63.8α. *Let $G = (V \cup \{s\}, E)$ be an undirected graph, where s has even and positive degree, and s is not incident with a bridge of G. Then s has two neighbours u and v such that the graph G' obtained from G by replacing su and sv by one new edge uv satisfies*

(63.35) $$\lambda_{G'}(x, y) = \lambda_G(x, y)$$

for all $x, y \in V$.

Proof. By induction on $|V| + \deg(s)$. For any $U \subseteq V$ with $\emptyset \neq U \neq V$, define

(63.36) $$R(U) := \max_{u \in U, v \in V \setminus U} \lambda_G(u, v),$$

and set $R(\emptyset) := R(V) := 0$. So $R(U) \leq d(U)$ for each $U \subseteq V$.

Let $\mathcal{P}$ be the collection of nonempty proper subsets U of V with $d(U) = R(U)$, and let $\mathcal{U}$ be the collection of nonempty proper subsets U of V with $d(U) \leq R(U) + 1$ (hence $\mathcal{P} \subseteq \mathcal{U}$).

Note that $u, v \in N(s)$ are as required in the lemma if and only if there is no $U \in \mathcal{U}$ containing both u and v. So we can assume that

(63.37) for each pair $u, v \in N(s)$ there is a $U \in \mathcal{U}$ containing u and v.

We first show:

(63.38) $|T| = 1$ for each $T \in \mathcal{P}$.

Suppose not. Consider the graph G/T obtained from G by contracting T (where T also denotes the vertex obtained by contracting T). By induction,

G/T has two edges su' and sv such that for the graph H obtained from G/T by replacing su' and sv by a new edge $u'v$, one has

(63.39) $\quad \lambda_H(x,y) = \lambda_{G/T}(x,y)$

for all $x, y \in V(G/T) \setminus \{s\}$. By symmetry of u' and v, we may assume that $v \neq T$, that is, $v \in V \setminus T$. Let $u := u'$ if $u' \neq T$ and choose $u \in T \cap N(s)$ if $u' = T$.

Then for all $Z \in \mathcal{U}$:

(63.40) $\quad$ if $T \subseteq Z$ or $T \cap Z = \emptyset$ then $u \notin Z$ or $v \notin Z$.

Indeed, as $R(Z) \geq d(Z) - 1$, there exist $x, y \in V$ such that Z splits x, y and $\lambda_G(x,y) \geq d(Z) - 1$. Since $T \subseteq Z$ or $T \cap Z = \emptyset$, we may assume that $x \notin T$. Define $y' := y$ if $y \notin T$, and $y' := T$ if $y \in T$. Define $Z' := Z$ if $T \cap Z = \emptyset$, and $Z' := (Z \setminus T) \cup \{T\}$ if $T \subseteq Z$. Suppose now $u, v \in Z$. Then $d_H(Z') = d_G(Z) - 2$. This gives the contradiction

(63.41)
$$\begin{aligned}\lambda_{G/T}(x,y') &\geq \lambda_G(x,y) \geq d_G(Z) - 1 > d_H(Z') \geq \lambda_H(x,y') \\ &= \lambda_{G/T}(x,y'),\end{aligned}$$

proving (63.40).

Now let $U \in \mathcal{U}$ contain u and v. By Lemma 61.6α, $R(T) + R(U)$ is at most $R(T \cap U) + R(T \cup U)$ or at most $R(T \setminus U) + R(U \setminus T)$.

If $R(T) + R(U) \leq R(T \cap U) + R(T \cup U)$, then

(63.42)
$$\begin{aligned}d(T) + d(U) &\geq d(T \cap U) + d(T \cup U) \geq R(T \cap U) + R(T \cup U) \\ &\geq R(T) + R(U) \geq d(T) + d(U) - 1,\end{aligned}$$

implying $R(T \cup U) \geq d(T \cup U) - 1$. So $T \cup U \in \mathcal{U}$ and $u, v \in T \cup U$, contradicting (63.40).

So $R(T) + R(U) \leq R(T \setminus U) + R(U \setminus T)$. Hence

(63.43)
$$\begin{aligned}d(T) + d(U) &\geq d(T \setminus U) + d(U \setminus T) \geq R(T \setminus U) + R(U \setminus T) \\ &\geq R(T) + R(U) \geq d(T) + d(U) - 1.\end{aligned}$$

So $d(T) + d(U) = d(T \setminus U) + d(U \setminus T)$, and hence $T \cap U$ contains no neighbours of s. So $u' \neq T$ (otherwise $u \in T \cap U \cap N(s)$). Hence $u' = u \in U \setminus T$. By (63.43) we also know $R(U \setminus T) \geq d(U \setminus T) - 1$. So $U \setminus T \in \mathcal{U}$ and $u, v \in U \setminus T$, contradicting (63.40). This proves (63.38).

Note that (63.38) implies

(63.44) $\quad \lambda_G(u,v) = \min\{\deg(u), \deg(v)\}$ for all $u, v \in V$,

since $\lambda_G(u,v) = d_E(U)$ for some $U \subseteq V$ splitting $\{u,v\}$. So $U \in \mathcal{P}$, and hence $|U| = 1$, implying (63.44).

Choose a vertex $t \in N(s)$ of minimum degree. Let $\mathcal{U}'$ be a minimal collection of inclusionwise maximal sets in $\mathcal{U}$ containing t such that $\bigcup \mathcal{U}' \supseteq N(s)$ (this exists by (63.37)). Note that for each $U \in \mathcal{U}$ one has $|E[U,s]| \leq \frac{1}{2}\deg(s)$, since otherwise $d(V \setminus U) \leq d(U) - 2$ (since $\deg(s)$ is even), and hence

(63.45) $\quad R(V \setminus U) \leq d(V \setminus U) \leq d(U) - 2 \leq R(U) - 1 = R(V \setminus U) - 1,$

a contradiction. Hence $|\mathcal{U}'| \geq 3$ (as $t \in U$ for each $U \in \mathcal{U}'$). Moreover,

(63.46) for each $U \in \mathcal{U}'$, there is a $v \in N(s)$ such that U is the only set in $\mathcal{U}'$ containing v.

(Otherwise we could delete U from $\mathcal{U}'$.)

Also,

(63.47) $R(U \setminus \{t\}) \geq R(U)$ for each $U \in \mathcal{U}'$.

Indeed, choose $x \in U$ and $y \in V \setminus U$ with $R(U) = \lambda_G(x, y)$. If $x \neq t$, then $R(U \setminus \{t\}) \geq \lambda_G(x, y) = R(U)$, as required. If $x = t$, then for any $u \in N(s) \cap (U \setminus \{t\})$,

(63.48)
$$R(U) = \lambda_G(t, y) = \min\{\deg(t), \deg(y)\} \leq \min\{\deg(u), \deg(y)\} \\ = \lambda_G(u, y) \leq R(U \setminus \{t\}).$$

This shows (63.47).

Moreover, for any distinct $X, Y \in \mathcal{U}'$ one has

(63.49)
$$R(X) + R(Y) \leq R(X \setminus Y) + R(Y \setminus X).$$

Suppose not. Then, by Lemma 61.6α we know $R(X) + R(Y) \leq R(X \cap Y) + R(X \cup Y)$, and by symmetry we can assume that $R(X) > R(X \setminus Y)$. Hence by (63.47), $X \cap Y \neq \{t\}$, and hence by (63.38), $X \cap Y \notin \mathcal{P}$. By the maximality of X and Y, $R(X \cup Y) \leq d(X \cup Y) - 2$. This gives the contradiction

(63.50)
$$R(X \cap Y) + R(X \cup Y) \leq (d(X \cap Y) - 1) + (d(X \cup Y) - 2) \\ \leq d(X) + d(Y) - 3 \leq R(X) + R(Y) - 1 \\ \leq R(X \cap Y) + R(X \cup Y) - 1,$$

proving (63.49).

This implies that for any distinct $X, Y \in \mathcal{U}'$ one has

(63.51) $|X \setminus Y| = |Y \setminus X| = 1$, and st is the only edge connecting $X \cap Y$ and $(V \cup \{s\}) \setminus (X \cup Y)$.

Indeed, by (63.49) (as st connects $X \cap Y$ and $X \cup Y$),

(63.52)
$$d(X) + d(Y) \\ = d(X \setminus Y) + d(Y \setminus X) + 2|E[X \cap Y, (V \cup \{s\}) \setminus (X \cup Y)]| \\ \geq d(X \setminus Y) + d(Y \setminus X) + 2 \geq R(X \setminus Y) + R(Y \setminus X) + 2 \\ \geq R(X) + R(Y) + 2 \geq d(X) + d(Y).$$

So we have equality throughout. Hence $X \setminus Y, Y \setminus X \in \mathcal{P}$, and therefore, by (63.38), $|X \setminus Y| = |Y \setminus X| = 1$. Moreover, $d(X \cap Y, V \setminus (X \cup Y)) = 1$, proving (63.51).

Now choose $X, Y, Z \in \mathcal{U}'$. Then (63.51) and (63.46) imply that $X \cap Y = X \cap Z = Y \cap Z$. So st is the only edge leaving $X \cap Y$, and hence st is a bridge. This contradicts the condition given in this lemma. ∎

Note that

(63.53) the graph G' arising in Lemma 63.8α again has no bridge incident with s,

as for any two neighbours x, y of s in G' one has $\lambda_{G'}(x,y) = \lambda_G(x,y) \geq 2$. The lemma therefore can be applied iteratively to yield:

Theorem 63.8. *Let $G = (V, E)$ be an undirected graph and let $r : V \times V \to \mathbb{Z}_+$ be symmetric. Let $x : V \to \mathbb{Z}_+$ be such that $x(K) \neq 1$ for each component K of G. Then G has an r-edge-connector F satisfying $\deg_F(v) = x_v$ for each $v \in V$ if and only if $x(V)$ is even and*

(63.54) $$x(U) + d_E(U) \geq r(u, v)$$

for all $U \subseteq V$ and all $u \in U$, $v \in V \setminus U$.

Proof. Necessity of (63.54) follows from the fact that $x(U) + d_E(U) \geq d_F(U) + d_E(U) \geq r(u, v)$. To see sufficiency, extend V by a new vertex s and, for each $v \in V$, x_v edges connecting s and v (parallel if $x_v \geq 2$). Let H be the extended graph. Then (63.54) implies

(63.55) $$d_H(U) \geq r(u, v)$$

for all $U \subseteq V$ and all $u \in U$, $v \in V \setminus U$. Hence, for all $u, v \in V$,

(63.56) $$\lambda_H(u, v) \geq r(u, v).$$

Now by iteratively splitting s as in Lemma 63.8α (cf. (63.53)), we obtain a set F of new edges such that adding F to G, the new graph G' satisfies

(63.57) $$\lambda_{G'}(u, v) = \lambda_H(u, v) \geq r(u, v)$$

for all $u, v \in V$. As moreover $\deg_F(v) = x_v$ for each $v \in V$, F is an r-edge-connector as required. ∎

The condition that $x(K) \neq 1$ for each component K cannot be deleted, as can be seen by taking $G = (V, \emptyset)$, $r := \mathbf{1}$, $x := \mathbf{1}$, with $|V| \geq 4$.

We next give the theorem of Frank [1990a,1992a] characterizing the minimum size $\gamma(G, r)$ of an r-edge-connector. To this end we can assume that r satisfies:

(63.58) (i) $r(u, v) = r(v, u) \geq \lambda_G(u, v)$ for all $u, v \in V$;
(ii) $r(u, w) \geq \min\{r(u, v), r(v, w)\}$ for all $u, v, w \in V$.

Define

(63.59) $$R(U) := \max_{u \in U, v \in V \setminus U} r(u, v) \text{ if } \emptyset \subset U \subset V, \text{ and}$$
$$R(\emptyset) := R(V) := 0.$$

Call a component K of G *marginal* if $K \neq V$, $r(u, v) = \lambda_G(u, v)$ for all $u, v \in K$, and $r(u, v) \leq 1$ for all $u \in K$ and $v \in V \setminus K$.

Theorem 63.9. *Let $G = (V, E)$ be an undirected graph and let $r : V \times V \to \mathbb{Z}_+$ satisfy* (63.58)*.*

(i) *If K is a marginal component of G, then*

(63.60) $$\gamma(G, r) = \gamma(G - K, r') + R(K),$$

where r' is the restriction of r to $(V \setminus K) \times (V \setminus K)$.

(ii) *If G has no marginal components, then $\gamma(G, r)$ is equal to the maximum value of*

(63.61) $$\lceil \tfrac{1}{2} \sum_{U \in \mathcal{P}} (R(U) - d_E(U)) \rceil$$

taken over all collections $\mathcal{P}$ of disjoint nonempty proper subsets of V.

Proof. We first show (i). Let K be a marginal component of G and define $\alpha := R(K)$. As K is marginal, $\alpha \leq 1$. The inequality

(63.62) $$\gamma(G, r) \leq \gamma(G - K, r') + \alpha$$

is easy, since an r-edge-connector for G can be obtained from an r'-edge-connector F for $G - K$: if $\alpha = 0$, then F is an r-edge-connector, and if $\alpha = 1$, we obtain an r-edge-connector by adding to F one edge connecting some pair $u \in K$, $v \in V \setminus K$ with $r(u, v) = 1$.

To see the reverse inequality, let F be a minimum-size r-edge-connector for G. Let $G' := (V, E \cup F)$. So G' is r-edge-connected.

If F contains no edges connecting K and $V \setminus K$, then $\alpha = 0$ and F contains an r'-edge-connector for $G - K$. Hence $\gamma(G, r) = |F| \geq \gamma(G - K, r') = \gamma(G - K, r') + \alpha$.

If F contains an edge uv with $u \in K$, $v \in V \setminus K$, then the graph H obtained from G' by contracting $(K \cup \{v\})$ to one vertex, is r'-edge-connected. Since edge $uv \in F$ is contracted, it implies that $G - K$ has an r'-edge-connector of size at most $|F| - 1$. So $\gamma(G - K, r') \leq |F| - 1 \leq \gamma(G, r) - \alpha$.

We next show (ii). Let G have no marginal components. Choose $x : V \to \mathbb{Z}_+$ such that $x(U) + d_E(U) \geq R(U)$ for each $U \subseteq V$, with $x(V)$ as small as possible. Let μ be the maximum value of (63.61). It suffices to show that $x(V) \leq 2\mu$, since then we can apply Theorem 63.8 (after increasing $x(v)$ by 1 for some $v \in V$ if $x(V)$ is odd). So assume $x(V) > 2\mu$. As $\mu > 0$ (otherwise $x = \mathbf{0}$), we know $x(V) > 2$.

Then

(63.63) $x(K) \neq 1$ for each component K of G.

For suppose $x(K) = 1$. We show that K is marginal, which is a contradiction. First, $K \neq V$, since $x(V) > 2$. Second, for each $u \in K$, $v \in V \setminus K$, we have $r(u, v) \leq x(K) + d_E(K) = x(K) \leq 1$. Third, to prove that $r(u, v) = \lambda_G(u, v)$ for $u, v \in K$, there is a subset U of K with $|U \cap \{u, v\}| = 1$, $\lambda_G(u, v) = d_E(U)$, and $x(U) = 0$. Then $r(u, v) \leq x(U) + d_E(U) = d_E(U) = \lambda_G(u, v)$. So K is marginal, contradicting our assumption. This proves (63.63).

By the minimality of x, there exists a collection $\mathcal{P}$ of nonempty proper subsets U of V satisfying $x(U) = R(U) - d_E(U)$, such that $\mathcal{P}$ covers $\{v \mid x_v \geq 1\}$. Choose $\mathcal{P}$ such that

$$\sum_{U \in \mathcal{P}} |U| \tag{63.64}$$

is as small as possible. Then

$$T \cap U = \emptyset \text{ for distinct } T, U \in \mathcal{P}. \tag{63.65}$$

For suppose $T \cap U \neq \emptyset$. Note that $T \not\subseteq U \not\subseteq T$, by the minimality of (63.64). Observe also that $T \cup U \neq V$, since otherwise we obtain the contradiction

$$\begin{aligned} 2\mu &< x(V) = x(T \cup U) \leq x(T) + x(U) \\ &= R(T) - d(T) + R(U) - d(U) \\ &= R(V \setminus T) - d(V \setminus T) + R(V \setminus U) - d(V \setminus U) \leq 2\mu. \end{aligned} \tag{63.66}$$

(by definition of μ, since $V \setminus T$ and $V \setminus U$ are disjoint).

By Lemma 61.6α, $R(T) + R(U)$ is at most $R(T \cap U) + R(T \cup U)$ or at most $R(T \setminus U) + R(U \setminus T)$.

If $R(T) + R(U) \leq R(T \cap U) + R(T \cup U)$, then

$$\begin{aligned} x(T) + x(U) &= R(T) - d(T) + R(U) - d(U) \\ &\leq R(T \cap U) - d(T \cap U) + R(T \cup U) - d(T \cup U) \\ &\leq x(T \cap U) + x(T \cup U) = x(T) + x(U), \end{aligned} \tag{63.67}$$

and hence we have equality throughout. This implies that $x(T \cup U) = R(T \cup U) - d_E(T \cup U)$, and hence replacing T and U by $T \cup U$ would decrease (63.64), a contradiction.

If $R(T) + R(U) \leq R(T \setminus U) + R(U \setminus T)$, then

$$\begin{aligned} x(T) + x(U) &= R(T) - d(T) + R(U) - d(U) \\ &\leq R(T \setminus U) - d(T \setminus U) + R(U \setminus T) - d(U \setminus T) \\ &\leq x(T \setminus U) + x(U \setminus T) \leq x(T) + x(U), \end{aligned} \tag{63.68}$$

implying equality throughout. This implies that $x(T \setminus U) = R(T \setminus U) - d_E(T \setminus U)$, and hence replacing T by $T \setminus U$ would decrease (63.64), a contradiction.

This proves (63.65), yielding the contradiction

$$2\mu < x(V) = \sum_{U \in \mathcal{P}} x(U) = \sum_{U \in \mathcal{P}} (R(U) - d(U)) \leq 2\mu, \tag{63.69}$$

which proves the theorem. ∎

Frank [1990a,1992a] also gave a polynomial-time algorithm to find a minimum-cost r-edge-connector if the cost of any new edge uv is given by $k(u) + k(v)$, for some function $k : V \to \mathbb{Q}_+$. This is done with the help of the following auxiliary result:

Theorem 63.10. *Let $G = (V, E)$ be an undirected graph and let $r : V \times V \to \mathbb{Z}_+$ be symmetric. Define $R(U)$ as in* (63.59). *Then*

(63.70) $$Q := \{x \in \mathbb{R}_+^V \mid x(U) \geq R(U) - d_E(U) \text{ for all } U \subseteq V\}$$

is a contrapolymatroid, with associated supermodular function given by, for $X \subseteq V$*:*

(63.71) $$g(X) := \max_{\mathcal{U}} \sum_{U \in \mathcal{U}} (R(U) - d_E(U)),$$

where the maximum ranges over collections $\mathcal{U}$ *of disjoint nonempty subsets of* X*.*

Proof. Clearly, for any $x \in \mathbb{R}_+^V$ one has $x \in Q$ if and only if $x(U) \geq g(X)$ for each $X \subseteq V$.

To see that g is supermodular, choose $X, Y \subseteq V$. Let

(63.72) $$g(X) = \sum_{U \in \mathcal{U}} (R(U) - d(U)) \text{ and } g(Y) = \sum_{T \in \mathcal{T}} (R(T) - d(T)),$$

where $\mathcal{U}$ and $\mathcal{T}$ are collections of disjoint nonempty subsets of X and of Y, respectively. The collections $\mathcal{U}$ and $\mathcal{T}$ together form a family $\mathcal{S}$ of nonempty subsets of V satisfying

(63.73) $$\sum_{S \in \mathcal{S}} \chi^S \leq \chi^{X \cap Y} + \chi^{X \cup Y} \text{ and } g(X) + g(Y) \leq \sum_{S \in \mathcal{S}} (R(S) - d(S)).$$

We now choose $\mathcal{S}$ such that (63.73) is satisfied and such that

(63.74) $$\sum_{S \in \mathcal{S}} |S|(|V \setminus S| + 1)$$

is as small as possible.

We claim that $\mathcal{S}$ is laminar; that is,

(63.75) if $T, U \in \mathcal{S}$, then $T \subseteq U$ or $U \subseteq T$ or $T \cap U = \emptyset$.

Suppose not. By Lemma 61.6α, $R(T) + R(U)$ is at most $R(T \cap U) + R(T \cup U)$ or at most $R(T \setminus U) + R(U \setminus T)$. If $R(T) + R(U) \leq R(T \cap U) + R(T \cup U)$, then replacing T and U by $T \cap U$ and $T \cup U$ maintains (63.73) but decreases (63.74) (by Theorem 2.1), contradicting the minimality assumption. If $R(T) + R(U) \leq R(T \setminus U) + R(U \setminus T)$, then replacing T and U by $T \setminus U$ and $U \setminus T$ maintains (63.73) but decreases (63.74) (again by Theorem 2.1), again contradicting the minimality condition. This proves (63.75).

Now let $\mathcal{P}$ be the collection of inclusionwise maximal elements in $\mathcal{S}$ and let $\mathcal{Q}$ be the collection of remaining sets in $\mathcal{S}$. (If a set occurs twice in $\mathcal{S}$, it is both in $\mathcal{P}$ and in $\mathcal{Q}$.) Then each set in $\mathcal{P}$ is contained in $X \cup Y$, and each set in $\mathcal{Q}$ is contained in $X \cap Y$. Moreover, both $\mathcal{P}$ and $\mathcal{Q}$ are collections of disjoint sets. Hence

(63.76) $$\begin{aligned} g(X \cup Y) + g(X \cap Y) &\geq \sum_{P \in \mathcal{P}} (R(P) - d(P)) + \sum_{Q \in \mathcal{Q}} (R(Q) - d(Q)) \\ &= \sum_{S \in \mathcal{S}} (R(S) - d(S)) \geq g(X) + g(Y); \end{aligned}$$

that is, g is supermodular. ∎

With this theorem, also good characterizations and polynomial-time algorithms can be obtained for the minimum size of an r-edge-connector satisfying prescribed lower and upper bounds on its degrees — see Frank [1990a,1992a].

Bang-Jensen, Frank, and Jackson [1995] extended these results to mixed graphs.

63.5. Making a directed graph k-vertex-connected

Let (V, A) and (V, B) be directed graphs. The set B is called a *k-vertex-connector* for D if the directed graph $(V, A \cup B)$ is k-vertex-connected. (Note that parallel edges will not help the vertex-connectivity.)

Since, for directed graphs, 1-vertex-connectors and 1-arc-connectors coincide, the problem of finding a minimum-size 1-vertex-connector for a given directed graph is addressed in Section 57.1.

Frank and Jordán [1995b] showed the following min-max relation for minimum-size k-vertex connector in directed graphs (which is a special case of Frank and Jordán's Theorem 60.5 above).

Call a pair (X, Y) of subsets of V a *good pair* if $X \neq \emptyset$, $Y \neq \emptyset$, $X \cap Y = \emptyset$, and D has no arc from X to Y. Call a collection $\mathcal{F}$ of good pairs a *good collection* if $X \cap X' = \emptyset$ or $Y \cap Y' = \emptyset$ for all distinct (X, Y), $(X', Y') \in \mathcal{F}$.

Theorem 63.11. *Let $D = (V, A)$ be a directed graph and let $k \in \mathbb{Z}_+$. Then the minimum size of a k-vertex-connector for D is equal to the maximum value of*

$$\sum_{(X,Y)\in\mathcal{F}} (k - |V \setminus (X \cup Y)|), \tag{63.77}$$

where $\mathcal{F}$ ranges over good collections of good pairs.

Proof. Let γ be the maximum value. The minimum is not less than γ, since for any $(X, Y) \in \mathcal{F}$, at least $k - |V \setminus (X \cup Y)|$ arcs from X to Y should be added, while such arcs do not run from X' to Y' for any other pair (X', Y') in $\mathcal{F}$ (as $X \cap X' = \emptyset$ or $Y \cap Y' = \emptyset$).

To see equality, we can assume that D is not k-vertex-connected. Then there exist disjoint nonempty subsets T and U of V such that D has no arc from T to U and such that $|V \setminus (T \cup U)| < k$.

If there exist $t \in T$ and $u \in U$ such that augmenting D with the arc (t, u), the maximum decreases, we are done by induction. So we can assume that no such pair t, u exists. Hence for each $t \in T$ and $u \in U$, there exists a good collection $\mathcal{F}_{t,u}$ of good pairs, with

$$\sum_{(X,Y)\in\mathcal{F}_{t,u}} (k - |V \setminus (X \cup Y)|) = \gamma, \tag{63.78}$$

and with $t \notin X$ or $u \notin Y$ for all $(X, Y) \in \mathcal{F}_{t,u}$.

Concatenating these collections $\mathcal{F}_{t,u}$ for all $t \in T$, $u \in U$, and adding the pair (T, U), we obtain a family $\mathcal{G}$ of good pairs satisfying:

(63.79) (i) for each $x \in T$, $y \in U$, there are at most $|T||U|$ pairs (X, Y) in $\mathcal{G}$ with $x \in X$ and $y \in Y$;
(ii) $\displaystyle\sum_{(X,Y)\in\mathcal{G}} (k - |V \setminus (X \cup Y)|) > \gamma|T||U|$.

Among all families $\mathcal{G}$ satisfying (63.79), we choose one minimizing

(63.80) $$\sum_{(X,Y)\in\mathcal{G}} (|X| + |V \setminus Y|)(|Y| + |V \setminus X|).$$

Then

(63.81) for all $(X, Y), (X', Y') \in \mathcal{G}$ one has $X \cap X' = \emptyset$ or $Y \cap Y' = \emptyset$ or $X \subseteq X', Y' \subseteq Y$ or $X' \subseteq X, Y \subseteq Y'$.

Suppose not. Replace (X, Y) and (X', Y') by $(X \cap X', Y \cup Y')$ and $(X \cup X', Y \cap Y')$. This maintains (63.79), while (63.80) decreases[19], contradicting our assumption. This proves (63.81).

Now consider the partial order $\leq$ on pairs (X, Y) of subsets of V, defined by $(X, Y) \leq (X', Y')$ if $X \subseteq X', Y' \subseteq Y$. For each pair (X, Y), let its ‘weight’ $w(X, Y)$ be the number of times (X, Y) occurs in $\mathcal{G}$, and let its ‘length’ $l(X, Y)$ be equal to $k - |V \setminus (X \cup Y)|$. Then by (63.79)(i), any chain has weight at most $|T||U|$. By (63.79)(ii), the sum of $l(X, Y)w(X, Y)$ over $(X, Y) \in \mathcal{G}$ is more than $\gamma|T||U|$. Hence, by the length-width inequality for partially ordered sets (Theorem 14.5), $\mathcal{G}$ contains an antichain $\mathcal{F}$ of length more than γ. Then $\mathcal{F}$ is a good collection by (63.81). This contradicts the definition of γ. ∎

The theorem implies that the minimum size of a k-vertex-connector for a given directed graph $D = (V, A)$ is equal to the minimum value of

(63.82) $$\sum_{u,v\in V} x_{u,v}$$

subject to

(63.83) (i) $x_{u,v} \geq 0$ for all $u, v \in V$,
(ii) $\displaystyle\sum_{u\in X}\sum_{v\in Y} x_{u,v} \geq k - |V \setminus (X \cup Y)|$ for all disjoint nonempty $X, Y \subseteq V$ with no arc from X to Y.

[19] This can be seen with Theorem 2.1: Make a copy $\widetilde{V}$ of V, and let $\widetilde{Y}$ be the set of copies of elements of Y. Define $Z_{X,Y} := X \cup (\widetilde{V} \setminus \widetilde{Y})$. Then $|X| + |V \setminus Y| = |Z_{X,Y}|$ and $|Y| + |V \setminus X| = |(V \cup \widetilde{V}) \setminus Z_{X,Y}|$. Moreover, for (X, Y) and (X', Y') we have $Z_{X\cap X',Y\cup Y'} = Z_{X,Y} \cap Z_{X',Y'}$ and $Z_{X\cup X',Y\cap Y'} = Z_{X,Y} \cup Z_{X',Y'}$. So the replacements decrease (63.80) by Theorem 2.1.

This can be seen by observing that Theorem 63.11 implies that this LP-problem has integer primal and dual solutions of equal value.

As Frank and Jordán [1995b] pointed out, this implies that a minimum-size k-vertex-connector can be found in polynomial time with the ellipsoid method, as follows.

Since the conditions (63.83) can be checked in polynomial time, the ellipsoid method (as we discuss below) implies that the minimum size of a k-arc-connector can be determined in polynomial time. Then an explicit minimum-size k-vertex-connector can be found by testing, for each pair $u, v \in V$, whether augmenting D by the new arc (u, v) decreases the minimum size of a k-arc-connector. If so, we add (u, v) to D and iterate.

The ellipsoid method applies, since given $x_{u,v} \geq 0$ $(u, v \in V)$, we can test if (63.83)(ii) holds. Indeed, let B be a set of new arcs forming a complete directed graph on V. Define a capacity function c on $A \cup B$ by: $c(a) := \infty$ for each $a \in A$ and $c(b) = x_{u,v}$ for each arc $b \in B$ from u to v. Then (63.83)(ii) is equivalent to: for each $s, t \in V$ there is an $s-t$ flow $f_{s,t}$ in $(V, A \cup B)$ subject to c of value k, such that for any vertex $v \neq s, t$, the amount of flow traversing v is at most 1 (since the set of arcs in B from X to Y, together with the vertices in $V \setminus (X \cup Y)$, form a mixed arc/vertex-cut separating s and t). As this can be tested in polynomial time, we have a polynomial-time test for (63.83).

In fact, we can transform the problem into a linear programming problem of polynomial size, by including the flow variables $f_{s,t}(a)$ (for $s, t \in V$ and $a \in A \cup B$), into the LP-problem. Thus the minimum size of a k-arc-connector can be described as the solution of a linear programming problem of polynomial size.

There is no combinatorial polynomial-time algorithm known to find a minimum-size k-vertex-connector for a given directed graph. (Frank and Jordán [1995a] describe a combinatorial polynomial-time algorithm for finding a minimum-size 2-vertex-connector for a strongly connected directed graph. Frank and Jordán [1999] extended it to a polynomial-time algorithm (for any fixed k) to find a minimum-size k-vertex-connector.)

Notes. Frank and Jordán [1995b] also showed that a directed graph $D = (V, A)$ has a k-vertex-connector B with all in- and outdegrees at most $k - \kappa(D)$ (where $\kappa(D)$ denotes the vertex-connectivity of D).

Frank [1994a] gave the following conjecture:

(63.84) (?) Let $D = (V, A)$ be a simple acyclic directed graph. Then the minimum size of a k-vertex-connector for D is equal to the maximum of $\sum_{v \in V} \max\{0, k - \deg^{\text{in}}(v)\}$ and $\sum_{v \in V} \max\{0, k - \deg^{\text{out}}(v)\}$. (?)

An $O(kn)$-time algorithm finding a minimum-size k-vertex-connector for a rooted tree was given by Masuzawa, Hagihara, and Tokura [1987]. Frank [1994a] observed that this result easily extends to branchings.

Approximation algorithms for the minimum size of a k-vertex-connector for a directed graph were given by Jordán [1993a].

63.6. Making an undirected graph k-vertex-connected

Let (V, E) and (V, F) be undirected graphs. The set F is called a *k-vertex-connector* for G if the graph $(V, E \cup F)$ is k-vertex-connected.

Trivially, the minimum size of a 1-vertex-connector for an undirected graph G is equal to one less than the number of components of G.

The minimum size of a 2-vertex-connector for undirected graphs was given by Eswaran and Tarjan [1976] and Plesník [1976]. To this end, call a block *pendant* if it contains exactly one cut vertex of G. Moreover, call a block *isolated* if it contains no cut vertex of G. So an isolated block is a component of G.

Theorem 63.12. *Let $G = (V, E)$ be a non-2-vertex-connected undirected graph, with p pendant blocks and q isolated blocks. Let d be the maximum number of components of $G - v$, maximized over $v \in V$. Then the minimum size of a 2-vertex-connector for G is equal to*

(63.85) $$k := \max\{d - 1, \lceil \tfrac{1}{2} p \rceil + q\}.$$

Proof. One needs at least $d-1$ edges, since for any $v \in V$, after deleting v the augmented graph should be connected. Any block containing no cut vertex should be incident with at least two new edges, and any block containing one cut vertex should be incident with at least one new edges. Hence the number of new edges is at least $\frac{1}{2}p + q$, and hence at least k.

To show that k can be attained, choose a counterexample G with k minimal. Then G is connected. Otherwise, we can choose two blocks B, B' from different components of G such that each of B, B' is pendant or isolated. We can choose a non-cut vertex from each of B, B', and connect them by a new edge to obtain graph G'. After that, k has decreased by exactly 1, and we can apply induction to G', implying the theorem.

So G is connected, and hence $q = 0$ (as G is not 2-vertex-connected). Moreover, $k \geq 2$, since otherwise $p \leq 2$, and we can add one edge to make G 2-vertex-connected.

Let U be the set of vertices v for which $G-v$ has at least three components and let W be the set of vertices v for which $G - v$ has $k + 1$ components. So $W \subseteq U$. Moreover, $|U| \geq 2$, since otherwise we can add $d - 1$ edges to make G connected.

We show:

(63.86) there exist two distinct pendant blocks B, B' such that each $B-B'$ path traverses all vertices in W and at least two vertices in U.

If $|W| \leq 1$, this is trivial. So we may assume that $|W| \geq 2$. Then, as $W \subseteq U$, it suffices to show that there exists a path traversing all vertices in W. If such a path would not exist, there exists a subset X of W with $|X| = 3$ that is not on a path. Then for each $v \in X$, one component K of $G - v$ contains $X \setminus \{v\}$. So for each $v \in X$, $G - v$ has k components disjoint from

X. Moreover, for distinct $v, v' \in X$, if K and K' are components of $G - v$ and $G - v'$ (respectively) each disjoint from X, then $K \cap K' = \emptyset$. Since for each $v \in X$ and each component K of $G - v$, $K \cup \{v\}$ contains at least one pendant block, we know $p \geq 3k \geq 3\lceil \frac{1}{2}p \rceil$, contradicting the fact that $p > 0$.

This shows (63.86). Now augment G by an edge connecting non-cut vertices in B and B', giving graph G'. As this augmentation decreases k (by the conditions given in (63.86)), we would obtain a counterexample with k smaller. ∎

This proof directly gives a polynomial-time algorithm to find a minimum-size 2-vertex-connector for G. Eswaran and Tarjan [1976] mention that a linear-time implementation of this algorithm was communicated to them in 1973 by R. Pecherer and A. Rosenthal — see Rosenthal and Goldner [1977]. (See also Hsu and Ramachandran [1991,1993].)

An equivalent form of Theorem 63.12 is:

Corollary 63.12a. *Let $G = (V, E)$ be a non-2-vertex-connected graph. Then G has a 2-vertex-connector of size at most γ if and only if for each vertex v, $G - v$ has at most $\gamma + 1$ components and*

$$(63.87) \qquad \sum_{U \in \mathcal{P}} (2 - |N(U)|) \leq 2\gamma$$

for each collection $\mathcal{P}$ of disjoint nonempty subsets U of V with $|U| \leq |V| - 3$.

Proof. Directly from Theorem 63.12. ∎

Jackson and Jordán [2001] showed that for each fixed k, a minimum-size k-vertex-connector for an undirected graph can be found in polynomial time.

Notes. Watanabe and Nakamura [1988,1993] give a characterization of the minimum size of a 3-vertex-connector, and Watanabe and Nakamura [1993] describe an $O(n(n+m)^2)$-time algorithm (for a sketch, see Watanabe and Nakamura [1988, 1990]). Hsu and Ramachandran [1991] gave a linear-time algorithm for this problem. Hsu [1992,2000] gave an almost-linear-time algorithm to find a minimum-size 4-vertex-connector for a 3-connected undirected graph.

Note that the natural extensions of Corollary 63.12a does not hold for k-vertex-connectors with $k \geq 4$, as is shown by the complete bipartite graph $K_{3,3}$.

For approximation algorithms, see Jordán [1993b,1995,1997a], Khuller and Thurimella [1993], Cheriyan and Thurimella [1996b,1999], Nutov and Penn [1997], Penn and Shasha-Krupnik [1997], and Jackson and Jordán [2000].

63.6a. Further notes

Corollary 53.6b implies the following characterization for connectivity augmentation, due to Frank [1979b].

Theorem 63.13. *Let $D = (V, A)$ be a digraph, let $r \in V$, and let $k \in \mathbb{Z}_+$ be such that D contains k disjoint r-arborescences. Moreover, let $D' = (V, A')$ and $l \in \mathbb{Z}_+^{A'}$. Then the minimum of $l(C)$ where $C \subseteq A'$ such that the digraph $(V, A \cup C)$ (taking arcs multiple) has $k + 1$ disjoint r-arborescences is equal to the maximum size t of a family of nonempty subsets $U_1, \ldots, U_t$ of $V \setminus \{r\}$ such that $d_D^{\text{in}}(U_j) = k$ for $j = 1, \ldots, t$ and such that each arc a of D' enters at most $l(a)$ of the U_j.*

Proof. Consider the digraph $D'' = (V, A'')$ with $A'' := A \cup A'$ (taking multiple arcs for arcs occurring both in A and in A'). Now the minimum in this corollary is equal to the minimum of $\sum_{a \in A'} l(a)x_a$ where $x \in \mathbb{Z}^{A''}$ satisfies

(63.88) $\quad 0 \leq x_a \leq 1$ if $a \in A'$,
$\quad x(\delta_{D'}^{\text{in}}(U)) \geq k + 1 - d_D^{\text{in}}(U)$ for each nonempty $U \subseteq V \setminus \{r\}$.

Since (63.88) is TDI by Corollary 53.6b, this minimum is equal to the maximum described in the present corollary. ∎

The problem of making a bipartite directed graph strongly connected while preserving bipartiteness is considered by Gabow and Jordán [1999,2000a]. Augmenting the arc-connectivity while preserving bipartiteness is studied by Gabow and Jordán [2000b]. Making a bipartite undirected graph k-edge-connected while preserving bipartiteness, and, more generally, edge-connectivity augmentation with partition constraints, is studied by Bang-Jensen, Gabow, Jordan, and Szigeti [1998, 1999].

For the 'successive augmentation problem', see Cheng and Jordán [1999]. For NP-completeness and approximation results for connectivity augmentation, see Frederickson and Ja'Ja' [1981,1982]. Frank and Király [2001] studied problems that combine graph orientation and connectivity augmentation.

Planar graph connectivity augmentation was considered by Provan and Burk [1999].

Ishii, Nagamochi, and Ibaraki [1997,1998b,1998a,1999,2000,2001] considered the problem of making an undirected graph both k-vertex- and l-edge-connected.

For surveys on connectivity augmentation, see Frank [1993a,1994a], Jordán [1994,1997b], and Nagamochi [2000].