Part III

Nonbipartite Matching and Covering

Part III: Nonbipartite Matching and Covering

Nonbipartite matching is a highlight of combinatorial optimization, thanks to pioneering work of Tutte and Edmonds. In particular the 1965 papers of Edmonds on nonbipartite matching opened up areas that were not accessible with the 'classical' methods based on flows, linear programming, and total unimodularity found in the 1950s. The papers are pioneering in polyhedral combinatorics, giving the first nontrivial characterizations of combinatorially defined polytopes.
The techniques are highly self-refining, and extend to b-matchings, b-factors, T-joins, shortest paths in undirected graphs, and the Chinese postman problem. Nonbipartite matching also applies to practical problems where an optimal pairing has to be found, like in seat or room assignment, crew planning, and two-processor scheduling.

Chapters:

Chapter 24

Cardinality nonbipartite matching

In this chapter we consider maximum-cardinality matching, with as key results Tutte's characterization of the existence of a perfect matching (implying the Tutte-Berge formula for the maximum-size of a matching) and Edmonds' polynomial-time algorithm to find a maximum-size matching.
As in Section 16.1, we call a path P an M*-augmenting path* if P has odd length and connects two vertices not covered by M, and its edges are alternatingly out of and in M. By Theorem 16.1, a matching M has maximum size if and only if there is no M-augmenting path.
We say that a matching M *covers* a vertex v if v is incident with an edge in M. If M does not cover v, we say that M *misses* v.
In this chapter, graphs can be assumed to be simple.

24.1. Tutte's 1-factor theorem and the Tutte-Berge formula

A basic result of Tutte [1947b] characterizes graphs that have a perfect matching. Berge [1958a] observed that it implies a min-max formula for the maximum size of a matching in a graph, the Tutte-Berge formula.

Call a component of a graph *odd* if it has an odd number of vertices. For any graph G, let

(24.1) $\quad o(G) :=$ number of odd components of G.

Let $\nu(G)$ denotes the maximum size of a matching. Then:

Theorem 24.1 (Tutte-Berge formula)**.** *For each graph* $G = (V, E)$*,*

(24.2) $$\nu(G) = \min_{U \subseteq V} \tfrac{1}{2}(|V| + |U| - o(G-U)).$$

Proof. To see $\leq$, we have for each $U \subseteq V$:

(24.3) $$\nu(G) \leq |U| + \nu(G-U) \leq |U| + \tfrac{1}{2}(|V \setminus U| - o(G-U)) = \tfrac{1}{2}(|V| + |U| - o(G-U)).$$

We prove the reverse inequality by induction on $|V|$, the case $V = \emptyset$ being trivial. We can assume that G is connected, since otherwise we can apply induction to the components of G.

First assume that there exists a vertex v covered by all maximum-size matchings. Then $\nu(G-v) = \nu(G)-1$, and by induction there exists a subset U' of $V \setminus \{v\}$ with

(24.4) $$\nu(G-v) = \tfrac{1}{2}(|V \setminus \{v\}| + |U'| - o(G-v-U')).$$

Then $U := U' \cup \{v\}$ gives equality in (24.2), since

(24.5) $$\begin{aligned}\nu(G) = \nu(G-v)+1 &= \tfrac{1}{2}(|V \setminus \{v\}| + |U'| - o(G-v-U')) + 1 \\ &= \tfrac{1}{2}(|V| + |U| - o(G-U)).\end{aligned}$$

So we can assume that there is no such v. In particular, $\nu(G) < \frac{1}{2}|V|$. We show that there exists a matching of size $\frac{1}{2}(|V|-1)$, which implies the theorem (taking $U := \emptyset$).

Indeed, suppose to the contrary that each maximum-size matching M misses at least two distinct vertices u and v. Among all such M, u, v, choose them such that the distance $\mathrm{dist}(u,v)$ of u and v in G is as small as possible.

If $\mathrm{dist}(u,v) = 1$, then u and v are adjacent, and hence we can augment M by the edge uv, contradicting the maximality of $|M|$. So $\mathrm{dist}(u,v) \geq 2$, and hence we can choose an intermediate vertex t on a shortest $u-v$ path. By assumption, there exists a maximum-size matching N missing t. Choose such an N with $|M \cap N|$ maximal.

By the minimality of $\mathrm{dist}(u,v)$, N covers both u and v. Hence, as M and N cover the same number of vertices, there exists a vertex $x \neq t$ covered by M but not by N. Let $x \in e = xy \in M$. Then y is covered by some edge $f \in N$, since otherwise $N \cup \{e\}$ would be a matching larger than N. Replacing N by $(N \setminus \{f\}) \cup \{e\}$ would increase its intersection with M, contradicting the choice of N. ∎

(This proof is based on the proof of Lovász [1979b] of Edmonds' matching polytope theorem.)

The Tutte-Berge formula immediately implies Tutte's 1-factor theorem. A *perfect matching* (or 1-*factor*) is a matching covering all vertices of the graph.

Corollary 24.1a (Tutte's 1-factor theorem)**.** *A graph $G = (V,E)$ has a perfect matching if and only if $G-U$ has at most $|U|$ odd components, for each $U \subseteq V$.*

Proof. Directly from the Tutte-Berge formula (Theorem 24.1), since G has a perfect matching if and only if $\nu(G) \geq \frac{1}{2}|V|$. ∎

24.1a. Tutte's proof of his 1-factor theorem

The original proof of Tutte [1947b] of his 1-factor theorem (Corollary 24.1a), with a simplification of Maunsell [1952], and smoothed by Halton [1966] and Lovász [1975d], is as follows.

Suppose that there exist graphs $G = (V, E)$ satisfying the condition, but not having a perfect matching. Fixing V, take such a graph G with G simple and $|E|$ as large as possible. Let $U := \{v \in V \mid v \text{ is adjacent to every other vertex of } G\}$. We show that each component of $G - U$ is a complete graph.

Suppose to the contrary that there are distinct $a, b, c \notin U$ with $ab, bc \in E$ and $ac \notin E$. By the maximality of $|E|$, adding ac to E makes that G has a perfect matching (since the condition is maintained under adding edges). So G has a matching M missing precisely a and c. As $b \notin U$, there exists a vertex d with $bd \notin E$. Again by the maximality of $|E|$, G has a matching N missing precisely b and d. Now each component of $M \triangle N$ contains the same number of edges in M as in N — otherwise there would exist an M- or N-augmenting path, and hence a perfect matching in G, a contradiction. So the component P of $M \triangle N$ containing d is a path starting at d, with first edge in M and last edge in N, and hence ending at a or c; by symmetry we may assume that it ends at a. Moreover, P does not traverse b. Then extending P by the edge ab gives an N-augmenting path, and hence a perfect matching in G — a contradiction.

So each component of $G - U$ is a complete graph. Moreover, by the condition, $G - U$ has at most $|U|$ odd components. This implies that G has a perfect matching, contradicting our assumption.

More proofs were given by Gallai [1950,1963b], Edmonds [1965d], Balinski [1970], Anderson [1971], Brualdi [1971d], Hetyei [1972,1999], Mader [1973], and Lovász [1975a,1979b].

24.1b. Petersen's theorem

The following theorem of Petersen [1891] is a consequence of Tutte's 1-factor theorem (a graph is *cubic* if it is 3-regular):

Corollary 24.1b (Petersen's theorem)**.** *A bridgeless cubic graph has a perfect matching.*

Proof. Let $G = (V, E)$ be a bridgeless cubic graph. By Tutte's 1-factor theorem, we should show that $G - U$ has at most $|U|$ odd components, for each $U \subseteq V$.

Each odd component of $G - U$ is left by an odd number of edges (as G is cubic), and hence by at least three edges (as G is bridgeless). On the other hand, U is left by at most $3|U|$ edges, since G is cubic. Hence $G - U$ has at most $|U|$ odd components. ∎

24.2. Cardinality matching algorithm

The idea of finding an M-augmenting path to increase a matching M is fundamental in finding a maximum-size matching. However, the simple trick for bipartite graphs, of orienting the edges based on the colour classes of the graph, does not extend to the nonbipartite case. Yet one could try to find an M-augmenting path by finding an 'M-alternating walk', but such a walk can run into a loop that cannot simply be deleted. It was Edmonds [1965d] who found the trick to resolve this problem, namely by 'shrinking' the loop (for which he introduced the term 'blossom'). Then applying recursion to a smaller graph solves the problem[1].

Let $G = (V, E)$ be a graph, let M be a matching in G, and let X be the set of vertices missed by M. A walk $P = (v_0, v_1, \ldots, v_t)$ is called *M-alternating* if for each $i = 1, \ldots, t-1$ exactly one of the edges $v_{i-1}v_i$ and v_iv_{i+1} belongs to M. Note that one can find a shortest M-alternating $X - X$ walk of positive length, by considering the auxiliary directed graph $D = (V, A)$ with

(24.6) $$A := \{(u, v) \mid \exists x \in V : ux \in E, xv \in M\}.$$

Then each M-alternating $X-X$ walk of positive length yields a directed $X-N(X)$ path in D, and vice versa (where $N(X)$ denotes the set of neighbours of X).

An M-alternating walk $P = (v_0, v_1, \ldots, v_t)$ is called an *M-flower* if t is odd, $v_0, \ldots, v_{t-1}$ are distinct, $v_0 \in X$, and $v_t = v_i$ for some even $i < t$. Then the circuit $(v_i, v_{i+1}, \ldots, v_t)$ is called an *M-blossom* (associated with the M-flower).

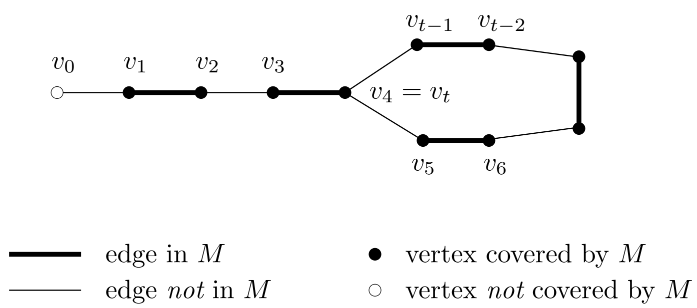


Figure 24.1
An M-flower

The core of the algorithm is the following observation. Let $G = (V, E)$ be a graph and let B be a subset of V. Denote by G/B the graph obtained by *contracting* (or *shrinking*) B to one new vertex, called B. That is, G/B has vertex set $(V \setminus B) \cup \{B\}$, and for each edge e of G an edge obtained from e by replacing any end vertex in B by the new vertex B. (We ignore loops that may arise.) We denote the new edge again by e. (So its ends are modified,

[1] The idea of applying shrinking recursively to matching problems was introduced by Petersen [1891], and was applied in an algorithmic way by Brahana [1917].

but not its name.) We say that the new edge is the *image* (or *projection*) of the original edge.

For any matching M, let M/B denote the set of edges in G/B that are images of edges in M not spanned by B. Obviously, if M intersects $\delta(B)$ in at most one edge, then M/B is a matching in G/B. In the following, we identify a blossom with its set of vertices.

Theorem 24.2. *Let B be an M-blossom in G. Then M is a maximum-size matching in G if and only if M/B is a maximum-size matching in G/B.*

Proof. Let $B = (v_i, v_{i+1}, \ldots, v_t)$.

First assume that M/B is not a maximum-size matching in G/B. Let P be an M/B-augmenting path in G/B. If P does not traverse vertex B of G/B, then P is also an M-augmenting path in G. If P traverses vertex B, we may assume that it enters B with some edge uB that is not in M/B. Then $uv_j \in E$ for some $j \in \{i, i+1, \ldots, t\}$.

(24.7) If j is odd, replace vertex B in P by $v_j, v_{j+1}, \ldots, v_t$.
If j is even, replace vertex B in P by $v_j, v_{j-1}, \ldots, v_i$.

In both cases we obtain an M-augmenting path in G. So M is not maximum-size.

Conversely, assume that M is not maximum-size. We may assume that $i = 0$, that is, $v_i \in X$, since replacing M by $M \triangle EQ$, where Q is the path $(v_0, v_1, \ldots, v_i)$, does not modify the theorem. Let $P = (u_0, u_1, \ldots, u_s)$ be an M-augmenting path in G. If P does not intersect B, then P is also an M/B-augmenting path in G/B. If P intersects B, we may assume that $u_0 \notin B$. (Otherwise replace P by its reverse.) Let u_j be the first vertex of P in B. Then $(u_0, u_1, \ldots, u_{j-1}, B)$ is an M/B-augmenting path in G/B. So M/B is not maximum-size. ∎

Another useful observation is:

Theorem 24.3. *Let $P = (v_0, v_1, \ldots, v_t)$ be a shortest M-alternating $X - X$ walk. Then either P is an M-augmenting path or $(v_0, v_1, \ldots, v_j)$ is an M-flower for some $j \leq t$.*

Proof. Assume that P is not a path. Choose $i < j$ with $v_j = v_i$ and with j as small as possible. So $v_0, \ldots, v_{j-1}$ are all distinct.

If $j - i$ would be even, we can delete $v_{i+1}, \ldots, v_j$ from P so as to obtain a shorter M-alternating $X - X$ walk. So $j - i$ is odd. If j is even and i is odd, then $v_{i+1} = v_{j-1}$ (as it is the vertex matched to $v_i = v_j$), contradicting the minimality of j.

Hence j is odd and i is even, and therefore $(v_0, v_1, \ldots, v_j)$ is an M-flower. ∎

We now describe an algorithm (the *matching-augmenting algorithm*) for the following problem:

(24.8) given: a matching M;
find: an M-augmenting path, if any.

Denote the set of vertices missed by M by X.

(24.9) If there is no M-alternating $X-X$ walk of positive length, there is no M-augmenting path.
If there exists an M-alternating $X-X$ walk of positive length, choose a shortest one, $P = (v_0, v_1, \ldots, v_t)$ say.
Case 1: P is a path. Then output P.
Case 2: P is not a path. Choose j such that $(v_0, \ldots, v_j)$ is an M-flower, with M-blossom B. Apply the algorithm (recursively) to G/B and M/B, giving an M/B-augmenting path P in G/B. Expand P to an M-augmenting path in G (cf. (24.7)).

The correctness of this algorithm follows from Theorems 24.2 and 24.3. It gives a polynomial-time algorithm to find a maximum-size matching, which is a basic result of Edmonds [1965d].

Theorem 24.4. *Given a graph, a maximum-size matching can be found in time $O(n^2m)$.*

Proof. The algorithm directly follows from algorithm (24.9), since, starting with $M = \emptyset$, one can iteratively apply it to find an M-augmenting path P and replace M by $M \triangle EP$. It terminates if there is no M-augmenting path, whence M is a maximum-size matching.

By using (24.6), path P in (24.9) can be found in time $O(m)$. Moreover, the graph G/B can be constructed in time $O(m)$. Since the recursion has depth at most n, an M-augmenting path can be found in time $O(nm)$. Since the number of augmentations is at most $\frac{1}{2}n$, the time bound follows. ∎

This implies for perfect matchings:

Corollary 24.4a. *A perfect matching in a graph (if any) can be found in time $O(n^2m)$.*

Proof. Directly from Theorem 24.4, as a perfect matching is a maximum-size matching. ∎

24.2a. An $O(n^3)$ algorithm

The matching algorithm described above consists of a series of matching augmentations. Each matching augmentation itself consists of a series of two steps performed alternatingly:

(24.10) finding an M-alternating walk, and
shrinking an M-blossom,

until the M-alternating walk is simple, that is, is an M-augmenting path.

Each of these two steps can be done in time $O(m)$. Since there are at most n shrinkings and at most n matching augmentations, we obtain the $O(n^2m)$ time bound.

If we want to save time we must consider speeding up both the walk-finding step and the shrinking step. In a sense, our description above gives a brute-force polynomial-time method. The $O(m)$ time bound for shrinking gives us time to construct the shrunk graph completely, by copying all vertices that are not in the blossom, by introducing a new vertex for the shrunk blossom, and by introducing for each original edge its 'image' in the shrunk graph. The $O(m)$ time bound for finding an M-alternating walk gives us time to find, after any shrinking, a walk starting just from scratch.

In fact, we cannot do much better if we explicitly construct the shrunk graph. But if we modify the graph only locally, by shrinking the M-blossom B and removing loops and parallel edges, this can be done in time $O(|B|n)$. Since the sum of $|B|$ over all M-blossoms B is $O(n)$, this yields a time bound of $O(n^2)$ for shrinking.

To reduce the $O(m)$ time for walk-finding, we keep data from the previous walk-search for the next walk-search, with the help of an M-alternating forest, defined as follows.

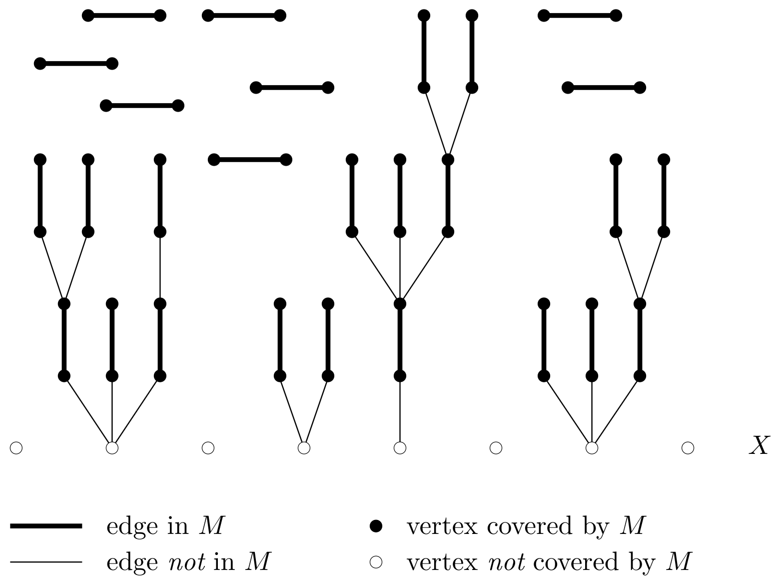


Figure 24.2
An M-alternating forest

Let $G = (V, E)$ be a simple graph and let M be a matching in G. Define X to be the set of vertices missed by M. An *M-alternating forest* is a subset F of E satisfying:

(24.11) F is a forest with $M \subseteq F$, each component of (V, F) contains either exactly one vertex in X or consists of one edge in M, and each path in F starting in X is M-alternating

(cf. Figure 24.2). For any M-alternating forest F, define

(24.12) $\text{even}(F) := \{v \in V \mid F \text{ contains an even-length } X - v \text{ path}\}$,
$\text{odd}(F) := \{v \in V \mid F \text{ contains an odd-length } X - v \text{ path}\}$,
$\text{free}(F) := \{v \in V \mid F \text{ contains no } X - v \text{ path}\}$.

Then each $u \in \text{odd}(F)$ is incident with a unique edge in $F \setminus M$ and a unique edge in M. Moreover:

(24.13) if there is no edge connecting $\text{even}(F)$ and $\text{even}(F) \cup \text{free}(F)$, then M is a maximum-size matching.

Indeed, if there is no such edge, $\text{even}(F)$ is a stable set in $G - \text{odd}(F)$. Hence, setting $U := \text{odd}(F)$:

(24.14) $o(G - U) \geq |\text{even}(F)| = |X| + |\text{odd}(F)| = (|V| - 2|M|) + |U|$,

and hence M has maximum size by (24.2).

Now algorithmically, we keep, next to E and M, an M-alternating forest F. We keep the set of vertices by a doubly linked list. We keep for each vertex v, the edges in E, M, and F, incident with v as doubly linked lists. We also keep the incidence functions $\chi^{\text{even}(F)}$ and $\chi^{\text{odd}(F)}$. Moreover, we keep for each vertex v of G one edge $e_v = vu$ with $u \in \text{even}(F)$, if such an edge exists.

Initially, $F := M$ and for each $v \in V$ we select an edge $e_v = vu$ with $u \in X$ (if any). The iteration is:

(24.15) Find a vertex $v \in \text{even}(F) \cup \text{free}(F)$ for which $e_v = vu$ exists.
Case 1: $v \in \text{free}(F)$. Add uv to F. Let vw be the edge in M incident with v. For each edge wx incident with w, set $e_x := wx$.
Case 2: $v \in \text{even}(F)$. Find the $X - u$ and $X - v$ paths P and Q in F.
Case 2a: P and Q are disjoint. Then P and Q form with uv an M-augmenting path.
Case 2b: P and Q are not disjoint. Then P and Q contain an M-blossom B. For each edge bx with $b \in B$ and $x \notin B$, set $e_x := Bx$. Replace G by G/B and remove all loops and parallel edges from E, M, and F.

The number of iterations is at most $|V|$, since, in each iteration, $|V| + |\text{free}(F)|$ decreases by at least 2 (one of these terms decreases by at least 2 and the other does not change). We end up either with a matching augmentation or with the situation that there is no edge connecting $\text{even}(F)$ and $\text{even}(F) \cup \text{free}(F)$, in which case M has maximum size by (24.13).

It is easy to update the data structure in Case 1 in time $O(n)$. In Case 2, the paths P and Q can be found in time $O(n)$, and hence in Case 2a, the M-augmenting path is found in time $O(n)$.

Finally, the data structure in Case 2b can be updated in $O(|B|n)$ time[2]. Also a matching augmentation in G/B can be transformed to a matching augmentation in G in time $O(|B|n)$. Since $|B|$ is bounded by twice the decrease in the number of vertices of the graph, this takes time $O(n^2)$ overall.

Hence a matching augmentation can be found in time $O(n^2)$, and therefore:

Theorem 24.5. *A maximum-size matching can be found in time $O(n^3)$.*

Proof. From the above. ∎

The first $O(n^3)$-time cardinality matching algorithm was published by Balinski [1969], and consists of a depth-first strategy to find an M-alternating forest, replacing shrinking by a clever labeling technique.

Bottleneck in a further speedup is storing the shrinking. With the disjoint set union data structure of Tarjan [1975] one can obtain an $O(nm\alpha(m,n))$-time algorithm (Gabow [1976a]). A special set union data structure of Gabow and Tarjan [1983,1985] gives an $O(nm)$-time algorithm. An $O(\sqrt{n}\,m)$-time algorithm was announced (with partial proof) by Micali and Vazirani [1980]. A proof was given by Blum [1990], Vazirani [1990,1994], and Gabow and Tarjan [1991] (cf. Peterson and Loui [1988]).

24.3. Matchings covering given vertices

Brualdi [1971d] derived from Tutte's 1-factor theorem the following extension of the Tutte-Berge formula:

Theorem 24.6. *Let $G = (V, E)$ be a graph and let $T \subseteq V$. Then the maximum size of a subset S of T for which there is a matching covering S is equal to the minimum value of*

(24.16) $\quad |T| + |U| - o_T(G - U)$

over $U \subseteq V$. Here $o_T(G-U)$ denotes the number of odd components of $G-U$ contained in T.

Proof. For any matching M in G and any $U \subseteq V$, at most $|U|$ odd components of $G - U$ can be covered completely by M. So M misses at least $o_T(G - U) - |U|$ vertices in T. This shows that the minimum is not less than the maximum.

To see equality, let μ be equal to the minimum. Let C be a set disjoint from V with $|C| = |V|$ and let $C' \subseteq C$ with $|C'| = |T| - \mu$. Make a new graph H by extending G by C, in such a way that C is a clique, each vertex in C'

[2] For each $Z \in \{E, M, F\}$, we scan the vertices b in B, and for $b \in B$ we scan the Z-neighbours w of b. If w does not belong to B and was not met as a Z-neighbour of an earlier scanned vertex in B, we replace bw by Bw in Z. Otherwise, we delete bw from Z.

is adjacent to each vertex in V, and each vertex in $C \setminus C'$ is adjacent to each vertex in $V \setminus T$.

If H has a perfect matching M, then M contains at most $|C'| = |T| - \mu$ edges connecting T and C (since T is not connected to $C \setminus C'$). Hence at least μ vertices in T are covered by edges in M spanned by V, as required.

So we may assume that H has no perfect matching. Then by Tutte's 1-factor theorem, there is a set W of vertices of H such that $H - W$ has at least $|W| + 2$ odd components (since $|V| + |C|$ is even).

If $C' \not\subseteq W$, then $H - W$ has only one component (since each vertex in C' is adjacent to every other vertex), a contradiction. If $C \subseteq W$, then $H - W$ has at most $|V|$ components, while $|W| + 2 \geq |C| + 2 = |V| + 2$, a contradiction.

So $C' \subseteq W$ and $C \setminus C' \not\subseteq W$. Then at most one component of $H - W$ is not contained in T (since $C \setminus C'$ is a clique and each vertex in $C \setminus C'$ is adjacent to each vertex in $V \setminus T$). Let $U := W \cap V$. Then

$$(24.17) \qquad \begin{aligned} o_T(G-U) &= o_T(H-W) \geq o(H-W) - 1 > |W| \geq |C'| + |U| \\ &= |T| - \mu + |U|, \end{aligned}$$

contradicting the definition of μ. ∎

(This theorem was also given by Las Vergnas [1975b].)

A consequence is a result of Lovász [1970c] on sets of vertices covered by matchings:

Corollary 24.6a. *Let $G = (V, E)$ be a graph and let T be a subset of V. Then G has a matching covering T if and only if T contains at most $|U|$ odd components of $G - U$, for each $U \subseteq V$.*

Proof. Directly from Theorem 24.6. ∎

(This theorem was also given by McCarthy [1975].)

24.4. Further results and notes

24.4a. Complexity survey for cardinality nonbipartite matching

$O(n^2 m)$	Edmonds [1965d] (cf. Witzgall and Zahn [1965])
$O(n^3)$	Balinski [1969] (also Gabow [1973,1976a], Karzanov [1976], Lawler [1976b])
$O(nm\alpha(m,n))$	Gabow [1976a]
$O(n^{5/2})$	Even and Kariv [1975], Kariv [1976] (also Bartnik [1978])
$O(\sqrt{n}\, m \log n)$	Even and Kariv [1975], Kariv [1976]

≫

continued

	$O(\sqrt{n}\, m \log\log n)$	Kariv [1976]
	$O(\sqrt{n}\, m + n^{1.5+\varepsilon})$	Kariv [1976] for each $\varepsilon > 0$
	$O(\sqrt{n}\, m)$	announced by Micali and Vazirani [1980], full proof in Blum [1990], Vazirani [1990,1994], and Gabow and Tarjan [1991](cf. Gabow and Tarjan [1983,1985])
*	$O(\sqrt{n}\, m \log_n \frac{n^2}{m})$	Goldberg and Karzanov [1995]

Here * indicates an asymptotically best bound in the table. (Kameda and Munro [1974] claim to give an $O(nm)$-time cardinality matching algorithm, but the proof contains some errors which I could not resolve.)

Gabow and Tarjan [1988a] observed that the method of Micali and Vazirani [1980] also implies that one can find, for given k, a matching of size at least $\nu(G) - \frac{n}{k}$ in time $O(km)$. They derived that a maximum-size matching M minimizing $\max_{e \in M} w(e)$ can be found in time $O(\sqrt{n \log n}\, m)$. (the 'bottleneck matching problem').

Mulmuley, Vazirani, and Vazirani [1987a,1987b] showed that 'matching is as easy as matrix inversion', which is especially of interest for the parallel complexity.

24.4b. The Edmonds-Gallai decomposition of a graph

There is a canonical set U that attains the minimum in (24.2). It has the property that the odd components of $G-U$ cover an inclusionwise minimal set of vertices, and is given by the *Edmonds-Gallai decomposition*, independently found by Edmonds [1965d] and Gallai [1963a,1964].

Let $G = (V, E)$ be a graph. The Edmonds-Gallai decomposition of G is the partition of V into $D(G)$, $A(G)$, and $C(G)$ defined as follows (recall that $N(U) := \{v \in V \setminus U \mid \exists u \in U\colon uv \in E\}$):

(24.18)
$$\begin{aligned} &D(G) := \{v \in V \mid \text{there exists a maximum-size matching missing } v\},\\ &A(G) := N(D(G)),\\ &C(G) := V \setminus (D(G) \cup A(G)). \end{aligned}$$

It yields a 'canonical' certificate of maximality of a matching:

Theorem 24.7. *$U := A(G)$ attains the minimum in* (24.2)*, $D(G)$ is the union of the odd components of $G-U$, and (hence) $C(G)$ is the union of the even components of $G-U$.*

Proof. *Case 1: $D(G)$ is a stable set.* Let M be a maximum-size matching and let X be the set of vertices missed by M. Then each vertex v in $A(G)$ is contained in an edge $uv \in M$ (as $v \notin D(G)$). We show that $u \in D(G)$. Assume that $u \notin D(G)$.

Since $v \in A(G) = N(D(G))$, there is an edge vw with $w \in D(G)$. Let N be a matching missing w. Then $M \triangle N$ contains a path component starting at a vertex in X and ending at w. Let $(v_0, v_1, \ldots, v_t)$ be this path, with $v_0 \in X$ and $v_t = w$. Then t is even and $v_i \in D(G)$ for each even i (because $M \triangle \{v_0v_1, v_2v_3, \ldots, v_{i-1}v_i\}$ is a maximum-size matching missing v_i). Hence, assuming $u \notin D(G)$, the edge vu is not on P. So extending P by wv and vu gives a path Q. Then $M \triangle Q$ is a maximum-size matching missing u. So $u \in D(G)$.

As this is true for any $v \in A(G)$, we see that part of M matches $A(G)$ and $D(G) \setminus X$. Hence

(24.19) $$o(G-U) \geq |D(G)| = |X| + |A(G)| = |V| - 2|M| + |U|.$$

So U attains the minimum in (24.2), and moreover $o(G-U) = |D(G)|$, that is, $D(G)$ is the union of the odd components of $G-U$.

Case 2: $D(G)$ spans some edge $e = uv$. Let M and N be maximum-size matchings missing u and v, respectively. Then $M \cup N$ contains a path component P starting at u. If it does not end at v, then $P \cup \{e\}$ forms an N-augmenting path, contradicting the maximality of N. So P ends at v, and hence $P \cup \{e\}$ gives an M-blossom B.

Let $G' := G/B$ and $M' := M/B$ and let X' be the set of vertices of G' missed by M'. By Theorem 24.2, $|M'| = \nu(G')$. Then

(24.20) $$D(G') = (D(G) \setminus B) \cup \{B\},$$

since $B \in D(G')$ and since for each $v \in V \setminus B$:

(24.21) $v \in D(G') \iff G'$ has an even-length M'-alternating $X' - v$ path $\iff G$ has an even-length M-alternating $X - v$ path $\iff v \in D(G)$.

This proves (24.20), which implies that $A(G') = A(G)$ and $C(G') = C(G)$. By induction, $D(G')$ is the union of the odd components of $G' - U$. Hence $D(G)$ is the union of the odd components of $G - U$ (since $B \subseteq D(G)$ by (24.20)). Also by induction, $|M'| = \frac{1}{2}(|V'| + |U| - o(G' - U))$. Hence $|M| = \frac{1}{2}(|V| + |U| - o(G-U))$, since $|V| - 2|M| = |V'| - 2|M'|$. ∎

So $U = A(G)$ is the unique set attaining the minimum in (24.2) for which the union of the odd components of $G - U$ is inclusionwise minimal.

Note that:

(24.22) for any U attaining the minimum in (24.2), each maximum-size matching M has exactly $\lfloor \frac{1}{2}|K| \rfloor$ edges contained in any component K of $G - U$, and each edge of M intersecting U also intersects some odd component of $G - U$.

This implies the following. Call a graph $G = (V, E)$ *factor-critical* if $G - v$ has a perfect matching for each vertex v.

Corollary 24.7a. *Let $G = (V, E)$ be a graph. Then each component K of $G[D(G)]$ is factor-critical.*

Proof. Directly from Theorem 24.7 and (24.22): if $v \in K$, then $v \in D(G)$, and hence $G - v$ has a maximum-size matching M missing v. By (24.22), M has $\lfloor \frac{1}{2}|K| \rfloor$ edges contained in K. So $K - v$ has a perfect matching. ∎

The Edmonds-Gallai decomposition can be found in polynomial time, since the set $D(G)$ of vertices missed by at least one maximum-size matching can be determined in polynomial time (with the cardinality matching algorithm). In fact, with the alternating forest approach of Section 24.2a one can find the Edmonds-Gallai decomposition in time $O(n^3)$. If we have a maximum-size matching, it takes $O(n^2)$ time.

24.4c. Strengthening of Tutte's 1-factor theorem

Tutte's 1-factor theorem can be (self-)refined as follows (this theorem also can be derived from Theorem 24.7 and Corollary 24.7a; we give a direct derivation from Tutte's 1-factor theorem):

Theorem 24.8. *A graph $G = (V, E)$ has a perfect matching if and only if for each $U \subseteq V$, the graph $G - U$ has at most $|U|$ factor-critical components.*

Proof. Necessity is easy, since each factor-critical component is odd. To see sufficiency, let the condition be satisfied, and suppose that G has no perfect matching. By Tutte's 1-factor theorem, there is a subset U of V such that $G - U$ has more than $|U|$ odd components. Choose an inclusionwise maximal such set U.

By the condition, at least one component K of $G-U$ is not factor-critical. That is, K contains a vertex v such that $K - v$ has no perfect matching. Then by Tutte's 1-factor theorem, there exists a subset U' of $K - v$ such that $K - v - U'$ has more than $|U'|$ odd components, and hence at least $|U'|+2$ odd components (since $K - v$ has an even number of vertices). Now define $U'' := U \cup U' \cup \{v\}$. Then $G - U''$ has more than $|U''|$ odd components. As $U'' \supset U$, this contradicts the maximality of U. ∎

24.4d. Ear-decomposition of factor-critical graphs

As mentioned, a graph $G = (V, E)$ is *factor-critical* if, for each $v \in V$, the graph $G - v$ has a perfect matching. Lovász [1972b] showed that all factor-critical graphs can be constructed by 'odd ear-decompositions' in the following sense. We say that graph H arises by *adding an odd ear* from G, if H arises from G by adding an odd-length path at two (not necessarily distinct) vertices of G. That is, if there is a path or circuit $(v_0, v_1, \ldots, v_t)$ in H with t odd, $v_1, \ldots, v_{t-1}$ each having degree 2, and $G = H - \{v_1, \ldots, v_{t-1}\}$.

It is easy to see that if H arises by adding an odd ear to a factor-critical graph G, then H is again factor-critical. Now each factor-critical graph arises in this way from the one-vertex graph:

Theorem 24.9. *A graph G is factor-critical if and only if there exists a series of graphs $G_0, \ldots, G_k$ with G_0 being a one-vertex graph, $G_k = G$, and G_i arising by adding an odd ear to G_{i-1} ($i = 1, \ldots, k$).*

Proof. For sufficiency, see above. To see necessity, fix, for each vertex v of G, a perfect matching M_v of $G-v$. Choose a vertex u of G. Let H be a maximal subgraph of G such that

(24.23)
(i) H arises by a series of odd ear addings from the one-vertex graph on u;
(ii) for each edge $e \in M_u$, if e intersects VH, then $e \in EH$.

Such a graph trivially exists, as the one-vertex graph on u satisfies (24.23).

If $EH = EG$ we are done, so assume $EH \neq EG$. As G is factor-critical, G is connected, and hence there is an edge $e = vw \in EG \setminus EH$ with $v \in VH$. Consider $M_w \cup M_u$. One of its components is an even-length $w-u$ path $P = (v_1, \ldots, v_t)$ with $v_1 = w$ and $v_t = u$. So $v_t \in VH$. Let j be the smallest index with $v_j \in VH$. Then j is odd, since otherwise $v_{j-1}v_j \in M_u$ with $v_{j-1} \notin VH$ and $v_j \in VH$, contradicting (24.23)(ii).

Let Q be the path $(v, v_1, \ldots, v_j)$. Then $H \cup Q$ arises by adding an odd ear to H, and moreover, it satisfies (24.23)(ii) again, contradicting the maximality of H. ∎

(This is the original proof of Lovász [1972b].)

As a consequence we have a recursive characterization of factor-critical graphs:

Corollary 24.9a. *Let $G = (V, E)$ be a graph with $|V| \geq 2$. Then G is factor-critical if and only if G has an odd circuit C with G/C factor-critical.*

Proof. To see sufficiency, let C be an odd circuit with G/C factor-critical. We show that G is factor-critical. Choose $v \in V$. If $v \in C$, let M' be a perfect matching of $G[C \setminus \{v\}]$. Since G/C is factor-critical, $G - C$ has a perfect matching M''. Then $M \cup M''$ is a perfect matching of $G - v$.

If $v \notin C$, let M'' be a perfect matching of $(G/C) - v$. In G this gives a matching covering all vertices in $V \setminus (C \cup \{v\})$ and exactly one vertex, u say, in C. Let M' be a perfect matching in $G[C \setminus \{u\}]$. Then $M' \cup M''$ is a perfect matching of $G - v$. This shows sufficiency.

Necessity is shown with Theorem 24.9. Let G be factor-critical. Consider an odd ear-decomposition of G, and let C be the first odd ear. Then the remaining ears form an odd ear-decomposition of G/C, and hence G/C is factor-critical. ∎

(Related results were given by Cornuéjols and Pulleyblank [1983].)

24.4e. Ear-decomposition of matching-covered graphs

A graph $G = (V, E)$ is called *matching-covered* if each edge of G belongs to a perfect matching of G. Matching-covered graphs can be constructed similarly to factor-critical graphs, but now starting from an even circuit (however, the decomposition does not characterize matching-covered graphs). This will be used in proving Theorem 29.11 on the maximum size of a join.

Theorem 24.10. *For each connected matching-covered graph G with at least four vertices there exists a series of graphs $G_0, \ldots, G_k$ with G_0 being an even circuit, $G_k = G$, and G_i arising by adding an odd ear to G_{i-1} ($i = 1, \ldots, k$).*

Proof. For each edge e of G, fix a perfect matching M_e of G containing e. Fix a perfect matching M of G. One easily checks that G contains an M-alternating even circuit C. Let H be a maximal subgraph of G such that

(24.24) (i) H arises by a series of odd ear addings from C;
(ii) for each edge $e \in M$, if e intersects VH, then $e \in EH$.

Such a graph trivially exists, as C satisfies (24.24).

If $EH = EG$ we are done, so assume $EH \neq EG$. As G is connected, there is an edge $e \in EG \setminus EH$ intersecting VH. Consider $M_e \cup M$. Then the component of $M_e \cup M$ containing e gives an odd ear that can be added to H, contradicting the maximality of H. ∎

A direct algorithmic proof was given by Little and Rendl [1989]. Little [1974] showed that in a matching-covered graph, any two edges belong to a circuit that is in the symmetric difference of two perfect matchings. Carvalho, Lucchesi, and Murty [1999] gave more results on ear-decompositions of matching-covered graphs.

24.4f. Barriers in matching-covered graphs

A *barrier* in a graph $G = (V, E)$ is a subset B of V such that $G - B$ has $|B|$ odd components. Note that if B is a barrier in a connected matching-covered graph G, then B is a stable set and each component of $G - B$ is odd.

Lovász and Plummer [1975,1986] showed:

Theorem 24.11. *Let B and C be barriers in a connected matching-covered graph $G = (V, E)$ with $B \cap C \neq \emptyset$. Then $B \cap C$ and $B \cup C$ are barriers again.*

Proof. We first show:

(24.25) if B and C are distinct barriers with $B \cap C \neq \emptyset$, then there exists a nonempty set D with $D \subseteq B \setminus C$ or $D \subseteq C \setminus B$ such that $B \triangle D$ and $C \triangle D$ are barriers again.

As B and C are stable sets, there is a path from $B \cap C$ to $B \triangle C$. Consider a shortest such path, say it runs from $B \cap C$ to $C \setminus B$. It implies that $G - B$ has a component K with a neighbour in $B \cap C$ and intersecting $C \setminus B$. Define $D := K \cap C$. We show that $B \cup D$ and $C \setminus D$ are barriers again.

Fix an edge e connecting $B \cap C$ and K. Let L be the component of $G - C$ incident with e. Let M be a perfect matching containing e. As e connects $K \cap L$ and $B \cap C$, all other edges in M incident with K are contained in K. So if some edge $f \in M$ leaves $K \cap L'$ for some component L' of $G - C$, and $f \neq e$, then f does not leave K. Hence f leaves L', implying $L' \neq L$ (otherwise, L is left by two edges in M). It also implies that f connects $K \cap L'$ and $K \cap C$ and that f is the only edge in M leaving $K \cap L'$. Moreover, each vertex in D is covered by an edge in M, and hence it is such an edge f. Hence the number of components L' of $G - C$ with $K \cap L'$ odd is equal to $|D| + 1$.

Now $B \cup D$ is a barrier, since $G[K \setminus D]$ has $|D|+1$ odd components. So $G-(B \cup D)$ has at least $|B| + |D|$ odd components, and hence $B \cup D$ is a barrier.

Hence, as G is matching-covered, each component of $G - B - D$ is odd. So each component of $G[K \setminus D]$ is odd, and therefore $G[K \setminus D]$ has exactly $|D| + 1$ components. So all but at most $|D| + 1$ components of $G - C$ are also components of $G - (C \setminus D)$. Hence the number of odd components of $G - (C \setminus D)$ is at least $|C| - |D| - 1$, and hence, by parity, at least $|C \setminus D|$. So $C \setminus D$ is a barrier. This proves (24.25).

Now to prove that $B \cup C$ is a barrier, we can assume that we have chosen B and C inclusionwise maximal barriers contained in $B \cup C$. Then $B = C$ by (24.25).

Similarly, to prove that $B \cap C$ is a barrier, we can assume that we have chosen B and C inclusionwise minimal barriers containing $B \cap C$. Again we have $B = C$ by (24.25). ∎

This has the following consequence due to Lovász [1972e] (cf. Kotzig [1960] (Theorem 31)):

Corollary 24.11a. *Any two distinct maximal barriers in a connected matching-covered graph are disjoint.*

Proof. Directly from Theorem 24.11. ∎

Since each singleton is a barrier, Corollary 24.11a implies that the maximal barriers in a connected matching-covered graph partition the vertex set of G. This gives the result of Kotzig [1959b] (Theorem 11):

Corollary 24.11b. *Let $G = (V, E)$ be a connected matching-covered graph. For $u, v \in V$ define $u \sim v$ by:*

(24.26) $u \sim v$ *if and only if* $G - u - v$ *has no perfect matching.*

Then $\sim$ is an equivalence relation.

Proof. Note that $u \sim v$ if and only if $\{u, v\}$ is contained in some barrier. So the corollary follows directly from Corollary 24.11a. ∎

For much more on barriers in matching-covered graphs, see Lovász and Plummer [1986].

24.4g. Two-processor scheduling

The following problem was considered by Fujii, Kasami, and Ninomiya [1969]. Suppose that we have to carry out certain jobs, where some of the jobs have to be done before other. We can represent this by a partially ordered set $(V, \leq)$ where V is the set of jobs and $x < y$ indicates that job x has to be done before job y. Each job takes one time-unit, say one hour.

Suppose now that there are two workers, each of which can do one job at a time. Alternatively, suppose that you have one machine, that can do at each moment two jobs simultaneously (a *two-processor*).

We wish to do all jobs within a minimum total time span. This problem can be solved with the matching algorithm as follows. Make a graph $G = (V, E)$, with vertex set V (the set of jobs) and with edge set

(24.27) $E := \{\{u, v\} \mid u \not\leq v \text{ and } v \not\leq u\}$.

(So (V, E) is the complementary graph of the 'comparability graph' associated with $(V, \leq)$.)

Consider now a possible schedule of the jobs. That is, we have a sequence $p_1, \ldots, p_t$, where each p_i is either a singleton vertex or an edge of G such that $p_1, \ldots, p_t$ partition V and such that if $u < v$ and $u \in p_i$ and $v \in p_j$, then $i < j$.

Now the pairs in this list should form a matching M in G. Hence $t = |V| - |M|$. In particular, t cannot be smaller than $|V| - \nu(G)$, where $\nu(G)$ is the matching number of G.

Fujii, Kasami, and Ninomiya [1969] showed that in fact one can always make a schedule with $t = |V| - \nu(G)$. For that it is sufficient to show:

Theorem 24.12. *G contains a maximum-size matching $M = \{e_1, \ldots, e_t\}$ such that if $u \in e_i$ and $v \in e_j$ with $u < v$, then $i < j$.*

Proof. The proof is by induction on $|V|$. Let M be a maximum-size matching in G. We may assume that M is a perfect matching, since otherwise we can delete all vertices missed by M, and apply induction.

Let $V^{\min}$ be the set of minimal elements of $(V \leq)$. If $V^{\min}$ contains an edge $uv \in M$ as a subset, we can delete u and v from V, and apply induction. So we may assume that each $s \in V^{\min}$ is contained in an edge $st \in M$ with $t \notin V^{\min}$. Choose an edge $st \in M$ with $s \in V^{\min}$ and with the height of t as small as possible. (The *height* of an element t is the maximum size of a chain in $(V, \leq)$ with maximum element t.) As $t \notin V^{\min}$ there exists an $s't' \in M$ with $s' \in V^{\min}$ and $s' < t$.

Now clearly ss' is an edge of G, as s and s' are minimal elements. Moreover, tt' is an edge of G. For if $t < t'$, then $s' < t < t'$, contradicting the fact that $s't' \in E$; and if $t' < t$, then t' would have smaller height than t.

So replacing st and $s't'$ in M by ss' and tt', we have $ss' \subseteq V^{\min}$, and so by deleting s and s' from V we can apply induction as before. ∎

The theorem implies that there is a linear extension $\preceq$ of $\leq$ and a maximum-size matching M in G such that if $uv \in M$, then u and v are neighbouring in $\preceq$.

Coffman and Graham [1972] gave a direct, $O(n^2)$-time algorithm. (Muntz and Coffman [1969] gave an algorithm for the two-processor scheduling problem if jobs may be interrupted and continued later.) This was improved to $O(m + n\alpha(m, n))$ by Gabow [1982] and to $O(m + n)$ by Gabow and Tarjan [1983,1985].

24.4h. The Tutte matrix and an algebraic matching algorithm

Tutte [1947b] observed the following. Let $G = (V, E)$ be a graph. Choose for each edge e an indeterminate x_e. Let M be a skew-symmetric[3] $V \times V$ matrix with $M_{u,v} = \pm x_e$ if $e = \{u, v\} \in E$, and $M_{u,v} = 0$ otherwise (including $u = v$) (the *Tutte matrix*). Then the rank of M is equal to twice the matching number of G.

Lovász [1979c] showed that substituting random integers for the x_e, gives an efficient randomized algorithm for finding the matching number of G. This idea was extended by Geelen [2000], who proved the following:

(24.28) Let M' arise from M by substituting the x_e by integers from $\{1, \ldots, n\}$, where $n := |V|$. If $\operatorname{rank}(M') < \operatorname{rank}(M)$, then there is an edge e of G and a number $b \in \{1, \ldots, n\}$ such that for the matrix M'' arising from

[3] A matrix M is *skew-symmetric* if $M^{\mathsf{T}} = -M$.

M' by resetting the $\pm x_e$ entries to $\pm b$, we have $\text{rank}(M'') > \text{rank}(M')$, or $\text{rank}(M'') = \text{rank}(M')$ and $D(M'') \supset D(M')$.

Here $D(A)$ denotes the set of $v \in V$ such that the $V \setminus \{v\} \times V \setminus \{v\}$ submatrix of A has the same rank as A.

(24.28) implies a polynomial-time algorithm to compute the matching number of G (and hence to find a maximum-size matching in G): start with an arbitrary matrix M' obtained by substituting the x_e by numbers in $\{1, \ldots, n\}$, and iteratively try to reset an entry to another number from $\{1, \ldots, n\}$, as long as it either increases the rank of M', or maintains the rank and increases $D(M')$. The final matrix has rank equal to the matching number of G.

L. Lovász (cf. Geelen [1995]) extended Tutte's result to the rank of any (not necessarily principal) submatrix of M. Geelen [1995] described the corresponding system of linear inequalities and proved its total dual integrality, generalizing Edmonds' matching polytope theorem.

24.4i. Further notes

Biedl, Bose, Demaine, and Lubiw [1999,2001] gave an $O(n \log^4 n)$ time algorithm to find a perfect matching in cubic bridgeless graphs (linear-time if the graph is moreover planar). Biedl [2001] gave a linear-time reduction of the general matching problem to the matching problem for cubic graphs.

Lower bounds on the maximum size of a matching were given by Nishizeki and Baybars [1979] for planar graphs and by Biedl, Demaine, Duncan, Fleischer, and Kobourov [2001] for several other classes of graphs.

Fulkerson, Hoffman, and McAndrew [1965] showed that any regular graph with an even number of vertices and with the property that each two vertex-disjoint odd circuits are connected by an edge, has a perfect matching (cf. Mahmoodian [1977], Berge [1978b,1981]). Other sufficient conditions were given by Anderson [1972], Sumner [1974a], Las Vergnas [1975a], and Chartrand, Goldsmith, and Schuster [1979].

Plesník [1972] showed that in a k-regular $(k-1)$-edge-connected graph with an even number of vertices, there is a perfect matching not containing $k-1$ prescribed edges (cf. Chartrand and Nebeský [1979]). For $k = 3$ this was proved by Schönberger [1934]. For general k, it can also be derived from Edmonds' perfect matching polytope theorem (Theorem 25.1 below). See also Plesník [1979].

Further studies on the structure of *matching-covered graphs* (graphs in which each edge belongs to a perfect matching) were made by Kotzig [1959a,1959b,1960], Hetyei [1964], Lovász [1970d,1972f,1972d,1972e,1983a], Little, Grant, and Holton [1975], Lovász and Plummer [1975], Gabow [1979], Edmonds, Lovász, and Pulleyblank [1982], Naddef [1982], and Szigeti [1998b].

Gabow, Kaplan, and Tarjan [1999,2001] gave fast algorithms to test if a given perfect matching is unique, to find it, and if it not unique to find another perfect matching.

Sumner [1974b,1976] studied sets U with $o(G-U) > |U|$. Weinstein [1963,1974] and Bollobás and Eldridge [1976] related the matching number to the minimum and maximum degree and the connectivity. Chvátal and Hanson [1976] evaluated the maximum number $f(n, b, d)$ of edges of a graph with n vertices having no vertex of degree $> d$ and no matching of size $> b$.

Implementing cardinality matching algorithms were studied by Burkard and Derigs [1980], Crocker [1993], and Mattingly and Ritchey [1993]. A simulated annealing approach was described by Sasaki and Hajek [1988].

Books covering nonbipartite matching algorithms include Christofides [1975], Lawler [1976b], Minieka [1978], Papadimitriou and Steiglitz [1982], Sysło, Deo, and Kowalik [1983], Tarjan [1983], Gondran and Minoux [1984], Derigs [1988a], Nemhauser and Wolsey [1988], Cook, Cunningham, Pulleyblank, and Schrijver [1998], Jungnickel [1999], Mehlhorn and Näher [1999], and Korte and Vygen [2000]. Surveys on matching algorithms were given by Galil [1983,1986a,1986b].

Motwani [1989,1994] investigated the expected running time of matching algorithms.

Gallai [1950], Tutte [1950], Kaluza [1953], Steffens [1977], and Aharoni [1984a, 1984c,1984d,1988] gave extensions to infinite graphs. The Edmonds-Gallai decomposition was extended to locally finite graphs by Bry and Las Vergnas [1982] (cf. Steffens [1985]).

The behaviour of a greedy heuristic for finding a large matching was investigated by Dyer and Frieze [1991], Dyer, Frieze, and Pittel [1993], and Aronson, Dyer, Frieze, and Suen [1994].

The standard work on matching theory is Lovász and Plummer [1986]. Other books discussing nonbipartite matching include Berge [1973b], Bondy and Murty [1976], Bollobás [1978,1979], Tutte [1984], and Diestel [1997]. Survey articles on matchings were given by Akiyama and Kano [1985b] and Lovász and Plummer [1986], Gerards [1995a], Pulleyblank [1995], and Cunningham [2002].

24.4j. Historical notes on nonbipartite matching

Petersen and Sylvester

Petersen [1891] was among the first to study perfect matchings (1-factors) in graphs, introducing several basic concepts and methods, like factors and alternating paths. He was motivated by finite basis theorems in invariant theory, especially by the question which polynomials form a finite basis. Petersen cooperated with J.J. Sylvester, who did similar studies, leading to an intensive correspondence on the topic in the years 1889-1890 — see Sabidussi [1992] (unfortunately, the letters of Petersen to Sylvester were not found).

In particular, they considered homogeneous polynomials of the form

$$\prod_{i<j}(x_i - x_j)^{r_{i,j}}, \tag{24.29}$$

and were interested in conditions under which such a polynomial can be factorized into other homogeneous polynomials of the same form. This is equivalent to characterizing the existence of k-factors in regular graphs. (Graph terms like 'factor' and 'degree' introduced by Petersen are motivated by this interpretation.)

In a letter of 18 October 1889, Sylvester expressed to Petersen the conjecture that each graph of minimum degree at least two has a 2-factor. He had checked it for graphs with up to 7 vertices, and said that he had 'not much doubt of being able to establish the proof for all values of n by the same process which has been successful for the earlier numbers'. Sylvester considered this as the most important theorem discovered hitherto in the science of chemical graphology, a field initiated by Sylvester [1878].

Two days later, Sylvester wrote a letter in which he restricted his conjecture to the case of regular graphs, and he was more doubtful on whether it is true. After a reply of Petersen, Sylvester gave in a letter of 27 October 1889 an example of a graph with 7 vertices, with degrees 2 and 3, not having a 2-factor. In this letter, Sylvester also remarked that as a consequence of his conjecture, each regular graph of odd order has a 2-factorization.

Then, in a letter of 8 November 1889, Sylvester observed that there is a cubic graph on 10 vertices that has no factorization (Figure 24.3). (A graph is *cubic* if it

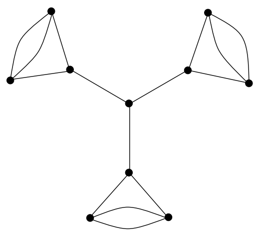


Figure 24.3
Sylvester's graph

is 3-regular.)

Subsequently, on 16 November 1889, Sylvester wrote to Petersen:

> Thanks for your interesting note—I also have a proof of the 'theorem of Ablation' for even equifrequencies.

Apparently, Petersen had written about his theorem that each regular graph of even degree has a 2-factorization, for which Sylvester also said to have a proof.

Next follows correspondence on the proofs the two have, with a lot of mutual misunderstanding. However, after hearing Petersen's proof at a visit of Petersen to Sylvester, at the end of December 1889, Sylvester became convinced of the correctness of Petersen's proof, and found it 'a very beautiful method'. On the other hand, Petersen remained very sceptical about Sylvester's proof, which Sylvester said was by induction on the number of vertices. They decided to publish their proofs separately. However, Sylvester did not publish on the topic; Petersen's proof appeared in the paper Petersen [1891].

Petersen's 1891 paper

In this paper, Petersen first observed that Gordan's finite basis theorem implies that for each n there exists a finite set $\mathcal{G}$ of regular graphs on n vertices (of nonzero degree) with the property that each regular graph on n vertices contains at least one graph in $\mathcal{G}$ as spanning subgraph (factor). (This result can also be proved by elementary means.) Petersen next puts as his goal to characterize all *primitive* graphs, that is, all regular graphs that have no other factors than itself and the 0-regular subgraph.

First, Petersen observed that a 2-regular graph is primitive if and only if at least one of its components is odd. Next, he showed that each 4-regular graph has a 2-factor. To this end, he made an Eulerian tour along all edges, colouring them alternatingly blue and red. The blue edges then form a 2-factor. He observed that similarly one can show more generally that each $2k$-regular graph with an even number of edges has a k-factor.

Next, Petersen showed that each $2k$-regular graph has a 2-factorization. His proof is by observing that the existence of a 2-factorization is invariant under replacing any two disjoint edges ab and cd by ac and bd (by using the result that each 4-regular graph has a 2-factorization).

This solves the factorization problem for k-regular graphs with k even. Petersen next considered the case of odd k. He gave an example of primitive k-regular graphs for arbitrary odd k. He showed that each k-regular graph on n vertices with $k > \frac{1}{2}n + 1$ has a perfect matching. To this end, he considered a matching M and observed that

(24.30) M has maximum size if and only if there is no M-augmenting path.

To formulate this, Petersen coloured the edges in M red, and all other edges blue. A *Wechselweg* (alternating path) is a path coloured alternatingly red and blue. Let $2n$ be the number of vertices of the graph and let α be the size of the matching (thus it misses $2n - 2\alpha$ vertices). Then:

> Wir sahen oben, dass α grösser gemacht werden konnte, wenn wir zwischen zwei von den $2n - 2\alpha$ Punkten einen Wechselweg *cabd* finden konnten; dasselbe gilt wenn wir zwischen zwei von den $2n - 2\alpha$ Punkten überhaupt einen Wechselweg finden können, denn verändert man die Farben der Seiten eines solches Weges, so wird die Anzahl der rothen Linien um eins vergrössert. Man beweist leicht, dass diese Bedingung auch notwendig is.[4]

This brought Petersen to propose an algorithm to find a 1-factor:

> Indem wir die α Linien aufs Geradewohl ausnehmen und dann mittelst Wechselwege α zu vergrössern suchen, können wir untersuchen, ob ein gegebener *graph* primitiv ist oder nicht;[5]

Petersen however preferred a direct characterization:

> es entsteht aber die Frage, ob die primitiven *graphs* sich nicht durch einfache Kennzeichen von den zerlegbaren scheiden.[6]

He conjectured:

[4] We saw above that α can be increased, if we could find an alternating path *cabd* between two of the $2n-2\alpha$ points; the same holds if we can find an alternating path at all between two of the $2n-2\alpha$ points, because if one changes the colours of the edges in such a path, then the number of red edges increases by one. One easily proves that this condition is also necessary.

[5] While we select the α edges arbitrarily and then try to increase α by alternating paths, we can investigate if a given *graph* is primitive or not;

[6] the question however arises if the primitive *graphs* are not distinguished from the factorizable by simple characteristics.

> Es spricht etwas dafür, dass ein primitiver *graph Blätter* haben muss, indem ein Blatt ein solcher Theil des *graphs* ist, der nur durch eine einzelne Linie mit dem übrigen Theil in Verbindung steht. Ich habe daher versucht dieses zu beweisen, habe aber die Schwierigkeiten so gross gefunden, dass ich die Untersuchung auf den *graph* dritten Grades beschränkt habe.[7]

Petersen [1891] described the cubic graph on 10 vertices found by Sylvester that has no 1-factor (Figure 24.3), which he called *Sylvester's graph*.

Petersen showed that each primitive cubic graph has at least three leaves. As mentioned, a *leaf* is a subset U of the vertices with $|\delta(U)| = 1$. (A graph is *cubic* if it is 3-regular.)

Again, Petersen showed his theorem with the help of studying alternating paths. Those edges that can be traversed in both directions by alternating paths starting at a 'free' vertex are called 'zweipfeilig' (two-arrow as adjective). He then reduced the problem by shrinking and stated:

> Wir ziehen jetzt jedes zweipfeiliges System in einen Punkt zusammen;[8]

Proofs and extensions of Petersen's theorem

Brahana [1917] gave a shorter proof of Petersen's theorem. He restricted the concept of leaf to a *minimal* set of vertices connected by only one edge to the remainder of the graph. (In fact, also Petersen's proof is valid for this restricted interpretation of leaf.)

Brahana's method is again based on augmenting paths and shrinking. Moreover, he used a reduction to smaller graphs by deleting two adjacent vertices u and v and connecting the two further vertices adjacent to u and v by new edges. This can be done in such a way that the number of leaves remains at most 2.

In fact, part of Brahana's method is algorithmic, and can be considered as a specialization of Edmonds' cardinality matching algorithm. Brahana needs to find a 1-factor, given a matching M of size $\frac{1}{2}n - 1$ (where n is the number of vertices). He described a depth-first method to find an M-augmenting path starting from a vertex missed by M. If it runs into a loop (a 'bicursal circuit'), it can be removed by shrinking:

> We continue this shrinking process as long as there are such bicursal circuits.

Also Errera [1921,1922], Frink [1925], Schönberger [1934], Kőnig [1936], and Baebler [1954] gave alternative proofs of Petersen's theorem (see also Sainte-Laguë [1926b]). The proof of Frink is 'by induction, no shrinking or counting processes being used.' He overlooked however some complications (in relation to the construction of a new 2-connected graph in the proof of his 'Theorem II') — they were resolved by Kőnig [1936]. The proof yields a polynomial-time algorithm to find a perfect matching in a 2-connected cubic graph.

Schönberger [1934] showed that in any 2-connected cubic graph each edge is in a perfect matching, and (more generally) for any two prescribed edges there is a perfect matching not containing these edges.

---

[7] Something speaks for it that a primitive *graph* must have *leaves*, while a leaf is such a part of the *graph* that is in connection with the remaining part only by one single edge. I therefore have tried to prove this, but have found the difficulties that big, that I have restricted the investigation to the *graph* of third degree.

[8] We now contract each two-arrow system to one point;

Baebler [1937] showed that each k-regular l-edge-connected graph, with k odd and l even, has an l-factor. His proof is based on shrinking.

Tutte

Tutte [1947b] characterized the graphs that have a perfect matching. His proof is essentially that given in Section 24.1a, defining a graph to be 'hyperprime' if it has no perfect matching, but adding any edge creates a perfect matching. He used 'pfaffians' in order to show that, in a hyperprime graph, each component of the subgraph induced by the set of vertices that are not adjacent to all other vertices, is complete. A combinatorial proof of this fact was given by Maunsell [1952].

Tutte's theorem was extended to arbitrary l-factors ($l \in \mathbb{Z}_+$) by Belck [1950] (see Chapter 33); the proof is by extension of Tutte's method. This in turn was generalized by Tutte [1952] to b-factors where $b \in \mathbb{Z}_+^V$. As an 'allied problem', Tutte [1952] considered perfect b-matchings, that is, functions $f \in \mathbb{Z}_+^E$ with $f(\delta(v)) = b(v)$ for each vertex v. The proof is by reduction to the b-factor case, by replacing each edge by several parallel edges.

Then in Tutte [1954b] it is realized that the b-factor and b-matching theorems can be reduced to the case $b = \mathbf{1}$ by splitting vertices and by the construction given in the proof of Theorem 32.1.

Gallai [1950] gave a short proof of Tutte's 1-factor theorem. He showed the following. Let G be a graph without a perfect matching, let M be a maximum-size matching in G, and let v be a vertex missed by M. Let U be the set of vertices u for which there is an M-alternating $v-u$ path of odd length. Then $G-U$ has more than $|U|$ odd components. Gallai [1950] also gave several characterizations for the existence of l-factors in regular graphs, and he considered the infinite case.

Also Tutte [1950] and Kaluza [1953] gave extensions to the infinite case. The main theorem of Ore [1957] is an alternative characterization of the existence of a b-factor. Berge [1958a] extended Tutte's 1-factor theorem to a min-max relation for the maximum size of a matching, the Tutte-Berge formula.

Kotzig [1959a,1959b,1960] studied the structure of matching-covered graphs, leading to a decomposition of any graph (cf. Ore [1959]).

Augmenting paths

Like Petersen, Berge [1957] observed that a matching M is maximum if and only if there is no M-augmenting path, and he suggested the following procedure for solving the cardinality matching problem:

> Construct a maximal matching V, and determine whether there exists an alternating chain W connecting two neutral points. (The procedure is known.) If such a chain exists, change V into $(V \setminus W) \cup (W \setminus V)$, and look again for a new alternating chain; if such a chain does not exist, V is maximum.

In Berge [1958b], a depth-first search approach to finding an augmenting path is sketched, however without shrinking, and not leading to a polynomial-time algorithm.

Also Norman and Rabin [1958,1959] found the augmenting path criterion for maximality of a matching (and similarly, for minimality of an edge cover):

> These results immediately lead to algorithms for a minimum cover and a maximum matching respectively.

Edmonds [1962] and Ray-Chaudhuri [1963] extended the augmenting path criterion to arbitrary hypergraphs.

Edmonds

Edmonds observed that Berge's proposal for finding an augmenting path (quoted above) does not lead to a polynomial-time algorithm. In his personal recollections, Edmonds [1991] stated:

> It is really hard for anyone to see that it isn't easy that when you've got a matching in a graph and you are starting at a deficient node, that you cannot just grow a tree looking for a Berge augmenting path.

Edmonds [1965d] argued:

> Berge proposed searching for augmenting paths as an algorithm for maximum matching. In fact, he proposed to trace out an alternating path from an exposed vertex until it must stop and, then, if it is not augmenting, to back up a little and try again, thereby exhausting possibilities.
> His idea is an important improvement over the completely naive algorithm. However, depending on what further directions are given, the task can still be one of exponential order, requiring an equally large memory to know when it is done.

In the summer of 1963, at a Workshop at the RAND Corporation, Edmonds discovered that shrinking leads to a polynomial-time algorithm to find a maximum-size matching in any graph. The result was described in the paper Edmonds [1965d] (received 22 November 1963), in which paper he also described his views on algorithms and complexity:

> For practical purposes computational details are vital. However, my purpose is only to show as attractively as I can that there is an efficient algorithm. According to the dictionary, "efficient" means "adequate in operation or performance". This is roughly the meaning I want — in the sense that it is conceivable for maximum matching to have no efficient algorithm. Perhaps a better word is "good".
> I am claiming, as a mathematical result, the existence of a *good* algorithm for finding a maximum cardinality matching in a graph.
> There is an obvious finite algorithm, but that algorithm increases in difficulty exponentially with the size of the graph. It is by no means obvious whether *or not* there exists an algorithm whose difficulty increases only algebraically with the size of the graph.

Moreover:

> For practical purposes the difference between algebraic and exponential order is often more crucial than the difference between finite and non-finite.

Edmonds described his algorithm, in terms of paths, trees, flowers, and blossoms, and concluded that the 'order of difficulty' is n^4 (more precisely, it is $O(n^2m)$).

In this paper, Edmonds also introduced the decomposition of any graph which is now called the Edmonds-Gallai decomposition. Also in 1963, Gallai submitted a paper (Gallai [1963a]), in which this decomposition is described implicitly, which was made more explicit in Gallai [1964].

In the Proceedings of the IBM Scientific Computing Symposium on Combinatorial Problems in March 1964 in Yorktown Heights, New York, at the end of Gomory [1966], the following discussion is reported:

J. Edmonds: I have a comment on the polyhedral approach to complete analysis, supplementing Professor Kuhn's remarks. I do not believe there is any reason for taking as a measure of the algorithmic difficulty of a class of combinatorial extremum problems the number of faces in the associated polyhedra. For example, consider the generalization of the assignment problem from bipartite graphs to arbitrary graphs. Unlike the case of bipartite graphs, the number of faces in the associated polyhedron increases exponentially with the size of the graph. On the other hand, there is an algorithm for this generalized assignment problem which has an upper bound on the work involved just as good as the upper bound for the bipartite assignment problem.
H.W. Kuhn: I could not agree with you more. That is shown by the unreasonable effectiveness of the Norman-Rabin scheme for solving this problem. Their result is unreasonable only in the sense that the number of faces of the polyhedron suggests that it ought to be a harder problem than it actually turned out to be. It is not impossible that some day we will have a practical combinatorial algorithm for this problem.
J. Edmonds: Actually, the amount of work in carrying out the Norman-Rabin scheme generally increases exponentially with the size of the graph.
The algorithm I had in mind is one I introduced in a paper submitted to the Canadian Journal of Mathematics (see Edmonds, 1965). This algorithm depends crucially on what amounts to knowing all the bounding inequalities of the associated convex polyhedron—and, as I said, there are many of them. The point is that the inequalities are known by an easily verifiable characterization rather than by exhaustive listing—so their number is not important.

Chapter 25

The matching polytope

As a by-product of his weighted matching algorithm (to be discussed in Chapter 26), Edmonds obtained a characterization of the matching polytope in terms of defining inequalities. It forms the first class of polytopes whose characterization does not simply follow just from total unimodularity, and its description was a breakthrough in polyhedral combinatorics.

25.1. The perfect matching polytope

The *perfect matching polytope* of a graph $G = (V, E)$ is the convex hull of the incidence vectors of the perfect matchings in G. It is denoted by $P_{\text{perfect matching}}(G)$:

(25.1) $$P_{\text{perfect matching}}(G) = \text{conv.hull}\{\chi^M \mid M \text{ perfect matching in } G\}.$$

So $P_{\text{perfect matching}}(G)$ is a polytope in $\mathbb{R}^E$.

Consider the following set of linear inequalities for $x \in \mathbb{R}^E$:

(25.2)
(i) $x_e \geq 0$ for each $e \in E$,
(ii) $x(\delta(v)) = 1$ for each $v \in V$,
(iii) $x(\delta(U)) \geq 1$ for each $U \subseteq V$ with $|U|$ odd.

In Section 18.1 we saw that if G is bipartite, the perfect matching polytope is fully determined by the inequalities (25.2)(i) and (ii). These inequalities are not enough for, say, K_3: taking $x_e := \frac{1}{2}$ for each edge e of K_3 gives a vector x satisfying (25.2)(i) and (ii) but not belonging to the perfect matching polytope of K_3 (as it is empty).

Edmonds [1965b] showed that for general graphs, adding (25.2)(iii) is enough. It is clear that for any perfect matching M in G, the incidence vector χ^M satisfies (25.2). So $P_{\text{perfect matching}}(G)$ is contained in the polytope determined by (25.2). The essence of Edmonds' theorem is that one needs no more inequalities.

Theorem 25.1 (Edmonds' perfect matching polytope theorem)**.** *The perfect matching polytope of any graph $G = (V, E)$ is determined by* (25.2).

Proof. Clearly, the perfect matching polytope is contained in the polytope Q determined by (25.2). Suppose that the converse inclusion does not hold. So we can choose a vertex x of Q that is not in the perfect matching polytope.

We may assume that we have chosen this counterexample such that $|V|+|E|$ is as small as possible. Hence $0 < x_e < 1$ for all $e \in E$ (otherwise, if $x_e = 0$, we can delete e, and if $x_e = 1$, we can delete e and its ends). So each degree of G is at least 2, and hence $|E| \geq |V|$. If $|E| = |V|$, each degree is 2, in which case the theorem is trivially true. So $|E| > |V|$. Note also that $|V|$ is even, since otherwise $Q = \emptyset$ (consider $U := V$ in (25.2)(iii)).

As x is a vertex, there are $|E|$ linearly independent constraints among (25.2) satisfied with equality. Since $|E| > |V|$, there is an odd subset U of V with $3 \leq |U| \leq |V| - 3$ and $x(\delta(U)) = 1$.

Consider the projections x' and x'' of x to the edge sets of the graphs $G/\overline{U}$ and G/U, respectively (where $\overline{U} := V \setminus U$). Here we keep parallel edges.

Then x' and x'' satisfy (25.2) for $G/\overline{U}$ and G/U, respectively, and hence belong to the perfect matching polytopes of $G/\overline{U}$ and G/U, by the minimality of $|V| + |E|$.

So $G/\overline{U}$ has perfect matchings $M'_1, \ldots, M'_k$ and G/U has perfect matchings $M''_1, \ldots, M''_k$ with

$$(25.3) \qquad x' = \frac{1}{k}\sum_{i=1}^{k} \chi^{M'_i} \text{ and } x'' = \frac{1}{k}\sum_{i=1}^{k} \chi^{M''_i}.$$

(Note that x is rational as it is a vertex of Q.)

Now for each $e \in \delta(U)$, the number of i with $e \in M'_i$ is equal to $kx'(e) = kx(e) = kx''(e)$, which is equal to the number of i with $e \in M''_i$. Hence we can assume that, for each $i = 1, \ldots, k$, M'_i and M''_i have an edge in $\delta(U)$ in common. So $M_i := M'_i \cup M''_i$ is a perfect matching of G. Then

$$(25.4) \qquad x = \frac{1}{k}\sum_{i=1}^{k} \chi^{M_i}.$$

Hence x belongs to the perfect matching polytope of G. ∎

Notes. This proof was given by Aráoz, Cunningham, Edmonds, and Green-Krótki [1983] and Schrijver [1983c], with ideas of Seymour [1979a]. For other proofs, see Balinski [1972], Hoffman and Oppenheim [1978], and Lovász [1979b]. A proof can also be derived from Edmonds' weighted matching algorithm (Chapter 26).

25.2. The matching polytope

The characterization of the perfect matching polytope implies Edmonds' matching polytope theorem. It characterizes the *matching polytope* of a graph $G = (V, E)$, denoted by $P_{\text{matching}}(G)$, which is the convex hull of the incidence vectors of the matchings in G:

(25.5) $$P_{\text{matching}}(G) = \text{conv.hull}\{\chi^M \mid M \text{ matching in } G\}.$$

Again, $P_{\text{matching}}(G)$ is a polytope in $\mathbb{R}^E$.

Corollary 25.1a (Edmonds' matching polytope theorem)**.** *For any graph $G = (V, E)$, the matching polytope is determined by:*

(25.6)
$$\begin{array}{rll} \text{(i)} & x_e \geq 0 & \text{for each } e \in E, \\ \text{(ii)} & x(\delta(v)) \leq 1 & \text{for each } v \in V, \\ \text{(iii)} & x(E[U]) \leq \lfloor \frac{1}{2}|U| \rfloor & \text{for each } U \subseteq V \text{ with } |U| \text{ odd.} \end{array}$$

Proof. Clearly, each vector x in the matching polytope satisfies (25.6). To see that the inequalities (25.6) are enough, let x satisfy (25.6). Make a copy $G' = (V', E')$ of G, and add edges vv' for each vertex $v \in V$, where v' is the copy of v in V'. This makes the graph $\widetilde{G} = (\widetilde{V}, \widetilde{E})$.

Define $\tilde{x}_e := \tilde{x}_{e'} := x_e$ for each $e \in E$, where e' is the copy of e in E', and $\tilde{x}(vv') := 1 - x(\delta(v))$ for each $v \in V$. Then by Theorem 25.1, $\tilde{x}$ belongs to the perfect matching polytope of $\widetilde{G}$, since $\tilde{x}$ satisfies (25.2) with respect to $\widetilde{G}$.

Indeed, for each $v \in V$ one has $\tilde{x}(\tilde{\delta}(v)) = \tilde{x}(\tilde{\delta}(v')) = 1$ (where $\tilde{\delta} := \delta_{\widetilde{G}}$). Moreover, consider any odd subset U of $\widetilde{V} = V \cup V'$, say $U = W \cup X'$ with $W, X \subseteq V$. Then $\tilde{x}(\tilde{\delta}(U)) \geq \tilde{x}(\tilde{\delta}(W \setminus X)) + \tilde{x}(\tilde{\delta}(X' \setminus W'))$. So we may assume that $W \cap X = \emptyset$, and by symmetry we may assume that W is odd, and hence that $X = \emptyset$. So it suffices to show that for any odd $U \subseteq V$ one has $\tilde{x}(\tilde{\delta}(U)) \geq 1$. Now

(25.7) $$\tilde{x}(\tilde{\delta}(U)) + 2\tilde{x}(\widetilde{E}[U]) = \sum_{v \in U} \tilde{x}(\tilde{\delta}(v)) = |U|,$$

and hence

(25.8) $$\tilde{x}(\tilde{\delta}(U)) = |U| - 2\tilde{x}(\widetilde{E}[U]) \geq |U| - 2\lfloor \tfrac{1}{2}|U| \rfloor = 1.$$

So by Theorem 25.1, $\tilde{x}$ belongs to the perfect matching polytope of $\widetilde{G}$, and hence x belongs to the matching polytope of G. ∎

25.3. Total dual integrality: the Cunningham-Marsh formula

With linear programming duality one can derive from Corollary 25.1a a min-max relation for the maximum weight of a matching:

Corollary 25.1b. *Let $G = (V, E)$ be a graph and let $w \in \mathbb{R}_+^E$ be a weight function. Then the maximum weight of a matching is equal to the minimum value of*

(25.9) $$\sum_{v \in V} y_v + \sum_{U \in \mathcal{P}_{\text{odd}}(V)} z_U \lfloor \tfrac{1}{2}|U| \rfloor,$$

where $y \in \mathbb{R}_+^V$ *and* $z \in \mathbb{R}_+^{\mathcal{P}_{\text{odd}}(V)}$ *satisfy*

(25.10) $$\sum_{v \in V} y_v \chi^{\delta(v)} + \sum_{U \in \mathcal{P}_{\text{odd}}(V)} z_U \chi^{E[U]} \geq w.$$

Proof. Directly with LP-duality from Corollary 25.1a. ∎

The constraints (25.6) determining the matching polytope in fact are totally dual integral, as was shown by Cunningham and Marsh [1978]. This implies that a stronger min-max relation holds than obtained by linear programming duality from the matching polytope inequalities: if w is integer-valued, then in Corollary 25.1b we can restrict y and z to integer vectors:

Theorem 25.2 (Cunningham-Marsh formula)**.** *In Corollary* 25.1b*, if w is integer, we can take y and z integer. We can take z moreover such that the collection $\{U \in \mathcal{P}_{\text{odd}}(V) \mid z_U > 0\}$ is laminar.*[9]

Proof. We prove the theorem by induction on $|E| + w(E)$. If $w(e) = 0$ for some $e \in E$, we can delete e and apply induction. So we may assume that $w(e) \geq 1$ for each $e \in E$.

First assume that there exists a vertex u of G covered by every maximum-weight matching. Let $w' := w - \chi^{\delta(u)}$. By induction, there exist integer y'_v, z'_U that are optimum with respect to w'. Now increasing y'_u by 1, gives y_v, z_U as required for w, since the maximum of $w'(M)$ over all matchings M is strictly less than the maximum of $w(M)$ over all matchings M, as each maximum-weight matching M contains an edge e incident with u.

So we may assume that for each vertex v there exists a maximum-weight matching missing v. Hence if $y \in \mathbb{R}_+^V$ and $z \in \mathbb{R}_+^{\mathcal{P}_{\text{odd}}(V)}$ satisfying (25.10) attain the minimum of (25.9), then $y = \mathbf{0}$. (If $y_u > 0$, then each maximum-weight matching covers u, by complementary slackness.)

Now choose z attaining the minimum (with $y = \mathbf{0}$) such that

(25.11) $$\sum_{U \in \mathcal{P}_{\text{odd}}(V)} z_U \lfloor \tfrac{1}{2}|U| \rfloor^2$$

is as large as possible. Let $\mathcal{F} := \{U \in \mathcal{P}_{\text{odd}}(V) \mid z_U > 0\}$. Then $\mathcal{F}$ is laminar. For suppose not. Let $U, W \in \mathcal{F}$ with $U \cap W \neq \emptyset$ and $U \not\subseteq W \not\subseteq U$. Then $|U \cap W|$ is odd. To see this, choose $v \in U \cap W$. Then there is a maximum-weight matching M missing v. Since $z_U > 0$, $E[U]$ contains $\lfloor \frac{1}{2}|U| \rfloor$ edges in M, and hence each vertex in $U \setminus \{v\}$ is covered by an edge in M contained in U. Similarly, each vertex in $W \setminus \{v\}$ is covered by an edge in M contained in

[9] A collection $\mathcal{F}$ of sets is called *laminar* if $U \cap W = \emptyset$ or $U \subseteq W$ or $W \subseteq U$ for all $U, W \in \mathcal{F}$.

W. Hence each vertex in $(U \cap W) \setminus \{v\}$ is covered by an edge in M contained in $U \cap W$. So $|(U \cap W) \setminus \{v\}|$ is even, and hence $|U \cap W|$ is odd.

Now let $\alpha := \min\{z_U, z_W\}$, and decrease z_U and z_W by α and increase $z_{U \cap W}$ and $z_{U \cup W}$ by α. This resetting maintains (25.10), does not change (25.9), but increases (25.11), contradicting our assumption.

This shows that $\mathcal{F}$ is laminar. Now suppose that z is not integer-valued, and let U be an inclusionwise maximal set in $\mathcal{F}$ with $z_U \notin \mathbb{Z}$. Let $U_1, \ldots, U_k$ be the inclusionwise maximal sets in $\mathcal{F}$ properly contained in U (possibly $k = 0$). As $\mathcal{F}$ is laminar, the U_i are disjoint. Let $\alpha := z_U - \lfloor z_U \rfloor$. Then decreasing z_U by α and increasing each z_{U_i} by α would maintain (25.10) (by the integrality of w), but would strictly decrease (25.9) (since $\sum_{i=1}^{k} \lfloor \frac{1}{2}|U_i| \rfloor < \lfloor \frac{1}{2}|U| \rfloor$). This contradicts the minimality of (25.9). ∎

(This proof follows the method given by Schrijver and Seymour [1977]. Other proofs were given by Hoffman and Oppenheim [1978], Schrijver [1983a,1983c], and Cook [1985].)

Note that the Cunningham-Marsh formula has the Tutte-Berge formula (Corollary 24.1) as special case. The previous theorem is equivalent to:

Corollary 25.2a. *System* (25.6) *is totally dual integral.*

Proof. This follows from Theorem 25.2. ∎

25.3a. Direct proof of the Cunningham-Marsh formula

We give a direct proof of the Cunningham-Marsh formula, as given in Schrijver [1983a] (generalizing the proof of Lovász [1979b] of Edmonds' matching polytope theorem). It does not use Edmonds' matching polytope theorem, which rather follows as a consequence.

Let $G = (V, E)$ be a graph. For each weight function $w \in \mathbb{Z}_+^E$, let ν_w denote the maximum weight of a matching. We must show that for each $w \in \mathbb{Z}_+^E$ there exist $y \in \mathbb{Z}_+^V$ and $z \in \mathbb{Z}_+^{\mathcal{P}_{\text{odd}}(V)}$ such that

(25.12) $$\sum_{v \in V} y_v + \sum_{U \in \mathcal{P}_{\text{odd}}(V)} z_U \lfloor \tfrac{1}{2}|U| \rfloor \leq \nu_w$$

and

(25.13) $$\sum_{v \in V} y_v \chi^{\delta(v)} + \sum_{U \in \mathcal{P}_{\text{odd}}(V)} z_U \chi^{E[U]} \geq w.$$

Suppose that G and w contradict this, with $|V| + |E| + w(E)$ as small as possible. Then G is connected (otherwise one of the components of G will form a smaller counterexample) and $w(e) \geq 1$ for each edge e (otherwise we can delete e). Now there are two cases.

Case 1: There is a vertex u covered by every maximum-weight matching. In this case, let $w' := w - \chi^{\delta(u)}$. Then $\nu_{w'} = \nu_w - 1$. Since $w'(E) < w(E)$, there are y' and z' satisfying (25.12) and (25.13) with respect to w'. Increasing y'_u by 1 gives y and z satisfying (25.12) and (25.13) with respect to w.

Case 2: No vertex is covered by every maximum-weight matching. Now let w' arise from w by decreasing all weights by 1. Let M be a matching with $w'(M) = \nu_{w'}$ and with $|M|$ as large as possible.

Then M does not cover all vertices, as, otherwise, for any matching N of maximum w-weight not covering all vertices:

(25.14) $$w'(N) = w(N) - |N| > w(N) - |M| \geq w(M) - |M| = w'(M) = \nu_{w'},$$

contradicting the definition of $\nu_{w'}$.

Suppose that M covers all but one vertex (in particular, $|V|$ is odd). Then

(25.15) $$\nu_w \geq w(M) = w'(M) + |M| = \nu_{w'} + \lfloor \tfrac{1}{2}|V| \rfloor.$$

Since $w'(E) < w(E)$, there are y' and z' satisfying (25.12) and (25.13) with respect to w'. Increasing z'_V by 1 gives y and z satisfying (25.12) and (25.13) with respect to w (by (25.15)), a contradiction.

So we know that M leaves at least two vertices in V uncovered. Let u and v be not covered by M. We can assume that we have chosen M, u, v under the additional condition that the distance $d(u,v)$ of u and v in G is as small as possible. Then $d(u,v) > 1$, since otherwise we could augment M by edge $\{u,v\}$, thereby increasing $|M|$ while not decreasing $w'(M)$. Let t be an internal vertex of a shortest $u-v$ path. Let N be a matching not covering t, with $w(N) = \nu_w$.

Let P be the component of $M \cup N$ containing t. Then P forms a path starting at t and not covering both u and v (as t is not covered by N and u and v are not covered by M). We can assume that P does not cover u. Now the symmetric differences $M' := M \triangle P$ and $N' := N \triangle P$ are matchings again, and $|M'| \leq |M|$ (as M covers t), implying

(25.16) $$\begin{aligned} w'(M') - w'(M) &= w(M') - |M'| - w(M) + |M| \geq w(M') - w(M) \\ &= w(N) - w(N') = \nu_w - w(N') \geq 0. \end{aligned}$$

So $w'(M') \geq w'(M) = \nu_{w'}$ and hence we have equality throughout. So $w(M') = w(M)$, $w'(M') = w'(M)$, and $|M'| = |M|$. However, M' does not cover t and u while $d(u,t) < d(u,v)$, contradicting our choice of M, u, v.

25.4. On the total dual integrality of the perfect matching constraints

System (25.2) determining the perfect matching polytope is generally not totally dual integral. Indeed, consider the complete graph $G = K_4$ on four vertices, with $w(e) := 1$ for each edge e; then the maximum weight of a perfect matching is 2, while the dual of optimizing $w^{\mathsf{T}}x$ subject to (25.2) is attained only by taking $y(\{v\}) = \frac{1}{2}$ for each vertex v.

However, consider the following system, again determining the perfect matching polytope (by Corollary 25.1a):

(25.17)
$$\begin{array}{rll} \text{(i)} & x_e \geq 0 & \text{for each } e \in E; \\ \text{(ii)} & x(\delta(v)) = 1 & \text{for each } v \in V; \\ \text{(iii)} & x(E[U]) \leq \lfloor \tfrac{1}{2}|U| \rfloor & \text{for each } U \subseteq V \text{ with } |U| \text{ odd.} \end{array}$$

Corollary 25.2b. *System* (25.17) *is totally dual integral.*

Proof. Directly from Corollary 25.2a, with Theorem 5.25. ∎

This implies a result stated by Edmonds and Johnson [1970]:

Corollary 25.2c. *The perfect matching inequalities* (25.2) *form a totally dual half-integral system.*

Proof. Let $w \in \mathbb{Z}^E$, and minimize $w^{\mathsf{T}}x$ subject to (25.2). As it is the same as minimizing $w^{\mathsf{T}}x$ subject to (25.17), by Corollary 25.2b there is an optimum dual solution $y \in \mathbb{Z}^V$, $z \in \mathbb{Z}_+^{\mathcal{P}_{\text{odd}}(V)}$. Since $x(E[U]) \leq \lfloor \frac{1}{2}|U| \rfloor$ is half of the sum of the inequalities $x(\delta(v)) = 1$ ($v \in U$) and $-x(\delta(U)) \leq -1$, we obtain the total dual half-integrality of (25.2). ∎

This can be strengthened to (Barahona and Cunningham [1989]):

Corollary 25.2d. *If $w \in \mathbb{Z}^E$ and $w(C)$ is even for each circuit C, then the problem of minimizing $w^{\mathsf{T}}x$ subject to* (25.2) *has an integer optimum dual solution.*

Proof. If $w(C)$ is even for each circuit, there is a subset T of V with $\{e \in E \mid w(e) \text{ is odd}\} = \delta(T)$. Now replace w by $\tilde{w} := w + \sum_{v \in T} \chi^{\delta(v)}$. Then $\tilde{w}(e)$ is an even integer for each edge e. Hence by Corollary 25.2c there is an optimum dual solution $\tilde{y} \in \mathbb{Z}^V$, $z \in \mathbb{Z}_+^{\mathcal{P}_{\text{odd}}(V)}$ for the problem of minimizing $\tilde{w}^{\mathsf{T}}x$ subject to (25.2). Now setting $y_v := \tilde{y}_v - 1$ if $v \in T$ and $y_v := \tilde{y}_v$ if $v \notin T$ gives an integer optimum dual solution for w. ∎

25.5. Further results and notes

25.5a. Adjacency and diameter of the matching polytope

Balinski and Russakoff [1974] and Chvátal [1975a] characterized adjacency on the matching polytope:

Theorem 25.3. *Let M and N be distinct matchings in a graph $G = (V, E)$. Then χ^M and χ^N are adjacent vertices of the matching polytope if and only if $M \triangle N$ is a path or circuit.*

Proof. To see necessity, let χ^M and χ^N be adjacent. Let P be any nontrivial component of $M \triangle N$ and let $M' := M \triangle P$ and $N' := N \triangle P$. So M' and N' are matchings again. Then

$$\text{(25.18)} \qquad \tfrac{1}{2}(\chi^M + \chi^N) = \tfrac{1}{2}(\chi^{M'} + \chi^{N'}).$$

As χ^M and χ^N are adjacent, it follows that $\{M', N'\} = \{M, N\}$. So $M' = N$ and $N' = M$, and therefore $M \triangle N = P$.

To see sufficiency, let $P := M \triangle N$ be a path or circuit. Suppose that χ^M and χ^N are not adjacent. Then there exists a matching $L \neq M, N$ that belongs to the smallest face of the matching polytope containing $x := \frac{1}{2}(\chi^M + \chi^N)$. As $x_e = 0$ for each edge $e \notin M \cup N$ and $x_e = 1$ for each edge $e \in M \cap N$, we know that $M \cap N \subseteq L \subseteq M \cup N$. Moreover, $x(\delta(v)) = 1$ for each vertex v covered both by M and by N. Hence each vertex v covered both by M and by N is covered by L. As P is a path or a circuit, it follows that $L = M$ or $L = N$, a contradiction. ∎

This has as consequence for the diameter:

Corollary 25.3a. *The diameter of the matching polytope of any graph $G = (V, E)$ is equal to the maximum size $\nu(G)$ of the matchings.*

Proof. First, by Theorem 25.3, for any two matchings M and N, the distance of χ^M and χ^N is at most the number of nontrivial components of $M \triangle N$. Since each such component contains at least one edge and since these edges are pairwise disjoint, this number is at most $\nu(G)$. So the diameter is at most $\nu(G)$.

Equality follows from the fact that $\emptyset$ and any matching M have distance $|M|$. This follows from the fact that if M and N are adjacent, then $\big||M| - |N|\big| \leq 1$ by Theorem 25.3. ∎

Another direct consequence concerns adjacency on the *perfect* matching polytope:

Corollary 25.3b. *Let M and N be perfect matchings in a graph $G = (V, E)$. Then χ^M and χ^N are adjacent vertices of the perfect matching polytope if and only if $M \triangle N$ is a circuit.*

Proof. Directly from Theorem 25.3. ∎

This in turn implies for the diameter of the perfect matching polytope:

Corollary 25.3c. *The perfect matching polytope of a graph $G = (V, E)$ has diameter at most $\frac{1}{2}|V|$ ($\frac{1}{4}|V|$ if G is simple).*

Proof. For any two perfect matching M, N, the symmetric difference has at most $\frac{1}{2}|V|$ components (each being a circuit). Hence Corollary 25.3b implies that χ^M and χ^N have distance at most $\frac{1}{2}|V|$.

If G is simple the bounds can be sharpened to $\frac{1}{4}|V|$, as each even circuit has at least four vertices. ∎

Padberg and Rao [1974] showed that if G is a complete graph with an even number $2n$ of vertices, then $P_{\text{perfect matching}}(G)$ has diameter at most 2. (This can be derived from Theorem 18.5, since any two perfect matchings belong to some $K_{n,n}$-subgraph of G, which subgraph gives a face of $P_{\text{perfect matching}}(G)$.)

25.5b. Facets of the matching polytope

Pulleyblank and Edmonds [1974] (cf. Pulleyblank [1973]) characterized which of the inequalities (25.6) give a facet of the matching polytope:

Let $G = (V, E)$ be a graph. Define

(25.19) $I := \{v \in V \mid \deg_G(v) \geq 3$, or $\deg_G(v) = 2$ and v is contained in no triangle, or $\deg_G(v) = 1$ and the neighbour of v also has degree $1\}$,
$\mathcal{T} := \{U \subseteq V \mid |U| \geq 3$, $G[U]$ is factor-critical and 2-vertex-connected$\}$.

(Recall that graph G is *factor-critical* if, for each vertex v of G, $G - v$ has a perfect matching.)

Consider the system

(25.20)
$$\begin{array}{rll} \text{(i)} & x_e \geq 0 & \text{for } e \in E, \\ \text{(ii)} & x(\delta(v)) \leq 1 & \text{for } v \in I, \\ \text{(iii)} & x(E[U]) \leq \lfloor \frac{1}{2}|U| \rfloor & \text{for } U \in \mathcal{T}. \end{array}$$

We first show:

Theorem 25.4. *Each inequality in* (25.6) *is a nonnegative integer combination of inequalities* (25.20)*.*

Proof. First consider a vertex $v \notin I$. If $\deg_G(v) = 1$, let u be the neighbour of v. Then $u \in I$ and

(25.21)
$$x(\delta(v)) = x(\delta(u)) - \sum_{e \in \delta(u) - \delta(v)} x_e.$$

If $\deg_G(v) = 2$ and v is contained in a triangle $G[U]$, then $x(\delta(v)) = x(E[U]) - x_e$, where e is the edge in $E[U]$ not incident with v.

Next consider a subset U of V with $|U|$ odd and $|U| \geq 3$. We show that $x(E[U]) \leq \lfloor \frac{1}{2}|U| \rfloor$ is a sum of constraints (25.20), by induction on $|U|$. If $U \in \mathcal{T}$ we are done. So assume that $U \notin \mathcal{T}$. Let $H := G[U]$. If H is not factor-critical, there is a vertex v such that $H - v$ has no perfect matching. Let $U' = U \setminus \{v\}$. Then $x(E[U']) \leq \lfloor \frac{1}{2}|U| \rfloor - 1$ for the incidence vector x of any matching, and hence also for each vector x in the matching polytope. By the total dual integrality of the matching constraints (Corollary 25.2a), this constraint is a sum of constraints (25.6), and hence, by induction, of constraints (25.20). So $x(E[U]) \leq \lfloor \frac{1}{2}|U| \rfloor$ is a sum of constraints (25.20), as $E[U] \subseteq E[U'] \cup \delta(v)$.

If H is factor-critical, it has a cut vertex v. Let $K_1, \ldots, K_t$ be the components of $H - v$ and let $U_i := K_i \cup \{v\}$ for each i. As H is factor-critical, each $|U_i|$ is odd. Hence $x(E[U]) \leq \lfloor \frac{1}{2}|U| \rfloor$ is a sum of the constraints $x(E[U_i]) \leq \lfloor \frac{1}{2}|U_i| \rfloor$. ∎

This implies that (25.20) is sufficient:

Corollary 25.4a. (25.20) *determines the matching polytope.*

Proof. Directly from Corollary 25.1a and Theorem 25.4. ∎

Another consequence is the result of Cunningham and Marsh [1978] that the irredundant system still is totally dual integral:

Corollary 25.4b. (25.20) *is TDI.*

Proof. Directly from Theorem 25.4, using the total dual integrality of system (25.6). ∎

(For a short proof of this result, see Cook [1985].)

Next we show that each inequality in (25.20) determines a facet. To this end, we first show:

Lemma 25.5α. *Let $G = (V, E)$ be a 2-vertex-connected factor-critical graph and let W be a proper subset of V with $|W|$ odd and ≥ 3. Then G has a matching of size $\lfloor \frac{1}{2}|V| \rfloor$ containing less than $\lfloor \frac{1}{2}|W| \rfloor$ edges in $E[W]$.*

Proof. Choose a vertex $v \in W$ that is adjacent to at least one vertex in $V \setminus W$. If v has no neighbour in W, choose $u \in W \setminus \{v\}$ and let M be a perfect matching in $G - u$. This matching has the required properties.

So we may assume that v has a neighbour in W. Make from G a graph G', by splitting v into two vertices v' and v'', where v' is adjacent to all vertices in W adjacent to v and where v'' is adjacent to all vertices in $V \setminus W$ adjacent to v.

If G' has a perfect matching M', then deleting the edge in M' covering v', and identifying v' and v'', gives a matching M in G with $|M| = \lfloor \frac{1}{2}|V| \rfloor$, but with $|M \cap E[W]| < \lfloor \frac{1}{2}|W| \rfloor$.

So we can assume that G' has no perfect matching. Then by Tutte's 1-factor theorem, there is a subset U of VG' such that $G' - U$ has more than $|U|$ odd components. Since the graph $G' \cup \{v'v''\}$ has a perfect matching[10] (as G is factor-critical), we know that $v', v'' \notin U$.

If $U = \emptyset$, G' has an odd component, contradicting the fact that G' is connected (since G is 2-vertex-connected) and has an even number of vertices. So $U \neq \emptyset$. Choose $u \in U$, and let M be a perfect matching in $G - u$. Then M yields a matching M' in G' missing u and exactly one of v', v''. So $G' \cup \{uv'\}$ or $G' \cup \{uv''\}$ has a perfect matching, contradicting the fact that $u \in U$ and $G' - U$ has more than $|U|$ odd components. ∎

This lemma is used in proving:

Theorem 25.5. *Each inequality in* (25.20) *determines a facet.*

Proof. We clearly cannot delete any inequality $x_e \geq 0$, since otherwise the vector x defined by $x_e := -1$ and $x_{e'} := 0$ for each $e' \neq e$ would be a solution. So it determines a facet.

Consider next an inequality

(25.22) $\quad x(\delta(v)) \leq 1$

for some $v \in I$. Let F be the set of vectors x in the matching polytope satisfying $x(\delta(v)) = 1$. Suppose that F is not a facet. Then there is a facet F' with $F' \supset F$. So F' is determined by one of the inequalities (25.20).

[10] By $G' \cup \{uv\}$ we denote the graph obtained from G' by adding edge uv.

If F' is determined by $x_e = 0$ for some $e \in E$, choose a matching M with $e \in M$ and covering v (the existence of such a matching follows from the definition of I). Then $\chi^M \in F \setminus F'$, a contradiction.

If F' is determined by $x(\delta(u)) = 1$ for some $u \in I$, then $u \neq v$ (since $F' \neq F$) and there is an edge e incident with u but not with v (since $u, v \in I$). Hence for matching $M := \{e\}$ we have $\chi^M \in F \setminus F'$, a contradiction.

If F' is determined by $x(E[U]) = \lfloor \frac{1}{2}|U| \rfloor$ for some $U \in \mathcal{T}$, then $\delta(u) \subseteq E[U]$ and $\lfloor \frac{1}{2}|U| \rfloor = 1$ (since $\chi^M \in F \subseteq F'$ for $M = \{e\}$, for each $e \in \delta(v)$). So $|U| = 3$. Since $F' \neq F$, U determines a triangle, contradicting the fact that $v \in I$.

Finally consider an inequality

$$x(E[U]) \leq \lfloor \tfrac{1}{2}|U| \rfloor \tag{25.23}$$

for some $U \in \mathcal{T}$. Let F be the set of vectors x in the matching polytope satisfying $x(E[U]) = \lfloor \frac{1}{2}|U| \rfloor$.

Suppose that F is not a facet, and let F' be a facet with $F' \supset F$.

First assume that F' is determined by $x_e = 0$ for some $e \in E$. If e is not spanned by U, there is a $v \in U$ such that $U \setminus \{v\}$ is not intersected by e. Let M be a perfect matching of $G[U] - v$. Then $\chi^{M \cup \{e\}} \in F \setminus F'$, a contradiction. If e is spanned by U, choose $v \in e$ and let M be a perfect matching of $G[U] - v$. Let $f \in M$ intersect e, and define $M' := (M \setminus \{f\}) \cup \{e\}$. Then $\chi^{M'} \in F \setminus F'$, a contradiction.

Next assume that F' is determined by $x(\delta(v)) = 1$ for some $v \in I$. Then, as $G[U]$ is factor-critical, there is a matching M with $|M \cap E[U]| = \lfloor \frac{1}{2}|U| \rfloor$ and $M \cap \delta(v) = \emptyset$. So $\chi^M \in F \setminus F'$, a contradiction.

Finally assume that F' is determined by $x(E[U']) = \lfloor \frac{1}{2}|U'| \rfloor$ for some $U' \in \mathcal{T}$. If $U' \not\subseteq U$, there is a matching M with $|M \cap E[U]| = \lfloor \frac{1}{2}|U| \rfloor$ missing at least two vertices in U' and hence $|M \cap E[U']| < \lfloor \frac{1}{2}|U'| \rfloor$. Then $\chi^M \in F \setminus F'$, a contradiction.

So $U' \subset U$. By Lemma 25.5α, $G[U]$ has a matching M of size $\lfloor \frac{1}{2}|U| \rfloor$ such that less than $\lfloor \frac{1}{2}|U'| \rfloor$ edges in M are spanned by U'. Then $\chi^M \in F \setminus F'$, a contradiction. ∎

(This proof is due to L. Lovász (cf. Cornuéjols and Pulleyblank [1982]). For another proof, see Cook [1985]. See also Giles [1978b].)

Edmonds, Lovász, and Pulleyblank [1982] gave an irredundant system of linear inequalities describing the *perfect* matching polytope. More on the combinatorial structure of the (perfect) matching polytope is given by Naddef and Pulleyblank [1981a].

25.5c. Polynomial-time solvability with the ellipsoid method

In Chapter 26 we shall describe Edmonds' strongly polynomial-time algorithm for the weighted matching problem. This algorithm gives as a by-product the inequalities describing the perfect matching polytope, as we shall see in Section 26.3b.

It turns out that conversely one can derive the strong polynomial-time solvability of the weighted matching problem from the description of the perfect matching polytope (albeit that the method is impractical).

Indeed, the weighted perfect matching problem is equivalent to the optimization problem over the perfect matching polytope. So, by the ellipsoid method, there exists a polynomial-time weighted perfect matching algorithm if and only if there exists a polynomial-time separation algorithm for the perfect matching polytope.

Such a polynomial-time algorithm indeed exists (and would follow conversely also with the ellipsoid method from the polynomial-time solvability of the weighted matching problem). A direct proof was given by Padberg and Rao [1982], and is as follows.

The separation problem for the perfect matching polytope is: given a graph $G = (V, E)$ and a vector $x \in \mathbb{R}^E_+$, decide if x belongs to the perfect matching polytope, and if not, find a separating hyperplane. To answer this question we can first check the constraints (25.2)(i)(ii) in polynomial time. If one of them is violated, it gives a separating hyperplane. If each of them is satisfied, we should check if $x(\delta(U)) < 1$ for some odd subset U of V. Considering x as a capacity function, we should find an odd cut of capacity less than 1. Here an *odd cut* is a cut $\delta(U)$ with $|U|$ odd.

Such a cut can be found in strongly polynomial time. For a graph $G = (V, E)$ and a tree $T = (V, F)$, a *fundamental cut determined by* T is a cut $\delta_E(W_f)$, where $f \in F$ and W_f is one of the components of $T - f$. Then:

Theorem 25.6. *Let $G = (V, E)$ be a graph with $|V|$ even, let $c \in \mathbb{R}^E_+$ be a capacity function, and let $T = (V, F)$ be a Gomory-Hu tree for G and c. Then one of the fundamental cuts determined by T is a minimum-capacity odd cut in G.*

Proof. For each $f \in F$, choose W_f as one of the two components of $T - f$. Let $\delta_G(U)$ be a minimum-capacity odd cut of G. Then U or $V \setminus U$ is equal to the symmetric difference of the W_f over $f \in \delta_F(U)$. Hence $|W_f|$ is odd for at least one $f \in \delta_F(U)$. So $\delta_G(W_f)$ is an odd cut. Let $f = uv$. As $\delta_G(W_f)$ is a minimum-capacity $u - v$ cut and as $\delta_G(U)$ is a $u - v$ cut, we have $c(\delta_G(W_f)) \leq c(\delta_G(U))$. So $\delta_G(W_f)$ is a minimum-capacity odd cut. ∎

This gives algorithmically:

Corollary 25.6a. *A minimum-capacity odd cut can be found in strongly polynomial time.*

Proof. This follows from Theorem 25.6, since a Gomory-Hu tree can be found in strongly polynomial time, by Corollary 15.15a. ∎

As the separation problem for the perfect matching polytope can be reduced to finding a minimum-capacity odd cut, this implies:

Corollary 25.6b. *The separation problem for the perfect matching polytope can be solved in strongly polynomial time.*

Proof. See above. ∎

Corollary 25.6c. *A minimum-weight perfect matching can be found in strongly polynomial time.*

Proof. This follows from Corollary 25.6b, with Theorem 5.11. ∎

25.5d. The matchable set polytope

Let $G = (V, E)$ be a graph. A subset U of V is called *matchable* if the graph $G[U]$ has a perfect matching. The *matchable set polytope* of G is the convex hull (in $\mathbb{R}^V$) of the incidence vectors of matchable sets.

Balas and Pulleyblank [1989] characterized the matchable set polytope as follows (where $N(U)$ is the set of neighbours of U and $o(G[U])$ is the number of odd components of $G[U]$):

Theorem 25.7. *The matchable set polytope of a graph $G = (V, E)$ is determined by:*

(25.24)
$$\begin{array}{lll} \text{(i)} & 0 \leq x_v \leq 1 & \text{for } v \in V, \\ \text{(ii)} & x(U) - x(N(U)) \leq |U| - o(G[U]) & \text{for } U \subseteq V. \end{array}$$

Proof. Each vector in the matchable set polytope of G satisfies (25.24), since the incidence vector of any matchable set satisfies (25.24), since if any odd component K of $G[U]$ is covered by a matching M, then M has an edge connecting K and $N(U)$.

To see the reverse, choose a counterexample with $|V| + |E|$ minimal, and let x be a vertex of the polytope determined by (25.24) that is not in the matchable set polytope.

Then $x_v > 0$ for each vertex v, since otherwise we can obtain a smaller counterexample by deleting v. Moreover, there exists at least one vertex v with $x_v < 1$, since otherwise $x = \chi^V$, while V is matchable (as follows from Tutte's theorem, using (25.24)(ii)).

Hence, since x is a vertex of the polytope determined by (25.24), at least one constraint in (25.24)(ii) is attained with equality for some U with $o(G[U]) \geq 1$ (for any other U, (ii) follows from (i)).

Choose such a U with U inclusionwise minimal. Let $\mathcal{K}$ be the collection of components of $G[U]$. Then

(25.25) $G[K]$ is factor-critical for each $K \in \mathcal{K}$.

Otherwise, if K is even, then

(25.26)
$$\begin{aligned} & x(U \setminus K) - x(N(U \setminus K)) \geq x(U) - x(K) - x(N(U)) \\ & \geq x(U) - |K| - x(N(U)) = |U| - o(G[U]) - |K| \\ & = |U \setminus K| - o(G[U \setminus K]), \end{aligned}$$

contradicting the minimality of U.

So K is odd. If $G[K]$ is not factor-critical, then by Tutte's 1-factor theorem, K has a nonempty subset C with $o(G[K] - C) \geq |C| + 1$. Then

(25.27)
$$\begin{aligned} & x(U \setminus C) - x(N(U \setminus C)) \geq x(U) - 2x(C) - x(N(U)) \\ & = |U| - o(G[U]) - 2x(C) \geq |U| - o(G[U]) - 2|C| \\ & = |U \setminus C| - o(G[U]) - |C| \geq |U \setminus C| - o(G[U]) - o(G[K \setminus C]) + 1 \\ & = |U \setminus C| - o(G[U \setminus C]). \end{aligned}$$

So we have equality by (25.24)(ii), contradicting the minimality of U. This shows (25.25).

Let $S := U \cup N(U)$. Let $G' := G - S$ and let x' be the restriction of x to $V \setminus S$. Then x' satisfies (25.24) with respect to G'. Indeed, (i) is trivial. To see (ii), choose a subset $U' \subseteq V \setminus S$. Then (since no edge connects U and U'):

$$(25.28)\qquad \begin{aligned} &x'(U') - x'(N_{G'}(U')) = x(U') - x(N(U') \setminus S) \\ &= x(U \cup U') - x(N(U \cup U')) - (x(U) - x(N(U))) \\ &\leq |U| + |U'| - o(G[U \cup U']) - (|U| - o(G[U])) = |U'| - o(G'[U']), \end{aligned}$$

as required.

Hence, by the minimality of G, x' belongs to the matchable set polytope of G'. Hence we are done if we have shown that the restriction of x to $G[S]$ belongs to the matchable set polytope of G.

Let H be the bipartite graph obtained from $G[S]$ by deleting all edges spanned by $N(U)$ and by contracting each $K \in \mathcal{K}$ to one vertex, u_K say. Define y on the vertices of H by: $y(v) := x(v)$ if $v \in N(U)$ and $y(u_K) := x(K) - |K| + 1$ for $K \in \mathcal{K}$.

Then y belongs to the matchable set polytope of H. To see this, we apply Theorem 21.30. Trivially $0 \leq y(v) \leq 1$ for each $v \in N(U)$. Moreover, $y(u_K) \geq 0$ for each $K \in \mathcal{K}$, since otherwise $x(K) < |K| - 1$ implying

$$(25.29)\qquad \begin{aligned} &x(U \setminus K) - x(N(U \setminus K)) \geq x(U) - x(K) - x(N(U)) \\ &= |U| - o(G[U]) - x(K) > |U| - o(G[U]) - |K| + 1 \\ &= |U \setminus K| - o(G[U \setminus K]), \end{aligned}$$

contradicting (25.24)(ii). The inequality $y(u_K) \leq 1$ follows from the fact that $x(K) \leq |K|$.

Now

$$(25.30)\qquad \sum_{K \in \mathcal{K}} y(u_K) = x(U) - |U| + |\mathcal{K}| = x(N(U)) = \sum_{v \in N(U)} y(v).$$

This implies, by Theorem 21.30, that if y is not in the matchable set polytope of H, then there exists a subcollection $\mathcal{L}$ of $\mathcal{K}$ with

$$(25.31)\qquad y(N(U')) < \sum_{K \in \mathcal{L}} y(u_K),$$

where $U' := \bigcup \mathcal{L}$. However, by (25.24) we have

$$(25.32)\qquad \begin{aligned} &\sum_{K \in \mathcal{L}} y(u_K) = \sum_{K \in \mathcal{L}} (x(K) - |K| + 1) = x(U') - |U'| + |\mathcal{L}| \\ &= x(U') - |U'| + o(G[U']) \leq x(N(U')) = y(N(U')). \end{aligned}$$

So y belongs to the matchable set polytope of H. Assuming that the restriction of x to S does not belong to the matchable set polytope of $G[S]$, there exists a vector $w \in \mathbb{R}^V$ with $w^{\mathsf{T}}x > w(Y)$ for each matchable set Y of $G[S]$ and with $w(v) = 0$ if $v \notin S$. For each $K \in \mathcal{K}$, let $v_K \in K$ minimize $w(v)$ over K. Define w' on the vertices of H by: $w'(v) := w(v)$ for $v \in N(U)$ and $w'(u_K) := w(v_K)$ for $K \in \mathcal{K}$. Since y belongs to the matchable set polytope of H, H has a matchable set Y' satisfying $w'(Y') \geq {w'}^{\mathsf{T}}y$. Let Y be the union of Y', of all K with $u_K \in Y'$, and of all $K \setminus \{v_K\}$. Since each $G[K]$ is factor-critical, Y is matchable. Moreover,

$$(25.33)\qquad \begin{aligned} w(Y) &= w'(Y') + \sum_{K \in \mathcal{K}} w(K \setminus \{v_K\}) \geq {w'}^{\mathsf{T}}y + \sum_{K \in \mathcal{K}} w(K \setminus \{v_K\}) \\ &= \sum_{v \in N(U)} w(v)x(v) + \sum_{K \in \mathcal{K}} w(v_K)(x(K) - |K| + 1) + \sum_{K \in \mathcal{K}} w(K \setminus \{v_K\}) \\ &\geq \sum_{v \in N(U)} w(v)x(v) + \sum_{K \in \mathcal{K}} \sum_{v \in K} (w(v) - w(v_K) + w(v_K)x(v)) \\ &\geq \sum_{v \in N(U)} w(v)x(v) + \sum_{K \in \mathcal{K}} \sum_{v \in K} w(v)x(v) = w^{\mathsf{T}}x \end{aligned}$$

(the last inequality follows from $(w(v) - w(v_K))(1 - x(v)) \geq 0)$, contradicting our assumption. ∎

Cunningham and Green-Krótki [1994] gave a combinatorial, polynomial-time separation algorithm for the matchable set polytope that implies a proof of Theorem 25.7. A combinatorial, strongly polynomial-time algorithm was given by Cunningham and Geelen [1996,1997]. Qi [1987] characterized adjacency of vertices on the matchable set polytope. Related work can be found in Barahona and Mahjoub [1994a].

25.5e. Further notes

We postpone a discussion of the dimension of the perfect matching polytope to Chapter 37.

Note that Edmonds' matching polytope theorem gives the linear inequalities determining the convex hull of all *symmetric* permutation matrices.

Hoffman and Oppenheim [1978] showed that for each graph $G = (V, E)$ and for each vertex x of the matching polytope of G, there exist $|E|$ linearly independent constraints among (25.6) satisfied by x with equality and yielding a matrix of determinant ± 1. This also implies the total dual integrality of the constraints (25.6).

Unlike in the bipartite case, the convex hull of incidence vectors of edge sets containing a perfect matching is not determined by linear inequalities with $0, 1$ coefficients (in the left-hand side), as was shown by Cunningham and Green-Krótki [1986]. They showed that for each integer $n > 0$ there exists a graph $G = (V, E)$ with $|V| = 2n+4$ such that the convex hull of the incidence vectors of supersets of perfect matchings has facet-inducing inequalities with coefficient set $\{0, 1, \ldots, n\}$. They also showed that for odd n a similar result holds for subsets of perfect matchings. So the polyhedra $P^{\uparrow}_{\text{perfect matching}}(G)$ and $P^{\downarrow}_{\text{perfect matching}}(G)$ are not determined by $0, 1$ inequalities.

Naddef and Pulleyblank [1981b] observed that Edmonds' perfect matching polytope theorem implies that any $(k-1)$-edge connected k-regular graph $G = (V, E)$ with an even number of vertices, is matching-covered. (This can be seen by showing that the all-$\frac{1}{k}$ vector in $\mathbb{R}^E$ belongs to the perfect matching polytope.)

Rispoli [1992] noticed that the 'monotonic diameter' of the perfect matching polytope of K_n is equal to $\lfloor \frac{n}{4} \rfloor$. So for any weight function w there is a polytopal path with monotonically increasing $w^{\mathsf{T}} x$ and leading from any vertex to a vertex maximizing $w^{\mathsf{T}} x$, of length at most $\lfloor \frac{n}{4} \rfloor$.

Chapter 26

Weighted nonbipartite matching algorithmically

In the previous chapter we gave good characterizations for the maximum-weight matching problem. In the present chapter we go over to the algorithmic side, and describe Edmonds' strongly polynomial-time algorithm for finding a minimum-weight perfect matching in any graph. It implies a strongly polynomial-time algorithm for finding a maximum-weight matching.
In this chapter, graphs can be assumed to be simple.

26.1. Introduction and preliminaries

As an extension of the cardinality matching algorithm, Edmonds [1965b] proved that also a maximum-*weight* matching can be found in strongly polynomial time. Equivalently, a minimum-weight perfect matching can be found in strongly polynomial time.

Like the cardinality matching algorithm, the weighted matching algorithm is based on shrinking sets of vertices. Unlike the cardinality matching algorithm however, for weighted matchings one has, at times, to 'deshrink' sets of vertices (the reverse operation of shrinking). For this purpose we have to keep track of the shrinking history throughout the iterations.

Let $G = (V, E)$ be a graph and let $w \in \mathbb{Q}^E$ be a weight function. We describe a strongly polynomial-time algorithm to find a minimum-weight perfect matching in G. We can assume that G has at least one perfect matching and that $w \geq \mathbf{0}$.

The algorithm is 'primal-dual'. The 'vehicle' carrying us to a minimum-weight perfect matching is a pair of a laminar[11] collection Ω of odd-size subsets of V and a function $\pi : \Omega \to \mathbb{Q}$ satisfying:

(26.1)
$$\begin{array}{lll} \text{(i)} & \pi(U) \geq 0 & \text{if } U \in \Omega \text{ and } |U| \geq 3, \\ \text{(ii)} & \displaystyle\sum_{\substack{U \in \Omega \\ e \in \delta(U)}} \pi(U) \leq w(e) & \text{for each } e \in E. \end{array}$$

[11] A collection Ω of sets is called *laminar* if $U \cap W = \emptyset$ or $U \subseteq W$ or $W \subseteq U$ for any $U, W \in \Omega$.

Condition (26.1) implies

$$w(M) \geq \sum_{U \in \Omega} \pi(U) \tag{26.2}$$

for each perfect matching M in G, since

$$\begin{aligned} w(M) &= \sum_{e \in M} w(e) \geq \sum_{e \in M} \sum_{\substack{U \in \Omega \\ e \in \delta(U)}} \pi(U) = \sum_{U \in \Omega} \pi(U)|M \cap \delta(U)| \\ &\geq \sum_{U \in \Omega} \pi(U). \end{aligned} \tag{26.3}$$

Hence M is a minimum-weight perfect matching if equality holds throughout in (26.3).

Notation. Let be given Ω and $\pi : \Omega \to \mathbb{Q}$. Define for any edge e:

$$w_\pi(e) := w(e) - \sum_{\substack{U \in \Omega \\ e \in \delta(U)}} \pi(U). \tag{26.4}$$

So (26.1)(ii) says that $w_\pi(e) \geq 0$ for each $e \in E$. Let E_π denote the set of edges e with $w_\pi(e) = 0$, and let $G_\pi = (V, E_\pi)$.

Throughout the algorithm we will have that $\{v\} \in \Omega$ for each $v \in V$. Hence, as Ω is laminar, the collection $\Omega^{\max}$ of inclusionwise maximal sets in Ω is a partition of V.

By G' we denote the graph obtained from G_π by shrinking all sets in $\Omega^{\max}$:

$$G' := G_\pi / \Omega^{\max}. \tag{26.5}$$

(So G' depends on Ω and π.) The vertex set of G' is $\Omega^{\max}$, with two distinct elements $U, U' \in \Omega^{\max}$ adjacent if and only if G_π has an edge connecting U and U'. We denote any edge of G' by the original edge in G.

Finally, for $U \in \Omega$ with $|U| \geq 3$, we denote by H_U the graph obtained from $G_\pi[U]$ by contracting each inclusionwise maximal proper subset of U that belongs to Ω.

26.2. Weighted matching algorithm

We keep a laminar collection Ω of odd-size subsets of V, a function $\pi : \Omega \to \mathbb{Q}$ satisfying (26.1), a matching M in G', and for each $U \in \Omega$ with $|U| \geq 3$, a Hamiltonian circuit C_U in H_U. We assume that G is simple and has at least one perfect matching.

Initially, we set $\Omega := \{\{v\} \mid v \in V\}$, $\pi(\{v\}) := 0$ for each $v \in V$, and $M := \emptyset$. The iteration is as follows. Let X be the set of vertices of G' missed by M. (In the algorithm, 'positive length' means: having at least one edge.)

(26.6) **Case 1: G' has an M-alternating $X - X$ walk of positive length.** Choose a shortest such walk P. If P is a path, it is an M-augmenting path in G'. Reset $M := M \triangle EP$ (*matching augmentation*) and iterate.

If P is not a path, it contains an M-flower (Theorem 24.3). Let C be the circuit in it. Add $U := \bigcup VC$ to Ω (*shrinking*), set $\pi(U) := 0$, $M := M \setminus EC$, and $C_U := C$, and iterate.

Case 2: G' has no M-alternating $X - X$ walk of positive length. Let $\mathcal{S}$ be the set of vertices U of G' for which G' has an odd-length M-alternating $X - U$ walk and let $\mathcal{T}$ be the set of vertices U of G' for which G' has an even-length M-alternating $X - U$ walk. Reset $\pi(U) := \pi(U) + \alpha$ if $U \in \mathcal{T}$ and $\pi(U) := \pi(U) - \alpha$ if $U \in \mathcal{S}$, where α is the largest value maintaining (26.1). If after this resetting $\pi(U) = 0$ for some $U \in \mathcal{S}$ with $|U| \geq 3$, delete U from Ω (*deshrinking*), extend M by the perfect matching of $C_U - v$, where v is the vertex of C_U covered by M, and iterate.

In Case 2, α is bounded, since $|\mathcal{T}| > |\mathcal{S}|$ if M is not perfect and since by (26.3), $\sum_{U \in \Omega} \pi(U)$ is bounded (as there exists at least one perfect matching by assumption).

The iterations stop if M is a perfect matching in G', and then we are done: using the C_U we can expand M to a perfect matching N in G with $w_\pi(N) = 0$ and $|N \cap \delta(U)| = 1$ for each $U \in \Omega$. Then N has equality throughout in (26.3), and hence it is a minimum-weight perfect matching.

As for estimating the number of iterations, it is good to observe that the laminarity of Ω implies (cf. Theorem 3.5)

(26.7) $|\Omega| \leq 2|V|$,

assuming $V \neq \emptyset$.

Theorem 26.1. *There are at most* $2|V|^2$ *iterations.*

Proof. There are at most $\frac{1}{2}|V|$ matching augmentations, since at each matching augmentation the size of X decreases by 2, and remains unchanged in any other iteration.

The further proof is based on the following observation:

(26.8) Any set U added to Ω ('shrinking') will not be removed from Ω ('deshrinking') before the next matching augmentation.

Indeed, after shrinking U, there exists an even-length M-alternating $X - U$ path. Until the next matching augmentation, this remains the case, or U is swallowed by a larger set that is shrunk. So U is not in $\mathcal{S}$ before the next matching augmentation, proving (26.8).

Consider any sequence of iterations between two consecutive matching augmentations. By (26.8), the number of deshrinkings is not more than the size of Ω at the start of the sequence. Similarly by (26.8), the number of shrinkings is not more than the size of Ω at the end of the sequence. So, by (26.7), both the number of shrinkings and the number of deshrinkings are at most $2|V|$.

If in Case 2 we do not deshrink, then there is an edge e connecting a vertex $U \in \mathcal{T}$ with a vertex $W \notin \mathcal{S}$ of G' for which $w_\pi(e)$ has decreased to 0. If $W \notin \mathcal{T}$, then after resetting π, $W \in \mathcal{S}$, and hence the number of vertices of G' not in $\mathcal{S} \cup \mathcal{T}$ decreases. If $W \in \mathcal{T}$, then, in the next iteration, Case 1 applies. So the number of Case 2 iterations in which we do not deshrink is at most $|V|$. This proves the theorem. ∎

This gives the theorem of Edmonds [1965b]:

Corollary 26.1a. *A minimum-weight perfect matching can be found in time $O(n^2m)$.*

Proof. By Theorem 26.1, since each iteration can be performed in time $O(m)$. ∎

This implies that also a maximum-weight matching can be found in time $O(n^2m)$:

Corollary 26.1b. *A maximum-weight matching can be found in time $O(n^2m)$.*

Proof. Let $G = (V, E)$ be a graph with weight function $w \in \mathbb{Q}^E$. Extend G as follows. Make copies G' and w' of G and w. Connect each $v \in V$ to its copy in V', by an edge of weight 0. Let M be a maximum-weight perfect matching in the extended graph. The restriction of M to the original edges is a maximum-weight matching in G. ∎

Notes. In fact, a bound of $\frac{3}{2}|V|$ can be shown in (26.7) (as the size of any set in Ω is odd), implying a bound of $|V|^2$ on the number of iterations in Theorem 26.1.

26.2a. An $O(n^3)$ algorithm

In the above description, we estimated the time required for any iteration by $O(m)$. This leaves time to find the walk in each iteration just from scratch, and to construct the graph $G' = G_\pi/\Omega$ from scratch, after any shrinking or deshrinking step.

Like in the cardinality case, we can speed this up (i) by using the result of the previous walk-search in the next walk-search, and (ii) by constructing the graph G' only in an implicit way. In this way we can reduce the time per iteration from $O(m)$ to $O(n)$ on average, leading to an overall time bound of $O(n^3)$.

Again we use M-alternating forests to reach this goal. Thus, next to Ω, π, M, and the C_U, we keep an M-alternating forest F in $G' := G_\pi/\Omega^{\max}$.

We do not keep the graph G'. Instead, we keep for each pair Y, Z of disjoint sets in Ω an edge e_{YZ} of G connecting Y and Z and minimizing $w_\pi(e_{YZ})$. We take e_{YZ} void if no such edge exists. We keep the e_{YZ} as lists: for each $Y \in \Omega$ we have a list containing the e_{YZ}.

Moreover, for each $Y \in \Omega$ we keep an edge e_Y with $e_Y = e_{YZ}$ for some $Z \in \text{even}(F)$ and with $w_\pi(e_{YZ})$ minimal. Again, if no such e_{YZ} exists, e_Y is void.

Finally, for each $v \in V$ we keep

(26.9) $$p(v) := \sum_{\substack{U \in \Omega \\ v \in U}} \pi(U).$$

Initially, we set $\Omega := \{\{v\} \mid v \in V\}$, $\pi(\{v\}) := 0$ and $p(v) := 0$ for each $v \in V$, and $M := \emptyset$, $F := \emptyset$. The e_{YZ} and e_Y are easily set.

Next we apply the following iteratively:

(26.10) Reset $\pi(U) := \pi(U) - \alpha$ for $U \in \text{odd}(F)$ and $\pi(U) := \pi(U) + \alpha$ for $U \in \text{even}(F)$, where α is the largest value maintaining (26.1). Update p accordingly. After that, at least one of the following three cases applies.
Case 1: $w_\pi(e_U) = 0$ for some $U \in \text{free}(F)$. Extend F by e_U and update the e_Y (*forest augmentation*).
Case 2: $w_\pi(e_U) = 0$ for some $U \in \text{even}(F)$. Let e_U connect vertices U and W in even(F). Let P and Q be the $X - U$ and the $X - W$ path in $(\Omega^{\max}, F)$, respectively.
Case 2a: Paths P and Q are disjoint. Then P and Q form with e_U an M-augmenting path, yielding a matching M' in G' with $|M'| = |M| + 1$. Reset $M := M'$, $F := M'$, and update the e_Y (*matching augmentation*).
Case 2b: Paths P and Q intersect. Then they contain (with e_U) an M-blossom B. Let T be the union of the sets (in $\Omega^{\max}$) forming the vertices of B. Add T to Ω, setting $C_T := B$ and $\pi(T) := 0$. Reset $F := F \setminus EB$ and $M \setminus EB$, and update the e_{YZ} and e_Y (*shrinking*).
Case 3: $\pi(U) = 0$ for some $U \in \text{odd}(F)$ with $|U| \geq 3$. Let v be the vertex in C_U covered by an edge in M and let u be the vertex in C_U covered by an edge in $F \setminus M$. Let P be the even-length $u - v$ path in C_U and let N be the matching in $C_U - v$. Delete U from Ω, reset $F := F \cup EP \cup N$ and $M := M \cup N$, and update the e_{YZ} and e_Y (*deshrinking*).

(In updating F and M, we update them as graphs on $\Omega^{\max}$.)

The number of iterations between any two matching augmentations is at most $|V|$, as may be proved similarly to the proof of Theorem 26.1 (replacing $\mathcal{S}$ by odd(F) and $\mathcal{T}$ by even(F)).

In the iteration (26.10), we can find the value α in $O(n)$ time, as it is the minimum of $w_\pi(e_U)$ over $U \in \text{free}(F)$, of $\frac{1}{2}w_\pi(e_U)$ over $U \in \text{even}(F)$, and of $\pi(U)$ over $U \in \text{odd}(F)$ with $|U| \geq 3$. So we can update π and p in $O(n)$ time. Also F and M can be updated in $O(n)$ time (as they have $O(n)$ edges).

Note that each time we need the value of $w_\pi(e)$ for some edge e (when determining α or the e_{YZ} and e_Y), then e connects two disjoint sets in $\Omega^{\max}$, and hence $w_\pi(e) = w(e) - p(u) - p(v)$. Note also that the resetting of π on $\Omega^{\max}$ changes no e_{YZ} and e_Y.

In Case 1, Ω, π, p, and the e_{YZ} are unchanged. The set $U \in \Omega^{\max}$ is moved from free(F) to odd(F), and a set $W \in \Omega^{\max}$ (the mate of U in M) is moved from free(F) to even(F). To update the e_Y, it suffices to scan the list of the e_{WZ}. This can be done in $O(n)$ time.

In Case 2a, Ω, π, p, and the e_{YZ} are unchanged. Since (in the new situation) $F = M$, we delete from even(F) and odd(F) all sets in $\Omega^{\max}$ covered by M. We can find the e_Y by scanning all e_{YZ}. We have $O(n^2)$ time for this, since there are only $\frac{1}{2}|V|$ matching augmentations.

In Case 2b, set T is inserted into $\Omega^{\max}$ and into even(F), and the sets in VB are removed from even(F) and odd(F). We need to find the e_{TZ}, which can be done by scanning the e_{YZ} for each $Y \in VB$. At the same time, the e_Z can be updated. This can be done in $O(|VB|n)$ time.

In Case 3, set U is removed from $\Omega^{\max}$ and from odd(F), and the sets in VC_U become members of $\Omega^{\max}$ and are inserted into even(F) or odd(F). This modifies no e_{YZ} (except that all e_{UZ} disappear). By scanning the e_{YZ} for each $Y \in VC_U$, we can update the e_Z. This can be done in $O(|VC_U|n)$ time.

Now, between any two matching augmentations, the sum of the $|VC_U|$ over the U added or removed is $O(n)$, since any set added will not be removed before the next matching augmentation (cf. (26.8)). So between any two matching augmentations, the iterations can be done in $O(n^2)$ time.

This gives the result of Gabow [1973] and Lawler [1976b]:

Theorem 26.2. *A minimum-weight perfect matching can be found in $O(n^3)$ time.*

Proof. See above. ∎

Several ingredients in this method can be implemented so as to require only $O(m)$ time between any two matching augmentations. However, reducing the time needed to administer Ω requires additional data structure — see the references in Section 26.3a.

26.3. Further results and notes

26.3a. Complexity survey for weighted nonbipartite matching

Complexity survey for weighted nonbipartite matching ($*$ indicates an asymptotically best bound in the table):

$O(n^4)$	Edmonds [1965b]
$O(n^3)$	Gabow [1973], Lawler [1976b]
$O(nm \log n)$	Galil, Micali, and Gabow [1982, 1986] (cf. Ball and Derigs [1983])
$O(n(m \log\log\log_{m/n} n + n \log n))$	Gabow, Galil, and Spencer [1984, 1989]
$O(n^{3/4} m \log W)$	Gabow [1985a,1985b]

≫

continued

*	$O(n(m + n \log n))$	Gabow [1990]
*	$O(m \log(nW)\sqrt{n\alpha(m,n) \log n})$	Gabow and Tarjan [1991]

Here W is the maximum absolute value of the weights, assuming they are integer.

Cunningham and Marsh [1978] gave a *primal* algorithm for weighted nonbipartite matching that takes $O(n^2 m)$ time (where, throughout the algorithm, there is a perfect matching at hand, the weight of which is improved iteratively). They state that it can be improved to $O(n^3)$. Derigs [1981] gave a shortest augmenting path method of running time $O(n^3)$. In Derigs [1988b] an $O(\min\{n^3, nm \log n\})$ algorithm is given based on successive improvement of a perfect matching by choosing an improving alternating circuit.

26.3b. Derivation of the matching polytope characterization from the algorithm

Edmonds' weighted matching algorithm directly yields the description of the perfect matching polytope. Indeed, one can derive from Edmonds' algorithm the following. Let $G = (V, E)$ be a graph and let $w \in \mathbb{Q}^E$ be a weight function. Then:

(26.11) the minimum weight of a perfect matching is equal to the maximum value of $\sum_{U \in \mathcal{P}_{\text{odd}}(V)} \pi(U)$ where π ranges over all functions $\pi : \mathcal{P}_{\text{odd}}(V) \to \mathbb{Q}$ satisfying (26.1),

where $\mathcal{P}_{\text{odd}}(V)$ denotes the collection of odd-size subsets of V.

To see this, we may assume that w is nonnegative: if μ is the minimum value of $w(e)$ over all edges e, decreasing each $w(e)$ by μ decreases both the maximum and the minimum by $\frac{1}{2}|V|\mu$.

That the minimum is not smaller than the maximum follows from (26.3). Equality follows from the fact that in the algorithm the final perfect matching and the final function π have equality throughout in (26.1). This shows (26.11).

It implies Edmonds' perfect matching polytope theorem: the perfect matching polytope of any graph $G = (V, E)$ is determined by (25.2). Indeed, by (weak) LP-duality, for any weight function $w \in \mathbb{Q}^E$, the minimum weight of a perfect matching is equal to the minimum of $w^{\mathsf{T}}x$ taken over the polytope determined by (25.2). Hence the two polytopes coincide.

26.3c. Further notes

Weber [1981] and Derigs [1985a] analyzed the sensitivity of minimum-weight perfect matchings to changing edge weights. White [1974] studied the maximum weight of a matching of size k, as a function of k.

An outstanding open problem is to formulate the weighted matching problem as a linear programming problem of size polynomial in the size of the graph, by extending the set of variables. That is, is the matching polytope of a graph $G = (V, E)$ equal to the projection of some polytope $\{x \mid Ax \leq b\}$ with A and b having size polynomial in $|V| + |E|$?

Yannakakis [1988,1991] showed that this is not possible in a symmetric fashion. (That is, for $G = K_n$ there is not a system $Ax \leq b$ which is invariant under each permutation of the vertex set.) For further partial results, see Yannakakis [1988, 1991], Gerards [1991], and Barahona [1993a,1993b].

Gabow, Kaplan, and Tarjan [1999,2001] gave fast algorithms to test uniqueness of a minimum-weight perfect matching.

For heuristics and fast approximation methods for the weighted matching problem if the weight function satisfies the triangle inequality (including matching points in Euclidean space), see Papadimitriou [1977b], Avis [1978,1981,1983], Supowit, Plaisted, and Reingold [1980], Iri, Murota, and Matsui [1981,1983], Reingold and Tarjan [1981], Bartholdi and Platzman [1983], Reingold and Supowit [1983], Supowit and Reingold [1983], Supowit, Reingold, and Plaisted [1983], Plaisted [1984], Grigoriadis and Kalantari [1986,1988], Grigoriadis, Kalantari, and Lai [1986], Imai [1986], Weber and Liebling [1986], Avis, Davis, and Steele [1988], Vaidya [1988, 1989a,1989b], Kalyanasundaram and Pruhs [1991,1993], Marcotte and Suri [1991], Goemans and Williamson [1992,1995a], Osiakwan and Akl [1994], Williamson and Goemans [1994], Jünger and Pulleyblank [1995], Arora [1997,1998], Varadarajan [1998], and Varadarajan and Agarwal [1999].

For studies of implementing weighted matching algorithms, see Cunningham and Marsh [1978], Burkard and Derigs [1980], Derigs [1981,1986a,1986b,1988b], Lessard, Rousseau, and Minoux [1989], Derigs and Metz [1991], Applegate and Cook [1993], and Cook and Rohe [1999].

Grötschel and Holland [1985] report on implementing a cutting plane algorithm for the weighted matching problem based on the simplex method (cf. Derigs and Metz [1991]). For an alternative approach, see Lessard, Rousseau, and Minoux [1989]. Derigs and Metz [1986b] showed how solving the matching problem fractionally can help in finding a shortest augmenting path.

Megiddo and Tamir [1978] gave an $O(n \log n)$ algorithm to find a maximum-weight matching in a graph $G = (V, E)$, if each weight $w(uv)$ is equal to $a(u) + b(v)$ for $u < v$, where the vertices are ordered by $<$ and where $a, b : V \to \mathbb{Q}$.

For weighted matching problems with side constraints, see Ball, Derigs, Hilbrand, and Metz [1990].

For a survey on weighted matching algorithms, see Galil [1983,1986a,1986b]. Books covering weighted nonbipartite matching algorithms include Christofides [1975], Lawler [1976b], Minieka [1978], Papadimitriou and Steiglitz [1982], Gondran and Minoux [1984], Derigs [1988a], Nemhauser and Wolsey [1988], Cook, Cunningham, Pulleyblank, and Schrijver [1998], Jungnickel [1999], and Korte and Vygen [2000].

Chapter 27

Nonbipartite edge cover

Edge cover is closely related to matching, through a construction described by Gallai. In this chapter we derive basic results on edge covers (min-max relation, polyhedral characterization, strongly polynomial-time algorithm) from the results on matchings given in the previous chapters.
In this chapter, graphs can be assumed to be loopless.

27.1. Minimum-size edge cover

With Gallai's theorem, the Tutte-Berge formula implies a formula for the edge cover number $\rho(G)$ (where $o(G[U])$ denotes the number of odd components of $G[U]$):

Theorem 27.1. *Let $G = (V, E)$ be a graph without isolated vertices. Then*

$$\rho(G) = \max_{U \subseteq V} \frac{|U| + o(G[U])}{2}. \tag{27.1}$$

Proof. By Gallai's theorem (Theorem 19.1) and the Tutte-Berge formula (Theorem 24.1),

$$\begin{aligned} \rho(G) &= |V| - \nu(G) = |V| - \min_{U \subseteq V} \frac{|V| + |U| - o(G-U)}{2} \\ &= \max_{U \subseteq V} \frac{|U| + o(G[U])}{2}. \end{aligned} \tag{27.2}$$

∎

This min-max relation is equivalent to: $\rho(G)$ is equal to the maximum value of

$$\sum_{U \in \mathcal{U}} \lceil \tfrac{1}{2}|U| \rceil, \tag{27.3}$$

where $\mathcal{U}$ is a collection of disjoint odd subsets of V such that no edge of G connects two distinct sets in $\mathcal{U}$.

By the method of Gallai's theorem, one can derive a minimum-size edge cover from a maximum-size matching M, just by adding for each vertex v

missed by M, an arbitrary edge incident with v. Hence a minimum-size edge cover can be found in polynomial time.

One can reduce the problem of finding a minimum-weight edge cover to that of finding a minimum-weight perfect matching, as described in Section 19.3. It gives the following result of Edmonds and Johnson [1970]:

Theorem 27.2. *A minimum-weight edge cover can be found in $O(n^3)$ time.*

Proof. From Corollary 26.1b, with the method of Section 19.3. ∎

27.2. The edge cover polytope and total dual integrality

The *edge cover polytope* of a graph $G = (V, E)$ is the convex hull of the incidence vectors of edge covers. We will show that the edge cover polytope is determined by

(27.4) (i) $0 \leq x_e \leq 1$ for each $e \in E$,
(ii) $x(E[U] \cup \delta(U)) \geq \lceil \frac{1}{2}|U| \rceil$ for each $U \subseteq V$ with $|U|$ odd,

and moreover, that this system is totally dual integral. The latter statement will be derived from the Cunningham-Marsh formula (Theorem 25.2), and is equivalent to:

Theorem 27.3. *Let $G = (V, E)$ be a graph without isolated vertices and let $w \in \mathbb{Z}_+^E$ be a weight function. Then the minimum weight of an edge cover is equal to the maximum value of*

$$\sum_{U \in \mathcal{P}_{\text{odd}}(V)} z_U \lceil \tfrac{1}{2}|U| \rceil, \tag{27.5}$$

where $z_U \in \mathbb{Z}_+$ for each $U \in \mathcal{P}_{\text{odd}}(V)$ such that

$$\sum_{U \in \mathcal{P}_{\text{odd}}(V)} z_U \chi^{E[U] \cup \delta(U)} \leq w. \tag{27.6}$$

Proof. We first show:

(27.7) in the Cunningham-Marsh formula one can assume that for each $v \in V$ there is an edge $e \in \delta(v)$ with $y_v + \sum_{U \ni v} z_U \leq w(e)$.

Indeed, by Theorem 25.2 we can take y, z such that $\mathcal{F} := \{U \mid z_U > 0\}$ is laminar. Now choose $v \in V$. Suppose that $y_v + \sum_{U \ni v} z_U > w(e)$ for each edge $e \in \delta(v)$. If no set in $\mathcal{F}$ covers v, then reducing y_v by 1 would maintain the conditions, contradicting the fact that y, z attain the minimum in the Cunningham-Marsh formula.

So some $T \in \mathcal{F}$ covers v. Choose an inclusionwise minimal set $T \in \mathcal{F}$ covering v. As $\mathcal{F}$ is laminar, $U \supseteq T$ for each $U \in \mathcal{F}$ containing v. Then for each edge $e = uv$ with $v \in e \subseteq T$ one has for each $U \in \mathcal{F}$: if $v \in U$, then $e \subseteq U$. So for each such edge $e = uv$,

$$y_u + y_v + \sum_{U \supseteq e} z_U \geq y_v + \sum_{U \ni v} z_U > w(e). \tag{27.8}$$

Hence, if we choose $s \in T \setminus \{v\}$, then decreasing z_T by 1 and increasing y_s and $z_{T \setminus \{v,s\}}$ by 1, gives again an optimum solution. Iterating this for all v, gives a solution as in (27.7).

We next show the theorem. For each vertex v, let e_v be an edge incident with v of minimum weight and let $\mu(v) := w(e_v)$. For each edge $e = uv$, define $w'(e) := \mu(u) + \mu(v) - w(e)$.

By the Cunningham-Marsh formula, there exists a matching M and $y_v \in \mathbb{Z}_+$ $(v \in V)$ and $z'_U \in \mathbb{Z}_+$ $(U \in \mathcal{P}_{\mathrm{odd}}(V))$ such that

$$\begin{aligned} &\text{(i) } y_u + y_v + \sum_{U \supseteq e} z'_U \geq w'(e) \text{ for each edge } e = uv; \\ &\text{(ii) } w'(M) = \sum_{v \in V} y_v + \sum_{U \in \mathcal{P}_{\mathrm{odd}}(V)} z'_U \lfloor \tfrac{1}{2}|U| \rfloor. \end{aligned} \tag{27.9}$$

We may assume that $z'_U = 0$ if $|U| = 1$. By (27.7) we may assume that for each $v \in V$:

$$y_v + \sum_{T \ni v} z'_T \leq w'(e) \tag{27.10}$$

for some edge e incident with v.

Let F be the edge cover obtained from M by adding the edge e_v for each vertex v missed by M. For each $U \in \mathcal{P}_{\mathrm{odd}}(V)$, define:

$$z_U := \begin{cases} \mu(v) - y_v - \sum_{T \ni v} z'_T & \text{if } U = \{v\}, \\ \qquad z'_U & \text{if } |U| \geq 3. \end{cases} \tag{27.11}$$

Clearly $z_U \geq 0$ if $|U| \geq 3$. If $U = \{v\}$, then let $e = uv \in \delta(v)$ satisfy satisfying (27.10). Hence

$$z_{\{v\}} = \mu(v) - y_v - \sum_{T \ni v} z'_T \geq \mu(v) - w'(e) = w(e) - \mu(u) \geq 0. \tag{27.12}$$

So z is nonnegative.

Now for each edge $e = uv$ one has:

$$\begin{aligned} \sum_{U \cap e \neq \emptyset} z_U &= z_{\{u\}} + z_{\{v\}} + \sum_{U \cap e \neq \emptyset} z'_U \\ &= \mu(u) - y_u - \sum_{U \ni u} z'_U + \mu(v) - y_v - \sum_{U \ni v} z'_U + \sum_{U \cap e \neq \emptyset} z'_U \\ &= \mu(u) + \mu(v) - y_u - y_v - \sum_{U \supseteq e} z'_U \leq \mu(u) + \mu(v) - w'(e) \\ &= w(e). \end{aligned} \tag{27.13}$$

Moreover,

$$\sum_U z_U\lceil\tfrac{1}{2}|U|\rceil = \sum_{v\in V}(\mu(v) - y_v - \sum_{U\ni v} z'_U) + \sum_U z'_U\lceil\tfrac{1}{2}|U|\rceil \tag{27.14}$$
$$= \sum_{v\in V}\mu(v) - \sum_{v\in V} y_v - \sum_U z'_U\lfloor\tfrac{1}{2}|U|\rfloor = \sum_{v\in V}\mu(v) - w'(M) = w(F).$$

∎

(The idea of using w' was given by J.F. Geelen.)

Equivalently, we can state:

Corollary 27.3a. *System* (27.4) *determines the edge cover polytope and is TDI.*

Proof. This is equivalent to Theorem 27.3. ∎

27.3. Further notes on edge covers

27.3a. Further notes

Inspired by Edmonds' algorithm for maximum-weight matching, White [1967] and Murty and Perin [1982] described minimum-weight edge cover algorithms based on blossoms.

White and Gillenson [1975] and Murty and Perin [1982] described a blossom-type algorithm to find a minimum-weight edge cover of given size k. Also White [1971] considered the problem of finding a minimum-weight edge cover of a given size, by parametrizing the weight function.

In fact, the convex hull of incidence vectors of edge covers F with $k \leq |F| \leq l$ is equal to the edge cover polytope intersected with $\{x \in \mathbb{R}^E \mid k \leq x(E) \leq l\}$. This can be proved similarly to the proof of Corollary 18.10a.

Hurkens [1991] characterized adjacency on the edge cover polytope and derived that its diameter is equal to $|E| - \rho(G)$. (This turns out to be harder to prove than the corresponding results for the matching polytope given in Section 25.5a.)

27.3b. Historical notes on edge covers

The nonbipartite edge cover problem was considered by Gallai [1959a] and Norman and Rabin [1959]. The latter were motivated by a problem of Roth [1958] related to minimizing the number of switches in a switching systems, for which they considered the problem of finding a minimum cover for a cubical complex.

Norman and Rabin [1959] showed that an edge cover F in a graph has minimum size if and only if there is no path P such that the end vertices of P are covered more than once by F, while all intermediate vertices are covered exactly once by F, and such that the edges of P are alternatingly in and out F, with the first and last edge in F. (Thus $F\triangle P$ is an edge cover of smaller size than F.)

Chapter 28

Edge-colouring

Edge-colouring means covering the edge set by matchings. The problem goes back to Tait [1878b], who showed that the four-colour conjecture is equivalent to the 3-edge-colourability of any bridgeless cubic planar graph. Nonbipartite edge-colouring is less tractable than in the special case of bipartite graphs. No tight min-max relation is known and finding a minimum edge-colouring is NP-complete. In this chapter we prove Vizing's theorem, which gives an almost tight min-max relation. Moreover, we consider the 'fractional' edge-colouring number, which approximates the edge-colouring number. It can be characterized and computed with the help of matching results. We also consider the related problem of packing edge covers.

28.1. Vizing's theorem for simple graphs

We recall some definitions and notation. Let $G = (V, E)$ be a graph. An *edge-colouring* is a partition of E into matchings. Each matching in an edge-colouring is called a *colour* or an *edge-colour*. A k*-edge-colouring* is an edge-colouring with k colours. G is k*-edge-colourable* if a k-edge-colouring exists. The smallest k for which G is k-edge-colourable is called the *edge-colouring number* of G, denoted by $\chi'(G)$. Since an edge-colouring of G is a vertex-colouring of the line-graph $L(G)$ of G, we have that $\chi'(G) = \chi(L(G))$.

Clearly $\chi'(G) \geq \Delta(G)$, where $\Delta(G)$ denotes the maximum degree of G. We saw that $\chi'(G) = \Delta(G)$ if G is bipartite (Kőnig's edge-colouring theorem (Theorem 20.1)). On the other hand, $\chi'(G) > \Delta(G)$ if $G = K_3$. It was proved by Holyer [1981] that deciding if $\chi(G) \leq 3$ is NP-complete.

Nevertheless, $\Delta(G)$ is a good estimate of the edge-colouring number as Vizing [1964,1965a] showed the following (our proof roots in Ehrenfeucht, Faber, and Kierstead [1984]):

Theorem 28.1 (Vizing's theorem for simple graphs)**.** $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$ *for any simple graph* G.

Proof. The inequality $\Delta(G) \leq \chi'(G)$ being trivial, we show $\chi'(G) \leq \Delta(G) + 1$. To prove this inductively, it suffices to show for any simple graph G:

(28.1) Let v be a vertex such that v and all its neighbours have degree at most k, while at most one neighbour has degree precisely k. Then if $G - v$ is k-edge-colourable, also G is k-edge-colourable.

We prove (28.1) by induction on k, the case $k = 0$ being trivial. We can assume that each neighbour u of v has degree $k-1$, except for one neighbour having degree exactly k, since otherwise we can add a new vertex w and an edge uw without violating the condition in (28.1).

Consider any k-edge-colouring of $G - v$. For $i = 1, \ldots, k$, let X_i be the set of neighbours of v that are missed by colour i. Choose the colouring such that $\sum_{i=1}^{k} |X_i|^2$ is minimized.

First assume that $|X_i| \neq 1$ for all i. Since all but one neighbour of v is in precisely two of the X_i, and one neighbour is in precisely one X_i, we have

$$(28.2) \qquad \sum_{i=1}^{k} |X_i| = 2\deg(v) - 1 < 2k.$$

Hence there exist i, j with $|X_i| < 2$ and $|X_j|$ odd. So $|X_i| = 0$ and $|X_j| \geq 3$. Consider the subgraph H made by all edges of colours i and j, and consider a component of H containing a vertex in X_j. This component is a path P starting in X_j. Exchanging colours i and j on P reduces $|X_i|^2 + |X_j|^2$, contradicting our minimality assumption.

So we can assume that $|X_k| = 1$, say $X_k = \{u\}$. Let G' be the graph obtained from G by deleting edge vu and deleting all edges of colour k. So $G' - v$ is $(k-1)$-edge-coloured. Moreover, in G', vertex v and all its neighbours have degree at most $k - 1$, and at most one neighbour has degree $k - 1$. So by the induction hypothesis, G' is $(k - 1)$-edge-colourable. Restoring colour k, and giving edge vu colour k, gives a k-edge-colouring of G. ∎

Notes. This theorem was also announced in an abstract of Gupta [1966].

The above proof implies the stronger result of Fournier [1973] that a simple graph G is $\Delta(G)$-edge-colourable if the maximum-degree vertices span no circuit (since this last condition implies that the maximum-degree vertices induce a forest as subgraph, and hence there exists a maximum-degree vertex v with at most one neighbour that has maximum degree).

Petersen [1898] gave the example of the (now-called) Petersen graph (Figure 28.1) which is 2-connected and cubic but not 3-edge-colourable. It was conjectured by Tutte [1966] that each 2-connected cubic graph without Petersen graph minor, is 3-edge-colourable. This conjecture was proved (using the 4-colour theorem) by the combined efforts of Robertson, Seymour, and Thomas [1997], Sanders, Seymour, and Thomas [2000], and Sanders and Thomas [2000].

Complexity. The proof gives a polynomial-time algorithm to find a $(\Delta+1)$-edge-colouring of a simple graph, in fact, $O(\Delta n^2)$-time. As we can assume that $\Delta n = O(m)$ (since we can merge vertices of degree at most $\frac{1}{2}\Delta$), this implies an $O(nm)$-time algorithm.

Gabow, Nishizeki, Kariv, Leven, and Terada [1985] gave algorithms finding a $(\Delta + 1)$-edge-colouring of a simple graph G of maximum degree Δ, with running

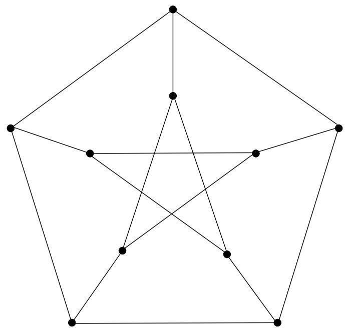


Figure 28.1
The Petersen graph

times $O(m\Delta \log n)$ and $O(m\sqrt{n \log n})$ (improving $O(nm)$ of Terada and Nishizeki [1982]).

28.2. Vizing's theorem for general graphs

In Theorem 28.1 we cannot delete the condition that G be simple: the graph G obtained from K_3 by replacing each edge by two parallel edges, has $\chi'(G) = 6$ and $\Delta(G) = 4$. However, Vizing's theorem can be extended so as to take also the nonsimple case into account. For any graph $G = (V, E)$ and $u, v \in V$, let $\mu(u, v)$ denote the number of edges connecting u and v, called the *multiplicity* of $\{u, v\}$. Let $\mu(G)$ denote the maximum of $\mu(u, v)$ over all distinct $u, v \in V$. Then Vizing [1964,1965a] showed (again, our proof roots in Ehrenfeucht, Faber, and Kierstead [1984]):

Theorem 28.2 (Vizing's theorem). *$\Delta(G) \leq \chi'(G) \leq \Delta(G) + \mu(G)$ for any graph G.*

Proof. The inequality $\Delta(G) \leq \chi'(G)$ being trivial, we show $\chi'(G) \leq \Delta(G) + \mu(G)$. To prove this inductively, it suffices to show for any graph G:

(28.3) Let v be a vertex of degree at most k such that each neighbour u of v satisfies $\deg(u) + \mu(u, v) \leq k + 1$, with equality for at most one neighbour. Then if $G - v$ is k-edge-colourable, also G is k-edge-colourable.

We prove (28.3) by induction on k, the case $k = 0$ being trivial. We can assume that for each vertex u in $N(v)$ (the set of neighbours of v) we have $\deg(u) + \mu(u, v) = k$, except for one satisfying $\deg(u) + \mu(u, v) = k + 1$, since otherwise we can add a new vertex w and an edge uw without violating the condition in (28.3).

Consider any k-edge-colouring of $G - v$. For $i = 1, \dots, k$, let X_i be the set of neighbours of v that are missed by colour i. Choose the colouring such that $\sum_{i=1}^{k} |X_i|^2$ is minimized.

First assume that $|X_i| \neq 1$ for all i. As each $u \in N(v)$ is in precisely $2\mu(u,v)$ of the X_i, except for one $u \in N(v)$ being in $2\mu(u,v) - 1$ of the X_i, we know

$$\sum_{i=1}^{k} |X_i| = -1 + 2 \sum_{u \in N(v)} \mu(u,v) = 2\deg(v) - 1 < 2k. \tag{28.4}$$

Hence there exist i, j with $|X_i| < 2$ and $|X_j|$ odd. So $|X_i| = 0$ and $|X_j| \geq 3$. Consider the subgraph H made by all edges of colours i and j, and consider a component of H containing a vertex in X_j. This component is a path P starting in X_j. Exchanging colours i and j on P reduces $|X_i|^2 + |X_j|^2$, contradicting our minimality assumption.

So we can assume that $|X_k| = 1$, say $X_k := \{u\}$. Let G' be the graph obtained from G by deleting one of the edges vu and deleting all edges of colour k. So $G' - v$ is $(k-1)$-edge-coloured. Moreover, in G', vertex v has degree at most $k-1$ and each neighbour w of v satisfies $\deg_{G'}(w) + \mu_{G'}(w,v) \leq k$, with equality for at most one neighbour. So by the induction hypothesis, G' is $(k-1)$-edge-colourable. Restoring colour k, and giving the deleted edge vu colour k, gives a k-edge-colouring of G. ∎

Notes. The proof of Theorem 28.2 in fact implies that the edge-colouring number of a graph G is at most

$$\max_{u \in V}(\deg(u) + \max\{1, \max_{\substack{v \in V \\ \deg(v) \geq \deg(u)}} \mu(u,v)\}), \tag{28.5}$$

where $\mu(u,v)$ is the number of edges connecting u and v (cf. Ore [1967]).

Other proofs of Vizing's theorem were given by Ore [1967], Fournier [1973], Berge and Fournier [1991], Misra and Gries [1992], Rao and Dijkstra [1992], and Chew [1997b].

28.3. NP-completeness of edge-colouring

Vizing's theorem gives us a close approximation to the edge-colouring number of a simple graph. The error is at most 1. However, it turns out to be NP-complete to determine the edge-colouring number precisely, even for cubic graphs, which was shown by Holyer [1981]:

Theorem 28.3. *It is* NP*-complete to decide if a given cubic graph is* 3*-edge-colourable.*

Proof. We show that the 3-satisfiability problem (3-SAT) can be reduced to the edge-colouring problem of graphs of maximum degree 3. One easily reduces this last problem to the edge-colouring problem for cubic graphs (by deleting iteratively all vertices of degree ≤ 1, next making a copy of the graph left, and adding an edge between each degree-2 vertex and its copy).

Consider the graph fragment, called the *inverting component*, given by the left-hand picture of Figure 28.2, where the right-hand picture gives its symbolic representation if we take it as part of larger graphs. The pairs a, b and c, d are called the *output pairs*.

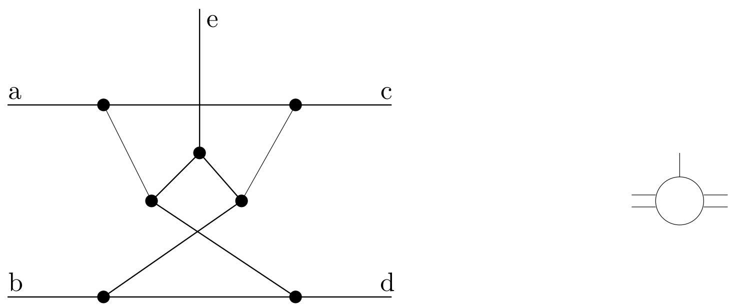
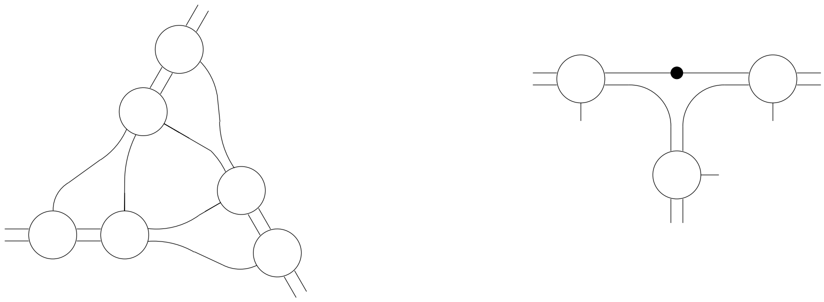


Figure 28.2
The *inverting component* and its symbolic representation.
This graph fragment has the property that a 3-colouring of the edges a, b, c, d, and e is extendible to a 3-edge-colouring of the fragment if and only if either a and b have the same colour while c, d, and e have three distinct colours, or c and d have the same colour while a, b, and e have three distinct colours.

Consider now an instance of the 3-satisfiability problem. From the inverting component we build larger graph fragments. A *splitting component* is given in Figure 28.3(a). For each variable u, occurring k times, as u or $\neg u$, we introduce a fragment Γ_u by concatenating $k-2$ splitting components. So Γ_u has k output pairs, and it has the property that in any colouring either all output pairs are monochromatic, or they all are nonmonochromatic.

For each clause C we introduce a component Δ_C given by Figure 28.3(b). If a variable u occurs in a clause C as u, we connect one of the output pairs of Γ_u with one of the output pairs of Δ_C. If a variable u occurs in a clause C as $\neg u$, we connect one of the output pairs of Γ_u with one side of an inverting component, and connect the other side of this inverting component with one of the output pairs of Δ_C.

In this way we can match up all output pairs of the Γ_u and those of the Δ_C. Deleting all loose ends, we obtain a graph G of maximum degree 3. Now, given the properties of the fragments, one easily checks that the input of the 3-satisfiability problem is satisfiable if and only if G is 3-edge-colourable. ∎



(a) (b)

Figure 28.3

Fragment (a) (the *splitting component*) has the property that for any 3-edge-colouring either all three output pairs are monochromatic or all are nonmonochromatic.

Fragment (b) has the property that a colouring of the output edges is extendible to a 3-edge-colouring of the fragment if and only if at least one of the output pairs is monochromatic.

Leven and Galil [1983] showed more generally that for each k, finding the edge-colouring number of a k-regular graph is NP-complete. (This does not seem to follow from the case $k = 3$.)

28.4. Nowhere-zero flows and edge-colouring

Let $D = (V, A)$ be a directed graph and let Γ be an additive abelian group. A *flow over* Γ is a function $f : A \to \Gamma$ such that for each $v \in V$:

(28.6) $\qquad f(\delta^{\text{in}}(v)) = f(\delta^{\text{out}}(v)).$

The flow is called *nowhere-zero* if all values of f are nonzero.

If G is an undirected graph, then a *flow over* Γ is a flow over Γ in some orientation of G. We say that an undirected graph G has a nowhere-zero flow over Γ if G has an orientation having a nowhere-zero flow over Γ.

Colouring the edges of an undirected graph is related to the problem of finding a nowhere-zero flow over a finite abelian group in the graph. This might be illustrated best by the following easy fact:

(28.7) $\qquad$ a cubic graph G is 3-edge-colourable $\Longleftrightarrow$ G has a nowhere-zero flow over $\mathrm{GF}(4)$.

Since $-x = x$ for each $x \in \mathrm{GF}(4)$, the orientation is irrelevant in this case.

Statement (28.7) implies that the four-colour theorem is equivalent to:

(28.8) each bridgeless cubic planar graph has a nowhere-zero flow over GF(4)

(since the four-colour theorem is equivalent to each bridgeless cubic planar graph being 3-edge-colourable (Tait [1878b])).

In studying nowhere-zero flows, the following theorem shows that for the existence of a nowhere-zero flow, only the size of the group is relevant (the equivalence (i)⇔(ii) was shown by Tutte [1947a], the equivalence (i)⇔(iii) by Tutte [1949], and the equivalence (iii)⇔(iv) by Minty [1967]):

Theorem 28.4. *Let $G = (V, E)$ be a graph and let $k \in \mathbb{Z}$ with $k \geq 1$. Then the following are equivalent:*

(28.9)
(i) *G has a nowhere-zero flow over some abelian group with precisely k elements;*
(ii) *G has a nowhere-zero flow over each abelian group with at least k elements;*
(iii) *G has a flow over $\mathbb{Z}$ taking values in the interval $[1, k-1]$ only;*
(iv) *G has an orientation $D = (V, A)$ with $d^{\text{in}}_A(U) \geq \frac{1}{k} d_E(U)$ for each $U \subseteq V$.*

Proof. The implication (ii)⇒(i) is trivial, while the implication (iii)⇒(i) is easy, by considering the integer values of (iii) as values in the group of integers mod k.

For any graph $G = (V, E)$ and any finite abelian group Γ, let $\phi_\Gamma(G)$ denote the number of nowhere-zero flows over Γ in G. Then for any nonloop edge e of G one has (where G/e is the graph obtained from G by contracting e):

(28.10) $\phi_\Gamma(G) = \phi_\Gamma(G/e) - \phi_\Gamma(G - e)$.

Moreover, if each edge of G is a loop, then:

(28.11) $\phi_\Gamma(G) = (|\Gamma| - 1)^{|E|}$.

This proves that if Γ and Γ' are finite abelian groups with $|\Gamma| = |\Gamma'|$, then $\phi_\Gamma(G) = \phi_{\Gamma'}(G)$. Hence G has a nowhere-zero flow over Γ if and only if G has a nowhere-zero flow over Γ'. Therefore:

(28.12) if G has a nowhere-zero flow over some abelian group of size k, then it has one over each abelian group of size k.

We now consider (i)⇒(iii). By (28.12), (i) implies that G has a nowhere-zero flow over the group of integers mod k. This implies that there is an orientation $D = (V, A)$ of G and a function $f : A \to \{1, \ldots, k-1\}$ such that for each $v \in V$:

(28.13) $f(\delta^{\text{in}}(v)) \equiv f(\delta^{\text{out}}(v)) \pmod{k}$.

We choose the orientation D and the function f such that the sum

(28.14) $$\sum_{v \in V} |f(\delta^{\text{in}}(v)) - f(\delta^{\text{out}}(v))|$$

is minimized. If the sum is 0, we are done. So assume that the sum is nonzero. Define

(28.15) $$U_+ := \{v \in V \mid f(\delta^{\text{in}}(v)) > f(\delta^{\text{out}}(v))\} \text{ and}$$
$$U_- := \{v \in V \mid f(\delta^{\text{in}}(v)) < f(\delta^{\text{out}}(v))\}.$$

Necessarily, there is a directed path P in D from U_- to U_+ (Theorem 11.1). Now reverse the orientation of each arc a on P to its reverse a^{-1}, and define $f(a^{-1}) := k - f(a)$. This maintains (28.13) but reduces the sum (28.14), a contradiction.

This proves (i)⇒(iii), and hence (i)⇔(iii). Since (iii) is maintained if we increase k, also (i) is maintained if we increase k. So with (28.12), (i) implies (ii) if (ii) is restricted to finite groups. Since each infinite abelian group has $\mathbb{Z}$ as subgroup or has arbitrarily large finite subgroups, (iii)⇒(ii) also follows for infinite groups.

The equivalence of (iii) and (iv) follows directly from Hoffman's circulation theorem (Theorem 11.2). ∎

This theorem implies that in studying the existence of nowhere-zero flows, we can restrict ourselves to the group $\mathbb{Z}_k$ with elements $0, \ldots, k-1$ and addition mod k. A *nowhere-zero k-flow* is a nowhere-zero flow over $\mathbb{Z}_k$.

It is easy to characterize the graphs having a nowhere-zero 2-flow: they are precisely the Eulerian graphs. As to larger values of k there are the following three famous conjectures of Tutte. The 5-*flow conjecture* (Tutte [1954a]):

(28.16) (?) each bridgeless graph has a nowhere-zero 5-flow, (?)

the 4-*flow conjecture* (Tutte [1966]):

(28.17) (?) each bridgeless graph without Petersen graph minor has a nowhere-zero 4-flow, (?)

and the 3-*flow conjecture* (W.T. Tutte, 1972 (cf. Bondy and Murty [1976], Unsolved problem 48)):

(28.18) (?) each 4-edge-connected graph has a nowhere-zero 3-flow. (?)

For planar graphs this is equivalent to the theorem of Grötzsch [1958] that each loopless triangle-free planar graph is 3-vertex-colourable.

It may be seen that a cubic graph G has a nowhere-zero 3-flow if and only if G is bipartite. This follows from the fact that the existence of such a flow implies that G has an orientation such that in each vertex the indegree and outdegree differ by a multiple of 3. Hence, one of them is 3, the other 0. Hence each arc is oriented from a source to a sink, and so G is bipartite. The reverse implication is easy, by orienting each edge from one colour class to the other.

Jaeger [1979] showed that each 4-edge-connected graph has a nowhere-zero 4-flow: a 4-edge-connected graph $G = (V, E)$ has two edge-disjoint spanning trees T_1 and T_2 (by Corollary 51.1a). For $i = 1, 2$, let C_i be the symmetric difference of all fundamental circuits of T_i. Then C_1 and C_2 are cycles covering E. This gives a nowhere-zero 4-flow.

Jaeger [1988] proposed a weakened version of the 3-flow conjecture, the *weak 3-flow conjecture*:

(28.19) (?) there exists a number k such that each k-edge-connected graph has a nowhere-zero 3-flow. (?)

By (28.8), the 4-flow conjecture implies the four-colour theorem. For cubic graphs, (28.17) was proved by Robertson, Seymour, and Thomas [1997], Sanders, Seymour, and Thomas [2000], and Sanders and Thomas [2000].

One should note that having a nowhere-zero 4-flow is equivalent to the existence of two cycles covering the edge set. In other words, there exist two disjoint T-joins, where T is the set of odd-degree vertices (see Chapter 29).

It was proved by Seymour [1981b] that each bridgeless graph has a nowhere-zero 6-flow. (Inspired by Seymour's method, Younger [1983] gave a polynomial-time algorithmic proof.)

Seymour's theorem improves an earlier result of Jaeger [1976,1979] that each bridgeless graph has a nowhere-zero 8-flow. This is equivalent to: each bridgeless graph contains three cycles covering all edges.

Jaeger [1984] offered a conjecture, the *circular flow conjecture*, that implies both the 3-flow and the 5-flow conjecture:

(28.20) (?) for each $k \geq 1$, each $4k$-connected graph has an orientation such that in each vertex, the indegree and the outdegree differ by an integer multiple of $2k + 1$. (?)

For $k = 1$, this is equivalent to the 3-flow conjecture. For $k = 2$, it implies the 5-flow conjecture: Let $G = (V, E)$ be a 3-edge-connected graph, and replace each edge by 3 parallel edges. The new graph, H say, is 9-edge-connected. If (28.20) is true for $k = 2$, H has an orientation such that in each vertex, the indegree and the outdegree differ by a multiple of 5. This can easily be transformed to a nowhere-zero 5-flow in G.[12]

More on the 3-flow conjecture can be found in Fan [1993] and Kochol [2001]. Jaeger [1979,1988] and Seymour [1995a] gave surveys on nowhere-zero flows, and a book on this topic was written by Zhang [1997b]. We continue discussing nowhere-zero flows in Section 38.8.

[12] Orient any edge e of G in the direction of the majority of the direction of the three parallel edges in H made from e, with flow equal to 3 if all three edges have the same orientation, and 1 otherwise.

28.5. Fractional edge-colouring

Determining the edge-colouring number of a graph is NP-complete, but with matching techniques one can determine a fractional version of it in polynomial time.

Let $G = (V, E)$ be a graph. The *fractional edge-colouring number* $\chi'^*(G)$ of G is defined as

$$\chi'^*(G) := \min\{\sum_{M\in\mathcal{M}} \lambda_M \mid \lambda \in \mathbb{R}_+^{\mathcal{M}}, \sum_{M\in\mathcal{M}} \lambda_M \chi^M = \mathbf{1}\}, \tag{28.21}$$

where $\mathcal{M}$ denotes the collection of all matchings in G.

So if we require the λ_M to be integer, this would define the edge-colouring number of G. Therefore, we have

$$\chi'^*(G) \leq \chi'(G). \tag{28.22}$$

The Petersen graph is an example of a graph G with $\chi'^*(G) = 3$ and $\chi'(G) = 4$. In Section 28.7 we shall see that $\chi'^*(G)$ can be computed in polynomial time.

$\chi'^*(G)$ can be characterized as follows. For any natural number $k \geq 1$, let G_k be the graph obtained from G by replacing each edge by k parallel edges. Then

$$\chi'^*(G) = \min_{k\geq 1} \frac{\chi'(G_k)}{k}. \tag{28.23}$$

This follows from the fact that the minimum in (28.21) is attained by rational λ_M. Then the minimum in (28.23) is attained by $k :=$ the l.c.m. of the denominators of the λ_M.

From Edmonds' matching polytope theorem (Corollary 25.1a), a characterization of the fractional edge-colouring number follows:

Theorem 28.5. *The fractional edge-colouring number $\chi'^*(G)$ satisfies:*

$$\chi'^*(G) = \max\{\Delta(G), \max_{U\subseteq V, |U|\geq 3} \frac{|E[U]|}{\lfloor \frac{1}{2}|U| \rfloor}\}. \tag{28.24}$$

Proof. Let μ be equal to the maximum in (28.24). Then $\chi'^*(G) \geq \mu$, since if λ_M attains minimum (28.21) and if vertex v has maximum degree, then

$$\begin{aligned} \chi'^*(G) &= \sum_M \lambda_M \geq \sum_M \lambda_M |M \cap \delta(v)| = \sum_{e\in\delta(v)} \sum_{M\ni e} \lambda_M \\ &= \sum_{e\in\delta(v)} 1 = \Delta(G). \end{aligned} \tag{28.25}$$

Moreover, for each $U \subseteq V$ with $|U| \geq 3$,

(28.26) $$\chi'^*(G) = \sum_M \lambda_M \geq \sum_M \lambda_M \frac{|M \cap E[U]|}{\lfloor \frac{1}{2}|U| \rfloor} = \frac{1}{\lfloor \frac{1}{2}|U| \rfloor} \sum_{e \in E[U]} \sum_{M \ni e} \lambda_M$$
$$= \frac{|E[U]|}{\lfloor \frac{1}{2}|U| \rfloor}.$$

To see that $\chi'^*(G) = \mu$, let x be the all-$\frac{1}{\mu}$ vector in $\mathbb{R}^E$. Then $x(\delta(v)) \leq 1$ for each $v \in V$ and $x(E[U]) \leq \lfloor \frac{1}{2}|U| \rfloor$ for each $U \subseteq V$ with $|U| \geq 3$. Hence x belongs to the matching polytope of G. So x is a convex combination of incidence vectors of matchings. Therefore $\mathbf{1} = \mu \cdot x = \sum_M \lambda_M \chi^M$ for some $\lambda_M \geq 0$ with $\sum_M \lambda_M = \mu$, showing that $\chi'^*(G) \leq \mu$. ∎

This implies for regular graphs:

Corollary 28.5a. *Let $G = (V, E)$ be a k-regular graph. Then $\chi'^*(G) = k$ if and only if $|\delta(U)| \geq k$ for each odd subset U of V.*

Proof. By Theorem 28.5, $\chi'^*(G) = k$ if and only if $|E[U]| \leq k\lfloor \frac{1}{2}|U| \rfloor$ for each subset U of V. This last is equivalent to $|\delta(U)| \geq k$ for each odd subset U of V. ∎

Call a graph $G = (V, E)$ a *k-graph* if G is regular of degree k and if $|\delta(U)| \geq k$ for each odd subset U of V. So by Corollary 28.5a, a k-regular graph G is a k-graph if and only if $\chi'^*(G) = k$.

28.6. Conjectures

Seymour [1979a] conjectures that

(28.27) (?) $\lceil \chi'^*(G) \rceil = \lceil \frac{1}{2}\chi'(G_2) \rceil$ (?)

for each graph G, where G_2 arises from G by replacing each edge by two parallel edges. Conjecture (28.27) is equivalent to the conjecture that, for each k,

(28.28) (?) for each k-graph G one has $\chi'(G_2) = 2k$ (?);

equivalently, for each k-graph G, the minimum (28.21) for $\chi'^*(G)$ is attained by half-integer λ_M. In other words, it is conjectured that any k-graph has $2k$ perfect matchings covering each edge exactly twice. (The equivalence of (28.27) and (28.28) can be seen as follows. The implication (28.27)⇒(28.28) is easy. To see the reverse implication, let G be any graph and define $k := \lceil \chi'^*(G) \rceil$. Make a disjoint copy G' of G, and connect each vertex v of G by $k - \deg_G(v)$ parallel edges to its copy v' in G'. This makes a k-regular graph H with $\chi'^*(H) = k$. So H is a k-graph, and hence by (28.28), $\chi'(H_2) = 2k$. Hence $\chi'(G_2) \leq 2k$, implying (28.27).)

Seymour called (28.28) the *generalized Fulkerson conjecture*, as it generalizes the special case $k = 3$ asked (but not conjectured) by Fulkerson [1971a]. This special case is called the 'Fulkerson conjecture'[13]. (By Corollary 28.5a, a cubic graph G has $\chi'^*(G) = 3$ if and only if G is bridgeless.) For a partial result, see Corollary 38.11e.

Berge [1979a] conjectured that the edges of any bridgeless cubic graph can be covered by 5 perfect matchings. This would follow from the Fulkerson conjecture.

A conjecture of Gol'dberg [1973] (and also of Seymour [1979a]) is that for each (possibly nonsimple) graph G one has

(28.29) (?) $\chi'(G) \leq \max\{\Delta(G) + 1, \lceil \chi'^*(G) \rceil\}$. (?)

(An equivalent conjecture was stated by Andersen [1977].)

As $\chi'(G) \geq \max\{\Delta(G), \lceil \chi'^*(G) \rceil\}$, validity of (28.29) would yield a tight (gap 1) bound for $\chi'(G)$ also for nonsimple graphs. In particular, if $\Delta(G) < \chi'^*(G)$, we would have equality in (28.29). Seymour [1979a] mentioned that he has shown that $\chi'(G) \leq \lceil \chi'^*(G) \rceil + 1$ for graphs G with $\chi'^*(G) \leq 6$.

Conjecture (28.29) would generalize Theorem 28.2 due to Vizing. For let $\mu(G)$ again denote the maximum multiplicity of any edge of $G = (V, E)$. Then for any subset U of V,

(28.30) $$|E[U]| \leq \tfrac{1}{2}\Delta(G[U])|U| \leq (\Delta(G[U]) + \mu(G))\tfrac{1}{2}(|U| - 1) \leq (\Delta(G) + \mu(G))\lfloor \tfrac{1}{2}|U| \rfloor$$

(The second inequality follows from $\Delta(G[U]) \leq \mu(G)(|U| - 1)$.) So with Theorem 28.5 we know that $\chi'^*(G) \leq \Delta(G) + \mu(G)$.

A well-known equivalent form of the four-colour theorem is that each bridgeless cubic planar graph is 3-edge-colourable. This equivalence was discovered by Tait [1878b]. Seymour [1981c] conjectures the following generalization:

(28.31) (?) each planar k-graph is k-edge-colourable. (?)

This was proved for $k = 4$ and $k = 5$ by Guenin [2002b].

A consequence of the 4-flow conjecture of Tutte [1966] is:

(28.32) each bridgeless cubic graph without Petersen graph minor is 3-edge-colourable.

This was proved jointly by Robertson, Seymour, and Thomas [1997], Sanders, Seymour, and Thomas [2000], and Sanders and Thomas [2000].

(28.31) and (28.32) made Lovász [1987] conjecture:

(28.33) (?) each k-graph without Petersen graph minor is k-edge-colourable. (?)

[13] Seymour [1979a] says that it was first conjectured by C. Berge, but that it is usually called Fulkerson's conjecture because the latter put it into print.

(An equivalent conjecture was given by Rizzi [1997,1999].) This is equivalent to stating that the incidence vectors of perfect matchings in a graph without Petersen graph minor, form a Hilbert base (cf. Section 5.18). It relates to Lovász's work on the perfect matching lattice — see Chapter 38.

Goddyn [1993] noted that (28.33) would not yield a full characterization, since also the perfect matchings of the Petersen graph form a Hilbert base. (This is due to the fact that the all-one vector does not belong to the perfect matching lattice of the Petersen graph.)

Notes. Seymour [1979a] conjectured that if $k \geq 4$, any k-graph $G = (V, E)$ has a perfect matching M such that $G - M$ is a $(k-1)$-graph. However, this was disproved by Rizzi [1997,1999], who showed that for any $k \geq 3$, there exists a k-graph in which any two perfect matchings intersect. Hence, for any $k \geq 3$ there exists a k-graph that cannot be decomposed into a k_1- and a k_2-graph for any $k_1, k_2 \geq 1$ with $k_1 + k_2 = k$.

Nishizeki and Kashiwagi [1990] showed that

(28.34) $$\chi'(G) \leq \max\{\tfrac{11}{10}\Delta(G) + \tfrac{4}{5}, \lceil \chi'^*(G) \rceil\},$$

and they gave a polynomial-time algorithm finding an edge-colouring fulfilling this bound. (This improves earlier results of Andersen [1977], Goldberg [1984], and Hochbaum, Nishizeki, and Shmoys [1986].)

Marcotte [1986,1990a,1990b,2001], Seymour [1990a], Lee and Leung [1993], and Caprara and Rizzi [1998] gave other partial results on conjecture (28.29).

28.7. Edge-colouring polyhedrally

Let $G = (V, E)$ be a graph and let Q be the polytope determined by

(28.35) $$\begin{array}{ll} x_e \geq 0 & (e \in E), \\ x(M) \leq 1 & (M \text{ matching}). \end{array}$$

So Q is the antiblocking polyhedron of the matching polytope. By the description of the matching polytope and by the theory of antiblocking polyhedra, Q is equal to the convex hull of the following set of vectors:

(28.36) $$\begin{array}{ll} \chi^S & S \text{ substar}, \\ \frac{1}{\lfloor \frac{1}{2} |\bigcup F| \rfloor} \chi^F & \text{for nonempty } F \subseteq E. \end{array}$$

Here a *substar* is any set S of edges with $S \subseteq \delta(v)$ for some $v \in V$. By $\bigcup F$ we denote the set of vertices covered by F.

Now the fractional edge-colouring number $\chi'^*(G)$ is equal to the maximum value of $\mathbf{1}^\mathsf{T} x$ over Q (by LP-duality). The ellipsoid method then gives:

Theorem 28.6. *The fractional edge-colouring number of a graph can be determined in polynomial time.*

Proof. The separation problem over Q is equivalent to the weighted matching problem, and hence is solvable in polynomial time. Therefore, with the ellipsoid method, also the optimization problem over Q is solvable in polynomial time. This gives the fractional edge-colouring number. ∎

For any weight function $w \in \mathbb{R}_+^E$, the maximum of $w^\mathsf{T} x$ where x ranges over the vectors (28.36), is equal to the minimum value of $\sum_M \lambda_M$ where $\lambda_M \geq 0$ for $M \in \mathcal{M}$ such that $\sum_M \lambda_M \chi^M = w$. Thus we have a min-max relation for the 'weighted fractional edge-colouring number'.

We should note that (generally) the matching polytope does not have the integer decomposition property, and (equivalently) that system (28.35) does not have the integer rounding property. Indeed, for the Petersen graph, the maximum of $\mathbf{1}^\mathsf{T} x$ over (28.35) is equal to 3. So it has a fractional optimum dual solution of value 3. However, there is no integer optimum dual solution, since the edges of the Petersen graph cannot be decomposed into three matchings.

28.8. Packing edge covers

The results on edge-colouring (which is essentially covering by matchings), can be dualized to packing edge covers, as observed by Gupta [1974] (where $\delta(G)$ denotes the minimum degree of G):

Theorem 28.7. *A simple graph $G = (V, E)$ has $\delta(G)-1$ disjoint edge covers.*

Proof. Make an auxiliary graph H as follows. For each $v \in V$, do the following. Make $\deg_G(v) - \delta(G)$ new vertices, and reconnect $\deg_G(v) - \delta(G)$ of the edges incident with v with the new vertices, in such a way that v has degree $\delta(G)$, while each new vertex has degree 1.

Then H has maximum degree $\delta(G)$ and there is a one-to-one mapping between the edges of G and those of H. By Vizing's theorem for simple graphs (Theorem 28.1), H has matchings $M_1, \ldots, M_{\delta(G)+1}$ partitioning E. We denote the corresponding edge sets in G also by M_i.

Then each vertex v of G is covered by all but at most one of the matchings $M_1, \ldots, M_{\delta(G)+1}$. Let U be the set of vertices of G missed by one of $M_1, \ldots, M_{\delta(G)-1}$. Then each vertex in U is covered by both $M_{\delta(G)}$ and $M_{\delta(G)+1}$. So $M_{\delta(G)} \cup M_{\delta(G)+1}$ forms a graph on V where each vertex in U has degree at least 2. Hence we can orient the edges in $M_{\delta(G)} \cup M_{\delta(G)+1}$ such that each vertex in U is head of at least one of the oriented edges.

Now for each $i = 1, \ldots, \delta(G) - 1$, add to M_i all edges in $M_{\delta(G)} \cup M_{\delta(G)+1}$ that are oriented towards a vertex missed by M_i. This gives $\delta(G) - 1$ disjoint edge covers. ∎

This can be formulated in terms of the *edge cover packing number* $\xi(G)$ of G, which is the maximum number of disjoint edge covers in G. Then, if G is simple,

(28.37) $\xi(G) \geq \delta(G) - 1.$

Gupta [1974] showed more generally for not necessarily simple graphs (where $\mu(G)$ denotes the maximum multiplicity of the edges of G):

Theorem 28.8. *Any graph G has $\delta(G) - \mu(G)$ disjoint edge covers.*

Proof. Let $\delta := \delta(G)$ and $\mu := \mu(G)$. Make an auxiliary graph H as follows. For each $v \in V$, do the following. Make $\deg_G(v) - \delta$ new vertices, and reconnect $\deg_G(v) - \delta$ of the edges incident with v with the new vertices, in such a way that v has degree δ, while each new vertex has degree 1.

Then H has maximum degree $\delta(G)$, and there is a one-to-one mapping between the edges of G and those of H. By Vizing's theorem (Theorem 28.2), H has matchings $M_1, \ldots, M_{\delta+\mu}$ partitioning E. We denote the corresponding edge sets in G also by M_i. Let

(28.38) $F := M_{\delta-\mu+1} \cup \cdots \cup M_{\delta+\mu}.$

Orient the edges in F such that each vertex v is entered by at least $\lfloor \frac{1}{2} \deg_F(v) \rfloor$ of the edges incident with v.

Consider any vertex v, and let v be missed by α of the $M_1, \ldots, M_{\delta-\mu}$. Let k be the number of $M_{\delta-\mu+1}, \ldots, M_{\delta+\mu}$ covering v. As v is covered by at least δ of the $M_1, \ldots, M_{\delta+\mu}$, we know $k + (\delta - \mu - \alpha) \geq \delta$, that is, $k \geq \mu + \alpha$. Since $k \leq 2\mu$, it follows that $\deg_F(v) \geq k = 2k - k \geq 2(\mu + \alpha) - 2\mu = 2\alpha$. Hence v is entered by at least α edges.

So for each $i = 1, \ldots, \delta - \mu$, if v is missed by M_i, then we can extend M_i by an edge in F oriented towards v. Doing this for each vertex v, we obtain $\delta - \mu$ disjoint edge covers. ∎

Equivalently, for any graph,

(28.39) $\xi(G) \geq \delta(G) - \mu(G).$

Gupta [1974] announced (without proof) and Fournier [1977] showed that for any graph $G = (V, E)$ and any $k \in \mathbb{Z}_+$, E can be partitioned into classes $E_1, \ldots, E_k$ such that each vertex v is covered by at least

(28.40) $\min\{k, \deg(v), \max\{k, \deg(v)\} - \mu(v)\}$

of the E_i, where $\mu(v)$ denotes the maximum multiplicity of the edges incident with v.

28.9. Further results and notes

28.9a. Shannon's theorem

Shannon [1949] gave the following upper bound on the edge-colouring number that can be better than Vizing's bound if G is not simple. As Vizing [1965a] observed, the bound can be derived from Vizing's theorem, as below.

Theorem 28.9. *The edge-colouring number $\chi'(G)$ of a graph $G = (V, E)$ is at most $\lfloor \frac{3}{2}\Delta(G) \rfloor$.*

Proof. Let G be a counterexample with a minimum number of edges. Define $k := \lfloor \frac{3}{2}\Delta(G) \rfloor$. So $\chi'(G) > k$ and by Vizing's theorem (Theorem 28.2), $\chi'(G) \leq \Delta(G) + \mu(G)$, where $\mu(G)$ is the maximum edge-multiplicity of G. Hence $\mu(G) > \frac{1}{2}\Delta(G)$.

Let u and v be vertices connected by $\mu(G)$ parallel edges. Choose one such edge, e say. By the minimality of G, $\chi'(G - e) \leq k$. Consider a k-edge-colouring of $G - e$. Let I_u and I_v be the sets of colours covering u and v respectively. Then $|I_u \cap I_v| \geq \mu(G) - 1$, since $\mu(G) - 1$ edges of $G - e$ connect u and v. Moreover, $|I_u| \leq \Delta(G) - 1$, since u has degree less than $\Delta(G)$ in $G - e$; similarly, $|I_v| \leq \Delta(G) - 1$. So

$$
\begin{aligned}
(28.41) \qquad |I_u \cup I_v| &= |I_u| + |I_v| - |I_u \cap I_v| \leq 2(\Delta(G) - 1) - (\mu(G) - 1) \\
&= 2\Delta(G) - \mu(G) - 1 < \tfrac{3}{2}\Delta(G) - 1 < k
\end{aligned}
$$

(since $\mu(G) > \frac{1}{2}\Delta(G)$), and hence at least one colour does not occur in $I_u \cup I_v$. This colour can be given to edge e to obtain a k-edge-colouring of G. ∎

The bound in this theorem is sharp, as is shown by a graph H on three vertices u, v, and w, with $\lceil \frac{1}{2}\Delta \rceil$ parallel arcs connecting u and v, $\lfloor \frac{1}{2}\Delta \rfloor$ parallel arcs connecting u and w, and $\lfloor \frac{1}{2}\Delta \rfloor$ parallel arcs connecting v and w. Then $\Delta(H) = \Delta$ and $\chi'(H) = \lfloor \frac{3}{2}\Delta \rfloor$.

Vizing [1965a] showed that any graph G with $\Delta(G) \geq 4$ and $\chi'(G) = \lfloor \frac{3}{2}\Delta(G) \rfloor$ contains this graph H as a subgraph.

The case $\Delta(G)$ even in Theorem 28.9 can be proved simpler as follows. We may assume that each degree of G is even (we can pair up the odd-degree vertices by new edges). Let $k := \frac{1}{2}\Delta(G)$. Make an Eulerian orientation of G. Split each vertex v into two vertices v' and v'', and replace any edge oriented from u to v, by an edge connecting u' and v''. In this way we obtain a bipartite graph H, of maximum degree k. Hence, by Kőnig's edge-colouring theorem, H has a k-edge-colouring. This yields a decomposition of the edges of G into classes $E_1, \ldots, E_k$ such that each graph $G_i = (V, E_i)$ has maximum degree 2. Hence each G_i is 3-edge-colourable, and therefore G is $3k$-colourable.

28.9b. Further notes

For simple *planar* graphs, if $\Delta(G) \geq 7$, then $\chi'(G) = \Delta(G)$ (for $\Delta \geq 8$, this was proved by Vizing [1965b], and for $\Delta = 7$ by Sanders and Zhao [2001] and Zhang [2000]). For $2 \leq \Delta \leq 5$ there exist simple planar graphs of maximum degree Δ with $\chi'(G) = \Delta + 1$. This is unknown for $\Delta = 6$ (and constitutes a question of Vizing

[1968]). For $\Delta \geq 8$, polynomial-time algorithms finding a Δ-edge-colouring of a simple planar graph were given by Terada and Nishizeki [1982] ($O(n^2)$), Chrobak and Yung [1989] ($O(n)$ if $\Delta \geq 19$), and Chrobak and Nishizeki [1990] ($O(n \log n)$ if $\Delta \geq 9$).

Kotzig [1957] showed the following theorem:

Theorem 28.10. *Let $G = (V, E)$ be a connected cubic graph with an even number of edges. Then G is 3-edge-colourable if and only if the line graph $L(G)$ of G is 4-edge-colourable.*

Proof. I. First assume that $L(G)$ is 4-edge-colourable, say with colours 0, 1, 2, and 3. We colour the edges of G with colours labeled by the three partitions of $\{0, 1, 2, 3\}$ into pairs. Consider an edge $e = uv$ of G. Let f_1 and f_2 be the two other edges incident with u and let g_1 and g_2 be the two other edges incident with v. Let i_1 and i_2 be the colours of the edges ef_1 and ef_2 of $L(G)$ and let j_1 and j_2 be the colours of the edges eg_1 and eg_2 of $L(G)$. Give e the colour labeled by the partition of $\{0, 1, 2, 3\}$ into the pairs $\{i_1, i_2\}$ and $\{j_1, j_2\}$. This gives a 3-edge-colouring of G.

II. Conversely, assume that G is 3-edge-colourable. We first show that $L(G)$ has a perfect matching. Indeed, there is a subset M of the edge set of $L(G)$ such that each vertex of $L(G)$ is covered an odd number of times. To see this, choose an arbitrary partition Π of the vertices of $L(G)$ into pairs, and for each pair $\{e, f\} \in \Pi$, we choose an $e - f$ path $P_{e,f}$ in $L(G)$. Then the symmetric difference of all these paths is a subset M as required.

Now choose such an M with $|M|$ as small as possible. We claim that each vertex of $L(G)$ is covered exactly once by M; that is, M is a perfect matching in $L(G)$. Suppose that vertex e of $L(G)$ is covered by three edges in M, say ee_1, ee_2, and ee_3. We can assume that e, e_1 and e_2 are pairwise adjacent in $L(G)$. Hence, replacing M by $M \triangle \{ee_1, ee_2, e_1e_2\}$, gives a subset M' covering each vertex an odd number of times, however with $|M'| < |M|$. This contradicts our assumption.

So M is a perfect matching in $L(G)$, forming our first colour 0. Let G be edge-coloured with colours 1, 2, and 3. Consider an edge e_1e_2 of $L(G)$ not having colour 0. Let e_0 be the third edge of G incident with the common vertex of e_1 and e_2. If e_0e_1 has colour 0, give e_1e_2 the colour of edge e_1. If e_0e_2 has colour 0, give e_1e_2 the colour of edge e_2. If neither e_0e_1 nor e_0e_2 has colour 0, give e_1e_2 the colour of edge e_0. It is straightforward to check that this gives a 4-edge-colouring of $L(G)$. ∎

For more on edge-colouring cubic graphs, see Kotzig [1975,1977].

McDiarmid [1972] observed that in any graph $G = (V, E)$, if $p \geq \chi'(G)$, then there is a p-edge-colouring with $\lfloor |E|/p \rfloor \leq |M| \leq \lceil |E|/p \rceil$ for each colour M. This can be proved in the same way as Theorem 20.8.

Meredith [1973] gave k-regular non-Hamiltonian non-k-edge colourable graphs with an even number of vertices, for each $k \geq 3$ (cf. Isaacs [1975]). Johnson [1966a] gave a short proof that any cubic graph is 4-edge-colourable.

Vizing [1965a] asked if the minimum number of colours of the edges of a graph can be obtained from any edge-colouring by iteratively swapping colours on a colour-alternating path or circuit and deleting empty colours.

Marcotte and Seymour [1990] observed that the following is a necessary condition for extending a partial k-edge colouring a graph $G = (V, E)$ to a complete k-edge colouring:

(28.42) $$|F| \leq \sum_{i=1}^{k} \mu_i(F) \text{ for each } F \subseteq E,$$

where $\mu_i(F)$ is the maximum size of a matching $M \subseteq F$ not covering any vertex covered by the ith colour. They studied graphs where this condition is sufficient as well.

Vizing [1965a] showed that if G is nonsimple and $\Delta(G) = 2\mu(G) + 1$, then $\chi'(G) \leq 3\mu(G)$ (where $\mu(G)$ is the maximum edge-multiplicity of G).

The *list-edge-colouring number* $\chi^l(G)$ of a graph $G = (V, E)$ is the minimum number k such that for each choice of sets L_e for $e \in E$ with $|L_e| = k$, one can select $l_e \in L_e$ for $e \in E$ such that for any two incident edges e, f one has $l_e \neq l_f$. Vizing [1976] conjectures that $\chi^l(G)$ is equal to the edge-colouring number of G, for each graph G (see Häggkvist and Chetwynd [1992]).

The *total colouring number* of a graph $G = (V, E)$ is a colouring of $V \cup E$ such that each colour consists of a stable set and a matching, vertex-disjoint. Behzad [1965] and Vizing [1968] conjecture that the total colouring number of a simple graph G is at most $\Delta(G) + 2$. Molloy and Reed [1998] showed that there exists a constant C such that the total colouring number of any simple graph is at most $\Delta(G) + C$. A polynomial-time algorithm finding a total colouring with $\Delta(G)$+poly($\log \Delta$) colours is given by Hind, Molloy, and Reed [1999].

More generally, Vizing [1968] conjectures that the total colouring number of a graph G is at most $\Delta(G) + \mu(G) + 1$, where $\mu(G)$ is the maximum size of a parallel class of edges. Partial results have been found by Kostochka [1977], Hind [1990, 1994], Kilakos and Reed [1992], and McDiarmid and Reed [1993].

For extensions of Vizing's theorem, see Vizing [1965b], Fournier [1973], Jakobsen [1973], Gol'dberg [1974], Fiorini [1975], Hilton [1975], Jakobsen [1975], Andersen [1977], Kierstead and Schmerl [1983], Kostochka [1983], Ehrenfeucht, Faber, and Kierstead [1984], Goldberg [1984], Hilton and Jackson [1987], Berge [1990], and Chew [1997a]. The fractional edge-colouring number $\chi'^*(G)$ was studied by Hilton [1975] and Stahl [1979]. A computational study based on fractional edge-colouring was made by Nemhauser and Park [1991]. Equitable edge-colourings were investigated by de Werra [1981].

Generalizations of edge-colouring were studied by Hakimi and Kariv [1986] and Nakano, Nishizeki, and Saito [1988,1990]. Fiorini and Wilson [1977,1978], Fiorini [1978], Jensen and Toft [1995], Nakano, Zhou, and Nishizeki [1995], and Zhou and Nishizeki [2000] gave surveys on edge-colouring and extensions.

28.9c. Historical notes on edge-colouring

Historically, studying edge-colouring was motivated by the equivalence of the four-colour conjecture with the 3-edge-colourability of planar bridgeless cubic graphs. The four-colour conjecture was raised in 1852 by F. Guthrie. An early attempt to prove the conjecture by Kempe [1879,1880] was shown by Heawood in 1890 to contain an error — see below.

Also Tait [1878a] studied the four-colour problem. He claimed without proof that each triangulated planar graph has two disjoint sets of edges such that each triangular face is incident with exactly one edge in each of these sets. From this he derived (correctly) that each loopless planar graph is 4-vertex-colourable. He also observed that his claim implies that each planar bridgeless cubic graph is 3-edge-colourable.

In a note following a note of Guthrie [1878] (describing the very early history of the four-colour problem, which note itself was a reaction to the article of Tait [1878a]), Tait [1878b] remarked that in his earlier paper

> I gave a series of proofs of the theorem that four colours suffice for a map. All of these were long, and I felt that, while more than sufficient to prove the truth of the theorem, they gave little insight into its real nature and bearings. A somewhat similar remark may, I think, be made about Mr Kempe's proof.

He therefore withdrew the former paper, and replaced it by the present note, in which, without proof, the following 'elementary theorem' is formulated:

> if an even number of points be joined, so that three (and only three) lines meet in each, these lines may be coloured with *three* colours only, so that no two conterminous lines shall have the same colour. (When an odd number of the points forms a group, connected by *one* line only with the rest, the theorem is not true.)

Tait next gave the now well-known construction of deriving 3-edge-colourability of bridgeless planar cubic graphs from the 4-vertex-colourability of planar loopless graphs. At that time, the error in Kempe's proof of the four-colour conjecture was not yet detected.

But Tait also said:

> The proof of the elementary theorem is given easily by induction; and then the proof that four colours suffice for a map follows almost immediately from the theorem, by an inversion of the demonstration just given.

Tait [1880] claimed that in Tait [1878b] the 3-edge-colourability of bridgeless planar cubic graphs had been shown:

> *If* 2n *points be joined by* 3n *lines, so that three lines, and three only, meet at each point, these lines can be divided (usually in many different ways) into three groups of* n *each, such that one of each group ends at each of the points.*

While Tait did not mention it explicitly, he restricted himself to *planar* cubic graphs, since he considered them equivalently as the skeletons of polytopes[14]. Also the figures given in Tait [1880] are planar (and also those in Tait [1884], where similar claims are made).

The validity of Kempe's proof of the four-colour conjecture was accepted until Heawood [1890] discovered a fatal error in Kempe's proof, and showed that it in fact gives a five-colour theorem for planar graphs. The error in his proof was acknowledged by Kempe [1889]. (For an account of the early history of the four-colour problem, see Biggs, Lloyd, and Wilson [1976].)

After that, the problem of the 3-edge-colourability of bridgeless planar cubic graphs was open again. At several occasions, this problem was advanced (cf. Goursat [1894]). It was only resolved in 1977 when Appel and Haken proved the four-colour theorem.

Petersen [1898] asserted that Tait had claimed to have proved the 3-edge-colourability of any (also nonplanar) bridgeless cubic graph. It motivated him to present, as a counterexample, the now-called *Petersen graph*, which is a bridgeless cubic graph that is not 3-edge-colourable:

[14] Tait's polytopes are 3-dimensional, since each vertex has degree 3.

j'ai réussi à construire un graphe où le théorème de Tait ne s'applique pas.[15]

Petersen [1898] drew the Petersen graph as follows:

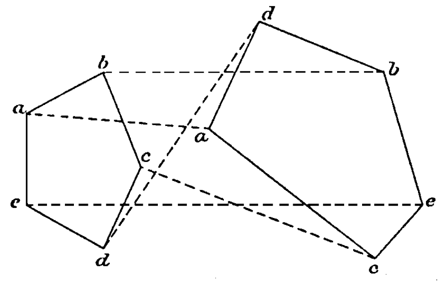


Figure 28.4

For another purpose, the Petersen graph was given earlier by Kempe [1886], who represented it as follows:

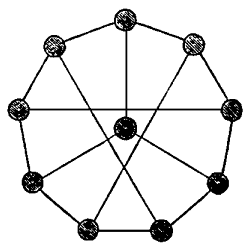


Figure 28.5

Sainte-Laguë [1926a] introduced the term *class* for the edge-colouring number of a graph. He noted (without exact argumentation) that Petersen's theorem on the existence of a perfect matching in a bridgeless cubic graph implies that each cubic graph has edge-colouring number 3 or 4.

[15] I have succeeded in constructing a graph where the theorem of Tait does not apply.

Chapter 29

T-joins, undirected shortest paths, and the Chinese postman

The methods for weighted matching also apply to shortest paths in undirected graphs (provided that each circuit has nonnegative length) and to the Chinese postman problem — more generally, to *T-joins*.

29.1. T-joins

Let $G = (V, E)$ be a graph and let $T \subseteq V$. A subset J of E is called a *T-join* if T is equal to the set of vertices of odd degree in the graph (V, J). So if a T-join exists, then $|T|$ is even. More precisely,

(29.1) G has a T-join if and only if $|K \cap T|$ is even for each component K of G.

T-joins are close to matchings. In fact, from Corollary 26.1a it can be derived that a shortest T-join can be found in strongly polynomial time. To see this, one should observe the following elementary graph-theoretical fact representing T-joins as sets of paths:

(29.2) each T-join is the edge-disjoint union of circuits and $\frac{1}{2}|T|$ paths connecting disjoint pairs of vertices in T;
the symmetric difference of a set of circuits and $\frac{1}{2}|T|$ paths connecting disjoint pairs of vertices in T is a T-join.

This is used in showing that a shortest T-join can be found in strongly polynomial time:

Theorem 29.1. *Given a graph $G = (V, E)$, a length function $l \in \mathbb{Q}^E$, and a subset T of V, a shortest T-join can be found in strongly polynomial time.*

Proof. First we dispose of negative lengths. Let N be the set of edges e with $l(e) < 0$, let U be the set of vertices v with $\deg_N(v)$ odd, let $T' := T \triangle U$, and let $l' \in \mathbb{Q}_+^E$ be defined by $l'(e) := |l(e)|$ for $e \in E$.

Then, if J' is a T'-join minimizing $l'(J')$, the set $J := J' \triangle N$ is a T-join minimizing $l(J)$. To see this, let $\widetilde{J}$ be any T-join. Then $\widetilde{J} \triangle N$ is a T'-join, and hence $l'(J') \le l'(\widetilde{J} \triangle N)$. Therefore, (29.3) $$l(J) = l'(J') + l(N) \leq l'(\widetilde{J} \triangle N) + l(N) = l(\widetilde{J}).$$

So we can assume $l \geq \mathbf{0}$. Now consider the complete graph K_T with vertex set T. For each edge st of K_T, determine a path P_{st} in G of minimum length, say, $w(st)$. Find a perfect matching M in K_T minimizing $w(M)$. Then the symmetric difference of the paths P_{st} for $st \in M$ is a shortest T-join in G. This follows directly from (29.2). ∎

(This method is due to Edmonds [1965e].)

Note that a T-join J has minimum length if and only if $l(C \cap J) \leq \frac{1}{2}l(C)$ for each circuit C. (This was observed essentially by Guan [1960].)

Theorem 29.1 implies that also a *longest* T-join can be found in strongly polynomial time:

Corollary 29.1a. *Given a graph $G = (V, E)$, a length function $l \in \mathbb{Q}^E$, and a subset T of V, a longest T-join can be found in strongly polynomial time.*

Proof. Apply Theorem 29.1 to $-l$. ∎

An application is finding a maximum-capacity cut in a planar graph $G = (V, E)$ (Orlova and Dorfman [1972][16], Hadlock [1975]): it amounts to finding a maximum-capacity $\emptyset$-join in the planar dual graph. (Barahona [1990] gave an $O(n^{3/2} \log n)$ time bound.)

Another consequence is:

Corollary 29.1b. *Given a graph $G = (V, E)$ and a length function $l : E \to \mathbb{Q}$, one can check if there is a negative-length circuit in strongly polynomial time.*

Proof. There is a negative-length circuit if and only if there exists an $\emptyset$-join J with $l(J) < 0$. So Theorem 29.1 gives the corollary. ∎

Complexity. With Dijkstra's shortest path method (Theorem 7.3) one derives from Theorem 26.2 that a shortest T-join can be found in $O(n^3)$ time. Generally, one has an $O(\mathrm{APSP}_+(n, m, L) + \mathrm{WM}(n, n^2, nL))$-time algorithm, where L is the maximum absolute value of the lengths of the edges in G (assuming they are integer), where $\mathrm{APSP}_+(n, m, L)$ is the time in which the all-pairs shortest paths problem can be solved, in an undirected graph, with n vertices and m edges and with nonnegative integer lengths at most L, and where $\mathrm{WM}(n, m, W)$ is the time in which a minimum-weight perfect matching can be found, in a graph with n vertices and m edges and with integer weights at most W in absolute value.

[16] Orlova and Dorfman observed that finding a maximum-size cut in a planar graph amounts to finding shortest paths connecting the odd-degree vertices in the dual graph, but described a branch-and-bound method for it, and did not state that it can be solved in polynomial time by matching techniques.

29.2. The shortest path problem for undirected graphs

In Chapter 8 we saw that a shortest path in a directed graph without negative-length directed circuits, can be found in strongly polynomial time. It implies a strongly polynomial-time shortest path algorithm for undirected graphs, provided that all lengths are nonnegative. This, because the reduction replaces each undirected edge uv by two directed edges (u, v) and (v, u) — which would create a negative-length directed circuit if uv has negative length.

However, Theorem 29.1 implies that one can find (in strongly polynomial-time) a shortest path in undirected graphs even if there are negative-length edges, provided that all circuits have nonnegative length:

Corollary 29.1c. *Given a graph $G = (V, E)$, $s, t \in V$, and a length function $l \in \mathbb{Q}^E$ such that each circuit has nonnegative length, a shortest $s-t$ path can be found in strongly polynomial time.*

Proof. Define $T := \{s, t\}$ and apply Theorem 29.1. By observation (29.2), a shortest T-join J can be partitioned into an $s-t$ path and a number of circuits. Since by assumption any circuit has nonnegative length, we can delete the circuits from J. ∎

Complexity. Since by Gabow [1990] the weighted matching problem is solvable in $O(n(m + n \log n))$ time, a shortest path in an undirected graph, without negative-length circuits, can be found in $O(n(m + n \log n))$ time. This can be derived as follows: If we want to find a shortest $s-t$ path, add to each vertex v a 'copy' v', for each edge uv add edges uv', $u'v$, and $u'v'$ (each with the same length as uv), and for each vertex v add an edge vv', of length 0. Let G' be the graph obtained. Then a minimum-weight perfect matching in $G' - s' - t'$ gives a shortest $s-t$ path in G.

Gabow [1983a] gave an $O(n \min\{m \log n, n^2\})$-time algorithm for the all-pairs shortest paths problem in undirected graphs.

29.3. The Chinese postman problem

Call a walk $C = (v_0, e_1, v_1, \ldots, e_t, v_t)$ in a graph G a *Chinese postman tour* if $v_t = v_0$ and each edge of G occurs at least once in C. The *Chinese postman problem*, first studied by Guan [1960] (and named by Edmonds [1965e]), is:

(29.4) given: a connected graph $G = (V, E)$ and a length function $l \in \mathbb{Q}_+^E$,
find: a shortest Chinese postman tour C.

By Euler's theorem, if each vertex has even degree, there is an Eulerian tour, that is, a walk traversing each edge *exactly* once. So in that case, any Eulerian tour is a shortest Chinese postman tour.

But if not all degrees are even, certain edges have to be traversed more than once. These edges form in fact a shortest T-join for $T := \{v \mid \deg_G(v)$ odd$\}$. This is the base of the following consequence of Theorem 29.1:

Corollary 29.1d. *The Chinese postman problem can be solved in strongly polynomial time.*

Proof. Let $T := \{v \mid \deg_G(v) \text{ odd}\}$. Find a shortest T-join J. Add to each edge e in J an edge e' parallel to e. This gives the Eulerian graph G'. Then any Eulerian tour in G' gives a shortest Chinese postman tour (by identifying any new edge e' with its parallel e).

To see this, note that obviously the Eulerian tour gives a Chinese postman tour C of length $l(E) + l(J)$. Suppose that there is a shorter tour C'. Let J' be the set of edges traversed an even number of times by C'. Then J' is a T-join, and so $l(J') \geq l(J)$. Hence $l(C') \geq l(E) + l(J') \geq l(E) + l(J) = l(C)$. ∎

Observe that a postman never has to traverse any street more than twice.

Complexity. The above gives an $O(n^3)$-time algorithm for the Chinese postman problem (more precisely, $O(k(m + n \log n) + k^3 + m)$, where k is the number of vertices of odd degree).

29.4. T-joins and T-cuts

There is an interesting min-max relation for the minimum size of T-joins. To this end, call, for any graph $G = (V, E)$ and any $T \subseteq V$, a subset C of E a *T-cut* if $C = \delta(U)$ for some $U \subseteq V$ with $|U \cap T|$ odd.

Trivially, each T-cut intersects each T-join. Moreover, each edge set C intersecting each T-join contains a T-cut (since otherwise each component of $(V, E \setminus C)$ has an even number of vertices in T, and hence there is a T-join disjoint from C). So the inclusionwise minimal T-cuts are exactly the inclusionwise minimal edge sets intersecting all T-joins. Hence the inclusionwise minimal T-joins are exactly the inclusionwise minimal edge sets intersecting all T-cuts.

Call a family $\mathcal{F}$ of cuts in $G = (V, E)$ *cross-free* if $\mathcal{F} = \{\delta(U) \mid U \in \mathcal{C}\}$ for some cross-free collection $\mathcal{C}$ of subsets of V; that is, a collection $\mathcal{C}$ with

(29.5) $\quad U \subseteq W$ or $W \subseteq U$ or $U \cap W = \emptyset$ or $U \cup W = V$

for all $U, W \in \mathcal{C}$.

The following min-max relation for minimum-size T-joins in bipartite graphs was proved by Seymour [1981d] — we give the simple proof due to Sebő [1987]:

Theorem 29.2. *Let $G = (V, E)$ be a bipartite graph and let $T \subseteq V$. Then the minimum size of a T-join is equal to the maximum number of disjoint T-cuts. The maximum is attained by a cross-free family of cuts.*

Proof. We may assume $T \neq \emptyset$. Let J be a minimum-size T-join in G. Define a length function $l : E \to \{+1, -1\}$ by: $l(e) := -1$ if $e \in J$ and $l(e) := +1$ if $e \notin J$. Then every circuit C has nonnegative length, since $J \triangle C$ is again a T-join, and hence $|J \triangle C| \geq |J|$, implying $l(C) = |C \setminus J| - |C \cap J| \geq 0$.

Let P be a minimum-length walk in G traversing no edge more than once. Choose P such that it traverses a minimum number of edges. So P is a path (as we can delete any circuit occurring in P). Let t be an end vertex of P and let f be the last edge of P.

Then $f \in J$, since otherwise we could make the walk shorter by deleting f from P. Moreover, $\deg_J(t) = 1$, as if J has another edge, e say, incident with t, then extending P by e would make the walk shorter.

We next show:

(29.6) Each circuit C traversing t and not traversing f has positive length.

If C has only vertex t in common with P, let e be the first edge of C. So $l(e) = 1$. Consider the walk $P' := P \cup (C - e)$. Then $l(P') \geq l(P)$ and hence $l(C - e) \geq 0$. So $l(C) > 0$.

If C has another vertex in common with P, let u be the last vertex on P with $u \neq t$ and traversed by C. Let P' be the $u - t$ part of P. Split C into two $u - t$ paths C' and C''. By the minimality of $|P|$, $l(P') < 0$. Hence, as $C' \cup P'$ and $C'' \cup P'$ are circuits, $l(C') > 0$ and $l(C'') > 0$. This implies $l(C) > 0$.

Now shrink $\{t\} \cup N(t)$ to one new vertex v_0, giving the graph G'. If $|T \cap (\{t\} \cup N(t))|$ is odd, let $T' := (T \setminus (\{t\} \cup N(t))) \cup \{v_0\}$, and otherwise let $T' := T \setminus (\{t\} \cup N(t))$. Let $J' := J \setminus \{f\}$.

Then J' is a minimum-size T'-join in G'. For suppose to the contrary that G' contains a circuit C' with $|C' \setminus J'| < |C' \cap J'|$. If C' comes from a circuit C in G not traversing t, we would have $|C \setminus J| < |C \cap J|$, a contradiction. So C' comes from a circuit C in G traversing t.

If C traverses f, then $|C' \setminus J'| - |C' \cap J'| = |C \setminus J| - |C \cap J| \geq 0$, a contradiction. If C does not traverse f, then, by (29.6), $l(C) > 0$, and hence $l(C) \geq 2$. So $|C' \setminus J'| = |C \setminus J| - 2 \geq |C \cap J| = |C' \cap J'|$, again a contradiction.

Now by induction (on $|V| + |T|$), G' has disjoint cross-free T'-cuts $D_1, \ldots, D_{|J'|}$. With the T-cut $\delta(t)$ this gives $|J|$ disjoint cross-free T-cuts in G. ∎

(For another, algorithmic proof, see Barahona [1990].)

This implies for not necessarily bipartite graphs (Lovász [1975a]):

Corollary 29.2a. *Let $G = (V, E)$ and let $T \subseteq V$ with $|T|$ even. Then the minimum size of a T-join is equal to half of the maximum number of T-cuts covering each edge at most twice. The maximum is attained by a cross-free family of cuts.*

Proof. Replace each edge of G by a path of length two, thus obtaining the bipartite graph G'. Applying Theorem 29.2 to G' gives the corollary. ∎

In general it is not true that the minimum size of a T-cut is equal to the maximum number of disjoint T-joins — see Section 29.11c.

Notes. Frank, Tardos, and Sebő [1984] sharpened Theorem 29.2 to the following. Let $G = (V, E)$ be a bipartite graph, with colour classes U and W, and let $T \subseteq V$. Then the minimum size of a T-join is equal to the maximum of

$$\sum_{S \in \Pi} q_T(S), \tag{29.7}$$

where Π ranges over all partitions of U. Here $q_T(S)$ denotes the number of components K of $G - S$ with $|K \cap T|$ odd. If G is arbitrary one takes the maximum of $\frac{1}{2}\sum_{S \in \Pi} q_T(S)$ over all partitions Π of V. (For extensions, see Kostochka [1994].)

29.5. The up hull of the T-join polytope

The last corollary implies a polyhedral result due to Edmonds and Johnson [1973] (also stated by Seymour [1979b]). Let $G = (V, E)$ be a graph and let $T \subseteq V$. The *T-join polytope*, denoted by $P_{T\text{-join}}(G)$, is the convex hull of the incidence vectors of T-joins. So it is a polytope in $\mathbb{R}^E$.

We first consider the 'up hull' of $P_{T\text{-join}}(G)$, that is,

$$P^{\uparrow}_{T\text{-join}}(G) := P_{T\text{-join}}(G) + \mathbb{R}^E_+, \tag{29.8}$$

which turns out to be determined by the system:

$$\begin{array}{llll} (29.9) & \text{(i)} & x_e \geq 0 & \text{for each } e \in E, \\ & \text{(ii)} & x(C) \geq 1 & \text{for each } T\text{-cut } C. \end{array}$$

Corollary 29.2b. *The polyhedron $P^{\uparrow}_{T\text{-join}}(G)$ is determined by* (29.9).

Proof. It is easy to see that $P^{\uparrow}_{T\text{-join}}(G)$ is contained in the polyhedron determined by (29.9). If the converse inclusion does not hold, there is a weight function $w \in \mathbb{Q}^E$ with $w > \mathbf{0}$ such that the minimum value of $w^{\mathsf{T}}x$ subject to (29.9) is less than the minimum weight α of any T-join. We may assume that each $w(e)$ is an even integer.

We make a new graph $G' = (V', E')$ as follows. Replace each edge $e = uv$ of G by a path from u to v of length $w(e)$. Then α is equal to the minimum size of a T-join in G'. Hence by Theorem 29.2, G' has α disjoint T-cuts. This gives a family of α T-cuts in G such that each edge e of G is in at most $w(e)$ of these T-cuts. Let y_C be the number of times that T-cut C occurs in this list. Then the y_C give a feasible dual solution to the problem of minimizing $w^{\mathsf{T}}x$ over (29.9), with value $\sum_C y_C = \alpha$. This contradicts our assumption that the minimum value of $w^{\mathsf{T}}x$ subject to (29.9) is less than α. ∎

(Gastou and Johnson [1986] gave a proof based on binary groups.)

By adding $x_e \leq 1$ for each $e \in E$ we obtain from (29.9) the system

(29.10) (i) $0 \leq x_e \leq 1$ for each $e \in E$,
(ii) $x(C) \geq 1$ for each T-cut C.

Corollary 29.2c. *The convex hull of the incidence vectors of edge sets containing a T-join as a subset is determined by* (29.10).

Proof. Directly from Corollary 29.2b, with Theorem 5.19. ∎

These systems are totally dual half-integral:

Corollary 29.2d. *Systems* (29.9) *and* (29.10) *are totally dual half-integral.*

Proof. This follows from the proof of Corollary 29.2b, observing that the y_C are integer if each w_e is an even integer. ∎

Generally these systems are not TDI, as is shown by taking $G = K_4$ and $T = V$ — see Section 29.11b.

Barahona [2002] gave an $O(n^6)$-time algorithm to decompose a vector in the up hull of the T-join polytope as a convex combination of incidence vectors of T-joins, added with a nonnegative vector.

29.6. The T-join polytope

In the previous section we considered the up hull of the T-join polytope. We can derive from it an inequality system determining the T-join polytope itself. Consider the following system of linear inequalities for $x \in \mathbb{R}^E$:

(29.11) (i) $0 \leq x_e \leq 1$ $(e \in E)$,
(ii) $x(\delta(U) \setminus F) - x(F) \geq 1 - |F|$ $(U \subseteq V, F \subseteq \delta(U),$ $|U \cap T| + |F|$ odd$)$.

Corollary 29.2e. *The T-join polytope is determined by* (29.11).

Proof. First, the incidence vector x of any T-join J satisfies (29.11). Indeed, if $U \subseteq V$, then $|\delta(U) \cap J| \equiv |U \cap T| \pmod 2$. Hence if $F \subseteq \delta(U)$ with $|U \cap T| + |F|$ odd, then $|\delta(U) \cap J| + |F|$ is odd, and hence if $x(F) = |F|$ one has $x(\delta(U) \setminus F) \geq 1$. This shows (29.11). So the T-join polytope is contained in the polytope determined by (29.11).

To see the reverse inclusion, choose a weight function $w \in \mathbb{Q}^E$. We show that the minimum value of $w^{\mathsf{T}}x$ subject to (29.11) is equal to $w(J)$ for some T-join J.

Define

(29.12) $\quad N := \{e \mid w(e) < 0\}$ and $T' := T \triangle \{v \mid \deg_N(v) \text{ odd}\}$.

Let $w'(e) := |w(e)|$ for each $e \in E$. Let J' be a T'-join minimizing $w'(J')$. By Corollary 29.2c, there exist λ_U for $U \subseteq V$ with $|U \cap T'|$ odd, such that

(29.13) $\quad$ (i) $\lambda_U \geq 0$ $\quad$ for each U with $|U \cap T'|$ odd, with equality if $|J' \cap \delta(U)| > 1$,

(ii) $\displaystyle\sum_{\substack{U \\ e \in \delta(U)}} \lambda_U \leq w'(e)$ $\quad$ for each $e \in E$, with equality if $e \in J'$.

Define $\mu, \nu : E \to \mathbb{R}_+$ by the conditions that $\mu(e)\nu(e) = 0$ for each $e \in E$ and that

(29.14) $$\nu - \mu + \sum_U \lambda_U(\chi^{\delta(U)\setminus N} - \chi^{\delta(U)\cap N}) = w.$$

So the $\nu(e)$, $\mu(e)$, and λ_U give a feasible dual solution to the problem of minimizing $w^{\mathsf{T}}x$ subject to (29.11) (taking $F := \delta(U) \cap N$).

Let $J := J' \triangle N$. So J is a T-join. We show that J, $\mu(e)$, $\nu(e)$, λ_U satisfy the complementary slackness conditions, thus finishing our proof.

First we show that if $e \in J$, then $\nu(e) = 0$. Indeed, if $e \in J \setminus N$, then $e \in J'$, and hence

(29.15) $$\sum_{U, e \in \delta(U)\setminus N} \lambda_U - \sum_{U, e \in \delta(U)\cap N} \lambda_U$$

is equal to $w'(e) = w(e)$ by (29.13)(ii), and hence $\nu(e) = 0$. If $e \in J \cap N$, then (29.15) is at least $-w'(e) = w(e)$ by (29.13)(ii), and hence $\nu(e) = 0$.

Second we show that if $e \notin J$, then $\mu(e) = 0$. If $e \notin J \cup N$, then (29.15) is at most $w'(e) = w(e)$ by (29.13)(ii), implying $\mu(e) = 0$. If $e \in N \setminus J$, then $e \in J'$, and hence (29.15) is equal to $-w'(e) = w(e)$ by (29.13)(ii), implying $\mu(e) = 0$.

Finally if $\lambda_U > 0$, then (as $J = J' \triangle N$ and $|J' \cap \delta(U)| = 1$ by (29.13)(i))

(29.16) $$\begin{aligned} &|J \cap (\delta(U) \setminus N)| - |J \cap (\delta(U) \cap N)| \\ &= |(J' \setminus N) \cap \delta(U)| - |(N \setminus J') \cap \delta(U)| \\ &= |J' \cap \delta(U)| - |N \cap \delta(U)| = 1 - |\delta(U) \cap N|. \end{aligned}$$ ∎

In Section 29.11b we show that (29.11) is TDI if and only if G is series-parallel.

29.7. Sums of circuits

Given a graph $G = (V, E)$, the *circuit cone* is the cone in $\mathbb{R}^E$ generated by the incidence vectors of circuits. Seymour [1979b] showed that this cone is determined by:

(29.17) (i) $x_e \geq 0$ for each $e \in E$,
(ii) $x(D) \geq 2x_e$ for each cut D and $e \in D$.

As J. Edmonds (cf. Seymour [1979b]) pointed out, this can be derived from (essentially) matching theory:

Corollary 29.2f. *The circuit cone is determined by* (29.17)*.*

Proof. Since the incidence vector x of any circuit satisfies (29.17), the circuit cone is contained in the cone determined by (29.17).

To see the converse inclusion, let x satisfy (29.17). To show that x belongs to the circuit cone, we may assume (by scaling) that $x(E) \leq 1$. It suffices to show that x belongs to the $\emptyset$-join polytope of G. Hence, by Corollary 29.2e, it suffices to show that $x(\delta(U)) - 2x(F) \geq 1 - |F|$ for each $U \subseteq V$ and $F \subseteq \delta(U)$ with $|F|$ odd. If $|F| = 1$, this follows from (29.17)(ii). If $|F| \geq 3$, then $x(\delta(U)) - 2x(F) \geq -x(E) \geq -1 \geq 1 - |F|$. ∎

(This proof is due to Aráoz, Cunningham, Edmonds, and Green-Krótki [1983]. Hoffman and Lee [1986] gave a 'different (but not shorter) proof'. Coullard and Pulleyblank [1989] gave a short elementary proof, together with decomposition results.)

Seymour [1979b] in fact characterized when a box has a nonempty intersection with the circuit cone:

Corollary 29.2g. *Let $G = (V, E)$ be a graph and let $l, u \in \mathbb{R}^E_+$ satisfying $l \leq u$. Then there exists an x in the circuit cone of G with $l \leq x \leq u$ if and only if*

(29.18) $u(D \setminus \{e\}) \geq l(e)$ *for each cut D and each $e \in D$.*

Proof. Necessity being trivial, we show sufficiency. Choose a counterexample with $\sum_{e \in E}(u_e - l_e)$ minimum. Suppose that $u_e > l_e$ for some edge e. Then there exist a cut D and $e \in D$ with $u(D \setminus \{e\}) = l(e)$ and there exist a cut D' containing e, and $f \in D' \setminus \{e\}$ with $u(D' \setminus \{f\}) = l(f)$. Then $f \notin D$, since otherwise $e, f \in D \cap D'$, implying

(29.19) $$u(D \triangle D') \leq u(D \setminus \{e, f\}) + u(D' \setminus \{e, f\}) \\ = l(e) - u(f) + l(f) - u(e) < 0.$$

Hence the cut $D \triangle D'$ satisfies

(29.20) $$u(D \triangle D' \setminus \{f\}) \leq u(D \setminus \{e\}) + u(D' \setminus \{e, f\}) = l(e) - u(e) + l(f) \\ < l(f),$$

contradicting (29.18).

So $u_e = l_e$ for each edge e, and hence the corollary follows from Corollary 29.2f. ∎

Let $G = (V, E)$ be a graph. A function $l : E \to \mathbb{R}$ is called *conservative* if $l(C) \geq 0$ for each circuit C. The conservative functions form a polyhedral convex cone, and Corollary 29.2f gives functions that generate this cone:

Corollary 29.2h. *The cone of conservative functions is generated by the nonnegative functions and by the functions l for which there is a subset U of V and an edge $e \in \delta(U)$ such that*

$$l = \chi^{\delta(U)\setminus\{e\}} - \chi^e. \tag{29.21}$$

Proof. Directly by polarity (cf. Section 5.7) from Corollary 29.2f. ∎

In Section 29.11b we show that system (29.17) is TDI if and only if G is series-parallel.

29.8. Integer sums of circuits

Seymour [1979b] gave the following characterization of integer sums of circuits in planar graphs. It is equivalent to saying that the incidence vectors of circuits in a planar graph form a Hilbert base. (We follow a proof suggested by A.V. Karzanov, which starts like Seymour's proof but does not use the four-colour theorem.)

Theorem 29.3. *Let $G = (V, E)$ be a planar graph and let $x \in \mathbb{R}^E$. Then x is a nonnegative integer combination of incidence vectors of circuits if and only if x is an integer vector in the circuit cone with $x(\delta(v))$ even for each vertex v.*

Proof. Necessity being easy, we show sufficiency. Consider a counterexample with

$$|V| + \sum_{e \in E}(x(e) + 1)^2 \tag{29.22}$$

minimal. Then G is connected (otherwise one of the components forms a counterexample with (29.22) smaller), $x(e) \geq 1$ for each $e \in E$ (otherwise we can delete e), and each vertex v has degree at least 3 (the degree is at least 2 by (29.17)(ii); if it is precisely 2, then x has the same value on the two edges incident with v (by (29.17)(ii)), and hence we can replace them by one edge).

Consider any edge e_0 with $x(e_0) \geq 2$ and $x(e_0)$ minimal. Let e_0 connect vertices p and q say. Let G' be the graph obtained from G by adding a new (parallel) edge f between p and q. Define $x'(e_0) := x(e_0) - 1$, $x'(f) := 1$, and

$x'(e) := x(e)$ for all other edges e of G'. Then condition (29.17) is maintained, but sum (29.22) decreases. So x' is a sum of circuits[17] in G'. If none of these circuits consist of e_0 and f, then x is a sum of circuits in G, a contradiction. So $\{e_0, f\}$ is one of the circuits. Therefore, in G, the vector

(29.23) $\qquad y := x - 2\chi^{e_0}$

is a sum of circuits, say

(29.24) $\qquad y = \sum_{C\in\mathcal{C}} \lambda_C \chi^{EC},$

where $\mathcal{C}$ is a collection of circuits and where the λ_C are positive integers. Let $\mathcal{C}_0$ be the collection of circuits in $\mathcal{C}$ traversing e_0, and let $\mathcal{C}_1 := \mathcal{C} \setminus \mathcal{C}_0$.

We construct a directed graph $D = (V, A)$. We say that a circuit C *generates* a pair (u, v) of distinct vertices if C traverses both u and v, in such a way that if C traverses e_0, then C traverses p, q, u, v cyclically in this order (possibly $u = q$ or $v = p$). The arc set A of D consists of all pairs (u, v) generated by at least one $C \in \mathcal{C}$. Then:

(29.25) $\qquad D$ contains a directed path from p to q.

For suppose not. Let U be the set of vertices reachable in D from p. So $q \notin U$, and no arc of D leaves U. Hence no $C \in \mathcal{C}_1$ intersects $\delta_E(U)$, and each $C \in \mathcal{C}_0$ intersects $\delta_E(U)$ precisely twice: once in e_0 and once elsewhere. So

(29.26) $\qquad x(\delta_E(U) \setminus \{e_0\}) = y(\delta_E(U) \setminus \{e_0\}) = y(e_0) < x(e_0),$

contradicting (29.17). This shows (29.25).

Now choose a shortest directed $p-q$ path P in D, say $P = (v_0, v_1, \ldots, v_k)$, with $v_0 = p$ and $v_k = q$. Let $\mathcal{C}'$ be an inclusionwise minimal subcollection of $\mathcal{C}$ with the property that each arc of P is generated by some C in $\mathcal{C}'$. Define $\mathcal{C}'_0 := \mathcal{C}' \cap \mathcal{C}_0$, and

(29.27) $\qquad z := 2\chi^{e_0} + \sum_{C\in\mathcal{C}'} \chi^{EC}.$

We show:

(29.28) $\qquad z = x$, $\mathcal{C}' = \mathcal{C}$, and $\lambda_C = 1$ for each $C \in \mathcal{C}$.

It suffices to show that $z = x$. Suppose $z \neq x$. Then, since $z \leq x$, by the minimality of (29.22), z is a sum of circuits. To see this, it suffices to show that (29.17) is satisfied by z. To this end, let $U \subseteq V$ and $e \in \delta(U)$. If $e \neq e_0$, then (29.17)(ii) follows since $z - 2\chi^{e_0}$ is a sum of circuits. If $e = e_0$, then we can assume that $p \in U$, $q \notin U$. Hence some arc (v_{i-1}, v_i) leaves U. Let $C' \in \mathcal{C}'$ generate (v_{i-1}, v_i). Then C' has at least two edges in $\delta(U)$, and at least four if $C' \in \mathcal{C}'_0$. Moreover, any $C \in \mathcal{C}'_0$ has at least two edges in $\delta(U)$. Hence

[17] By a ‘sum of circuits’ we mean a sum of incidence vectors of circuits.

(29.29) $$z(\delta(U) \setminus \{e_0\}) = \sum_{C \in \mathcal{C}'} |EC \cap \delta(U) \setminus \{e_0\}| \geq |\mathcal{C}'_0| + 2 = z(e_0).$$

So (29.17) is satisfied by z. Hence z is a sum of circuits. But then also x is a sum of circuits, since

(29.30) $$\begin{aligned} x - z = y - \sum_{C \in \mathcal{C}'} \chi^{EC} &= \sum_{C \in \mathcal{C}} \lambda_C \chi^{EC} - \sum_{C \in \mathcal{C}'} \chi^{EC} \\ &= \sum_{C \in \mathcal{C} \setminus \mathcal{C}'} \lambda_C \chi^{EC} + \sum_{C \in \mathcal{C}'} (\lambda_C - 1) \chi^{EC}. \end{aligned}$$

This contradicts our assumption, proving (29.28).

Then:

(29.31) each vertex v is traversed by at most two circuits in $\mathcal{C}_1$.

Otherwise, there exist three arcs on P generated by circuits in $\mathcal{C}_1$ traversing v. Hence there exist arcs (v_{i-1}, v_i) and (v_{j-1}, v_j) on P generated by circuits C and C' in $\mathcal{C}_1$ traversing v, with $i < j - 1$. This implies that we can make P shorter (by replacing $v_i, v_{i+1}, \ldots, v_{j-1}$ by v), a contradiction. This shows (29.31).

Consider now any vertex $v \neq p, q$ and any $f \in \delta(v)$ with $x(f) \geq 2$. By the choice of e_0 we know $x(f) \geq x(e_0)$. Hence, using (29.31),

(29.32) $$2x(f) \leq x(\delta(v)) \leq 2|\mathcal{C}_0| + 4 = 2(y(e_0) + 2) = 2x(e_0) \leq 2x(f).$$

So we have equality throughout. In particular, v is traversed by precisely two circuits in $\mathcal{C}_1$, and $x(f) = x(e_0)$.

It follows that, for any $i = 1, \ldots, k-1$, the arcs (v_{i-1}, v_i) and (v_i, v_{i+1}) are generated by circuits in $\mathcal{C}_1$ (by taking $v = v_i$). Trivially, if $k = 1$, the arc (v_0, v_1) is not generated by any circuit in $\mathcal{C}_0$, and hence by some circuit in $\mathcal{C}_1$. Therefore, by the minimality of $\mathcal{C}$, $\mathcal{C}_0 = \emptyset$ and $\mathcal{C}_1 = \mathcal{C}$. Hence $y(e_0) = 0$, and so $x(e_0) = 2$. Therefore, $x(e) \in \{1, 2\}$ for each edge e.

Since each vertex $v \neq p, q$ is traversed by precisely two circuits in $\mathcal{C}$, we know that v is incident with at most one edge e with $x(e) = 2$. Since any e with $x(e) = 2$ can play the role of e_0, this also holds for $v \in \{p, q\}$. So

(29.33) the edges e with $x(e) = 2$ form a matching M in G.

Consider the path P above. Let arc (v_{i-1}, v_i) be generated by circuit $C_i \in \mathcal{C}$, for $i = 1, \ldots, k$. By the minimality of k, C_i and C_j are vertex-disjoint if $j > i + 1$. Let D_1 be the union of the EC_i for odd i, and let D_2 be the union of the EC_i for even i. So (for each $i = 1, 2$) D_i consists of vertex-disjoint circuits, and $D_1 \cap D_2 = M \setminus \{e_0\}$.

This is used in proving:

(29.34) each nonempty cut D contained in M is odd.

Indeed, by symmetry we may assume that $e_0 \in D$. Then $D \setminus \{e_0\} = D \cap D_1$ (since $D \setminus \{e_0\} \subseteq M \setminus \{e_0\} \subseteq D_1$ and since $e_0 \notin D_1$). Moreover, $|D \cap D_1|$ is even, since D_1 is a disjoint union of circuits.

This proves (29.34), which implies that

(29.35) $G - M$ has at most two components,

since if K and L are components with $K \cup L \neq V$, then at least one of $\delta_E(K)$, $\delta_E(L)$, and $\delta_E(K \cup L)$ is nonempty and even, contradicting (29.34).

Moreover:

(29.36) M forms a cut in G.

Otherwise, M has an edge spanned by a component of $G - M$. Hence G has a circuit C with $|C \cap M| = 1$ By symmetry, we may assume that $C \cap M = \{e_0\}$. Then $C \triangle D_1$ and $C \triangle D_2$ form cycles whose incidence vectors add up to x. Hence x is a sum of circuits, a contradiction. So we have (29.36).

Now let K_1 and K_2 be the components of $G - M$. They are connected Eulerian graphs. Since M forms a cut, we can assume that the attachments of M at K_1 and at K_2 are at the outer boundaries B_1 of K_1 and B_2 of K_2. By the planarity of G, the attachments of M occur in the same order on B_1 as on B_2. So $\chi^{EB_1} + \chi^{EB_2} + 2\chi^M$ is a sum of circuits. Since $EK_1 \setminus EB_1$ and $EK_2 \setminus EB_2$ are cycles, this gives x as a sum of circuits. ∎

(The proof of Seymour [1979b] of Theorem 29.3 uses the four-colour theorem. Fleischner and Frank [1990] showed that a method of Fleischner [1980] gives a proof not using the four-colour theorem. Also Alspach and Zhang [1993] gave a proof not using the four-colour theorem.)

In Theorem 29.3 we cannot delete the planarity condition, as is shown by the Petersen graph: fix a perfect matching M, and set $x_e := 2$ if $e \in M$ and $x_e := 1$ if $e \notin M$. Alspach, Goddyn, and Zhang [1994] (extending Alspach and Zhang [1993]) proved that the Petersen graph is the critical example:

Theorem 29.4. *For any graph $G = (V, E)$, the following are equivalent:*

(29.37) (i) *each integer vector x in the circuit cone with $x(\delta(v))$ even for each vertex v is a nonnegative integer combination of incidence vectors of circuits;*
(ii) *G has no Petersen graph minor.*

(This was generalized to binary matroids by Fu and Goddyn [1999] — see Section 81.9.)

Seymour [1979b] conjectures that each *even* integer vector x in the circuit cone is a nonnegative integer combination of incidence vectors of circuits. A special case of this is the *circuit double cover conjecture* (it was asked by Szekeres [1973] and conjectured by Seymour [1979b]): each bridgeless graph has circuits such that each edge is covered by precisely two of them. Thus Theorem 29.4 implies that the circuit double cover conjecture is true for graphs without Petersen graph minor.

It has been proved that for any even integer $k \geq 4$, each bridgeless graph has circuits such that each edge is covered by precisely k of them. (For $k = 6$ by Jaeger [1979] and for $k = 4$ by Fan [1992] — hence any even $k \geq 4$ follows.)

This relates to the 4-*flow conjecture* of Tutte [1966], which generalizes the four-colour theorem:

(29.38) (?) The edges of any bridgeless graph without Petersen graph minor can be covered by two Eulerian subgraphs. (?)

(It is called the 4-flow conjecture, since it is equivalent to saying that for each bridgeless graph $G = (V, E)$ without Petersen graph minor, there is an orientation $D = (V, A)$ of G and a function $f : A \to \{1, 2, 3\}$ with $f(\delta^{\mathrm{in}}(v)) = f(\delta^{\mathrm{out}}(v))$ for each $v \in V$ — see Section 28.4.)

Conjecture (29.38) was proved for 4-edge-connected graphs by Jaeger [1979], and for cubic graphs jointly by Robertson, Seymour, and Thomas [1997], Sanders, Seymour, and Thomas [2000], and Sanders and Thomas [2000].

(29.38) is equivalent to:

(29.39) (?) Any bridgeless graph without Petersen graph minor has two disjoint T-joins, where T is the set of vertices of odd degree (?)

(since J is a T-join if and only if $E \setminus T$ yields an Eulerian graph).

It is NP-complete to decide if a graph has two disjoint T-joins, since for cubic graphs it is equivalent to 3-edge-colourability (cf. Theorem 28.3).

Related work can be found in Zhang [1993c]. Surveys on the circuit double cover conjecture were given by Jaeger [1985], Jackson [1993], and Zhang [1993a,1993b,1997b], and on integer decomposition of the circuit cone (and more general decompositions) by Goddyn [1993].

29.9. The T-cut polytope

The T*-cut polytope* $P_{T\text{-cut}}(G)$ — the convex hull of the incidence vectors of T-cuts — is a 'hard' polytope, even if $|T| = 2$, since finding a maximum cut separating two given vertices in a graph is NP-complete. However, the up hull of the T-cut polytope:

(29.40) $$P^{\uparrow}_{T\text{-cut}}(G) := P_{T\text{-cut}}(G) + \mathbb{R}^E_+$$

is tractable, as follows directly with the theory of blocking polyhedra from the results above on the up hull of the T-join polytope, and is determined by:

(29.41)
(i) $x_e \geq 0$ for each $e \in E$,
(ii) $x(J) \geq 1$ for each T-join J.

Theorem 29.5. *The up hull $P^{\uparrow}_{T\text{-cut}}(G)$ of the T-cut polytope of G is determined by* (29.41).

Proof. Directly with the theory of blocking polyhedra from Corollary 29.2b. ∎

This implies that the following system:

(29.42) (i) $0 \leq x_e \leq 1$ for each $e \in E$,
(ii) $x(J) \geq 1$ for each T-join J,

describes a convex hull as follows.

Corollary 29.5a. *The convex hull of the incidence vectors of edge sets containing a T-cut is determined by* (29.42).

Proof. Directly from Theorem 29.5 with Theorem 5.19. ∎

(For a direct derivation from Edmonds' perfect matching polytope theorem, see Seymour [1979a].)

In general, (29.41) is not TDI, not even totally dual half-integral (Seymour [1979a]). Seymour [1977b] characterized pairs of G, T for which (29.41) is TDI — see Section 29.11c.

Rizzi [1997] showed that the minimal TDI-system for the up hull of the T-cut polytope can have arbitrarily large coefficients and right-hand sides.

29.10. Finding a minimum-capacity T-cut

Like in Section 25.5c we can find a minimum-capacity T-cut by constructing a Gomory-Hu tree (for a graph $G = (V, E)$ and a tree $H = (V, F)$, a *fundamental cut* is a cut $\delta_E(W_f)$, where $f \in F$ and W_f is a component of $H - f$):

Theorem 29.6. *Let $G = (V, E)$ be a graph and let $T \subseteq V$ with $|T|$ even. Let $c \in \mathbb{R}^E_+$ be a capacity function and let $H = (V, F)$ be a Gomory-Hu tree. Then one of the fundamental cuts of H is a minimum-capacity T-cut in G.*

Proof. For each $f \in F$, choose W_f to be one of the two components of $H - f$. Let $\delta_G(U)$ be a minimum-capacity T-cut of G. So $|U \cap T|$ is odd.

Then U or $V \setminus U$ is equal to the symmetric difference of the W_f over $f \in \delta_F(U)$. Hence $|W_f \cap T|$ is odd for at least one $f \in \delta_F(U)$. So $\delta_G(W_f)$ is a T-cut.

Let $f = uv$. As $\delta_G(W_f)$ is a minimum-capacity $u-v$ cut and as $\delta_G(U)$ is a $u-v$ cut, we have $c(\delta_G(W_f)) \leq c(\delta_G(U))$. So $\delta_G(W_f)$ is a minimum-capacity T-cut. ∎

This gives algorithmically (Padberg and Rao [1982]):

Corollary 29.6a. *A minimum-capacity T-cut can be found in strongly polynomial time.*

Proof. This follows from Theorem 29.6, since a Gomory-Hu tree can be found in strongly polynomial time, by Corollary 15.15a. ∎

Barahona and Conforti [1987] showed that a cut $\delta(U)$ with $T \cap U$ and $T \setminus U$ even and nonempty, and of minimum capacity, can be found in strongly polynomial time.

Barahona [2002] gave a combinatorial strongly polynomial-time algorithm to solve the dual of maximizing $c^{\mathsf{T}}x$ over (29.41) (yielding a fractional packing of T-joins).

29.11. Further results and notes

29.11a. Minimum-mean length circuit

Let $G = (V, E)$ be an undirected graph and let $l \in \mathbb{Q}^E$ be a length function. The *mean length* of a circuit C is equal to $l(C)/|C|$. Barahona [1993b] showed (using an argument of Cunningham [1985c]) that a minimum-mean length circuit in an undirected graph can be found in strongly polynomial time, by solving at most m Chinese postman problems.

Theorem 29.7. *A minimum-mean length circuit in an undirected graph can be found in strongly polynomial time.*

Proof. Let $G = (V, E)$ be an undirected graph and let $l \in \mathbb{Q}^E$ be a length function. Note that by adding a constant γ to all edge-lengths, the collection of minimum-mean length circuits does not change (as the mean length of any circuit increases by exactly γ). So we can assume that there exists a circuit C with $l(C) < 0$.

The algorithm is as follows:

(29.43) Find a minimum-length $\emptyset$-join J.
If $l(J) = 0$, output a circuit of length 0, and stop.
If $l(J) < 0$, add $\gamma := -l(J)/|J|$ to all edge-lengths, and iterate.

We first show that the algorithm stops; in fact, in at most $|E| + 1$ iterations. To this end, consider two subsequent iterations. Let l and l' be two subsequent length functions and let J and J' be the shortest $\emptyset$-joins found. So $l'(e) = l(e) - l(J)/|J|$ for all $e \in E$. If $l'(J') < 0$, then $|J'| < |J|$, since

$$0 > l'(J') = l(J') - \frac{l(J)}{|J|}|J'| \geq l(J) - \frac{l(J)}{|J|}|J'| = l(J)(1 - \frac{|J'|}{|J|}) \tag{29.44}$$

(note that $l(J) < 0$ and $l(J') \geq l(J)$). This shows that the algorithm stops after at most $|E| + 1$ iterations.

As throughout the iterations, the collection of minimum-mean length circuits is invariant, a minimum-mean length circuit for the final length function, is also a minimum-mean length circuit for the initial length function. Hence the output is correct.

Finally, for the 0-length circuit C in the final iteration we can take any circuit contained in the $\emptyset$-join J found in the one but last iteration (as J has length 0 in the last iteration). ∎

Barahona [1993b] also showed that, conversely, the minimum-length T-join problem can be solved by solving $O(m^2 \log n)$ minimum-mean length circuit problems, as follows. Let $l \in \mathbb{Q}^E$ be a length function. Start with any T-join J. Find a minimum-mean length circuit C for the length function l' given by: $l'(e) := -l(e)$ if $e \in T$ and $l'(e) := l(e)$ otherwise. If $l'(C) \geq 0$, then J is a T-join minimizing $l(J)$. Otherwise, reset $T := T \triangle C$, and iterate.

(We note here that Guan [1960] proposed to find a circuit C minimizing $l'(C)$ and iteratively reset T as above, until $l'(C) \geq 0$. It is however NP-complete to find such a circuit, and moreover, no polynomial upper bound on the number of iterations is known.)

Barahona [1993b] also observed that the minimum-mean length circuit problem can be solved by solving a 'compact' linear programming problem (that is, one in which the number of variables and constraints is bounded by a polynomial in the size of the graph).

This follows from the fact that, for any graph $G = (V, E)$, the convex hull of

$$\{\frac{1}{|C|}\chi^C \mid C \text{ circuit}\} \tag{29.45}$$

(where χ^C is the incidence vector of C in $\mathbb{R}^E$) consists of all vectors x in the circuit cone of G satisfying $\mathbf{1}^{\mathsf{T}} x = 1$; moreover, by Corollary 29.2f, x belongs to the circuit cone of G if and only if for each edge $e = st$ there exists an $s-t$ flow $y \leq x$ in $G-e$ of value x_e. Here the flow is described on the directed graph obtained from $G - e$ by replacing each edge uv by two arcs (u, v) and (v, u). As the flows are determined by flow conservation constraints (next to the negativity and capacity constraints), this yields a compact linear program.

A minimum mean-weight circuit therefore can be found in polynomial time with any polynomial-time LP-algorithm.

29.11b. Packing T-cuts

System (29.9) generally is not TDI, as is shown by taking $G = K_4$ and $T = VK_4$. This example is the critical example, since Seymour [1977b] showed that if system (29.9) is not TDI, then G, T contains K_4, VK_4 as a 'minor' — see Corollary 29.9b below. To prove this, we follow the approach of Frank and Szigeti [1994] using the results of Sebő [1988b].

Each polyhedron is determined by a TDI-system, albeit not necessarily the minimal system defining the polyhedron. Sebő [1988b] showed that system (29.9) can be extended as follows to a TDI-system defining the up hull of the T-join polytope.

Let $G = (V, E)$ be a graph and let T be an even-size subset of V. Call a set B of edges a T-*border* if there exists a partition $\mathcal{P} = (U_1, \ldots, U_k)$ of V such that

$|U_i \cap T|$ is odd for each i and such that B is equal to the set of edges connecting distinct classes of $\mathcal{P}$. The *value* $\mathrm{val}(B)$ of the T-border B is, by definition, half of the number of components K of $G - B$ with $|K \cap T|$ odd. (This is at least $\frac{1}{2}k$.) So a T-border is a T-cut if and only if $\mathrm{val}(B) = 1$. Moreover, each T-join intersects any T-border B in at least $\mathrm{val}(B)$ edges. Hence the minimum size of any T-join is at least the maximum total value of any packing of T-borders. (The *total value* of a collection of T-borders is the sum of the values of the T-borders in the collection.) Sebő [1988b] showed that the minimum and maximum are equal:

Theorem 29.8. *Let $G = (V, E)$ be a graph and let $T \subseteq V$. Then the minimum size of a T-join is equal to the maximum total value of a packing of T-borders.*

Proof. Choose a counterexample with $|V|$ as small as possible. Then G is connected.

By Corollary 29.2a, it suffices to show that the maximum total value of a packing of T-borders is at least half of the maximum size of a 2-packing of T-cuts[18] Choose a maximum-size 2-packing of T-cuts $\delta(U_1), \ldots, \delta(U_k)$, which by Corollary 29.2a we may assume to be cross-free. We must find a packing of T-borders of total value $\frac{1}{2}k$.

We choose the U_i such that

(29.46) $$\sum_{i=1}^{k} |U_i|$$

is as small as possible. In particular, $|U_i| \leq |V \setminus U_i|$ for each i.

For each such 2-packing we have

(29.47) $\delta(U_i) \neq \delta(U_j)$ if $i \neq j$,

since otherwise we can contract the edges in $\delta(U_i)$ to obtain G', T' and apply induction. We obtain a packing of T'-borders in G', of total value $\frac{1}{2}(k-2)$. Together with the T-border $B := \delta(U_i)$ this gives a packing of T-borders in G of total value $\frac{1}{2}k$. This shows (29.47).

We next show

(29.48) $|U_i| = 1$ for each i.

Suppose not. Choose an inclusionwise minimal set U_i with $|U_i| > 1$. So for any j, if $U_j \subset U_i$, then $U_j = \{t\}$ for some $t \in T \cap U_i$. Moreover, for each $t \in T \cap U_i$, there is a j with $U_j = \{t\}$, since otherwise we could reset $U_i := \{t\}$, contradicting the minimality of the sum (29.46). Then $U_i \subseteq T$, since otherwise we can replace U_i by $T \cap U_i$, again contradicting the minimality of the sum (29.46). It follows that the union of the $\delta(t)$ for $t \in U_i$ forms a T-border B of value $\frac{1}{2}(|U_i| + 1)$. Contracting the edges in B gives G', T' say. Applying induction to G', T' (in which there exists a 2-packing of T'-cuts of size $k - (|U_i| + 1)$), gives a packing of T'-borders in G' of total value $\frac{1}{2}(k - |U_i| - 1)$. Adding B, gives a packing of T-borders in G of total value $\frac{1}{2}k$.

So we can assume that $|U_i| = 1$ for each i. Then the union of the $\delta(U_i)$ for $i = 2, \ldots, k$ forms a T-border of value $\frac{1}{2}k$. ∎

This theorem bears upon the system

[18] A 2-*packing* is a family of sets such that no element is in more than two of them.

(29.49) (i) $x_e \geq 0$ for each $e \in E$,
(ii) $x(B) \geq \text{val}(B)$ for each T-border B.

Since each inequality (29.9)(ii) occurs among (29.49), and since, conversely, each inequality (29.49)(ii) is a half-integer sum of inequalities (29.9)(ii), the two systems (29.9) and (29.49) define the same polyhedron — namely $P^{\uparrow}_{T\text{-join}}(G)$. In fact:

Corollary 29.8a. *System* (29.49) *is TDI.*

Proof. For any weight function $w \in \mathbb{Z}_+^E$ we can replace any edge $e = uv$ by a $u-v$ path of length $w(e)$, contracting e if $w(e) = 0$. Applying Theorem 29.8 to the new graph gives an integer optimum dual solution to the problem of minimizing $w^{\mathsf{T}}x$ subject to (29.49). ∎

We next use Theorem 29.8 to show that system (29.9) is TDI if G, T contains no K_4, VK_4 as a 'minor'. We follow the line of proof given by Frank and Szigeti [1994]. We first prove the following.

Call a graph $G = (V, E)$ *bicritical* if $G - u - v$ has a perfect matching for each pair of distinct vertices u and v. Call a graph $G = (V, E)$ *oddly contractible* to K_4 if V can be partitioned into four odd sets V_1, V_2, V_3, V_4 such that $G[V_i \cup V_j]$ is connected for all i, j (also if $i = j$). The following result is due to A. Sebő (cf. Frank and Szigeti [1994]):

Theorem 29.9. *A bicritical graph with at least four vertices is oddly contractible to K_4.*

Proof. Let $G = (V, E)$ be a bicritical graph with $|V| \geq 4$. This immediately implies that G is connected and has a perfect matching M. Moreover,

(29.50) for all $u, v \in V$ with $u \neq v$ there is an odd-length M-alternating $u-v$ path $P_{u,v}$ with first and last edge not in M.

To see this, first assume that $uv \in M$. Then there is a perfect matching N not containing uv (since there exists an edge uw with $w \neq v$ (by the connectedness of G), and hence the perfect matching of $G = \{u, w\}$ together with uw forms a perfect matching). Let C be the circuit in $M \cup N$ containing uv. Then $C - uv$ is a path as required in (29.50).

If $uv \notin M$, let u' and v' be such that $uu' \in M$ and $vv' \in M$. Let N be a perfect matching in $G - u' - v'$. Then $(M \cup N) \setminus \{uu', vv'\}$ contains a $u-v$ path as required. This shows (29.50).

Now (29.50) implies:

(29.51) there exists an odd-length M-alternating circuit $C = (v_0, v_1, \ldots, v_t)$.

(So t is odd, and $v_i v_{i+1} \in M$ if and only if i is odd.) To see (29.51), choose edges $uv \in M$ and $vw \notin M$. Then $P_{u,w}$ does not traverse v (otherwise uv is on $P_{u,w}$). So $C := EP_{u,w} \cup \{uv, vw\}$ is a circuit as required in (29.51).

Let w be such that $wv_0 \in M$. Let K be the component of $G - VC$ containing w. So $N(K) \subseteq VC$. We first show that $|N(K)| \geq 3$. Indeed, first we have $v_0 \in N(K)$. Let s be the first vertex in P_{w,v_1} contained in VC. Then $s \neq v_0$, since otherwise $v_0 w \in EP_{w,v_1}$. Let s' be such that $ss' \in M$. So $s' \neq v_0$. Let r be the first vertex in

$P_{w,s'}$ contained in VC. Again $r \neq v_0$. Moreover, $r \neq s$, since otherwise $ss' \in EP_{w,s'}$ (implying that the last edge of $P_{w,s'}$ is in M, a contradiction). As $v_0, s, r \in N(K)$, we have $|N(K)| \geq 3$.

As K is the union of w with a number of edges in M, $|K|$ is odd. Similarly, any other component of $G - VC$ is even. As C is an odd circuit, VC can be partitioned into three paths with an odd number of vertices, each containing a neighbour of K. Hence G is oddly contractible to K_4. ∎

We define deletion, contraction, and minor for pairs G, T. Let $G = (V, E)$ be a graph, $T \subseteq V$, and $e = uv \in E$. We say that $G - e, T$ arises from G, T by *deleting* e. Let G/e be the graph obtained from G by contracting e. Denote the new vertex to which e is contracted by v^e. Define $T' := T \setminus \{u, v\}$ if $|T \cap \{u, v\}|$ is even, and $T' := (T \setminus \{u, v\}) \cup \{v^e\}$ if $|T \cap \{u, v\}|$ is odd. Then we say that $G/e, T'$ arises from G, T by *contracting* e.

We say that the pair G', T' is a *minor* of the pair G, T if G', T' arises from G, T by a series of deletions and contractions of edges, and of deletions of vertices not in T. Then the following is a special case of a more general hypergraph theorem of Seymour [1977b] (Theorem 80.1):

Corollary 29.9a. *Let $G = (V, E)$ be a graph and let $T \subseteq V$ with $|T|$ even, such that K_4, VK_4 is not a minor of G, T. Then the minimum size of a T-join is equal to the maximum number of disjoint T-cuts.*

Proof. By Theorem 29.8, the minimum size of a T-join is equal to the maximum total value of a packing of T-borders. Consider such an optimum packing, with the number of T-borders as large as possible. If each T-border is a T-cut, we are done. So assume that one of the T-borders, B say, has value at least 2. Let $\mathcal{P} = (U_1, \ldots, U_k)$ be a partition of V with $|U_i \cap T|$ odd for each i and such that B is the union of the $\delta(U_i)$.

Let $G' = (V', E'), T'$ be obtained from G, T by contracting each U_i to one vertex. So $T' = V'$. As G', T' contains no K_4, VK_4 as a minor, G is not bicritical, by Theorem 29.9. Hence there are distinct $u, v \in V'$ such that $G' - u - v$ has no perfect matching. By Tutte's 1-factor theorem this implies that there is a subset U of V' with $u, v \in U$ and with $o(G' - U) \geq |U|$. Take such a U with $|U|$ maximal. Then each component of $G' - U$ is odd. (Otherwise, we can add an element of some even component to U, contradicting the maximality of $|U|$.)

For each component K of $G' - U$, the set of edges of G' incident with K form a V'-border in G' of value $\frac{1}{2}(|K|+1)$. So G' has a packing of V'-borders of total value $|V' \setminus U| + o(G' - U) \geq |V'| = k$. Since $|U| \geq 2$ (as $u, v \in U$), we have $o(G' - U) \geq 2$, so there are at least two such components. Hence the packing contains at least two V'-borders. Decontracting the U_i gives a decomposition of B into a packing of at least two T-borders, of total value k. This contradicts the maximality of the number of T-borders in the original packing. ∎

This can be formulated equivalently in terms of total dual integrality. Note that total dual integrality of system (29.9) is closed under taking minors: deletion of an edge e corresponds to intersection with the hyperplane $H := \{x \mid x_e = 0\}$, while contracting e corresponds to projecting on H. Hence total dual integrality of (29.9) can be characterized by forbidden minors; in fact, there is only one forbidden minor:

Corollary 29.9b. *System* (29.9) *is totally dual integral if and only if* G, T *has no minor* K_4, VK_4.

Proof. To see necessity, it suffices to show that if $G = K_4$ and $T = VK_4$, then (29.9) is not TDI. Taking $w_e := 1$ for each $e \in EG$, the minimum weight of a T-join equals 2, while each two T-cuts intersect, implying that there is no integer optimum dual solution.

To see sufficiency, let K_4, VK_4 not be a minor of $G = (V, E), T$. Let $w \in \mathbb{Z}_+^E$. Let G', T' arise from G, T by replacing each edge e by a path of length w_e, contracting e if $w_e = 0$. Then also G', T' has no minor K_4, VK_4. Moreover, the minimum weight k of a T-join in G is equal to the minimum size of a T'-join in G'. By Corollary 29.9a, G' contains a T'-cut packing of size k. So G contains k T-cuts such that each edge e of G is in at most $w(e)$ of them. This gives an integer optimum dual solution to the problem of minimizing $w^{\mathsf{T}}x$ subject to (29.9). ∎

This implies a characterization of series-parallel graphs:

Corollary 29.9c. *The following are equivalent for any graph* $G = (V, E)$*:*

(29.52)
(i) *G is series-parallel;*
(ii) (29.9) *is TDI for each choice of T;*
(iii) (29.11) *is TDI for each choice of T;*
(iv) (29.11) *is TDI for some choice of T;*
(v) (29.17) *is TDI.*

Proof. The equivalence of (i) and (ii) follows from Corollary 29.9b, since a graph G is series-parallel if and only if G has no K_4 minor. The implication (iii)⇒(iv) is direct.

We next show (v)⇒(ii). Let (29.17) be TDI. Choose $T \subseteq V$ and $w \in \mathbb{Z}_+^E$. Let J be a T-join minimizing $w(J)$. Define $\tilde{w}(e) := w(e)$ if $e \in E \setminus J$ and $\tilde{w}(e) := -w(e)$ if $e \in J$. Then $\emptyset$ is a $\tilde{w}$-minimal $\emptyset$-join. Since (29.17) is TDI, there exist $\lambda_{U,e} \in \mathbb{Z}_+$ for $U \subseteq V$ and $e \in \delta(U)$ with

$$\tilde{w} \geq \sum_{U,e} \lambda_{U,e}(\chi^{\delta(U)\setminus\{e\}} - \chi^e). \tag{29.53}$$

Choose the $\lambda_{U,e}$ such that $\sum_{U,e} \lambda_{U,e}$ is minimized. Then

$$\text{if } \lambda_{U,e} \geq 1 \text{ and } \lambda_{U',e'} \geq 1, \text{ then } e' \notin \delta(U) \setminus \{e\}. \tag{29.54}$$

Otherwise, if $e \in \delta(U')$, then

$$(\chi^{\delta(U)\setminus\{e\}} - \chi^e) + (\chi^{\delta(U')\setminus\{e'\}} - \chi^{e'}) \tag{29.55}$$

is nonnegative, and hence we can decrease $\lambda_{U,e}$ and $\lambda_{U',e'}$ by 1, without violating (29.53), contradicting our minimality assumption.

If $e \notin \delta(U')$, then $e \in \delta(U \triangle U')$. Also, (29.55) is at least

$$\chi^{\delta(U\triangle U')\setminus\{e\}} - \chi^e, \tag{29.56}$$

and hence we can decrease $\lambda_{U,e}$ and $\lambda_{U',e'}$ by 1, and increase $\lambda_{U\triangle U',e}$ by 1, without violating (29.53), again contradicting our minimality assumption.

This shows (29.54). So there are no two $\lambda_{U,e} \geq 1$ and $\lambda_{U',e'} \geq 1$ such that the vectors $\chi^{\delta(U)\setminus\{e\}} - \chi^e$ and $\chi^{\delta(U')\setminus\{e'\}} - \chi^{e'}$ have opposite signs in some position. The minimality of $\sum \lambda_{U,e}$ then implies that $\sum \lambda_{U,e} = -\tilde{w}(J) = w(J)$ and that $J \cap \delta(U) = \{e\}$ for each U, e with $\lambda_{U,e} \geq 1$. So each such $\delta(U)$ is a T-cut. Moreover,

(29.57) $$w \geq \sum_{U,e} \lambda_{U,e} \chi^{\delta(U)}.$$

So we have an integral dual solution for the problem of minimizing $w^{\mathsf{T}} x$ over (29.9). This proves (v)$\Rightarrow$(ii).

We next show the reverse implication (ii)$\Rightarrow$(v). Let (29.9) be TDI for each choice of T. To prove that (29.17) is TDI, choose $w \in \mathbb{Z}^E$, such that minimizing $w^{\mathsf{T}} x$ over (29.17) is finite — that is (as (29.17) determines the circuit cone) $w(C) \geq 0$ for each circuit C.

Define $J := \{e \in E \mid w(e) < 0\}$ and $T := \{v \in V \mid \deg_J(v) \text{ is odd}\}$. Moreover, $\tilde{w}(e) := |w(e)|$ for $e \in E$. Then J is a T-join minimizing $\tilde{w}(J)$ (as $w(C) \geq 0$ for each circuit C). Hence, as (29.9) is TDI, there exist $\lambda_U \in \mathbb{Z}_+$ for U with $T \cap U$ odd, such that

(29.58) $$\sum_U \lambda_U \chi^{\delta(U)} \leq \tilde{w} \text{ and } \sum_U \lambda_U = \tilde{w}(J).$$

For each U with $\lambda_U \geq 1$ one has $|J \cap \delta(U)| = 1$; let e_U be the edge in $J \cap \delta(U)$. Then

(29.59) $$w \geq \sum_U \lambda_U (\chi^{\delta(U)\setminus e_U} - \chi^{e_U}),$$

proving total dual integrality of (29.17).

Finally we show (v)$\Leftrightarrow$(iii)$\Leftrightarrow$(iv). Consider any $T \subseteq V$ and any vertex χ^J of the T-join polytope, determined by T-join J. Total dual integrality of (29.11) in χ^J means that the following system is TDI:

(29.60)
$$\begin{array}{ll} x_e \geq 0 & \text{for each } e \in E \setminus J, \\ x_e \leq 1 & \text{for each } e \in J, \\ x(H) - x(F) \geq 1 - |F|, & \text{for each } U \subseteq V \text{ and partition } F, H \text{ of} \\ & \delta(U) \text{ with } |U \cap T| + |F| \text{ odd and} \\ & |H \cap J| + |F \setminus J| = 1. \end{array}$$

The condition $|H \cap J| + |F \setminus J| = 1$ implies that there exists an edge $e \in \delta(U)$ with $F = (\delta(U) \cap J) \triangle \{e\}$ and $H = (\delta(U) \setminus J) \triangle \{e\}$.

Setting $\tilde{x}_e := 1 - x_e$ if $e \in J$ and $\tilde{x}_e := x_e$ if $e \in E \setminus J$, (29.60) is equivalent to:

(29.61)
$$\begin{array}{l} \tilde{x}_e \geq 0 \text{ for } e \in E, \\ \tilde{x}(H \setminus J) + |H \cap J| - \tilde{x}(H \cap J) - \tilde{x}(F \setminus J) - |F \cap J| + \tilde{x}(F \cap J) \geq 1 - |F| \end{array}$$

for each U, F, H as described in (29.60). The second line in (29.61) is equivalent to:

(29.62) $$\tilde{x}(H \triangle (J \cap \delta(U))) - \tilde{x}(F \triangle (J \cap \delta(U))) \geq 1 - |F \triangle (J \cap \delta(U))|.$$

and hence to

(29.63) $$\tilde{x}(\delta(U) \setminus \{e\}) - \tilde{x}_e \geq 0,$$

where $\{e\} := F \triangle (J \cap \delta(U))$. As this equivalence holds for any fixed T, this proves both (iv)$\Rightarrow$(v) and (v)$\Rightarrow$(iii). ∎

(Korach [1982] gave an algorithmic proof of this corollary.)

Sebő [1988b] also characterized the minimal TDI-system for the polyhedron $P^{\uparrow}_{T\text{-join}}(G)$. Call a T-border B *reduced* if $B = \delta(U_1) \cup \cdots \cup \delta(U_k)$ for some partition $\mathcal{P} = (U_1, \ldots, U_k)$ of V such that $|U_i \cap T|$ is odd and $G[U_i]$ is connected for each i and such that the graph obtained by contracting each U_i to one vertex is bicritical. Then the following is a minimal TDI-system for connected graphs:

(29.64) (i) $x_e \geq 0$ for each edge e for which $\{e\}$ is not a T-cut,
(ii) $x(B) \geq \mathrm{val}(B)$ for each reduced T-border B.

Sebő [1993c] showed that for each fixed k, the problem of finding a maximum integer packing of T-cuts subject to a capacity constraint is polynomial-time solvable if $|T| = k$. The method uses that integer linear programming is polynomial-time solvable in fixed dimension (Lenstra [1983]).

29.11c. Packing T-joins

In the previous section we considered packing T-cuts, which relates to the total dual integrality of system (29.9). We now consider packing T-joins, which relates to the total dual integrality of system (29.41).

System (29.41) generally is not TDI. Indeed, let G be the graph $K_{2,3}$ and let $T_0 := VK_{2,3} \setminus \{v_0\}$, where v_0 is one of the two vertices of degree 3 in $K_{2,3}$. Then the minimum size of a T_0-cut in $K_{2,3}$ is equal to 2, while there are no two disjoint T_0-joins. This again is the critical example, as follows again from a more general hypergraph theorem of Seymour [1977b] (Theorem 80.1). For this special case, we follow the line of proof given by Codato, Conforti, and Serafini [1996].

Theorem 29.10. *Let $G = (V, E)$ be a graph and let $T \subseteq V$, such that $K_{2,3}, T_0$ is not a minor of G, T. Then the minimum size of a T-cut is equal to the maximum number of disjoint T-joins.*

Proof. Let G, T form a counterexample, with $|V| + |E|$ as small as possible. Let k be the minimum size of a T-cut. Then trivially G is connected. Moreover:

(29.65) any T-cut C of size k satisfies $C = \delta(t)$ for some $t \in T$.

Indeed, let $C = \delta(U)$ for $U \subseteq V$ with $|U \cap T|$ odd and $|C| = k$. Assume that $1 < |U| < |V| - 1$. Then $G[U]$ is connected, since otherwise there would exist a T-cut smaller than k. Similarly, $G - U$ is connected.

Now contract U to one vertex v', yielding minor G', T' of G, T. The minimum size of a T'-cut in G' equals k. As $|VG'| < |VG|$, we know that G' has k disjoint T'-joins. Each of them intersects $\delta_{G'}(v')$ in exactly one edge (as it is a T'-cut of size k).

We can contract $V \setminus U$ to one vertex v'', yielding minor G'', T'' of G, T. Again, G'' has k disjoint T''-joins, each intersecting $\delta_{G''}(v'')$ in exactly one edge.

Using the one-to-one correspondence between $\delta_{G'}(v')$ and $\delta_{G''}(v'')$, we can glue the two collections of joins together, to obtain k disjoint T-joins in G, contradicting our assumption. This gives (29.65).

Let $T' := \{t \in T \mid \deg(t) = k\}$. Then (29.65) implies that

(29.66) each edge of G intersects T'.

Otherwise we could delete the edge without decreasing the minimum size of a T-cut, by (29.65). This would give a smaller counterexample, contradicting our assumption.

We next have:

(29.67) $|V \setminus T'| \geq 2$.

For suppose $|V \setminus T'| \leq 1$. We know that G has $k-1$ disjoint T-joins (by the minimality of $|V|+|E|$ — otherwise deleting any edge would give a smaller counterexample). Let F be the union of these T-joins. Then $\deg_F(v)$ is even if $v \notin T$ while $\deg_F(v) \equiv k-1 \pmod 2$ if $v \in T$. Hence $\deg_{E\setminus F}(v)$ is odd for each $v \in T'$. As $|V \setminus T'| \leq 1$ it follows that $E \setminus F$ is a T-join, and hence G would have k disjoint T-joins. This contradicts our assumption, and proves (29.67).

Then

(29.68) there is no subset U of T' with $|U| \leq 2$ and $G-U$ disconnected.

Suppose not. If $|U|=1$, let $U=\{t\}$ for $t \in T'$. Then $|K \cap T|$ is odd for some component K of $G-t$. As $G-t$ is disconnected, $|\delta(K)| < \deg(t) = k$, contradicting the fact that $\delta(K)$ is a T-cut.

If $|U|=2$, let $U=\{t,t'\}$ for $t,t' \in T'$. Choose a component K of $G-U$ not contained in T'. Let l (l', respectively) be the number of edges connecting K and t (K and t', respectively). If $|K \cap T|$ is odd, then $l+l' = |\delta(K)| > k$ and hence $|\delta(K \cup \{t,t'\})| \leq (k-l)+(k-l') < k$, a contradiction. If $|K \cap T|$ is even, then $l'+(k-l) = |\delta(K \cup \{t\})| > k$, and similarly $l+(k-l') > k$, a contradiction. This proves (29.68).

Now choose $u \in V \setminus T'$. As $N(u) \subseteq T'$ (by (29.66)), by (29.68) we know $|N(u)| \geq 3$. Choose a component K of $G' := G - (\{u\} \cup N(u))$, with $|N(K)|$ as small as possible. (K exists by (29.67).) If possible, we take K such that moreover $|K \cap T|$ is odd.

Again by (29.68), $|N(K)| \geq 3$. Choose $t_1,t_2,t_3 \in N(K)$. Then

(29.69) for any component $L \neq K$ of G' with $N(L)=\{t_1,t_2,t_3\}$ one has $|L \cap T|$ even.

For suppose that $|L \cap T|$ is odd. By the minimality of $|N(K)|$, we know $N(K) = \{t_1,t_2,t_3\}$. Moreover, $|K \cap T|$ is odd. Let k_i be the number of edges connecting K and t_i and let l_i be the number of edges connecting L and t_i, for $i=1,2,3$. Then $k_1+k_2+k_3 = |\delta(K)| \geq k$, and similarly $l_1+l_2+l_3 \geq k$. This gives the contradiction

(29.70) $$k < |\delta(K \cup L \cup \{t_1,t_2,t_3\})| \leq (k-k_1-l_1)+(k-k_2-l_2)+(k-k_3-l_3) \leq k$$

(the first inequality follows from (29.65)). This shows (29.69).

Now contract the union of $\{u\} \cup (N(u) \setminus \{t_1,t_2,t_3\})$ and all components $L \neq K$ of G' with $N(L) \neq \{t_1,t_2,t_3\}$ to one vertex u'. Moreover, contract the union of $\{t_1\}$ and all components $L \neq K$ of G' with $N(L)=\{t_1,t_2,t_3\}$ to one vertex t_1'. Finally contract K to one vertex u''. This gives minor G'',T'' of G,T.

So G'' has vertices u',u'',t_1',t_2,t_3, with each of u',u'' adjacent to each of t_1',t_2,t_3 (possibly there are more adjacencies). Each of t_1',t_2,t_3 belongs to T''. As $|T''|$ is even, exactly one of u',u'' belongs to T''. Hence G,T has minor $K_{2,3},T_0$, a contradiction. ∎

This implies the characterization:

Corollary 29.10a. *System* (29.41) *is TDI if and only if* $K_{2,3}, T_0$ *is not a minor of* G, T.

Proof. Necessity follows from the fact that total dual integrality of (29.41) is maintained under taking minors (contraction of an edge e corresponds to intersecting the polytope with the hyperplane $x_e = 0$, and deletion of e corresponds to projecting on it), while the minimum size of a T_0-cut in $K_{2,3}$ is 2, and $K_{2,3}$ has no two disjoint T_0-joins.

To see sufficiency, let $w \in \mathbb{Z}_+^E$. Replace any edge $e = uv$ of G by $w(e)$ parallel edges connecting u and v, yielding the graph G'. Then the minimum weight of a T-cut in G is equal to the minimum size of a T-cut in G'. By Theorem 29.10, this minimum size is equal to the maximum number of disjoint T-joins in G'. These T-joins give an integer optimum dual solution to the problem of minimizing $w^{\mathsf{T}}x$ subject to (29.41). ∎

Generally, system (29.41) is not totally dual half-integral, as is shown by the following example of Seymour [1979a]. Let $G' = (V', E')$ be a connected bridgeless cubic graph with $\chi'(G) = 4$ and with an even number of edges. (For instance, G' is the Petersen graph with one vertex replaced by a triangle (in such a way that the three vertices adjacent to it in the Petersen graph, now each are adjacent to one of the vertices in the triangle).)

Let $G = (V, E)$ be obtained from G' by replacing each edge by a path of length 2. So $|V|$ is even.

Then trivially the minimum size of a V-cut is equal to 2. However, the maximum number of V-joins covering each edge at most twice is equal to 3. For suppose that there exist four V-joins $J_1, \ldots, J_4$ covering each edge at most twice. Since each edge of G is incident with a vertex of degree two, each edge of G is covered exactly twice by the J_i. For $i = 1, 2, 3$, let $C_i := J_i \triangle J_4$. Then each C_i is a vertex-disjoint union of circuits, and each edge of G is in exactly two of the C_i. Then the complements of the C_i form edge-disjoint V'-joins in G. This would yield a 3-edge-colouring of G' — a contradiction.

If we replace each edge of G by two parallel edges, thus obtaining an Eulerian graph, the minimum size of a V-cut equals 4, whereas the maximum number of disjoint V-joins is 3.

If Seymour's 'generalized Fulkerson conjecture' (see Section 28.5) is true, there exists a $\frac{1}{4}$-integer packing (that is, the minimum size of a T-cut is equal to one quarter of the maximum size of a 4-packing of T-joins); in other words, the total dual quarter-integrality of the T-join constraints (29.41) follows — we give the proof of Seymour [1979a] of this derivation.

Proof that the generalized Fulkerson conjecture implies the total dual quarter-integrality of the T-join constraints. Let $G = (V, E)$ be a graph and let $T \subseteq V$. Let k be the minimum size of a T-cut. We must show that the generalized Fulkerson conjecture implies:

(29.71) there exist T-joins $J_1, \ldots, J_{4k}$ covering each edge of G at most four times.

First assume that $T = V$. We show:

(29.72) if each vertex of G has even degree, then there exist V-joins $J_1, \ldots, J_{2k}$ covering each edge of G at most twice.

To see this, assume that each vertex of G has even degree. So k is even. If $k \leq 2$, (29.72) is trivial. (If $k = 2$ there exists a V-join J; then the complement $E \setminus J$ is a V-join again.) So we can assume that $k \geq 4$.

For each $v \in V$, let G_v be a $(k-1)$-edge-connected graph with $\deg_G(v) + 1$ vertices, one of degree k and all other vertices of degree $k - 1$. (Such graphs G_v exist: If $k = 4$, take any cubic 3-edge-connected graph on $\deg_G(v) + 2$ vertices (for instance, by taking a circuit on $\deg_G(v) + 2$ vertices, and making opposite vertices adjacent), and contract an arbitrary edge of it. If $k \geq 6$, add a Hamiltonian circuit to the graph for the case $k - 2$.)

We take the G_v vertex-disjoint. Now transform G to a graph H, by replacing each vertex v by G_v, and making each edge of G which was incident with v, incident instead with one of the $\deg_G(v)$ vertices of G_v of degree $k - 1$, in such a way that the resulting graph H is k-regular.

We show that H is a k-graph, by showing

(29.73) $|\delta_H(U)| \geq k$ for each $U \subseteq VH$ with $|U|$ odd.

To see this, assume $|\delta_H(U)| < k$. Observe that $|\delta_H(U)|$ is even, as k is even and H is k-regular. Hence $|\delta_H(U)| \leq k - 2$. Since each G_v is $(k-1)$-edge-connected, for each $v \in V$ we know that either $VG_v \subseteq U$ or $VG_v \cap U = \emptyset$. Define

(29.74) $X := \{v \in V \mid VG_v \subseteq U\}$.

Then $|\delta_H(U)| = |\delta_G(X)|$. Moreover, $|X|$ is odd as $|VG_v|$ is odd for each $v \in V$. Therefore $|\delta_G(X)| \geq k$ and hence $|\delta_H(U)| \geq k$. This shows (29.73).

Then by the generalized Fulkerson conjecture, there exist perfect matchings $M_1, \ldots, M_{2k}$ in H covering each edge of H exactly twice. Projecting these matchings to the original edges of G, gives V-joins as required in (29.72).

Now, for $T = V$, (29.71) follows from (29.72) by replacing each edge of G by two parallel edges. The case of general T can be reduced to the case $T = V$ as follows. Let T be arbitrary. For each vertex $v \in V \setminus T$, make a new vertex v', connected by k parallel edges with v. This gives the graph $G' = (V', E')$. Then the minimum size of a V'-cut in G' is equal to k. Hence by (29.71) there exist V'-joins $J'_1, \ldots, J'_{4k}$ in G' covering each edge of G' at most four times. Restricting the J'_i to the edges of G, gives V-joins in G as required.

Cohen and Lucchesi [1997] showed that conjecture (29.72) is equivalent to: if all T-cuts have the same parity, then the maximum size of a 2-packing of T-joins is equal to twice the minimum size of a T-cut. They also showed that this is true if $|T| \leq 8$; more strongly, that if $|T| \leq 8$ and all T-cuts have the same parity, then the maximum number of disjoint T-joins is equal to the minimum size of a T-cut.

29.11d. Maximum joins

Let $G = (V, E)$ be a graph. Call a subset J of E a *join* if $|J \cap C| \leq \frac{1}{2}|C|$ for each circuit C; that is, $|J \triangle C| \geq |C|$ for each circuit C. This can be expressed in terms of the length function $l_J : E \to \{-1, +1\}$, defined by

(29.75) $$l_J(e) := \begin{cases} -1 & \text{if } e \in J, \\ +1 & \text{if } e \notin J. \end{cases}$$

So

(29.76) $\quad l_J(F) = |F \triangle J| - |J|$

for each $F \subseteq E$. Then a set J is a join if and only if $l_J(C) \geq 0$ for each circuit C. Note also that

(29.77) $\quad$ a set J is a join if and only if it is a minimum-size T-join for $T := \{v \in V \mid \deg_J(v) \text{ odd}\}$.

Frank [1990b,1993b] gave a min-max relation for the maximum size of a join. By Corollary 29.2a and (29.77), the maximum size of a join is equal to the maximum size of a fractional packing of T-cuts, taken over $T \subseteq V$ with $|T \cap K|$ even for each component K of G. This, however, is not a min-max relation.

A min-max relation can be described in terms of ear-decomposition. Let $G = (V, E)$ be an undirected graph. An *ear* of G is a path or circuit P in G, of length ≥ 1, such that all internal vertices of P have degree 2 in G. The path may consist of a single edge — so any edge of G is an ear.

If I is the set of internal vertices of an ear P, we say that G arises from $G - I$ by *adding ear*. An *ear-decomposition* of G is a series of graphs $G_0, G_1, \ldots, G_k$, where $G_0 = K_1$, $G_k = G$, and G_i arises from G_{i-1} by adding an ear ($i = 1, \ldots, k$).

A graph $G = (V, E)$ has an ear-decomposition if and only if G is 2-edge-connected (see Theorem 15.17). Moreover, the number of ears in any ear-decomposition is equal to $|E| - |V| + 1$. Then the min-max relation for maximum-size join in 2-connected graphs is formulated as:

Theorem 29.11. *Let $G = (V, E)$ be a 2-edge-connected graph. Then the maximum size of a join is equal to the minimum value of*

(29.78) $$\sum_{i=1}^{k} \lfloor \tfrac{1}{2} |EP_i| \rfloor$$

taken over all ear-decompositions $(P_1, \ldots, P_k)$ of G.

Proof. We first show that the maximum is not more than the minimum. Let J be a join in G and let $\Pi = (P_1, \ldots, P_k)$ be an ear-decomposition of G. Let $G' = (V', E')$ be the graph made by $P_1, \ldots, P_{k-1}$ and let $J' := J \cap E'$. By induction we know

(29.79) $$|J'| \leq \sum_{i=1}^{k-1} \lfloor \tfrac{1}{2} |EP_i| \rfloor.$$

If $|J \cap EP_k| \leq \lfloor \frac{1}{2} |EP_k| \rfloor$ we are done. So assume that $|J \cap EP_k| > \lfloor \frac{1}{2} |EP_k| \rfloor$; that is, $l_J(P_k) < 0$. Let u and v be the end vertices of P_k. Let Q be a $u - v$ path in G' minimizing $l_{J'}(Q)$. So $l_J(P_k) + l_J(Q) \geq 0$ (since J is a maximum-size join). Since $l_{J'}(Q) = |J' \triangle EQ| - |J'|$, Q minimizes $|J' \triangle EQ|$.

Then $J'' := J' \triangle EQ$ is again a join in G', since for any circuit C in G':

(29.80) $\quad |J'' \triangle C| = |J' \triangle (EQ \triangle C)| \geq |J' \triangle EQ| = |J''|$

(since Q minimizes $|J' \triangle EQ|$). Moreover,

(29.81) $$|J''| - |J'| = |J' \triangle EQ| - |J'| = l_J(Q) \geq -l_J(P_k)$$
$$= |J \cap EP_k| - |EP_k \setminus J| \geq |J \cap EP_k| - \lfloor \tfrac{1}{2} |EP_k| \rfloor.$$

Hence, by induction applied to J'',

(29.82) $$|J| = |J'| + |J \cap EP_k| \leq |J''| + \lfloor \tfrac{1}{2}|EP_k| \rfloor \leq \sum_{i=1}^{k} \lfloor \tfrac{1}{2}|EP_i| \rfloor.$$

This shows that the maximum is not more than the minimum. To see equality, for any graph G let $\beta(G)$ be the maximum size of a join in G. For any ear-decomposition $\Pi = (P_1, \ldots, P_k)$, let $\sigma(\Pi) := \sum_{i=1}^{k} \lfloor \frac{1}{2}|EP_i| \rfloor$. Let $\pi(G)$ be the minimum of $\sigma(\Pi)$ over all ear-decompositions Π of G. So we must prove $\beta(G) = \pi(G)$. Call an ear-decomposition Π *optimum* if it minimizes $\sigma(\Pi)$.

We first show:

(29.83) Let $U \subseteq V$ with $G[U]$ 2-edge-connected. Then $\pi(G) \leq \pi(G[U]) + \pi(G/U)$.

To see this, first observe that if $G[U]$ has a Hamiltonian circuit C, then an optimum ear-decomposition Π' of $G[U]$ is obtained by first taking C, and next adding the remaining edges as ears. Now in any optimum ear-decomposition Π'' of G/U, we can insert Π' at the first ear of Π'' containing the vertex into which U is contracted (by splitting C appropriately). In this way we obtain an ear-decomposition Π of G with $\sigma(\Pi) \leq \sigma(\Pi') + \sigma(\Pi'')$.

If $G[U]$ has no Hamiltonian circuit, let Π' be an optimum ear-decomposition of $G[U]$. Let C be its first ear. By the above, $\pi(G) \leq \pi(G[VC]) + \pi(G/VC)$. Also, by induction, $\pi(G/VC) \leq \pi((G[U])/VC) + \pi(G/U)$. As C is the first ear of Π', we have $\pi(G[VC]) + \pi((G[U])/VC) = \pi(G[U])$. Combining, we get $\pi(G) \leq \pi(G[U]) + \pi(G/U)$, showing (29.83).

Next we state:

(29.84) if G is factor-critical, then $\pi(G) \leq \lfloor \frac{1}{2}|VG| \rfloor$.

This follows directly from Theorem 24.9, since $\lfloor \frac{1}{2}|EP_i| \rfloor$ is at most $\frac{1}{2}$ the number of internal vertices of P_i.

In particular, it follows that if G is factor-critical, then $\beta(G) = \pi(G)$, as G has a join of size $\lfloor \frac{1}{2}|VG| \rfloor$, namely a matching. So we can assume that G is not factor-critical.

A graph G is called *matching-covered* if each edge of G is contained in a perfect matching. By Theorem 24.10,

(29.85) if G is matching-covered and 2-edge-connected, then $\pi(G) \leq \frac{1}{2}|VG|$.

For any subset W of V let H_W be the graph obtained from $G[W \cup N(W)]$ by deleting all edges in $N(W)$ and contracting all edges in W. (H_W may have parallel edges.) So H_W is a bipartite graph with colour classes $N(W)$ and $\kappa(W) :=$ the set of components of $G[W]$.

(29.86) There is a nonempty subset W of V such that each component of $G[W]$ is factor-critical and such that H_W is 2-edge-connected and matching-covered.

To see this, we first observe that there is a nonempty subset X of V such that each component of $G[X]$ is factor-critical and such that H_X has a matching M covering $N(X)$. Indeed, if G has no perfect matching, then we can take $X := D(G)$ (= the set of vertices v for which G has a maximum-size matching missing v). By Corollary

24.7a, X has the required properties. If G has a perfect matching, call it M. Choose $u \in V$, and let $X := D(G-u)$. Then X has the required properties (note that the vertex matched in M to u belongs to $D(G-u)$).

Having X and M, orient the edges in M in the direction from $\kappa(X)$ to $N(X)$, and all other edges of H_X in the direction from $N(X)$ to $\kappa(X)$. This gives a directed graph, that has (like any directed graph) a strong component L such that no arc enters L. Let W be the union of those components of $G[X]$ whose contraction belong to L. Since no arc leaves L, for any edge $e = uv \in M$, if $u \in N(X)$ and $u \in L$, then $v \in W$. Conversely, if $v \in W$, then $u \in L$. For let $v \in K \in L$. As G is 2-edge-connected, there exists an edge $f \neq e$ leaving K. As $K \in L$ and no arc enters L, both ends of f belong to L. As L is strongly connected, f belongs to a directed circuit. Necessarily, e is in this directed circuit. So both ends of e are in L.

Hence the edges of M intersecting W, form a perfect matching M' in H_W, and so $|N(W)| = |\kappa(W)|$. Moreover, consider any edge e of H_W not in M. In H_X, e is oriented from $N(W)$ to $\kappa(W)$, and hence, as L is a strong component, it is contained in a directed circuit. This directed circuit forms an M'-alternating circuit in H_W, implying that e belongs to a perfect matching in H_W. So H_W is matching-covered. Finally H_W is 2-edge-connected, as it has a strongly connected orientation, since L is a strong component. This shows (29.86).

Define $U := W \cup N(W)$. Then (29.83), (29.84), (29.85), and (29.86) imply

$$\begin{aligned}(29.87)\qquad \pi(G) &\leq \pi(G/U) + \pi(G[U]) \leq \pi(G/U) + \pi(H_W) + \sum_{K\in\kappa(W)} \pi(G[K]) \\ &\leq \pi(G/U) + \tfrac{1}{2}|VH_W| + \sum_{K\in\kappa(W)} \lfloor \tfrac{1}{2}|K| \rfloor \leq \pi(G/U) + \tfrac{1}{2}|U|.\end{aligned}$$

On the other hand, we have

$$(29.88)\qquad \beta(G) \geq \beta(G/U) + \tfrac{1}{2}|U|.$$

Indeed, let $G' := G/N(W)$. Then trivially, $\beta(G) \geq \beta(G')$. The contracted $N(W)$ forms a cut vertex v_0 in G', and so $\beta(G')$ is equal to the sum of the $\beta(G'[K \cup \{v_0\}])$ over all components K of $G - v_0$. Now for each component K of $G[W]$ we have $\beta(G'[K \cup \{v_0\}]) \geq \frac{1}{2}(|K|+1)$, since $G'[K \cup \{v_0\}]$ has a perfect matching (as K is factor-critical), which is a join. Since $G[W]$ has $|N(W)|$ components, this proves (29.88).

Hence the theorem follows by induction. ∎

The proof gives a polynomial-time algorithm to find a maximum-size join and an ear-decomposition minimizing (29.78).

In Section 24.4d we saw that a graph is factor-critical if and only if it has an ear-decomposition with odd ears only. This can be generalized to (where G/F arises from G by contracting all edges in F):

Theorem 29.12. *Let $G = (V, E)$ be a 2-edge-connected graph. Then the minimum number of even ears in an ear-decomposition of G is equal to the minimum size of a subset F of E with G/F factor-critical.*

Proof. First let $P_1, \ldots, P_k$ be an ear-decomposition of G. Choose one edge from each even ear. This gives a set F with G/F factor-critical, by Theorem 24.9.

Conversely, let $F \subseteq E$ with G/F factor-critical and $|F|$ minimum. By Theorem 24.9, G/F has an ear-decomposition $(P_1, \ldots, P_k)$ with odd ears only. Then we can partition F into $F_1, \ldots, F_k$ such that $P_1 \cup F_1, \ldots, P_k \cup F_k$ is an ear-decomposition of G. This ear-decomposition has at most $|F|$ even ears. ∎

We can derive from this a characterization of the maximum size of a join in any graph:

Corollary 29.12a. *Let $G = (V, E)$ be a connected graph. Then the maximum size $\beta(G)$ of a join is equal to*

$$\tfrac{1}{2}(\phi(G) + |V| - 1). \tag{29.89}$$

where $\phi(G)$ is the minimum size of a subset F of E with G/F factor-critical.

Proof. If G has a cut edge e, the corollary follows by applying induction to G/e, since $\beta(G) = \beta(G/e) + 1$ and $\phi(G) = \phi(G/e) + 1$.

So we can assume that G is 2-edge-connected, and then the corollary follows from Theorem 29.11, with Theorem 29.12. Note that

$$\begin{aligned}\sum_{i=1}^{k} \lfloor \tfrac{1}{2}|EP_i| \rfloor &= \tfrac{1}{2}(\text{number of even ears} + \sum_{i=1}^{k}(|EP_i| - 1)) \\ &= \tfrac{1}{2}(\text{number of even ears} + |V| - 1).\end{aligned} \tag{29.90}$$

∎

For 2-edge-connected bipartite graphs we have:

Corollary 29.12b. *Let $G = (V, E)$ be a 2-edge-connected bipartite graph, with colour classes U and W. Then the maximum size of a join is equal to the minimum number of edges oriented towards U in any strongly connected orientation of G.*

Proof. To see that the maximum is not more than the minimum, consider any strongly connected orientation of G, yielding the directed graph D. By Theorem 6.9, D has an ear-decomposition $(P_1, \ldots, P_k)$. Any ear P_i contains at least $\lfloor \frac{1}{2}|EP_i| \rfloor$ edges oriented towards U. So the sum (29.78) is at most the total number of edges oriented towards U. Hence by Theorem 29.11, the maximum is not more than the minimum.

To see equality, consider an ear-decomposition $P_1, \ldots, P_k$ of G minimizing (29.78). In any ear P_i, we can orient the edges so as to obtain a directed path, with exactly $\lfloor \frac{1}{2}|EP_i| \rfloor$ edges oriented towards U. This gives a strongly connected orientation with $\sum_i \lfloor \frac{1}{2}|EP_i| \rfloor$ edges oriented towards U. So Theorem 29.11 gives equality. ∎

We can derive some more min-max relations for bipartite graphs. Seymour [1981d] observed that Theorem 29.2 is equivalent to:

Theorem 29.13. *Let $G = (V, E)$ be bipartite and let $J \subseteq E$. Then J is a join if and only if there exist $|J|$ disjoint cuts each intersecting J in exactly one edge.*

Proof. By Theorem 29.2, using (29.77). ∎

This implies a max-max relation for the maximum size of a join in bipartite graphs:

Corollary 29.13a. *Let G be bipartite. Then the maximum size of a join is equal to the maximum number of disjoint nonempty cuts.*

Proof. Directly from Theorem 29.13. ∎

Hence, with Corollary 29.12b, a result of D.H. Younger follows (cf. Frank [1993b]):

Corollary 29.13b. *Let G be a 2-edge-connected bipartite graph, with colour classes U and W. Then the minimum number of edges oriented towards U in any strongly connected orientation of G is equal to the maximum number of disjoint nonempty cuts in G.*

Proof. From Corollaries 29.13a and 29.12b. ∎

Frank, Tardos, and Sebő [1984] showed the following. Let G be a 2-edge-connected bipartite graph, with colour classes U and W. Then the minimum number of edges oriented towards U in any strongly connected orientation of G is equal to the maximum value of

$$\sum_{S\in\Pi} \kappa(G-S), \tag{29.91}$$

ranging over all partitions Π of U, where $\kappa(H)$ denotes the number of components of H.

For an extension, see Kostochka [1994]. Szigeti [1996] gave a weighted version, based on matroids. Fraenkel and Loebl [1995] showed that it is NP-complete to find the maximum size of a subset J of the edge set E of a graph G with $l_J(C) < \frac{1}{2}|EC|$ for each circuit C (even if G is planar and bipartite). Connected joins were investigated by Sebő and Tannier [2001].

29.11e. Odd paths

We saw in Section 29.2 that the problem of finding a shortest $s-t$ path in an undirected graph $G=(V,E)$, with length function $l : E \to \mathbb{Q}$ can be solved in polynomial time, if each circuit has nonnegative length. This is by reduction to the weighted matching problem.

As J. Edmonds (cf. Grötschel and Pulleyblank [1981]) observed, another problem reducible to the weighted matching problem is: given a graph $G=(V,E)$ and a length function $l : E \to \mathbb{Q}_+$, find a shortest odd $s-t$ path. Here a path is *odd* if it has an odd number of edges.

This reduction is as follows: make a copy $G'=(V',E')$ of G, and a copy $l' : E' \to \mathbb{Q}_+$ of l, add edges vv' for each $v \in V$ (where v' is the copy of v), each of length 0. Call the extended graph H. Then a minimum-length odd $s-t$ path in G can be found by finding a minimum-length perfect matching M in $H-s'-t'$: let N be the perfect matching $\{vv' \mid v \in V\}$ in H; then the component of $M \cup N$ containing s and t gives a shortest odd $s-t$ path in G.

Next consider the following polyhedron Q in $\mathbb{R}^E$:

(29.92) $$Q := \text{conv.hull}\{\chi^P \mid P \text{ odd } s-t \text{ path}\} + \mathbb{R}_+^E$$

and its blocking polyhedron

(29.93) $$B(Q) = \{x \in \mathbb{R}_+^E \mid x(P) \geq 1 \text{ for each odd } s-t \text{ path } P\}.$$

By the above method, one can optimize over Q in polynomial time. Hence, with the ellipsoid method, one can decide if a given $x \in \mathbb{Q}^E$ belongs to Q or not, and if not, find a separating facet. This also implies that for given capacity function $c : E \to \mathbb{Q}_+$, one can find in polynomial time a fractional packing of odd $s-t$ paths subject to c, of maximum value (by minimizing $c^\mathsf{T}x$ over $B(Q)$).

Schrijver and Seymour [1994] considered the problem (raised by Grötschel [1984]) of finding an explicit system of inequalities describing Q; equivalently, of describing the vertices of $B(Q)$.

Call a subset F of E *odd-blocking* if each odd $s-t$ path contains an edge in F. For each $F \subseteq E$, define $h_F \in \mathbb{Z}_+^E$ as follows, where $e = uv \in E$ and $W_F := \{s,t\} \cup \{v \in V \mid v \text{ is incident with at least one edge in } E \setminus F\}$:

(29.94) $$h_F(e) := \begin{cases} 2 & \text{if } u,v \in W_F \text{ and } e \in F, \\ 1 & \text{if exactly one of } u,v \text{ belongs to } W_F, \\ 0 & \text{otherwise.} \end{cases}$$

In other words,

(29.95) $$h_F = \sum_{v \in W_F} \chi^{\delta(v) \cap F}.$$

In particular, $h_F(e) = 0$ if $e \notin F$.

Note that for each $x \in \mathbb{Z}_+^E$ one has:

(29.96) $h_F^\mathsf{T}x \geq 1$ for each odd-blocking F $\iff$ there exists an odd $s-t$ path P with $\chi^P \leq x$ $\iff$ $h_F^\mathsf{T}x \geq 2$ for each odd-blocking F.

Then Schrijver and Seymour [1994] proved:

(29.97) Let $l : E \to \mathbb{Z}_+$ be a length function such that each circuit and each $s-t$ path has even length. Then the minimum length of an odd $s-t$ path is equal to the maximum value of $2k$ for which there exist odd-blocking sets $F_1, \ldots, F_k$ with $h_{F_1} + \cdots + h_{F_k} \leq l$.

This implies:

(29.98) Let $l : E \to \mathbb{Z}_+$ be a length function. Then the minimum length of an odd $s-t$ path is equal to the maximum value of k for which there exist odd-blocking $F_1, \ldots, F_k$ with $\frac{1}{2}h_{F_1} + \cdots + \frac{1}{2}h_{F_k} \leq l$.

This can be formulated in terms of LP-duality. Let $\mathcal{F}$ be the collection of odd-blocking sets and let H be the $\mathcal{F} \times E$ matrix whose Fth row equals h_F (for $F \in \mathcal{F}$). Then (29.98) states that for $l : E \to \mathbb{Z}_+$:

(29.99) $$\min\{l^\mathsf{T}x \mid x \in \mathbb{Z}_+^E, (\tfrac{1}{2}H)x \geq \mathbf{1}\} = \max\{y^\mathsf{T}\mathbf{1} \mid y \in \mathbb{Z}_+^{\mathcal{F}}, y^\mathsf{T}(\tfrac{1}{2}H) \leq l^\mathsf{T}\}.$$

Equivalently, the system

(29.100) $$\begin{array}{ll} x_e \geq 0 & e \in E, \\ \frac{1}{2}h_F^\mathsf{T}x \geq 1 & F \text{ odd-blocking}, \end{array}$$

determines Q and is TDI. Hence:

(29.101) each vertex of $B(Q)$ is equal to $\frac{1}{2}h_F$ for some odd-blocking $F \subseteq E$

(this implies the conjecture of W.J. Cook and A. Sebő that the vertices of $B(Q)$ are half-integer).

Minimizing $c^{\mathsf{T}}x$ over $B(Q)$ then gives the following. Let $G = (V, E)$ be an undirected graph, let $s, t \in V$, and let $c : E \to \mathbb{R}_+$. Then the maximum value of a fractional packing of odd $s - t$ paths subject to c is equal to the minimum value of

(29.102) $$\frac{1}{2} \sum_{v \in W_F} c(\delta(v) \cap F),$$

taken over odd-blocking $F \subseteq E$.

L. Lovász asked for the complexity of the following combination of two of the problems above: given a graph $G = (V, E)$, vertices $s, t \in V$, and a length function $l : E \to \mathbb{Q}$, such that each circuit has nonnegative length, find a shortest odd $s - t$ path.

29.11f. Further notes

Complexity survey for all-pairs shortest paths in undirected graphs without negative-length circuits ($*$ indicates an asymptotically best bound in the table):

$*$	$O(nm \log n)$	Gabow [1983a]
$*$	$O(n^3)$	Gabow [1983a]

(The algorithm proposed by Bernstein [1984] fails (for instance, for a graph with four vertices).)

Karzanov [1986] gave an $O(|T|m \log n + |T|^3 \log |T|)$-time algorithm to find a shortest T-join and a maximum fractional packing of T-cuts.

It is easy to see that the vertices of $P^{\uparrow}_{T\text{-join}}(G)$ are the incidence vectors of the inclusionwise minimal T-joins (that is, those T-joins that are a forest). Indeed, consider a T-join J. If J contains another T-join J' as subset, then $\chi^{J'} \leq \chi^J$, and hence χ^J is not a vertex of $P^{\uparrow}_{T\text{-join}}(G)$. Conversely, if χ^J is not a vertex, then $\chi^J \geq x$ for some convex combination x of incidence vectors T-joins. Each of these T-joins J' satisfies $\chi^{J'} \leq \chi^J$, and hence $J' \subseteq J$.

Similarly, an inequality $x(C) \geq 1$ for a T-cut C determines a facet if and only if C is an inclusionwise minimal T-cut.

Giles [1981] showed that two inclusionwise minimal T-joins J and J' give adjacent vertices of the polyhedron $P^{\uparrow}_{T\text{-join}}(G)$ if and only if $J \cup J'$ contains exactly one circuit. It implies that the distance of J and J' in $P^{\uparrow}_{T\text{-join}}(G)$ is at most $|J \setminus J'|$ — this implies the Hirsch conjecture for $P^{\uparrow}_{T\text{-join}}(G)$.

Gerards [1992b] showed the following. For any graph H, an *odd-H* is a subdivision of H such that each odd circuit of H becomes an odd circuit in the subdivision. In other words, the edges of H that become an even-length path form a cut in H. The *prism* is the complement of the 6-circuit C_6. Let $G = (V, E)$ be a graph not containing an odd-K_4 or an odd-prism as subgraph. Then for each $T \subseteq V$, the minimum size of a T-join is equal to the maximum number of disjoint T-cuts. This generalizes Corollary 29.9c and Theorem 29.2.

Call a graph $G = (V, E)$ a *Seymour graph* if for each subset T of V for which there exists a T-join, the minimum-size of a T-join is equal to the maximum number of disjoint T-cuts. Ageev, Kostochka, and Szigeti [1995,1997] showed that G is a Seymour graph if and only if for each length function $l \in \mathbb{Z}^E$ with $l(C) \geq 0$ for each circuit C, and for each pair of circuits C_1 and C_2 with $l(C_1) = 0$ and $l(C_2) = 0$, the graph formed by $C_1 \cup C_2$ is neither an odd-K_4 nor an odd-prism. (Here sufficiency was proved by A. Sebő.)

Seymour [1981d] characterized for which pairs G, T with $|T| = 4$, the minimum size of a T-join is equal to the maximum number of disjoint T-cuts. In fact, let $T = \{t_1, t_2, t_3, t_4\}$ and let $k \in \mathbb{Z}_+$. Then there is a packing of T-cuts of size k if and only if

(29.103)
$$\begin{aligned} &\mathrm{dist}(t_1, t_2) + \mathrm{dist}(t_3, t_4) \geq k, \\ &\mathrm{dist}(t_1, t_3) + \mathrm{dist}(t_2, t_4) \geq k, \\ &\mathrm{dist}(t_1, t_4) + \mathrm{dist}(t_2, t_3) \geq k, \end{aligned}$$

such that if equality holds in each of these inequalities, then $\mathrm{dist}(t_1, t_2)+\mathrm{dist}(t_1, t_3)+\mathrm{dist}(t_2, t_3)$ is even.

Korach [1982] characterized such pairs for $|T| = 6$, and gave a polynomial-time algorithm recognizing them.

The existence of T-joins satisfying given upper bounds on the degrees can be characterized by reduction to Tutte's 1-factor theorem (cf. Ning [1987]).

Middendorf and Pfeiffer [1990b,1993] showed that it is NP-complete to decide, for given planar graph $G = (V, E)$ and $T \subseteq V$, if the minimum size of a T-join is equal to the maximum number of disjoint T-cuts. As a minimum-size T-join can be found in polynomial time, it follows that it is NP-complete to determine a maximum packing of T-cuts. (Related results are given by Korach and Penn [1992], Korach [1994], and Granot and Penn [1995].)

The *directed* Chinese postman problem can be solved as a minimum-cost circulation problem (see Section 12.5b). The *mixed* Chinese postman problem (with directed and undirected edges) however is NP-complete (Papadimitriou [1976]). Guan [1984] derived from this that the *windy* (or *asymmetric*) *postman problem* (where the length of an edge may depend on the direction in which it is traversed) is NP-complete.

Edmonds and Johnson [1973] showed that the mixed Chinese postman problem in which each vertex has even total degree is polynomial-time solvable. (The *total degree* of a vertex v is the total number of edges (directed and undirected) incident with v.) Similarly, Guan and Pulleyblank [1985] and Win [1989] showed that the windy postman problem is solvable in polynomial time if the graph is Eulerian (by reduction to a minimum-cost circulation problem). More on the windy postman can be found in Grötschel and Win [1992], Pearn and Li [1994], and Raghavachari and Veerasamy [1999b].

For approximation algorithms for the mixed postman problem, see Frederickson [1979] and Raghavachari and Veerasamy [1998,1999a]. Further work on the mixed postman problem is reported in Kappauf and Koehler [1979], Minieka [1979], Brucker [1981], Christofides, Benavent, Campos, Corberán, and Mota [1984], Ralphs [1993], and Nobert and Picard [1996].

An extension of the Edmonds-Gallai decomposition to T-joins was given by Sebő [1990b] (cf. Sebő [1986,1997]). Goemans and Williamson [1992,1995a] gave a fast 2-approximative algorithm for finding a shortest T-join.

Benczúr and Fülöp [2000] give fast algorithms for finding minimum-size T-cut, with generalization to directed graphs.

Tobin [1975] studied finding a negative-length circuit with Edmonds' algorithm. For more on packing T-joins, see Rizzi [1997]. For surveys on T-joins and T-cuts, see Sebő [1988a] and Frank [1996a].

29.11g. On the history of the Chinese postman problem

In a paper in Chinese in *Acta Mathematica Sinica*, entitled (in translation) 'Graphic programming using odd or even points', Guan [1960] introduced the problem of finding a shortest postman route:

> When the author was plotting a diagram for a mailman's route, he discovered the following problem: "A mailman has to cover his assigned segment before returning to the post office. The problem is to find the shortest walking distance for the mailman."

(In a footnote it is mentioned that 'In postal service, a mailman's route is called a segment'.) Next:

> This problem can be reduced to the following: "Given a connected graph in the plane, we are to draw a continuous graph (repetition permitted) from a given point and back minimizing the number of repeated arcs."

So Guan restricted himself to planar graphs. He observed that a postman never has to traverse any edge more than twice. Hence the problem amounts to finding a minimum-length set J of edges such that adding a parallel edge to each of them, gives an Eulerian graph. He next gave an algorithm, which consist of starting with any such set J, and next iteratively improving it by finding a circuit C such that the length of $J \cap C$ is larger than half of the length of C, and replacing J by $J \triangle C$. As in each iteration the length of J decreases, the method finds a shortest route after a finite number of steps.

In a review in Mathematical Reviews of the article of Guan [1960], Fulkerson [1964a] observed:

> Unfortunately, the construction involves examining all simple cycles to see whether the minimality test is met or not, and this is easier said than done.

Therefore, Edmonds [1965e] announced a better method in an abstract for the 27th National Meeting of the Operations Research Society of America (May 1965 in Boston):

> We present an algorithm which does not involve examining simple cycles. It is "good" in the sense that the amount of work in applying it is at worst moderately algebraic, relative to the size of the graph, rather than exponential. It combines two earlier known algorithms: (1) the well-known "shortest path" algorithm, (2) a recent algorithm for "maximum matching".

The name of the problem seems to occur first in the title of this abstract: 'The Chinese Postman's Problem' (where 'The Chinese's Postman Problem' would be more appropriate).

Chapter 30

2-matchings, 2-covers, and 2-factors

The results on matchings are strongly self-refining, as was pointed out by Tutte [1952,1954b] and Edmonds and Johnson [1970,1973]. In this chapter we see a first instance of this phenomenon. By splitting vertices, results on 2-matchings can be derived from those on ordinary matchings. 2-matchings are of interest for the traveling salesman problem.

30.1. 2-matchings and 2-vertex covers

Let $G = (V, E)$ be an undirected graph. A 2-*matching* is a vector $x \in \mathbb{Z}_+^E$ satisfying $x(\delta(v)) \leq 2$ for each vertex v. A 2-*vertex cover* is a vector $y \in \mathbb{Z}_+^V$ such that $y_u + y_v \geq 2$ for each edge uv of G. Defining the *size* of a vector as the sum of its entries, we denote:

(30.1) $\nu_2(G) :=$ the maximum size of a 2-matching in G,
$\tau_2(G) :=$ the minimum size of a 2-vertex cover in G.

Note that

(30.2) $\tau_2(G) = \min\{|V \setminus S| + |N(S)| \mid S \subseteq V, S \text{ stable set}\}$,

since for a minimum-size 2-vertex cover y, the set $S := \{v \in V \mid y_v = 0\}$ is a stable set, while $N(S) = \{v \in V \mid y_v = 2\}$, and since $\chi^{V \setminus S} + \chi^{N(S)}$ is a 2-vertex cover for each stable set S.

Note also that

(30.3) $\nu(G) \leq \frac{1}{2}\nu_2(G) \leq \frac{1}{2}\tau_2(G) \leq \tau(G)$.

The following is a special case of a theorem of Gallai [1957,1958a,1958b] (cf. Theorem 31.7), and can be derived from Kőnig's matching theorem.

Theorem 30.1. $\nu_2(G) = \tau_2(G)$ *for any graph* G*. That is, the maximum size of a* 2*-matching is equal to the minimum size of a* 2*-vertex cover.*

Proof. Make for each vertex v of G a new vertex v', and replace each edge uv of G by two edges $u'v$ and uv'. This makes the bipartite graph H. By Kőnig's matching theorem (Theorem 16.2), H has a vertex cover C and a matching M with $|C| = |M|$. For any edge $e = uv$ of G let $x_e := |\{u'v, uv'\} \cap M|$ and for any vertex v of G let $y_v := |\{v, v'\} \cap C|$. Then x is a 2-matching and y is a 2-vertex cover with $x(E) = |M| = |C| = y(V)$. ∎

This construction was given by Nemhauser and Trotter [1975]. It also yields a polynomial-time reduction of the problems of finding a maximum-size 2-matching and a minimum-size 2-vertex cover to the problems of finding a minimum-size matching and a maximum-size vertex cover in a bipartite graph — hence these problems are polynomial-time solvable.

Call a 2-matching x *perfect* if $x(\delta(v)) = 2$ for each vertex v. So a 2-matching x is perfect if and only if $x(E) = |V|$. Theorem 30.1 implies a characterization of the existence of a perfect 2-matching (Tutte [1952]):

Corollary 30.1a. *Let $G = (V, E)$ be a graph. Then G has a perfect 2-matching if and only if $|N(S)| \geq |S|$ for each stable set S.*

Proof. Directly from Theorem 30.1, since G has a perfect 2-matching $\iff$ $\nu_2(G) \geq |V| \iff \tau_2(G) \geq |V|$. With (30.2), this last is equivalent to the condition of the present corollary. ∎

As finding a perfect 2-matching can be reduced to finding a maximum-size 2-matching, it is polynomial-time solvable.

30.2. Fractional matchings and vertex covers

Any vector $x \in \mathbb{R}^E$ satisfying

(30.4) (i) $x_e \geq 0$ for $e \in E$,
(ii) $x(\delta(v)) \leq 1$ for $v \in V$,

is called a *fractional matching*. The maximum size $x(E)$ of a fractional matching is called the *fractional matching number*, denoted by $\nu^*(G)$. By linear programming duality, $\nu^*(G)$ is equal to the *fractional vertex cover number* $\tau^*(G)$ — the minimum size of a *fractional vertex cover*, which is any solution $y \in \mathbb{R}^V$ of

(30.5) (i) $0 \leq y_v \leq 1$ for $v \in V$,
(ii) $y_u + y_v \geq 1$ for $uv \in E$.

The equality $\nu^*(G) = \tau^*(G)$ also follows from Theorem 30.1, since trivially

(30.6) $\frac{1}{2}\nu_2(G) \leq \nu^*(G) \leq \tau^*(G) \leq \frac{1}{2}\tau_2(G)$.

(An extension to infinite graphs was given by Aharoni and Ziv [1990].)

30.3. The fractional matching polytope

Let $G = (V, E)$ be a graph. The *fractional matching polytope* of G is the polytope determined by (30.4). Balinski [1965] showed:

Theorem 30.2. *Each vertex of the fractional matching polytope of G is half-integer.*

Proof. Let x be a vertex of the fractional matching polytope. We can assume that $x_e > 0$ for each edge e, since if $x_e = 0$ we can apply induction to $G - e$. Hence we can assume also that $x_e < 1$ for each edge e; equivalently, that each vertex of G has degree at least two.

As x is a vertex, there are $|E|$ constraints among (30.4)(ii) satisfied with equality. So $|E| \leq |V|$, implying that G is 2-regular. Then $x_e = \frac{1}{2}$ for each $e \in E$, as it is a solution to setting (30.4)(ii) to equality, and as the solution must be unique (as x is a vertex). ∎

Balinski [1965] also observed that the support of any vertex x of the fractional matching polytope can be partitioned into a matching M, with $x_e = 1$ for $e \in M$, and a set of odd circuits, vertex-disjoint and disjoint from M, with $x_e = \frac{1}{2}$ for each edge e in any of the odd circuits.

30.4. The 2-matching polytope

The 2-*matching polytope* of G is the convex hull of the 2-matchings in G. Theorem 30.2 implies a characterization of the 2-matching polytope (Edmonds [1965b]):

Corollary 30.2a. *The* 2*-matching polytope is determined by:*

(30.7)
$$\begin{array}{rll} \text{(i)} & x_e \geq 0 & \text{for } e \in E, \\ \text{(ii)} & x(\delta(v)) \leq 2 & \text{for } v \in V. \end{array}$$

Proof. Directly from Theorem 30.2, since it implies that the vertices of the polytope determined by (30.7) are integer, and hence are 2-matchings. ∎

Given a graph $G = (V, E)$, the *perfect* 2*-matching polytope* of G is the convex hull of the perfect 2-matchings in G. As the perfect 2-matching polytope is a face of the 2-matching polytope (if nonempty), Corollary 30.2a implies (Edmonds [1965b]):

Corollary 30.2b. *The perfect* 2*-matching polytope is determined by*

(30.8)
$$\begin{array}{rll} \text{(i)} & x_e \geq 0 & \text{for } e \in E, \\ \text{(ii)} & x(\delta(v)) = 2 & \text{for } v \in V. \end{array}$$

Proof. Directly from Corollary 30.2a. ∎

Pulleyblank [1987] related the vertices of the 2-matching polytope with the Edmonds-Gallai decomposition of the graph.

Similar results as for fractional matchings and 2-matchings hold for fractional vertex covers and 2-vertex covers. We discuss them in Section 64.6.

30.5. The weighted 2-matching problem

Given a graph $G = (V, E)$ and a weight function $w \in \mathbb{Q}^E$, the *weight* of a 2-matching x is $w^{\mathsf{T}} x$. The weighted 2-matching problem is strongly polynomial-time solvable:

Theorem 30.3. *A maximum-weight* 2*-matching can be found in time* $O(n^3)$.

Proof. Make the bipartite graph H as in the proof of Theorem 30.1, with weight function $w'(u'v) := w'(uv') := w(uv)$ for each edge uv of G. Then a maximum-weight matching in the new graph gives a maximum-weight 2-matching in the original graph. So Theorem 17.4 gives the present theorem. ∎

One can derive similarly from Egerváry's theorem a characterization of the maximum weight of a 2-matching, given by Gallai [1957,1958a,1958b]. Given $w : E \to \mathbb{Z}_+$, call a vector $y : V \to \mathbb{Z}_+$ a *w-vertex cover* if $y_u + y_v \geq w(e)$ for each edge $e = uv$.

Theorem 30.4. *Let* $G = (V, E)$ *be a graph and let* $w \in \mathbb{Z}_+^E$. *Then the maximum weight* $w^{\mathsf{T}} x$ *of a* 2*-matching* x *is equal to the minimum size of a* $2w$*-vertex cover.*

Proof. It is easy to see that the maximum cannot be larger than the minimum. To see equality, make the bipartite graph H as in the proof of Theorem 30.1, with weight $w'(u'v) := w'(uv') := w(uv)$ for each edge uv of G. Then the maximum w-weight of a 2-matching in G is equal to the maximum w'-weight of a matching in H. By Theorem 17.1, the latter is equal to the minimum of $y'(V \cup V')$ where $y' : V \cup V' \to \mathbb{Z}_+$ with $y'(u) + y'(v') \geq w(uv)$ and $y'(u') + y'(v) \geq w(uv)$ for each edge uv of G. Defining $y_v := y'_v + y'_{v'}$ for each $v \in V$, we obtain y as required. ∎

System (30.7) is generally not totally dual integral: if $G = (V, E)$ is the complete graph K_3 on three vertices, and $w(e) := 1$ for each $e \in E$, then the maximum weight of a 2-matching is equal to 3, while there is no integer dual solution of odd value (when considering the dual of maximizing $w^{\mathsf{T}} x$ subject to (30.7)).

However, *half*-integrality holds:

Corollary 30.4a. *System* (30.7) *is totally dual half-integral.*

Proof. This is equivalent to Theorem 30.4. ∎

Pulleyblank [1973,1980] showed that (30.7) can be extended to a TDI system as follows:

Corollary 30.4b. *The following system is totally dual integral:*

(30.9)
$$\begin{array}{rll} \text{(i)} & x_e \geq 0 & \text{for } e \in E, \\ \text{(ii)} & x(\delta(v)) \leq 2 & \text{for } v \in V, \\ \text{(iii)} & x(E[U]) \leq |U| & \text{for } U \subseteq V. \end{array}$$

Proof. Choose $w \in \mathbb{Z}_+^E$. By Corollary 30.4a, the problem of maximizing $w^\mathsf{T} x$ over (30.7) has an optimum dual solution $y \in \frac{1}{2}\mathbb{Z}_+^V$. Let $y'_v := \lfloor y_v \rfloor$ and $T := \{v \in V \mid y_v \notin \mathbb{Z}\}$. Let $z_T := 1$ and $z_U := 0$ for each $U \subseteq V$ with $U \neq T$. Then y', z is an integer optimum dual solution of the problem of maximizing $w^\mathsf{T} x$ over (30.9). ∎

Corollary 30.4a gives the total dual half-integrality of the perfect 2-matching constraints (30.8):

Corollary 30.4c. *System* (30.8) *is totally dual half-integral.*

Proof. Directly from Corollary 30.4a. ∎

More strongly, one has:

Corollary 30.4d. *Let $w \in \mathbb{Z}^E$ with $w(C)$ even for each circuit C. Then the problem of minimizing $w^\mathsf{T} x$ subject to* (30.8) *has an integer optimum dual solution.*

Proof. As $w(C)$ is even for each circuit, there is a subset U of V with $\{e \in E \mid w(e) \text{ odd}\} = \delta(U)$. Now replace w by $w' := w + \sum_{v \in U} \chi^{\delta(v)}$. Then $w'(e)$ is an even integer for each edge e. Hence by Corollary 30.4c there is an integer optimum dual solution y'_v $(v \in V)$ for the problem of minimizing $w'^\mathsf{T} x$ subject to (30.8). Now setting $y_v := y'_v - 1$ if $v \in U$ and $y_v := y'_v$ if $v \notin U$ gives an integer optimum dual solution y for w. ∎

30.5a. Maximum-size 2-matchings and maximum-size matchings

Uhry [1975] gave the following relation between maximum-size 2-matchings and maximum-size matchings:

Theorem 30.5. *For each maximum-size* 2*-matching* x *in a graph* G*, there exists a maximum-size matching* M *missing each vertex* v *with* $x(\delta(v)) = 0$.

Proof. Let x be a maximum-size 2-matching in G and let M be a maximum-size matching covering a minimum number of vertices v with $x(\delta(v)) = 0$. Suppose that M covers a vertex u with $x(\delta(u)) = 0$. To prove the theorem, we can assume that x has inclusionwise minimal support. This implies that the edges e with $x_e = 1$ form a collection of vertex-disjoint odd circuits.

Let N be the matching consisting of those edges e with $x_e = 2$. Let P be the component of $M \cup N$ containing u. Then P is a path starting at u, and ending at, say, w. If P has even length, then $M \triangle P$ is a maximum-size matching covering fewer vertices v with $x(\delta(v)) = 0$ than M does — a contradiction. So P has odd length, and hence, since x is a maximum-size 2-matching, w belongs to the vertex set of some odd circuit C consisting of edges e with $x_e = 1$. However, in that case we can augment x, by redefining $x_e := 0$ if $e \in P \cap N$, $x_e := 2$ if $e \in P \cap M$, and $x_e := 0$ or 2 alternatingly on the edges of C. ∎

Uhry [1975] (cf. Pulleyblank [1987]) related maximum-size 2-matchings and maximum-size matchings further by:

Theorem 30.6. *Let* x *be a maximum-size* 2*-matching with the set* $\{e \mid x_e = 1\}$ *inclusionwise minimal. Then the support of* x *contains a maximum-size matching* M *of* G.

Proof. As the set $F := \{e \mid x_e = 1\}$ is inclusionwise minimal, it forms a collection $\mathcal{C}$ of vertex-disjoint odd circuits. So $x(\delta(v)) = 0$ or 2 for each vertex v. By Theorem 30.5, we can assume that $x(\delta(v)) = 2$ for each $v \in V$, since deleting all vertices v with $x(\delta(v)) = 0$ does not decrease the maximum size of a matching.

Let M be a maximum-size matching containing a minimum number of edges e with $x_e = 0$. Let N be the matching consisting of those edges e with $x_e = 2$. Consider any component P of $M \cup N$. Then P is not a circuit or an even path of positive length, since otherwise $M \triangle P$ is a maximum-size matching having fewer edges e with $x_e = 0$ than M has — a contradiction. So if P is not a singleton, it is a path of odd length; let it connect vertices u and w. Since P is not M-augmenting, both u and w are vertices on odd circuits in $\mathcal{C}$, say on C_u and C_w respectively. If $C_u \neq C_w$, we can modify x so as to decrease the set of edges e with $x_e = 1$. So $C_u = C_w$.

It follows that each $C \in \mathcal{C}$ contains an even number of vertices covered by M, and hence an odd number of vertices missed by M. Hence

$$(30.10) \qquad 2|M| \leq |V| - |\mathcal{C}| = 2|N| + \sum_{C \in \mathcal{C}} (|C| - 1).$$

Therefore, by augmenting N with a matching of size $\frac{1}{2}(|C| - 1)$ contained in C, for each circuit $C \in \mathcal{C}$, we obtain a matching M' with $|M'| \geq |M|$ contained in the support of x. ∎

(Theorem 30.6 was generalized in (30.88).) Related results were obtained by Balas [1981].

Mühlbacher, Steinparz, and Tinhofer [1984] showed that if x is a vertex of the 2-matching polytope maximizing $|\{e \in E \mid x_e = 2\}|$, then the vector $(i_3(x), i_5(x), \ldots)$ is lexicographically maximal, where $i_k(x)$ is the number of circuits in the support of x of size k. For related work, see Mühlbacher [1979] and Hell and Kirkpatrick [1981].

30.6. Simple 2-matchings and 2-factors

Call a 2-matching x *simple* if x is a 0,1 vector. So we can identify simple 2-matchings with subsets F of E satisfying $\deg_F(v) \leq 2$ for each $v \in V$.

A construction of Tutte [1954b] gives the following characterization of the maximum size of a simple 2-matching, with the help of the Tutte-Berge formula ($E[K,S]$ denotes the set of edges connecting K and S):

Theorem 30.7. *Let $G = (V,E)$ be a graph. The maximum size of a simple 2-matching is equal to the minimum value of*

$$(30.11)\qquad |V| + |U| - |S| + \sum_K \lfloor \tfrac{1}{2}|E[K,S]| \rfloor,$$

where U and S are disjoint subsets of V, with S a stable set, and where K ranges over the components of $G - U - S$.

Proof. To see that the maximum is not more than the minimum, let F be a simple 2-matching and let U and S be disjoint subsets of V, with S a stable set. Then F has at most $2|U|$ edges incident with U. Moreover, for each component K of $G-U-S$, the number of edges in F spanned by $K \cup S$ is at most $|K| + \lfloor \frac{1}{2}|E[K,S]| \rfloor$, since

$$(30.12)\qquad \begin{aligned} 2|F \cap E[K \cup S]| &= 2|F \cap E[K]| + 2|F \cap E[K,S]| \\ &\leq 2|F \cap E[K]| + |F \cap E[K,S]| + |E[K,S]| \leq 2|K| + |E[K,S]|. \end{aligned}$$

Hence

$$(30.13)\qquad |F| \leq 2|U| + \sum_K (|K| + \lfloor \tfrac{1}{2}|E[K,S]| \rfloor)$$

(where K ranges over the components of $G-U-S$), giving that F is at most (30.11).

To see the reverse inequality, make a graph $G' = (V', E')$ as follows. For each vertex v of G, introduce vertices v' and v'' of G'. For each edge $e = uv$ of G, introduce vertices $p_{e,u}$ and $p_{e,v}$ and edges

$$(30.14)\qquad u'p_{e,u}, u'p_{e,u}, p_{e,u}p_{e,v}, v'p_{e,v}, v''p_{e,v}.$$

This defines all vertices and edges of G'.

Now:

$$(30.15)\qquad \nu_2^{\mathrm{s}}(G) = \nu(G') - |E|,$$

where $\nu(G')$ denotes the maximum size of a matching in G' and $\nu_2^{\mathrm{s}}(G)$ denotes the maximum size of a simple 2-matching in G. In this proof we only need $\geq$ in (30.15). This inequality holds as there is a maximum-size matching M in G' with the property that for each edge $e = uv$ of G, both vertices $p_{e,u}$ and $p_{e,v}$ of G' are covered by M. Then the edges e of G for which edge $p_{e,u}p_{e,v}$ does not belong to M, form a simple 2-matching N in G with $|N| = |M| - |E|$. So we have $\geq$ in (30.15).

By the Tutte-Berge formula (Theorem 24.1), there is a subset X of V' such that the number $o(G' - X)$ of odd components of $G' - X$ is at least $|V'| - 2\nu(G') + |X|$. We take X inclusionwise minimal with this property.

Then for each $v \in V$, if one of v', v'' does not belong to X, then both do not belong to X. For suppose $v' \in X$ and $v'' \notin X$. As v' and v'' have the same set of neighbours in G', removing v' from X, decreases X by 1 and decreases the number of odd components of $o(G' - X)$ by at most one. So we would obtain a smaller set X as required, contradicting the minimality assumption.

Consider any vertex v of G and any edge $e = uv$ of G with $p_{e,v} \in X$. Then the three neighbours of $p_{e,v}$ in G' belong to three different odd components of $G' - X$. (Otherwise, removing $p_{e,v}$ from X decreases X by 1, and decreases $o(G' - X)$ be at most 1, contradicting the minimality of X.) Hence $p_{e,u}, v', v'' \notin X$, and moreover $p_{f,v} \in X$ for each edge f of G incident with v.

Let U be the set of $v \in V$ for which $v', v'' \in X$ and let S be the set of $v \in V$ for which $p_{e,v} \in X$ for each edge e of G incident with v. So U and S are disjoint, and S is a stable set.

Then $|X| = 2|U| + |\delta(S)|$. Let κ denote the number of components K of $G - U - S$ with $|E[K, S]|$ odd. Then

(30.16) $$o(G' - X) = 2|S| + |E[U, S]| + \kappa.$$

Hence we have

(30.17) $$\begin{aligned} \nu_2^{\mathrm{s}}(G) &\geq \nu(G') - |E| \geq \tfrac{1}{2}(|V'| + |X| - o(G' - X)) - |E| \\ &= |V| + |U| + \tfrac{1}{2}|\delta(S)| - |S| - \tfrac{1}{2}|E[U, S]| - \tfrac{1}{2}\kappa \\ &= |V| + |U| - |S| + \sum_K \lfloor \tfrac{1}{2}|E[K, S]| \rfloor \end{aligned}$$

(where K ranges over the components of $G - U - S$), as required. ∎

A 2-*factor* is a simple perfect 2-matching. Equivalently, it is a subset F of E with $\deg_F(v) = 2$ for each $v \in V$.

Theorem 30.7 implies the following result of Belck [1950] (also Gallai [1950] announced a characterization of the existence of a 2-factor):

Corollary 30.7a. *A graph $G = (V, E)$ has a 2-factor if and only if*

(30.18) $$|S| \leq |U| + \sum_K \lfloor \tfrac{1}{2}|E[K, S]| \rfloor$$

for each pair of disjoint subsets U, S of V, with S a stable set, where K ranges over the components of $G - U - S$.

Proof. Directly from Theorem 30.7. ∎

This implies a classical result of Petersen [1891]:

Corollary 30.7b. *Each $2k$-regular graph has a 2-factor.*

Proof. Let $G = (V, E)$ be $2k$-regular. We check (30.18). Let U and S be disjoint subsets of V, with S a stable set. Let l be the number of components K of $G-U-S$ with $|E[K, S]|$ odd. Then for each such component K we have $|E[K, U]| \geq 1$ (since G is Eulerian). Hence $|E[U, S]| \leq 2k|U| - l$. Therefore,

$$
\begin{aligned}
(30.19)\qquad 2k|S| &= |\delta(S)| = |E[U,S]| + \sum_K |E[K,S]| \\
&\leq 2k|U| - l + \sum_K |E[K,S]| = 2k|U| + \sum_K 2\lfloor \tfrac{1}{2}|E[K,S]|\rfloor \\
&\leq 2k(|U| + \sum_K \lfloor \tfrac{1}{2}|E[K,S]|\rfloor)
\end{aligned}
$$

(where K ranges over the components of $G - U - S$), and (30.18) follows. ∎

The construction above gives also a reduction of finding a maximum-weight simple 2-matching to finding a maximum-weight matching — hence it can be done in strongly polynomial time. This implies that also a minimum-weight 2-factor can be found in strongly polynomial time.

(Grötschel and Holland [1987] gave computational results on a cutting plane method to find a minimum-weight 2-factor.)

30.7. The simple 2-matching polytope and the 2-factor polytope

Given a graph $G = (V, E)$, the *simple 2-matching polytope* is the convex hull of the simple 2-matchings in G. It can be characterized as follows (Edmonds [1965b]):

Theorem 30.8. *The simple 2-matching polytope is determined by*

$$
\begin{array}{llll}
(30.20) & \text{(i)} & 0 \leq x_e \leq 1 & (e \in E), \\
 & \text{(ii)} & x(\delta(v)) \leq 2 & (v \in V), \\
 & \text{(iii)} & x(E[U]) + x(F) \leq |U| + \lfloor \tfrac{1}{2}|F| \rfloor & (U \subseteq V, F \subseteq \delta(U), \\
 & & & F \text{ matching}, |F| \text{ odd}).
\end{array}
$$

Proof. It is easy to show that each simple 2-matching x satisfies (30.20). Condition (iii) follows from

(30.21) $$x(E[U]) + x(F) \leq x(E[U]) + \tfrac{1}{2}x(\delta(U)) + \tfrac{1}{2}x(F) \leq |U| + \tfrac{1}{2}|F|$$

if x is a simple 2-matching.

To show that (30.20) is enough to determine the simple 2-matching polytope, we first show that (30.20) implies an extended version of (30.20)(iii), where we delete the condition that F be a matching. This can be seen by induction on $|F|$. Indeed, suppose that F contains edges f_1, f_2 incident with a vertex v. Let $F' := F \setminus \{f_1, f_2\}$. Then, if $v \in U$, setting $U' := U \setminus \{v\}$:

(30.22) $$x(E[U]) + x(F) \leq x(E[U']) + x(F') + x(\delta(v)) \leq |U'| + \tfrac{1}{2}|F'| + 2 = |U| + \tfrac{1}{2}|F|.$$

If $v \notin U$, setting $U' := U \cup \{v\}$:

(30.23) $$x(E[U]) + x(F) \leq x(E[U']) + x(F') \leq |U'| + \tfrac{1}{2}|F'| = |U| + \tfrac{1}{2}|F|.$$

So we can delete in (iii) the requirement that F be a matching.

We now prove that the conditions determine the simple 2-matching polytope. Let $G' = (V', E')$ be as in the proof of Theorem 30.7. Let x satisfy (30.20). Define $x' \in \mathbb{R}^{E'}$ by

(30.24) $$x'(u'p_{e,u}) := x'(u''p_{e,u}) := x'(v'p_{e,v}) := x'(v''p_{e,v}) := \tfrac{1}{2}x_e \text{ and } x'(p_{e,u}p_{e,v}) := 1 - x_e,$$

for any edge $e = uv$ of G. We show that x' belongs to the matching polytope of G'.

That is, by Edmonds' matching polytope theorem (Corollary 25.1a), we should check

(30.25)
(i) $x'(e') \geq 0$ for $e' \in E'$,
(ii) $x'(\delta'(v')) \leq 1$ for $v' \in V'$,
(iii) $x'(E'[Y]) \leq \lfloor \tfrac{1}{2}|Y| \rfloor$ for $Y \subseteq V'$ with $|Y|$ odd,

where $\delta' := \delta_{G'}$ and where $E'[Y]$ is the set of edges in E' spanned by Y.

Trivially we have (30.25)(i) and (ii) by (30.20)(i) and (ii). To prove (30.25)(iii), let Y violate (30.25)(iii). We first show that if one of v', v'' belongs to Y, then both belong to Y. For suppose that $v' \in Y$ and $v'' \notin Y$. Let $Y_1 := Y \setminus \{v'\}$ and $Y_2 := Y \cup \{v''\}$. Then

(30.26) $$x'(E'[Y]) = \tfrac{1}{2}(x'(E'[Y_1]) + x'(E'[Y_2])) \leq x'(E'[Y_1]) + \tfrac{1}{2}x'(\delta'(Y_1)) = \tfrac{1}{2}\sum_{u \in Y_1} x'(\delta'(u)) \leq \tfrac{1}{2}|Y_1| = \lfloor \tfrac{1}{2}|Y| \rfloor,$$

a contradiction.

We choose Y with $|Y| + |\delta'(Y)|$ minimal. Then:

(30.27)
(i) if $u', v' \in Y$, then $p_{e,u} \in Y$ and $p_{e,v} \in Y$,
(ii) if $p_{e,u} \in Y$, then $u' \in Y$.

To see (30.27)(i), first suppose that $u', v' \in Y$ and $p_{e,u} \notin Y$. Define $Y' := Y \cup \{p_{e,u}, p_{e,v}\}$. Then $|Y'| + |\delta'(Y')| < |Y| + |\delta'(Y)|$, and hence Y' satisfies inequality (30.25)(iii). Therefore, (30.28) $$x'(E'[Y]) \leq x'(E'[Y']) - x'(\delta'(p_{e,u})) \leq \lfloor \tfrac{1}{2}|Y'| \rfloor - 1 \leq \lfloor \tfrac{1}{2}|Y| \rfloor.$$

This contradicts our assumption that Y violates (30.25)(iii).

To see (30.27)(ii), let $p_{e,u} \in Y$ and $u' \notin Y$. Define $Y' := Y \setminus \{p_{e,u}, p_{e,v}\}$. Again $|Y'| + |\delta'(Y')| < |Y| + |\delta'(Y)|$, and hence Y' satisfies inequality (30.25)(iii). If $p_{e,v} \notin Y$, then

(30.29) $$x'(E'[Y]) = x'(E'[Y']) \leq \lfloor \tfrac{1}{2}|Y'| \rfloor \leq \lfloor \tfrac{1}{2}|Y| \rfloor.$$

If $p_{e,v} \in Y$, then

(30.30) $$x'(E'[Y]) \leq x'(E'[Y']) + x'(\delta'(p_{e,v})) \leq \lfloor \tfrac{1}{2}|Y'| \rfloor + 1 = \lfloor \tfrac{1}{2}|Y| \rfloor.$$

Both (30.29) and (30.30) contradict our assumption that Y does not satisfy (30.25)(iii). This proves (30.27).

Let $U := \{v \in V \mid v', v'' \in Y\}$ and let F be the set of those edges $e = uv$ in $\delta(U)$ with $u \in U$, $v \notin U$, and $p_{e,u} \in Y$. Then $x'(E'[Y]) = x(E[U]) + |E[U]| + x(F)$ and $|Y| = 2|U| + 2|E[U]| + |F|$. Hence (30.20)(iii) implies (30.25)(iii).

So x' is a convex combination of incidence vectors of matchings in G'. Each such vector y satisfies $y(\delta'(v')) = 1$ for each vertex $v' = p_{e,u}$ (as x' satisfies this equality). Hence each such matching corresponds to a simple 2-matching in G, and we obtain x as convex combination of simple 2-matchings in G. ∎

Given a graph $G = (V, E)$, the 2-*factor polytope* is the convex hull of (the incidence vectors of) 2-factors in G. Then:

Corollary 30.8a. *The* 2*-factor polytope is determined by*

(30.31)
$$\begin{array}{rlll} \text{(i)} & 0 \leq x_e \leq 1 & (e \in E), \\ \text{(ii)} & x(\delta(v)) = 2 & (v \in V), \\ \text{(iii)} & x(\delta(U) \setminus F) - x(F) \geq 1 - |F| & (U \subseteq V,\ F \subseteq \delta(U), \\ & & F \text{ matching, } |F| \text{ odd}). \end{array}$$

Proof. Directly from Theorem 30.8, since (30.31)(ii) implies $x(E[U]) = |U| - \frac{1}{2}x(\delta(U))$. ∎

Notes. Grötschel [1977a] characterized the facets of the simple 2-matching polytope and of the 2-factor polytope of the complete graph K_n. Rispoli and Cosares [1998] showed that the diameter of the 2-factor polytope of a complete graph is at most 6. Rispoli [1994] showed that the 'monotonic diameter' of the 2-factor polytope is equal to $\lfloor \frac{1}{2}n \rfloor$ if $n \geq 5$ and $n \neq 8, 9$, and to $\lfloor \frac{1}{2}n \rfloor - 1$ if $n = 3, 4, 8, 9$.

Boyd and Carr [1999] showed that if $G = (V, E)$ is a complete graph and $l : E \to \mathbb{R}_+$ satisfies the triangle inequality, then the minimum value of $l^{\mathsf{T}}x$ over (30.31) is at most $\frac{4}{3}$ times the minimum value of $l^{\mathsf{T}}x$ over (30.31)(i)(ii). They also show that the factor $\frac{4}{3}$ is best possible.

30.8. Total dual integrality

Consider the system

(30.32)
$$\begin{array}{rll} \text{(i)} & 0 \leq x_e \leq 1 & (e \in E), \\ \text{(ii)} & x(\delta(v)) \leq 2 & (v \in V), \\ \text{(iii)} & x(E[U]) + x(F) \leq |U| + \lfloor \frac{1}{2}|F| \rfloor & (U \subseteq V, F \subseteq \delta(U), \\ & & F \text{ matching}). \end{array}$$

(So $|F|$ is not required to be odd.)

It is a special case of Theorem 32.3 (cf. Cook [1983b]) that system (30.32) is TDI (the restriction in (30.32) that F is a matching follows from (30.22) and (30.23)). This implies that (30.31) is totally dual half-integral. This also gives:

(30.33) Let $w \in \mathbb{Z}^E$ with $w(C)$ even for each circuit C. Then the problem of minimizing $w^{\mathsf{T}}x$ subject to (30.31) has an integer optimum dual solution.

To see this, notice that if $w(C)$ is even for each circuit, there is a subset U of V with $\{e \in E \mid w(e) \text{ odd}\} = \delta(U)$. Now replace w by $w' := w + \sum_{v \in U} \chi^{\delta(v)}$. Then $w'(e)$ is an even integer for each edge e. Hence there is an integer optimum dual solution y'_v $(v \in V)$, for the problem of minimizing ${w'}^{\mathsf{T}}x$ subject to (30.31). Now setting $y_v := y'_v - 1$ if $v \in U$ and $y_v := y'_v$ if $v \notin U$ gives an integer optimum dual solution for w.

30.9. 2-edge covers and 2-stable sets

Let $G = (V, E)$ be an undirected graph. A 2-*edge cover* is a vector $x \in \mathbb{Z}_+^E$ satisfying $x(\delta(v)) \geq 2$ for each vertex v. A 2-*stable set* is a vector $y \in \mathbb{Z}_+^V$ such that $y_u + y_v \leq 2$ for each edge uv of G. Defining the *size* of a vector as the sum of its entries, we denote:

(30.34)
$$\begin{array}{l} \rho_2(G) := \text{the minimum size of a 2-edge cover in } G, \\ \alpha_2(G) := \text{the maximum size of a 2-stable set in } G. \end{array}$$

Note that if G has no isolated vertices, then:

(30.35)
$$\alpha_2(G) = \max\{|V| + |U| - |N(U)| \mid U \subseteq V, U \text{ stable set}\}$$

and that

(30.36)
$$\alpha(G) \leq \tfrac{1}{2}\alpha_2(G) \leq \tfrac{1}{2}\rho_2(G) \leq \rho(G).$$

Gallai's theorem (Theorem 19.1) can be extended to 2-matchings and 2-stable sets, which was published also in Gallai [1959a]:

Theorem 30.9. *For any graph $G = (V, E)$ without isolated vertices:*

(30.37) $\alpha_2(G) + \tau_2(G) = \nu_2(G) + \rho_2(G) = 2|V|$.

Proof. Let x be a minimum-size 2-vertex cover. Then $x_v \leq 2$ for each vertex v. Define $y_v := 2 - x_v$ for each vertex v. Then y is a 2-stable set, and hence $\alpha_2(G) \geq y(V) = 2|V| - x(V) = 2|V| - \tau_2(G)$.

Conversely, let y be a maximum-size 2-stable set. Then $y_v \leq 2$ for each vertex v. Define $x_v := 2 - y_v$ for each vertex v. Then x is a 2-vertex cover, and hence $\tau_2(G) \leq x(V) = 2|V| - y(V) = 2|V| - \alpha_2(G)$. This shows that $\alpha_2(G) + \tau_2(G) = 2|V|$.

To see that $\nu_2(G) + \rho_2(G) = 2|V|$, let x be a minimum-size 2-edge cover. For each $v \in V$, reduce $x(\delta(v))$ by $x(\delta(v)) - 2$, by reducing x_e on edges $e \in \delta(v)$. We obtain a 2-matching y of size

(30.38) $$y(E) \geq x(E) - \sum_{v \in V} (x(\delta(v)) - 2) = 2|V| - x(E) = 2|V| - \rho_2(G).$$

Hence $\nu_2(G) \geq 2|V| - \rho_2(G)$.

Conversely, let y be a maximum-size 2-matching. For each $v \in V$, increase $y(\delta(v))$ by $2 - y(\delta(v))$, by increasing y_e on edges $e \in \delta(v)$. We obtain a 2-edge cover x of size

(30.39) $$x(E) \leq y(E) + \sum_{v \in V} (2 - y(\delta(v))) = 2|V| - y(E) = 2|V| - \nu_2(G).$$

Hence $\rho_2(G) \leq 2|V| - \nu_2(G)$. ∎

This implies the following, which is a special case of a theorem of Gallai [1957,1958a,1958b] (cf. Theorem 30.11) (and can be derived alternatively from the Kőnig-Rado edge cover theorem):

Corollary 30.9a. *$\alpha_2(G) = \rho_2(G)$ for any graph G without isolated vertices. That is, the maximum size of a 2-stable set is equal to the minimum size of a 2-edge cover.*

Proof. Directly from Theorems 30.1 and 30.9. ∎

These reductions also imply the polynomial-time solvability of the problems of finding a minimum-size 2-edge cover and a maximum-size 2-stable set.

30.10. Fractional edge covers and stable sets

Any vector $x \in \mathbb{R}^E$ satisfying

(30.40)
(i) $x_e \geq 0$ for $e \in E$,
(ii) $x(\delta(v)) \geq 1$ for $v \in V$,

is called a *fractional edge cover*. The minimum size $x(E)$ of a fractional edge cover is called the *fractional edge cover number* and is denoted by $\rho^*(G)$. By linear programming duality, $\rho^*(G)$ is equal to the *fractional stable set number* $\alpha^*(G)$ — the maximum size of a *fractional stable set*, which is any solution $y \in \mathbb{R}^V$ of

(30.41) (i) $0 \leq y_v \leq 1$ for $v \in V$,
(ii) $y_u + y_v \leq 1$ for $uv \in E$.

The equality $\rho^*(G) = \alpha^*(G)$ also follows from Corollary 30.9a, since trivially

(30.42) $\frac{1}{2}\rho_2(G) \geq \rho^*(G) \geq \alpha^*(G) \geq \frac{1}{2}\alpha_2(G)$.

30.11. The fractional edge cover polyhedron

Let $G = (V, E)$ be a graph. The *fractional edge cover polyhedron* of G is the polyhedron determined by (30.40). Balinski [1965] showed:

Theorem 30.10. *Each vertex of the fractional edge cover polyhedron of G is half-integer.*

Proof. Let x be a vertex of the fractional edge cover polyhedron. We can assume that $x_e > 0$ for each edge e, since if $x_e = 0$ we can apply induction to $G - e$. Moreover, we can assume that G is connected and has at least three vertices.

As x is a vertex, there are $|E|$ constraints among (30.40)(ii) satisfied with equality. Define $U := \{v \mid x(\delta(v)) = 1\}$. So $|E| \leq |V|$. If there exists an end vertex v in U, with neighbour u say, then $u \in U$ and there is no other edge incident with u (otherwise it would have $x_e = 0$), implying the theorem. So no such end vertex exists.

If G is a tree, then there is at most one vertex w with $x(\delta(w)) \neq 1$, implying the existence of an end vertex v and a neighbour u of v with $u, v \in U$.

So G is not a tree, and hence $|E| = |V|$ and $U = V$. Since G has no end vertex, G is a circuit. Then $\frac{1}{2} \cdot \mathbf{1}$ satisfies all constraints that x satisfies. So $x = \frac{1}{2} \cdot \mathbf{1}$, as x is a vertex. ∎

30.12. The 2-edge cover polyhedron

Theorem 30.10 implies a characterization of the 2-*edge cover polyhedron* of G, which is, by definition, the convex hull of the 2-edge covers in G:

Corollary 30.10a. *The 2-edge cover polyhedron is determined by*

(30.43) (i) $x_e \geq 0$ for $e \in E$,
(ii) $x(\delta(v)) \geq 2$ for $v \in V$.

Proof. Directly from Theorem 30.10, since it implies that the vertices of the polyhedron determined by (30.43) are integer, and hence 2-edge covers. ∎

Similar results as for fractional edge covers and 2-edge covers hold for fractional stable sets and 2-stable sets. We discuss them in Section 64.5.

30.13. Total dual integrality of the 2-edge cover constraints

Finding a minimum-weight 2-edge cover is easily reduced to the minimum-weight edge cover problem, by splitting vertices. Gallai [1957,1958a,1958b] characterized the minimum weight as follows. Given $w : E \to \mathbb{Z}_+$, a *w-stable set* is a function $y : V \to \mathbb{Z}_+$ with $y_u + y_v \leq w(e)$ for each edge $e = uv$.

Theorem 30.11. *Let $G = (V, E)$ be a graph without isolated vertices and let $w \in \mathbb{Z}_+^E$. Then the minimum weight $w^\mathsf{T} x$ of a 2-edge cover x is equal to the maximum size of a $2w$-stable set.*

Proof. From Egerváry's theorem (Theorem 17.1). ∎

This is equivalent to the following result:

Corollary 30.11a. *System* (30.43) *is totally dual half-integral.*

Proof. Choose $w \in \mathbb{Z}_+^E$. Then the minimum weight $w^\mathsf{T} x$ of a 2-edge cover is equal to

(30.44) $$\max\{2y(V) \mid y \in \tfrac{1}{2}\mathbb{Z}_+^V, y_u + y_v \leq w(e) \text{ for each } e = uv \in E\},$$

by Theorem 30.11. ∎

System (30.43) can be extended to a TDI system as follows:

Corollary 30.11b. *The following system is totally dual integral:*

(30.45) (i) $x_e \geq 0$ for $e \in E$,
(ii) $x(\delta(v)) \geq 2$ for $v \in V$,
(ii) $x(E[U] \cup \delta(U)) \geq |U|$ for $U \subseteq V$.

Proof. Choose $w \in \mathbb{Z}_+^E$. By Corollary 30.11a, the problem of minimizing $w^\mathsf{T} x$ over (30.43) has an optimum dual solution $y \in \frac{1}{2}\mathbb{Z}_+^V$. Define $y'_v := \lfloor y_v \rfloor$ for $v \in V$, and $T := \{v \in V \mid y_v \notin \mathbb{Z}\}$. Define $z_T := 1$ and $z_U := 0$ for each $U \subseteq V$ with $U \neq T$. Then y', z is an integer optimum dual solution for the problem of minimizing $w^\mathsf{T} x$ over (30.45). ∎

30.14. Simple 2-edge covers

Call a 2-edge cover x *simple* if x is a 0,1 vector. Thus we can identify simple 2-edge covers with subsets F of E satisfying $\deg_F(v) \geq 2$ for each $v \in V$. A 2-edge cover exists if and only if all degrees are at least 2. Define

(30.46) $\nu_2^{\mathrm{s}}(G) :=$ the maximum size of a simple 2-matching,
$\rho_2^{\mathrm{s}}(G) :=$ the minimum size of a simple 2-edge cover.

Again there is a relation between $\nu_2^{\mathrm{s}}(G)$ and $\rho_2^{\mathrm{s}}(G)$ similar to Gallai's theorem (Theorem 19.1):

Theorem 30.12. *For any graph $G = (V, E)$ of minimum degree at least 2 one has:*

(30.47) $$\nu_2^{\mathrm{s}}(G) + \rho_2^{\mathrm{s}}(G) = 2|V|.$$

Proof. Let M be a maximum-size simple 2-matching. For each $v \in V$, add to M $2 - \deg_M(v)$ edges incident with v. We can do this in such a way that we obtain a simple 2-edge cover F with

(30.48) $$|F| \leq |M| + \sum_{v \in V} (2 - \deg_M(v)) = 2|V| - |M|.$$

So $\rho_2^{\mathrm{s}}(G) \leq 2|V| - |M| = 2|V| - \nu_2^{\mathrm{s}}(G)$.

To see the reverse inequality, let F be a minimum-size simple 2-edge cover. For each $v \in V$, delete from F $\deg_F(v) - 2$ edges incident with v. We obtain a simple 2-matching M with

(30.49) $$|M| \geq |F| - \sum_{v \in V} (\deg_F(v) - 2) = 2|V| - |F|.$$

So $\nu_2^{\mathrm{s}}(G) \geq 2|V| - |F| = 2|V| - \rho_2^{\mathrm{s}}(G)$, which shows (30.47). ∎

This implies a min-max relation for minimum-size simple 2-edge cover:

Corollary 30.12a. *Let $G = (V, E)$ be a graph of minimum degree at least 2. Then the minimum size of a simple 2-edge cover is equal to the maximum value of*

(30.50) $$|V| - |U| + |S| - \sum_K \lfloor \tfrac{1}{2} |E[K, S]| \rfloor,$$

where U and S are disjoint subsets of V, with S a stable set, and where K ranges over the components of $G - U - S$.

Proof. Directly from Theorems 30.7 and 30.12. ∎

These reductions also imply the polynomial-time solvability of the problem of finding a minimum-size simple 2-edge cover.

Given a graph $G = (V, E)$, the *simple* 2-*edge cover polytope* is the convex hull of the simple 2-edge covers in G. A special case of Theorem 34.9 below is that the simple 2-edge cover polytope is determined by

(30.51)
$$\begin{array}{rll} \text{(i)} & 0 \leq x_e \leq 1 & (e \in E), \\ \text{(ii)} & x(\delta(v)) \geq 2 & (v \in V), \\ \text{(iii)} & x(E[U]) + x(F) \geq |U| + \lceil \frac{1}{2}|F| \rceil & (U \subseteq V, F \subseteq \delta(U), \\ & & |F| \text{ odd}). \end{array}$$

We refer to Theorem 34.10 for the total dual integrality of the following system (Cook [1983b]):

(30.52)
$$\begin{array}{rll} \text{(i)} & 0 \leq x_e \leq 1 & (e \in E), \\ \text{(ii)} & x(\delta(v)) \geq 2 & (v \in V), \\ \text{(iii)} & x(E[U]) + x(F) \geq |U| + \lceil \frac{1}{2}|F| \rceil & (U \subseteq V, F \subseteq \delta(U)). \end{array}$$

Theorem 34.11 implies that a minimum-weight simple 2-edge-cover can be found in strongly polynomial time.

30.15. Graphs with $\nu(G) = \tau(G)$ and $\alpha(G) = \rho(G)$

Kőnig's matching theorem states that the matching number $\nu(G)$ is equal to the vertex cover number $\tau(G)$ for each bipartite graph G. A graph G therefore is said to have the *Kőnig property* if $\nu(G) = \tau(G)$. Deming [1979b] and Sterboul [1979] characterized the class of graphs with the Kőnig property.

Note that by Gallai's theorem (Theorem 19.1), for any graph G without isolated vertices:

(30.53)
$$\nu(G) = \tau(G) \iff \alpha(G) = \rho(G)$$

(where $\alpha(G)$ and $\rho(G)$ denote the stable set and edge cover number of G, respectively).

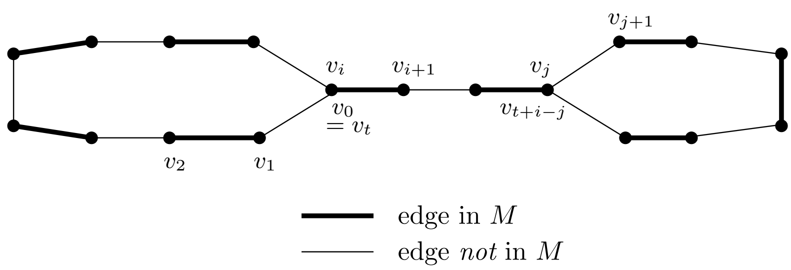


Figure 30.1
An M-posy
The two circuits may intersect.

To characterize graphs G with $\nu(G) = \tau(G)$, Sterboul defined, for any graph $G = (V, E)$ and any matching M in G, an *M-posy* to be an even-length M-alternating closed walk $(v_0, v_1, \ldots, v_t)$, with $v_{i-1}v_i \in M$ if i is even, such that there exist $i < j$ with i odd and j even, $v_1, \ldots, v_j$ all distinct, $v_{j+1}, \ldots, v_t$ all distinct, and

(30.54) $\quad v_i = v_t, v_{i+1} = v_{t-1}, \ldots, v_j = v_{t+i-j}.$

Lemma 30.13α. *If there exists an even-length M-alternating closed walk $C = (v_0, v_1, \ldots, v_t)$ with $v_i = v_j$ for i, j of different parity, then there exists an M-posy.*

Proof. Let C be a shortest such closed walk, covering a minimum number of edges (in this order of priority). Then

(30.55) $\quad$ there exist no three distinct $h, i, k \geq 1$ with $v_h = v_i = v_k$,

since otherwise we may assume that h and i have the same parity. Leaving out one of the $v_h - v_i$ parts of C gives a shorter such closed walk.

Choose h, i of different parity with $v_h = v_i$ and with $|h - i|$ minimal. We may assume that $h = 0$ and that $v_0v_1 \notin M$. Choose $j, k \geq i$ of different parity with $v_j = v_k$ and $j < k$, and with $k - j$ minimal. (Such j, k exist, as $v_i = v_t$.) Then j is even and k is odd, since otherwise $v_{j+1} = v_{k-1}$ (as it is the vertex matched to $v_j = v_k$). Moreover, $j - i = t - k$ and

(30.56) $\quad v_i = v_t, v_{i+1} = v_{t-1}, \ldots, v_j = v_k.$

Otherwise, resetting the $v_k - v_t$ part of C to the $v_j - v_i$ part of C^{-1} or conversely, we obtain again a shortest such closed walk, however covering a fewer number of edges, a contradiction.

Then C is an M-posy, since $v_1, \ldots, v_j$ are all distinct and $v_{j+1}, \ldots, v_t$ are all distinct. If say $v_a = v_b$ with $1 \leq a < b \leq j$, then $b \leq i$ (since otherwise $v_a = v_b = v_l$ for some $l > b$, contradicting (30.55)). So by the minimality of $|h - i|$, $a \equiv b \pmod 2$. Hence, deleting the $v_a - v_b$ part from C gives a shortest such walk, a contradiction. ∎

This is used in proving:

Theorem 30.13. *Let $G = (V, E)$ be a graph. Then the following are equivalent:*

(30.57) (i) *G has the Kőnig property, that is $\nu(G) = \tau(G)$;*
(ii) *for some maximum-size matching M there is no M-flower and no M-posy;*
(iii) *for each maximum-size matching M there is no M-flower and no M-posy.*

Proof. The implication (iii)⇒(ii) is trivial, and the implication (i)⇒(iii) is easy: suppose $\nu(G) = \tau(G)$, let M be a maximum-size matching and let U

be a minimum-size vertex cover. Then each edge in M has exactly one vertex in U. Suppose that $P = (v_0, \ldots, v_t)$ is an M-flower or an M-posy. Then for each odd k, exactly one of v_k and v_{k+1} belongs to U, while for each even k at least one of v_k and v_{k+1} belongs to U. If $v_t \notin U$, then $v_k \in U$ for each even k. Since $v_t = v_j$ for some even j, it follows that $v_t \in U$. If $v_0 \notin U$, then $v_k \in U$ for each odd k. Since $v_j \in U$ and j is even, we have $v_0 \in U$. So v_0 is covered by M, and hence P is an M-posy. So $v_0 = v_i$ for some odd i. So $v_i \in U$ for some odd i, a contradiction.

It remains to prove (ii)⇒(i). Let M be a maximum-size matching in G and let X be the set of vertices missed by M. Then there is no M-alternating $X - X$ walk (since M has maximum size and since there is no M-flower (cf. Theorem 24.3)). Let U be the set of vertices v for which there is an M-alternating $X - v$ walk and let Z be the set of vertices v for which there exists an odd-length M-alternating $X - v$ walk. Then Z intersects each edge intersecting U, while $|Z|$ is equal to the number of edges in M contained in U.

So we can apply induction to $G - U$ if $U \neq \emptyset$. Hence we may assume that $U = \emptyset$. Equivalently, $X = \emptyset$, that is, M is a perfect matching. Choose $e = uv \in M$. By Lemma 30.13α, $G - u$ has no $M \setminus \{e\}$-flower or $G - v$ has no $M \setminus \{e\}$-flower. By symmetry, we may assume that $G - v$ has no $M \setminus \{e\}$-flower. Since G has no M-posy, $G - v$ has no $M \setminus \{e\}$-posy. Hence, by induction:

(30.58) $$\nu(G) = \nu(G - v) + 1 = \tau(G - v) + 1 \geq \tau(G).$$

Hence $\nu(G) = \tau(G)$. ∎

This implies a characterization due to Lovász [1974] ((i)⇔(ii) below) and Lovász and Plummer [1986] ((i)⇔(iii) below), based on the minimum size $\tau_2(G)$ of a 2-vertex cover studied in Section 30.1:

Corollary 30.13a. *For any graph G, the following are equivalent:*

(30.59)
(i) $\nu(G) = \tau(G)$,
(ii) $\tau_2(G) = 2\tau(G)$,
(iii) *the edges e for which there exists a maximum-size 2-matching x with $x_e \geq 1$, form a bipartite graph.*

Proof. The implication (i)⇒(ii) follows from (30.3). To see (ii)⇒(iii), let U be a minimum-size vertex cover and let x be a maximum-size 2-matching. Then, using Theorem 30.1,

(30.60) $$\tau_2(G) = \nu_2(G) = \sum_{e \in E} x_e \leq \sum_{e \in E} x_e |e \cap U| = \sum_{v \in U} x(\delta(v)) \leq 2|U| = 2\tau(G),$$

and hence we have equality throughout. So $e \in \delta(U)$ if $x_e \geq 1$. As this is true for each maximum-size 2-matching x, we have (iii).

We finally show (iii)⇒(i), which we derive from Theorem 30.13. Suppose that (iii) holds, and let M be a maximum-size matching. If there would exist any M-flower or M-posy, then we can find a 2-matching of size at least $2|M|$ such that M and the support of the 2-matching contains an odd circuit. For an M-flower this is trivial. For an M-posy $(v_0, \ldots, v_t)$, let

$$(30.61) \qquad x := 2\chi^M - \sum_{h=1}^{t} (-1)^h \chi^{v_{h-1}v_h}.$$

Then x is a 2-matching of size $2|M|$. However, the support of x together with M contains an odd circuit. This contradicts (iii). ∎

Note that characterization (iii) can be checked in polynomial time. By Theorem 30.9 and its proof method, we know that (i), (ii), and (iii) are also equivalent to each of:

(30.62)
(iv) $\alpha(G) = \rho(G)$,
(v) $\alpha_2(G) = 2\alpha(G)$,
(vi) the edges e for which there exists a minimum-size 2-edge cover x with $x_e \geq 1$, form a bipartite graph.

More on the Kőnig property can be found in Korach [1982], Bourjolly, Hammer, and Simeone [1984], and Bourjolly and Pulleyblank [1989], and related results in Tipnis and Trotter [1989].

30.16. Excluding triangles

Let $G = (V, E)$ be a graph. Call a 2-matching x *triangle-free* if $x(ET) \leq 2$ for each triangle T in G. (A *triangle* is a subgraph isomorphic to K_3.) The *triangle-free 2-matching polytope* is the convex hull of the triangle-free 2-matchings.

In order to characterize the triangle-free 2-matching polytope, Cornuéjols and Pulleyblank [1980a] (cf. Cook [1983b], Cook and Pulleyblank [1987]) showed the following:

Theorem 30.14. *Let $G = (V, E)$ be a simple graph and let $\mathcal{T}$ be a collection of triangles in G. Then the following system is totally dual integral:*

$$(30.63) \qquad \begin{array}{rll} \text{(i)} & x_e \geq 0 & \text{for each } e \in E, \\ \text{(ii)} & \tfrac{1}{2}x(\delta(v)) \leq 1 & \text{for each } v \in V, \\ \text{(iii)} & x(ET) \leq 2 & \text{for each } T \in \mathcal{T}. \end{array}$$

Proof. Let $w \in \mathbb{Z}_+^E$ and consider the problem dual to maximizing $w^{\mathsf{T}}x$ over (30.63):

(30.64) $$\begin{aligned} &\text{minimize } \sum_{v\in V} y_v + 2\sum_{T\in\mathcal{T}} z_T \\ &\text{subject to } \tfrac{1}{2}\sum_{v\in V} y_v\chi^{\delta(v)} + \sum_{T\in\mathcal{T}} z_T\chi^{ET} \geq w, \end{aligned}$$

with $y \in \mathbb{R}_+^V$ and $z \in \mathbb{R}_+^{\mathcal{T}}$. We must show that there exists an integer optimum solution y, z. We take a counterexample with $|E|+w(E)$ minimal. This implies that G is connected. Moreover, $w(e) \geq 1$ for each edge e, since otherwise we could delete e.

On the other hand, $w(e) \leq 2$ for each edge e. To see this, let y, z be any optimum solution. If $y_u \geq 2$ for some vertex u, we can reset $w(e) := w(e) - 1$ for each $e \in \delta(u)$. By resetting, the optimum value decreases by at least 2 (since resetting $y_u := y_u - 2$ gives a feasible solution for the new w, with objective value 1 less than the original objective value). By the minimality of G, w, for the new w there is an integer optimum solution y, z. Resetting $y_u := y_u + 2$ then gives an integer optimum solution for the original w.

So we can assume that $y_v < 2$ for each vertex v, and similarly, that $z_T < 1$ for each $T \in \mathcal{T}$.

Choose an optimum solution y, z with $\sum_{T\in\mathcal{T}} z_T$ minimal. Let $\mathcal{T}_+ := \{T \in \mathcal{T} \mid z_T > 0\}$. Then:

(30.65) no two triangles in $\mathcal{T}_+$ have an edge in common.

For suppose that $T_1, T_2 \in \mathcal{T}_+$ have $ET_1 \cap ET_2 = \{e\}$, say $e = v_1v_2$. Resetting $z_{T_i} := z_{T_i} - \varepsilon$ and $y_{v_i} := y_{v_i} + 2\varepsilon$ for $i = 1, 2$, for $\varepsilon > 0$ small enough, gives again an optimum solution. However, $\sum_{T\in\mathcal{T}} z_T$ decreases, contradicting our assumption. This proves (30.65).

This implies that $w(e) \leq 2$ for each edge e, since $y_v < 2$ and $z_T < 1$.

Next:

(30.66) for any $T \in \mathcal{T}_+$ and any $v \in VT$ one has either $0 < y_v < 1$ for each $v \in VT$ and $w(e) = 1$ for each $e \in ET$, or $1 < y_v < 2$ for each $v \in VT$ and $w(e) = 2$ for each $e \in ET$.

Let $VT = \{v_1, v_2, v_3\}$. First assume that $\frac{1}{2}y_{v_1} + \frac{1}{2}y_{v_2} + z_T > w(v_1v_2)$. Then after resetting $y_{v_3} := y_{v_3} + 2\varepsilon$ and $z_T := z_T - \varepsilon$ we obtain again an optimum solution, for $\varepsilon > 0$ small enough. However, $\sum_{T\in\mathcal{T}} z_T$ decreases, contradicting our assumption. So $\frac{1}{2}y_{v_1} + \frac{1}{2}y_{v_2} + z_T = w(v_1v_2)$, and similarly for any other pair from v_1, v_2, v_3. This implies

(30.67) $$y_{v_1} = w(v_1v_2) + w(v_1v_3) - w(v_2v_3) - z_T,$$

and similarly for v_2 and v_3. So if $w(e) = 1$ for each $e \in ET$, then $0 < y_v < 1$ for each $v \in VT$. Similarly, if $w(e) = 2$ for each $e \in ET$, then $1 < y_v < 2$ for each $v \in VT$. If not all three edges of T have the same weight, (30.67) implies that there is a vertex v in T with $y_v > 2$ or $y_v < 0$, a contradiction. This proves (30.66).

Now consider resetting

(30.68)
$$
\begin{array}{lll}
y_v & := y_v - \varepsilon & \text{if } 0 < y_v < 1, \\
y_v & := y_v + \varepsilon & \text{if } 1 < y_v < 2, \\
z_T & := z_T + \varepsilon & \text{if } T \in \mathcal{T}_+ \text{ and } w(e) = 1 \text{ for each edge } e \text{ in } T, \\
z_T & := z_T - \varepsilon & \text{if } T \in \mathcal{T}_+ \text{ and } w(e) = 2 \text{ for each edge } e \text{ in } T.
\end{array}
$$

If we choose ε close enough to 0 (positive or negative), we obtain again a feasible solution of (30.64), by (30.65) and (30.66), using the integrality of w. Moreover, the objective value changes linearly in ε. However, as y, z is an optimum solution, the objective value cannot decrease. Hence there is no change in the objective value at all. That is, for any ε close enough to 0, we obtain again an optimum solution. Therefore, by choosing ε appropriately, we can decrease the number of noninteger values of y_v, z_T. ∎

This theorem implies (in fact, is equivalent to) the following TDI result:

Corollary 30.14a. *Let $G = (V, E)$ be a simple graph and let $\mathcal{T}$ be a collection of triangles in G. Then the following system is totally dual integral:*

(30.69)
$$
\begin{array}{rll}
\text{(i)} & x_e \geq 0 & \text{for each } e \in E, \\
\text{(ii)} & x(\delta(v)) \leq 2 & \text{for each } v \in V, \\
\text{(iii)} & x(ET) \leq 2 & \text{for each } T \in \mathcal{T}, \\
\text{(iv)} & x(E[U]) \leq |U| & \text{for each } U \subseteq V.
\end{array}
$$

Proof. Let $w \in \mathbb{Z}_+^E$. Let μ be the maximum value of $w^{\mathsf{T}}x$ over (30.69), This is equal to the maximum value of $w^{\mathsf{T}}x$ over (30.63) (since (30.69)(iv) follows from (i) and (ii)).

Consider an integer optimum solution y_v ($v \in V$), z_T ($T \in \mathcal{T}$) of the problem dual to maximizing $w^{\mathsf{T}}x$ over (30.63). Define $y'_v := \lfloor \frac{1}{2} y_v \rfloor$ for $v \in V$ and $T := \{v \in V \mid \frac{1}{2} y_v \notin \mathbb{Z}\}$. Define $a_U := 1$ if $U = T$ and $a_U := 0$ for any other subset U of V.

Then y', a, z is an integer feasible solution of the problem dual to maximizing $w^{\mathsf{T}}x$ over (30.69), as w is integer. Moreover, it is optimum, since

(30.70)
$$
\sum_{v \in V} 2y'_v + \sum_{U \subseteq V} a_U |U| + \sum_{T \in \mathcal{T}} 2z_T = \sum_{v \in V} y_v + \sum_{T \in \mathcal{T}} 2z_T = \mu.
$$
∎

The theorem implies the following characterization of the triangle-free 2-matching polytope, given by Cornuéjols and Pulleyblank [1980a] and J.F. Maurras (cf. Cornuéjols and Pulleyblank [1980b]):

Corollary 30.14b. *Let $G = (V, E)$ be a graph. The triangle-free 2-matching polytope is determined by:*

(30.71)
$$
\begin{array}{rll}
\text{(i)} & x_e \geq 0 & \text{for each } e \in E, \\
\text{(ii)} & x(\delta(v)) \leq 2 & \text{for each } v \in V, \\
\text{(iii)} & x(ET) \leq 2 & \text{for each triangle } T \text{ in } G.
\end{array}
$$

Proof. Theorem 30.14 implies that the polytope determined by (30.63) is integer (as the right-hand sides are integer). Since (30.71) determines the same polytope, the corollary follows. ∎

In fact, there is a sharper consequence, where we just consider an arbitrary subcollection $\mathcal{T}$ of the triangles:

Corollary 30.14c. *Let $G = (V, E)$ be a graph and let $\mathcal{T}$ be a collection of triangles in G. Then the following inequalities determine an integer polytope:*

$$\text{(30.72)} \qquad \begin{array}{rll} \text{(i)} & x_e \geq 0 & \text{for each } e \in E, \\ \text{(ii)} & x(\delta(v)) \leq 2 & \text{for each } v \in V, \\ \text{(iii)} & x(ET) \leq 2 & \text{for each triangle } T \in \mathcal{T}. \end{array}$$

Proof. Similar to the proof of the previous corollary. ∎

Cornuéjols and Pulleyblank [1980a] also showed that the inequalities (30.72)(i) and (iii) are all necessary, while (ii) is necessary unless $\deg_G(v) = 2$ and v is in a triangle in $\mathcal{T}$ (assuming that G is connected and has at least three vertices). They also gave a polynomial-time algorithm to find a maximum-weight triangle-free 2-matching.

Moreover, they showed the following. A *triangle cluster* is a graph defined recursively as follows: any one-vertex graph is a triangle cluster; if G is a triangle cluster and v is a vertex of G, then by introducing two new vertices u, u' and adding edges vu, vu' and uu', we obtain again a triangle cluster.

For any graph G, let $\beta(G)$ denote the number of components of G that are triangle clusters. This is used in the following min-max relation for maximum-size triangle-free 2-matching (Cornuéjols and Pulleyblank [1980a]):

Theorem 30.15. *The maximum size of a triangle-free 2-matching in a graph $G = (V, E)$ is equal to the minimum value of $|V| + |U| - \beta(G - U)$ taken over $U \subseteq V$.*

Proof. To see that the maximum is not more than the minimum, let x be a maximum-size triangle-free 2-matching in G. Let $U \subseteq V$ and let W be the set of vertices of $G - U$ that are in triangle cluster components. Consider any component K of $G - U$ that is a triangle cluster. Then the edges of K can be partitioned into $\frac{1}{2}(|K| - 1)$ triangles. Hence $x(E[K]) \leq |K| - 1$, and therefore

$$\text{(30.73)} \qquad \sum_{v \in K} x(\delta(v)) = 2x(E[K]) + x(\delta(K)) \leq 2(|K| - 1) + x(\delta(K)).$$

Summing over all components K that are triangle cluster, we see that

$$\text{(30.74)} \qquad \sum_{v \in W} x(\delta(v)) \leq 2|W| - 2\beta(G - U) + x(\delta(W)).$$

Moreover,

(30.75) $$x(\delta(W)) \leq x(\delta(U)) \leq \sum_{v \in U} x(\delta(v)) \leq 2|U|.$$

This implies

(30.76) $$\begin{aligned} 2x(E) &= \sum_{v \in W} x(\delta(v)) + \sum_{v \in V \setminus W} x(\delta(v)) \\ &\leq 2|W| - 2\beta(G-U) + 2|U| + 2|V \setminus W| \\ &= 2(|V| + |U| - \beta(G-U)). \end{aligned}$$

This shows that the maximum is not more than the minimum.

To see the reverse inequality, let $\mathcal{T}$ denote the set of triangles in G. By Theorem 30.14, the maximum size of a triangle-free 2-matching is equal to the minimum value of

(30.77) $$\sum_{v \in V} y_v + 2 \sum_{T \in \mathcal{T}} z_T$$

where $y_v \in \mathbb{Z}_+$ (for $v \in V$) and $z_T \in \mathbb{Z}_+$ (for $T \in \mathcal{T}$) such that

(30.78) $$\tfrac{1}{2} \sum_{v \in V} y_v \chi^{\delta(v)} + \sum_{T \in \mathcal{T}} z_T \chi^{ET} \geq \mathbf{1}.$$

Choose y, z attaining this minimum, with

(30.79) $$\sum_{T \in \mathcal{T}} z_T \text{ as small as possible.}$$

Clearly, $y_v \leq 2$ for each $v \in V$ and $z_T \leq 1$ for each $T \in \mathcal{T}$. Let $\mathcal{T}_+ := \{T \in T \mid z_T = 1\}$.

Then we have:

(30.80) if $T \in \mathcal{T}_+$ and $v \in T$, then $y_v = 0$.

Indeed, suppose $y_v \geq 1$, and let u and u' be the two other vertices in T. Then resetting $z_T := 0$, $y_u := y_u + 1$, and $y_{u'} := y_{u'} + 1$, we obtain y, z again attaining the minimum value (30.77), contradicting our minimality assumption (30.79). This shows (30.80).

Let F be the set of edges contained in some $T \in \mathcal{T}_+$. Then

(30.81) each component of the graph (V, F) is a triangle cluster.

If not, there exist distinct $T_1, \ldots, T_k \in \mathcal{T}_+$ and distinct $v_1, \ldots, v_k \in V$, such that, taking $v_0 := v_k$,

(30.82) $$v_{i-1}v_i \in T_i$$

for $i = 1, \ldots, k$, and such that $k > 1$. Then resetting $z_{T_i} := 0$ and $y_{v_i} := 2$ for $i = 1, \ldots, k$, we obtain y, z again attaining the minimum value (30.77), contradicting our minimality assumption (30.79). This shows (30.81).

Now let $W := \{v \in V \mid y_v = 0\}$. Then each edge contained in W is contained in some $T \in \mathcal{T}_+$, and hence, by (30.81), each component of $G[W]$

is a triangle cluster. Let k be the number of components of $G[W]$. Then $\sum_{T\in\mathcal{T}} z_T = \frac{1}{2}(|W|-k)$.

Define $U := N(W)$. Then $y_v = 2$ for each $v \in U$, since each edge e connecting W and U should satisfy (30.78). Therefore, (30.77) is at least

(30.83) $$|V|-|W|+|U|+2\cdot\tfrac{1}{2}(|W|-k) = |V|+|U|-k \geq |V|+|U|-\beta(G-U),$$

proving the theorem. ∎

This characterizes the existence of a triangle-free perfect 2-matching:

Corollary 30.15a. *A graph $G = (V, E)$ has a triangle-free perfect 2-matching if and only if $G - U$ has at most $|U|$ components that are triangle clusters, for each $U \subseteq V$.*

Proof. Directly from Theorem 30.15. ∎

Cornuéjols and Pulleyblank [1980b] gave a polynomial-time algorithm to find a triangle-free perfect b-matching. Cook [1983b] and Cook and Pulleyblank [1987] characterized the facets and the minimal TDI-system for the triangle-free 2-matching polytope.

30.16a. Excluding higher polygons

Cornuéjols and Pulleyblank [1983] considered excluding higher polygons. For any collection P of graphs, call a graph G *P-critical* if $G \notin P$ while $G - v \in P$ for each vertex v of G. Let P_k be the collection of graphs that have a perfect 2-matching in which each circuit has length larger than k. Then for each k and each graph $G = (V, E)$:

(30.84) If G is P_k-critical, then G is factor-critical,

and

(30.85) V can be partitioned into edges and subsets U with $G[U]$ P_k-critical if and only if for each $S \subseteq V$, the graph $G-S$ has at most $|S|$ P_k-critical components.

This generalizes Theorem 24.8 and (30.86) below.

Corollary 30.14b does not extend to 2-matchings excluding triangles and pentagons, as is shown by the example given in Figure 30.2. (The sum of the values is at most 4 on each pentagon, but it does not belong to the convex hull of the 2-matchings without pentagons, since the sum of the values is equal to $\frac{20}{3}$, but there is no pentagon-free 2-matching of size ≥ 7.)

30.16b. Packing edges and factor-critical subgraphs

Cornuéjols, Hartvigsen, and Pulleyblank [1982] and Cornuéjols and Hartvigsen [1986] discovered an interesting direction of extensions of the results on matchings. Let $G = (V, E)$ be a graph. Call a subset U of V *factor-critical* if $G[U]$ is factor-critical; that is, if for each $v \in U$, the set $U \setminus \{v\}$ is matchable.

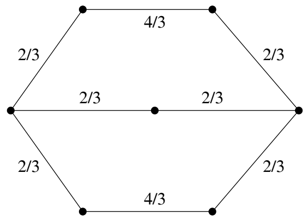


Figure 30.2

Let $\mathcal{F}$ be a collection of factor-critical subsets of V. An $\mathcal{F}$*-matching* is a collection of disjoint subsets from $E \cup \mathcal{F}$. It is *perfect* if it covers V. Call a subset U of V $\mathcal{F}$*-critical* if $G[U]$ has no perfect $\mathcal{F}$-matching but for each $v \in U$, the graph $G[U] - v$ has one. Cornuéjols, Hartvigsen, and Pulleyblank [1982] showed that

(30.86) if U is $\mathcal{F}$-critical, then U is factor-critical.

Then Cornuéjols and Hartvigsen [1986] proved the following extension of Tutte's 1-factor theorem (Theorem 24.1a):

(30.87) G has a perfect $\mathcal{F}$-matching if and only if for each $U \subseteq V$, the graph $G - U$ has at most $|U|$ $\mathcal{F}$-critical components.

Call an $\mathcal{F}$-matching $\mathcal{M}$ *maximum* if it maximizes $\sum_{U \in \mathcal{M}} |U|$. Cornuéjols and Hartvigsen [1986] also showed:

(30.88) Let $\mathcal{M}$ be a maximum $\mathcal{F}$-matching containing a minimum number of sets in $\mathcal{F}$. Let M be a matching containing $\mathcal{M} \cap E$ and having $\lfloor \frac{1}{2}|U| \rfloor$ edges in any $U \in \mathcal{M} \cap \mathcal{F}$. Then M is a maximum-size matching in G.

They also described an extension of the Edmonds-Gallai decomposition theorem. Cornuéjols, Hartvigsen, and Pulleyblank [1982] gave a polynomial-time algorithm to find a maximum $\mathcal{F}$-matching. Related results were obtained by Kirkpatrick and Hell [1978,1983] and Hell and Kirkpatrick [1984,1986].

30.16c. 2-factors without short circuits

Hartvigsen [1984] showed that a maximum size simple 2-matching without triangles can be found in polynomial time. He also gave good characterization for the existence of a 2-factor without triangles.

On the other hand, Cornuéjols and Pulleyblank [1980a] showed with a method of C.H. Papadimitriou that the problem of finding a 2-factor without circuits of length at most 5, is NP-complete. The complexity of deciding if a 2-factor exists without circuits of length at most 4 is not known.

Vornberger [1980] showed the NP-completeness of finding a maximum-weight 2-factor without circuits of length at most 4. The complexity status of finding a maximum-weight 2-factor without circuits of length at most 3 is unknown. Hell, Kirkpatrick, Kratochvíl, and Kříž [1988] and Cunningham and Wang [2000] give related results.

Chapter 31

b-matchings

b-matchings form an extension of 2-matchings and can be handled again by applying splitting techniques to ordinary matchings.

31.1. *b*-matchings

Let $G = (V, E)$ be a graph and let $b \in \mathbb{Z}_+^V$. A *b-matching* is a function $x \in \mathbb{Z}_+^E$ satisfying

(31.1) $\quad x(\delta(v)) \leq b(v)$

for each $v \in V$. This is equivalent to: $Mx \leq b$, where M is the $V \times E$ incidence matrix of G.

In (31.1), we count multiplicities: if e is a loop at v, then x_e is added twice at v. (This is consistent with our definition of $\delta(v)$ as a *family* of edges, in which each loop at v occurs twice.)

It is convenient to consider the graph G_b arising from G by splitting each vertex v into $b(v)$ copies, and by replacing any edge uv by $b(u)b(v)$ edges connecting the $b(u)$ copies of u with the $b(v)$ copies of v. More formally, $G_b = (V_b, E_b)$, where

(31.2) $\quad V_b := \{q_{v,i} \mid v \in V, 1 \leq i \leq b(v)\},$
$\quad E_b := \{q_{u,j}q_{v,i} \mid uv \in E, 1 \leq j \leq b(u), 1 \leq i \leq b(v), q_{u,j} \neq q_{v,i}\}.$

The condition $q_{u,j} \neq q_{v,i}$ is relevant only if $u = v$, that is, if there is a loop at u.

This construction was given by Tutte [1954b], and yields a min-max relation for maximum-size b-matching (where again the *size* of a vector is the sum of its components):

Theorem 31.1. *Let $G = (V, E)$ be a graph and let $b \in \mathbb{Z}_+^V$. Then the maximum size of a b-matching is equal to the minimum value of*

(31.3) $\quad b(U) + \sum_K \lfloor \tfrac{1}{2} b(K) \rfloor$

taken over $U \subseteq V$, where K ranges over the components of $G-U$ spanning at least one edge[19].

Proof. To see that the maximum is not more than the minimum, consider a b-matching x and a subset U of V. Then the sum of x_e over the edges e intersecting U is at most $b(U)$. The sum of x_e over the edges e contained in some component K of $G-U$ is at most $\lfloor \frac{1}{2}b(K)\rfloor$.

Equality is derived from the Tutte-Berge formula (Theorem 24.1). Let G_b be the graph described in (31.2). Then the maximum size of a b-matching in G is equal to the maximum size of a matching in G_b. By the Tutte-Berge formula, this is equal to the minimum value of

$$\text{(31.4)} \qquad \tfrac{1}{2}(|V_b| + |U'| - o(G_b - U'))$$

over $U' \subseteq V_b$ (where $o(H)$ denotes the number of odd components of a graph H).

Let U' attain this minimum. We may assume that if U' misses at least one copy of some vertex v of G, it misses all copies of v (since deleting all copies does not increase (31.4)). Hence there is a subset U of V such that U' is equal to the set of copies of vertices in U. We take $v \in U$ if $b(v) = 0$.

Let I_U be the set of isolated (hence loopless) vertices of $G-U$. Then $o(G_b - U')$ is equal to $b(I_U)$ plus the number of components K of $G-U$ that span at least one edge and have $b(K)$ odd. Setting k to the number of such components, (31.4) is equal to

$$\text{(31.5)} \qquad \begin{aligned} &\tfrac{1}{2}(b(V) + b(U) - o(G_b - U')) = b(U) + \tfrac{1}{2}(b(V\setminus U) - o(G_b - U')) \\ &= b(U) + \tfrac{1}{2}(b(V\setminus U) - b(I_U) - k), \end{aligned}$$

which is equal to (31.3). ∎

This theorem directly gives a characterization of the existence of a *perfect b-matching*, that is a b-matching having equality in (31.1) for each $v \in V$. This characterization is due to Tutte [1952]. By I_U we denote the set of isolated, loopless vertices of $G-U$.

Corollary 31.1a. *Let $G = (V,E)$ be a graph and let $b \in \mathbb{Z}_+^V$. Then there exists a perfect b-matching if and only if for each $U \subseteq V$, $G-U-I_U$ has at most $b(U) - b(I_U)$ components K with $b(K)$ odd.*

Proof. Directly from Theorem 31.1, by observing that a perfect b-matching exists if and only if the minimum value of (31.3) is at least $\frac{1}{2}b(V)$. ∎

31.2. The *b*-matching polytope

By a similar construction we can derive a characterization of the b-matching polytope. Given a graph $G = (V,E)$ and $b \in \mathbb{Z}_+^V$, the *b-matching polytope* is

[19] So K may consist of one vertex with a loop attached.

the convex hull of the b-matchings. The inequalities describing the b-matching polytope were announced by Edmonds [1965b] (cf. Pulleyblank [1973], Edmonds [1975]):

Theorem 31.2. *Let $G = (V, E)$ be a graph and let $b \in \mathbb{Z}_+^V$. Then the b-matching polytope is determined by the inequalities*

(31.6)
$$\begin{array}{rll} \text{(i)} & x_e \geq 0 & \text{for } e \in E, \\ \text{(ii)} & x(\delta(v)) \leq b(v) & \text{for } v \in V, \\ \text{(iii)} & x(E[U]) \leq \lfloor \frac{1}{2} b(U) \rfloor & \text{for } U \subseteq V \text{ with } b(U) \text{ odd.} \end{array}$$

Proof. The inequalities (31.6) are trivially valid for the vectors in the b-matching polytope. To see that they determine the b-matching polytope, let x satisfy (31.6). We may assume that $b \geq \mathbf{1}$.

Again consider the graph $G_b = (V_b, E_b)$ obtained by splitting each vertex v into $b(v)$ copies (cf. (31.2)). For any edge $e' = u'v'$ of G_b, with u' and v' copies of u and v in G, define $x'(e') := x_e/b(u)b(v)$, where $e := uv$. We show that x' belongs to the matching polytope of G_b, which implies the theorem.

By Edmonds' matching polytope theorem, it suffices to show that x' satisfies:

(31.7)
$$\begin{array}{rll} \text{(i)} & x'(e') \geq 0 & \text{for each edge } e' \in E_b, \\ \text{(ii)} & x'(\delta'(u')) \leq 1 & \text{for each vertex } u' \in V_b, \\ \text{(iii)} & x'(E'[U']) \leq \lfloor \frac{1}{2} |U'| \rfloor & \text{for each } U' \subseteq V_b \text{ with } |U'| \text{ odd.} \end{array}$$

Clearly (i) holds. To see (31.7)(ii), let u' be a vertex of G_b, being a copy of vertex u of G. Then

(31.8)
$$x'(\delta'(u')) = x(\delta(u))/b(u) \leq 1,$$

since for any edge $e = uv$ of G one has that

(31.9)
$$\sum_{v'} x'(u'v') = \sum_{v'} x(uv)/b(u)b(v) = x(uv)/b(u),$$

where v' ranges over the copies of v in G_b. So summing over all neighbours v' of u' gives $x(\delta(u))/b(u)$.

To see (31.7)(iii), choose $U' \subseteq V_b$ with $|U'|$ odd. Note that x satisfies (31.6)(iii) for *all* subsets U of V, since if $b(U)$ is even, then $x(E[U]) \leq \frac{1}{2}\sum_{v \in U} x(\delta(v)) \leq \frac{1}{2} b(U)$ by (31.6)(ii).

For any vertex v of G let B_v denote the set of copies of v in G_b. We show (31.7)(iii) by induction on the number of $v \in V$ for which U' 'splits' B_v, that is, for which

(31.10)
$$B_v \cap U' \neq \emptyset \text{ and } B_v \not\subseteq U'.$$

If this number is 0, (31.7)(iii) follows from (31.6)(iii). If this number is nonzero, choose a vertex v satisfying (31.10). Let $U_1 := U' \setminus B_v$ and $U_2 := U' \cup B_v$. So by induction we know

$$(31.11) \qquad x'(E'[U_1]) \leq \tfrac{1}{2}|U_1| \text{ and } x'(E'[U_2]) \leq \tfrac{1}{2}|U_2|.$$

Moreover, (31.8) implies:

$$(31.12) \qquad x'(E'[U_1]) + x'(E'[U_2]) \leq \sum_{u' \in U_1} x'(\delta'(u')) \leq |U_1|.$$

(This uses the fact that $B_v = U_2 \setminus U_1$ is a stable set in G_b.) Now define $\lambda := |B_v \cap U'|/b(v)$ and $\mu := |B_v \setminus U'|/b(v)$. So $\lambda + \mu = 1$ and

$$(31.13) \qquad x'(E'[U']) = \lambda x'(E'[U_2]) + \mu x'(E'[U_1]).$$

If $\lambda \leq \frac{1}{2}$, then, by (31.11) and (31.12):

$$(31.14) \qquad \begin{aligned} x'(E'[U']) &= (\mu - \lambda)x'(E'[U_1]) + \lambda(x'(E'[U_1]) + x'(E'[U_2])) \\ &\leq \tfrac{1}{2}(\mu - \lambda)|U_1| + \lambda|U_1| = \tfrac{1}{2}|U_1| \leq \lfloor \tfrac{1}{2}|U'| \rfloor. \end{aligned}$$

(The last inequality holds as $U_1 \subset U'$.)

If $\lambda > \frac{1}{2}$, then, by (31.11) and (31.12):

$$(31.15) \qquad \begin{aligned} x'(E'[U']) &= (\lambda - \mu)x'(E'[U_2]) + \mu(x'(E'[U_1]) + x'(E'[U_2])) \\ &\leq (\lambda - \mu)\tfrac{1}{2}|U_2| + \mu|U_1| = \tfrac{1}{2}|U_1| + \tfrac{1}{2}(\lambda - \mu)|U_2 \setminus U_1| \\ &= \tfrac{1}{2}|U_1| + \tfrac{1}{2}(\lambda - \mu)b_v = \tfrac{1}{2}|U_1| + \tfrac{1}{2}(|B_v \cap U'| - |B_v \setminus U'|) \\ &\leq \tfrac{1}{2}|U_1| + \tfrac{1}{2}(|B_v \cap U'| - 1) \leq \lfloor \tfrac{1}{2}|U'| \rfloor. \end{aligned}$$

(The last inequality holds as $U' = U_1 \cup (B_v \cap U')$.)

Thus we have (31.7)(iii). ∎

(This theorem follows also from the proof of the total dual integrality of the constraints (31.17) in Theorem 31.3 below.)

Given a graph $G = (V, E)$ and $b \in \mathbb{Z}_+^V$, the *perfect b-matching polytope* is the convex hull of the perfect b-matchings in G. As it is a face of the b-matching polytope (if nonempty), the previous theorem implies:

Corollary 31.2a. *Let $G = (V, E)$ be a graph and let $b \in \mathbb{Z}_+^V$. Then the perfect b-matching polytope is determined by the inequalities*

$$(31.16) \qquad \begin{array}{rll} \text{(i)} & x_e \geq 0 & \text{for } e \in E, \\ \text{(ii)} & x(\delta(v)) = b(v) & \text{for } v \in V, \\ \text{(iii)} & x(\delta(U)) \geq 1 & \text{for } U \subseteq V \text{ with } b(U) \text{ odd.} \end{array}$$

Proof. Directly from Theorem 31.2. ∎

(For a direct proof of this Corollary also based on considering the graph G_b obtained from G by splitting each vertex v into $b(v)$ copies, see Aráoz, Cunningham, Edmonds, and Green-Krótki [1983].)

Hurkens [1988] characterized adjacency on the b-matching polytope and showed that the diameter of the b-matching polytope is equal to the maximum size of a b-matching.

31.3. Total dual integrality

System (31.6) generally is not totally dual integral: if $G = (V, E)$ is the complete graph K_3 on three vertices, and $b(v) := 2$ for each $v \in V$ and $w(e) := 1$ for each $e \in E$, then the maximum weight of a b-matching is equal to 3, while there is no integer dual solution of odd value (when considering the dual of optimizing $w^{\mathsf{T}}x$ subject to (31.6)).

However, if we extend (31.6)(iii) to *all* subsets U of V, the system is totally dual integral, as was shown by Pulleyblank [1980]. So the system becomes:

(31.17)
$$\begin{array}{rll} \text{(i)} & x_e \geq 0 & \text{for } e \in E, \\ \text{(ii)} & x(\delta(v)) \leq b(v) & \text{for } v \in V, \\ \text{(iii)} & x(E[U]) \leq \lfloor \frac{1}{2} b(U) \rfloor & \text{for } U \subseteq V. \end{array}$$

It is equivalent to the following result:

Theorem 31.3. *Let $G = (V, E)$ be a graph, let $b \in \mathbb{Z}_+^V$ and let $w \in \mathbb{Z}_+^E$. Then the maximum weight $w^{\mathsf{T}}x$ of a b-matching x is equal to the minimum value of*

(31.18)
$$\sum_{v \in V} y_v b(v) + \sum_{U \subseteq V} z(U) \lfloor \tfrac{1}{2} b(U) \rfloor,$$

where $y \in \mathbb{Z}_+^V$ and $z \in \mathbb{Z}_+^{\mathcal{P}(V)}$ satisfy

(31.19)
$$\sum_{v \in V} y_v \chi^{\delta(v)} + \sum_{U \subseteq V} \chi^{E[U]} \geq w.$$

Proof. By Theorem 31.2 and LP-duality, the maximum weight of a b-matching is equal to the minimum of (31.18) over $y \in \mathbb{R}_+^V$ and $z \in \mathbb{R}_+^{\mathcal{P}(V)}$ satisfying (31.19). Suppose that this minimum is strictly smaller than if we restrict y and z to integer-valued functions. Then there exists a $t \in \mathbb{Z}_+$ such that the minimum with y and z restricted to values in $2^{-t}\mathbb{Z}_+$ is strictly smaller than when restricting y and z to values in $\mathbb{Z}_+$, because we can slightly increase any value of y_v and $z(U)$ to a dyadic vector. Choose t with this property as small as possible. By replacing w by $2^{t-1}w$, we may assume that $t = 1$.

It therefore is enough to show that for each $y \in \frac{1}{2}\mathbb{Z}_+^V$ and $z \in \frac{1}{2}\mathbb{Z}_+^{\mathcal{P}(V)}$ satisfying (31.19), there exist $y' \in \mathbb{Z}_+^V$ and $z' \in \mathbb{Z}_+^{\mathcal{P}(V)}$ satisfying (31.19) such that

(31.20)
$$\sum_{v \in V} y'_v(v) + \sum_{U \subseteq V} z'(U) \lfloor \tfrac{1}{2} b(U) \rfloor \leq \sum_{v \in V} y_v b(v) + \sum_{U \subseteq V} z(U) \lfloor \tfrac{1}{2} b(U) \rfloor.$$

We show this by induction on $w(E)$. More precisely, we consider a counterexample $y \in \frac{1}{2}\mathbb{Z}_+^V$ and $z \in \frac{1}{2}\mathbb{Z}_+^{\mathcal{P}(V)}$ with smallest $w(E)$. Then necessarily

(31.21)
$$y \in \{0, \tfrac{1}{2}\}^V \text{ and } z \in \{0, \tfrac{1}{2}\}^{\mathcal{P}(V)},$$

since if $y_v \geq 1$ for some vertex v we can reduce $w(e)$ by 1 for each $e \in \delta(v)$ and reduce y_v by 1, to obtain a counterexample with smaller $w(E)$. Similarly, if $z(U) \geq 1$ for some $U \subseteq V$ we can reduce $w(e)$ by 1 for each $e \in E[U]$ and reduce $z(U)$ by 1, to obtain a counterexample with smaller $w(E)$.

Put on $y \in \{0, \frac{1}{2}\}^V$ and $z \in \{0, \frac{1}{2}\}^{\mathcal{P}(V)}$ the additional requirements that, first, $y(V)$ is as large as possible, and, second, that

(31.22) $$\sum_{U \subseteq V} z(U)|U||V \setminus U|$$

is as small as possible.

Let $S := \{v \in V \mid y_v = \frac{1}{2}\}$ and $\mathcal{F} := \{U \subseteq V \mid z(U) = \frac{1}{2}\}$. We first show that $\mathcal{F}$ is laminar; that is,

(31.23) if $U, W \in \mathcal{F}$, then $U \cap W = \emptyset$ or $U \subseteq W$ or $W \subseteq U$.

Indeed, suppose that $U \cap W \neq \emptyset$, $U \not\subseteq W$, and $W \not\subseteq U$ for some $U, W \in \mathcal{F}$.

If $b(U \cap W)$ is odd, then decreasing $z(U)$ and $z(W)$ by $\frac{1}{2}$, and increasing $z(U \cap W)$ and $z(U \cup W)$ by $\frac{1}{2}$, would not increase (31.18) (since $\lfloor \frac{1}{2} b(U \cap W) \rfloor + \lfloor \frac{1}{2} b(U \cup W) \rfloor \leq \lfloor \frac{1}{2} b(U) \rfloor + \lfloor \frac{1}{2} b(W) \rfloor$), would maintain (31.19) (since $\chi^{E[U \cap W]} + \chi^{E[U \cup W]} \geq \chi^{E[U]} + \chi^{E[W]}$), would leave $y(V)$ unchanged, but would decrease (31.22), contradicting the minimality of (31.22).

If $b(U \cap W)$ is even, then resetting

(31.24) $$z(U) := z(U) - \tfrac{1}{2},\ z(W) := z(W) - \tfrac{1}{2},\ z(U \setminus W) := z(U \setminus W) + \tfrac{1}{2},$$ $$z(W \setminus U) := z(W \setminus U) + \tfrac{1}{2}, \text{ and } y_v := y_v + \tfrac{1}{2} \text{ for each } v \in U \cap W,$$

would not increase (31.18) (since $\lfloor \frac{1}{2} b(U \setminus W) \rfloor + \lfloor \frac{1}{2} b(W \setminus U) \rfloor + b(U \cap W) \leq \lfloor \frac{1}{2} b(U) \rfloor + \lfloor \frac{1}{2} b(W) \rfloor$), would maintain (31.19) (since $\chi^{E[U \setminus W]} + \chi^{E[W \setminus U]} + \sum_{v \in U \cap W} \chi^{\delta(v)} \geq \chi^{E[U]} + \chi^{E[W]}$), but would increase $y(V)$, contradicting the maximality of $y(V)$.

This shows (31.23). Suppose $\mathcal{F} \neq \emptyset$. Then choose an inclusionwise minimal set $U \in \mathcal{F}$ with the property that there exist an even number of sets $W \in \mathcal{F}$ with $W \supset U$. Let $U_1, \ldots, U_k$ be the inclusionwise maximal proper subsets of U with $U_i \in \mathcal{F}$ (possibly $k = 0$). By the choice of U, none of the U_i contain properly a set in $\mathcal{F}$. Then

(31.25) $$\lfloor \tfrac{1}{2} b(U) \rfloor + \sum_{i=1}^{k} \lfloor \tfrac{1}{2} b(U_i) \rfloor \geq b(U \cap S) + \sum_{i=1}^{k} 2 \lfloor \tfrac{1}{2} b(U_i \setminus S) \rfloor$$

or

(31.26) $$\lfloor \tfrac{1}{2} b(U) \rfloor + \sum_{i=1}^{k} \lfloor \tfrac{1}{2} b(U_i) \rfloor \geq b(U \setminus S) + \sum_{i=1}^{k} 2 \lfloor \tfrac{1}{2} b(U_i \cap S) \rfloor,$$

as

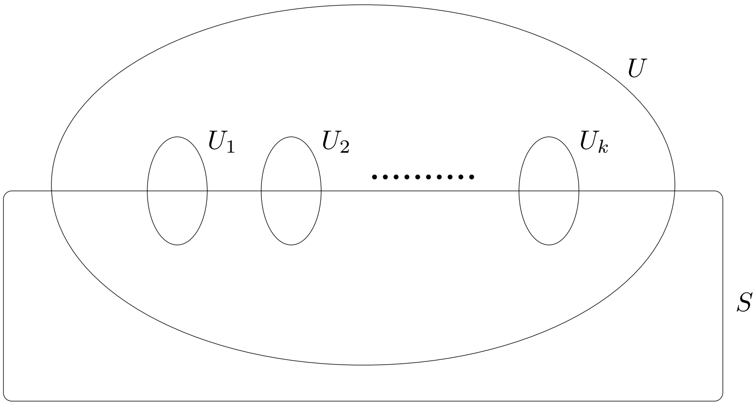


Figure 31.1

$$(31.27)\qquad b(U)+2\sum_{i=1}^{k}\lfloor\tfrac{1}{2}b(U_i)\rfloor$$
$$\geq b(U\cap S)+b(U\setminus S)+2\sum_{i=1}^{k}\lfloor\tfrac{1}{2}b(U_i\setminus S)\rfloor+2\sum_{i=1}^{k}\lfloor\tfrac{1}{2}b(U_i\cap S)\rfloor.$$

If (31.25) holds, then resetting $y_v := y_v + \frac{1}{2}$ for each $v \in U \cap S$, $z(U) := z(U)-\frac{1}{2}$, and $z(U_i) := z(U_i)-\frac{1}{2}$, $z(U_i\setminus S) := z(U_i\setminus S)+1$ for each $i = 1,\ldots,k$ would not increase (31.18) (by (31.25)) and would maintain (31.19): on edges not spanned by U, the left-hand side of (31.19) does not decrease; on edges spanned by U the contribution of the nonmodified variables is integer, and

$$(31.28)\qquad \lfloor\tfrac{1}{2}\big(\sum_{v\in U\cap S}\chi^{\delta(v)}+\chi^{E[U]}+\sum_{i=1}^{k}\chi^{E[U_i]}\big)\rfloor \leq \sum_{v\in U\cap S}\chi^{\delta(v)}+\sum_{i=1}^{k}\chi^{E[U_i\setminus S]}.$$

By the maximality of $y(V)$ it follows that $U \cap S = \emptyset$. Hence, after resetting we have $z(U_i) = 1$ for each $i = 1,\ldots,k$. If $k > 0$ we contradict (31.21). So $k = 0$, and therefore (as $z(U)$ decreases) (31.18) decreases, contradicting the minimality of (31.18).

If (31.26) holds, then resetting $y_v := y_v + \frac{1}{2}$ for each $v \in U \setminus S$, $z(U) := z(U)-\frac{1}{2}$, and $z(U_i) := z(U_i)-\frac{1}{2}$, $z(U_i\cap S) := z(U_i\cap S)+1$ for each $i = 1,\ldots,k$ would not increase (31.18) (by (31.26)) and would maintain (31.19), since now

$$(31.29)\qquad \lfloor\tfrac{1}{2}\big(\sum_{v\in U\cap S}\chi^{\delta(v)}+\chi^{E[U]}+\sum_{i=1}^{k}\chi^{E[U_i]}\big)\rfloor \leq \tfrac{1}{2}\sum_{v\in U}\chi^{\delta(v)}+\sum_{i=1}^{k}\chi^{E[U_i\cap S]}.$$

By the maximality of $y(V)$ it follows that $U \setminus S = \emptyset$, that is, $U \subseteq S$. Hence, after resetting we have $z(U_i) = 1$ for each $i = 1,\ldots,k$. If $k > 0$ we again contradict (31.21). So $k = 0$, and therefore (as $z(U)$ decreases) (31.18) decreases, again contradicting the minimality of (31.18).

So $\mathcal{F} = \emptyset$. Now setting $z'_S := 1$ and $y' := \mathbf{0}$ gives (31.20). ∎

(This is the proof method followed by Schrijver and Seymour [1977]. For a related proof, see Hoffman and Oppenheim [1978]. See also Cook [1983b].)

This theorem can be formulated equivalently in terms of total dual integrality:

Corollary 31.3a. *System* (31.17) *is TDI.*

Proof. Directly from Theorem 31.3. ∎

If we restrict the subsets U to odd-size subsets, the system is totally dual half-integral — a result stated by Pulleyblank [1973] and Edmonds [1975]:

Corollary 31.3b. *System* (31.6) *is totally dual half-integral.*

Proof. This follows from Corollary 31.3a, by using the fact that inequality (31.17)(iii) for $|U|$ even, is a half-integer sum of inequalities (31.6)(i) and (ii). ∎

Next considering the perfect b-matching polytope, generally (31.16) is not TDI. However:

Corollary 31.3c. *System* (31.16) *with* (31.16)(iii) *replaced by* (31.17)(iii) *is TDI.*

Proof. Directly from Corollary 31.3a with Theorem 5.25. ∎

This implies for the original system (Edmonds and Johnson [1970]):

Corollary 31.3d. *System* (31.16) *is totally dual half-integral.*

Proof. Consider an inequality $x(E[U]) \leq \lfloor \frac{1}{2} b(U) \rfloor$ in (31.17). If $b(U)$ is odd, this inequality is half of the sum of the inequalities $x(\delta(v)) \leq b(v)$ for $v \in U$ and of $-x(\delta(U)) \leq -1$. If $b(U)$ is even, this inequality is half of the sum of the inequalities $x(\delta(v)) \leq b(v)$ for $v \in U$ and of $-x_e \leq 0$ for $e \in \delta(U)$. ∎

In fact (Barahona and Cunningham [1989]):

Corollary 31.3e. *Let $w \in \mathbb{Z}^E$ with $w(C)$ even for each circuit C. Then the problem of minimizing $w^{\mathsf{T}} x$ subject to* (31.16) *has an integer optimum dual solution.*

Proof. As $w(C)$ is even for each circuit, there is a subset U of V with $\{e \in E \mid w(e) \text{ odd}\} = \delta(U)$. Now replace w by $w' := w + \sum_{v \in U} \chi^{\delta(v)}$. Then $w'(e)$ is an even integer for each edge e. Hence by Corollary 31.3d there is an integer optimum dual solution y'_v ($v \in V$), z_U ($U \subseteq V$, $b(U)$ odd) for the problem of minimizing ${w'}^{\mathsf{T}}x$ subject to (31.16). Now setting $y_v := y'_v - 1$ if $v \in U$ and $y_v := y'_v$ if $v \notin U$ gives an integer optimum dual solution for w. ∎

31.4. The weighted b-matching problem

We now consider the problem of finding a maximum-weight b-matching. Here, for a graph $G = (V, E)$, $b \in \mathbb{Z}_+^V$, and a weight function $w \in \mathbb{Q}^E$, the *weight* of a b-matching x is $w^{\mathsf{T}}x$.

It should be noted that the method of reducing a b-matching problem to a matching problem by replacing each vertex v by $b(v)$ copies, does not yield a polynomial-time algorithm for the weighted b-matching problem. W.H. Cunningham and A.B. Marsh, III (with suggestions of W.R. Pulleyblank, K. Truemper, and M.R. Rao — cf. Marsh [1979]) and Gabow [1983a] gave polynomial-time algorithms for the weighted b-matching problem. Padberg and Rao [1982] showed, with a method similar to that described in Section 25.5c, that one can test the constraints (31.16) in polynomial time, thus yielding the polynomial-time solvability of the maximum-weight b-matching problem (with the ellipsoid method).

Gerards [1995a] attributed the following method, leading to a strongly polynomial-time algorithm, to J. Edmonds. It extends a similar approach of Anstee [1987], and amounts to reducing the b-matching problem to a bipartite b-matching problem and a nonbipartite $\mathbf{1}$-matching problem.

First there is the following observation.

Lemma 31.4α. *Let $G = (V, E)$ be a graph and let $b, b' \in \mathbb{Z}_+^V$ with $\|b - b'\|_1 = 1$. Let x be a b-matching and let x' be a b'-matching. Then there exists a $y \in \mathbb{Z}^E$ such that $\|y\|_\infty \leq 2$ and such that $x + y$ is a b'-matching and $x' - y$ is a b-matching.*

Proof. By symmetry we may assume that there exists a $u \in V$ such that $b'(u) = b(u) + 1$ and $b'(v) = b(v)$ if $v \neq u$. Hence x is a b'-matching. If x' is a b'-matching, we are done (taking $y = \mathbf{0}$). So we may assume that x' is not a b-matching, that is, $x'_u = b'(u)$. Then there exists a walk $P = (v_0, e_1, v_1, \ldots, e_t, v_t)$ in G such that

(31.30) (i) $v_0 = u$, $x'_{e_i} > x_{e_i}$ if i is odd, $x'_{e_i} < x_{e_i}$ if i is even, and each edge e is traversed at most $|x'_e - x_e|$ times,
(ii) $x'(\delta(v_t)) < x(\delta(v_t))$ if t is even, and $x'(\delta(v_t)) > x(\delta(v_t))$ if t is odd (if $v_t = v_0$ and t is odd, then $x'(\delta(v_t)) \geq x(\delta(v_t)) + 2$).

The existence of such a path follows by taking a longest path satisfying (31.30)(i).

We now assume that P is a shortest path satisfying (31.30). Then no vertex is traversed more than twice (otherwise we can shortcut P), hence no edge is traversed more than twice. Let y_e be the number of times P traverses e, if $x'_e \geq x_e$, and let y_e be *minus* the number of times P traverses e, if $x'_e < x_e$. Then $x + y$ is a b'-matching, $x' - y$ is a b-matching, and $\|y\|_\infty \leq 2$. ∎

This implies a sensitivity result for maximum-weight b-matchings if we vary b:

Lemma 31.4β. *Let $G = (V, E)$, let $b, b' \in \mathbb{Z}_+^V$ and let a weight function $w \in \mathbb{R}^E$ be given. Then for any maximum-weight b-matching x there exists a maximum-weight b'-matching x' satisfying*

(31.31) $\quad \|x - x'\|_\infty \leq 2\|b - b'\|_1.$

Proof. We may assume that $\|b - b'\|_1 = 1$. Let x be a maximum-weight b-matching and let x' be a maximum-weight b'-matching. By Lemma 31.4α, we know that there exists an integer vector y with $x + y$ a b'-matching, $x' - y$ a b-matching, and $\|y\|_\infty \leq 2$. Since $x' - y$ is a b-matching and since x is a maximum-weight b-matching, we have $w^\mathsf{T} x \geq w^\mathsf{T}(x' - y)$, and hence $w^\mathsf{T}(x + y) \geq w^\mathsf{T} x'$. Since x' is a maximum-weight b'-matching, it follows that $x'' := x + y$ is a maximum-weight b'-matching with $\|x'' - x\|_\infty = \|y\|_\infty \leq 2$. ∎

This is used in showing the strong polynomial-time solvability of the weighted b-matching problem:

Theorem 31.4. *Given a graph $G = (V, E)$, $b \in \mathbb{Z}_+^V$, and a weight function $w \in \mathbb{Q}^E$, a maximum-weight b-matching can be found in strongly polynomial time.*

Proof. I. First consider the case that b is even. Make a bipartite graph H as follows. Make a new vertex v' for each $v \in V$. Let H have edges $u'v$ and uv' for each edge uv of G. Define $\tilde{b}(v) := \tilde{b}(v') := \frac{1}{2}b(v)$ for each $v \in V$. Define a weight $\tilde{w}(u'v) := \tilde{w}(uv') := w(uv)$ for each edge uv of G.

Find a maximum-weight $\tilde{b}$-matching $\tilde{x}$ in H. This can be done in strongly polynomial time by Theorem 21.9. Defining $x(uv) := \tilde{x}(u'v) + \tilde{x}(uv')$ for each edge uv of G, gives a maximum-weight b-matching x in G. Indeed, if there would be a b-matching in G of larger weight than that of x, then there is a half-integer $\tilde{b}$-matching in H of larger weight than that of $\tilde{x}$. This contradicts the fact that in a bipartite graph a maximum-weight b-matching is also a maximum-weight fractional b-matching (by Theorem 21.1).

II. Next consider the case of arbitrary b. Define $b' := 2\lfloor \frac{1}{2}b \rfloor$. Since b' is even, by part I of this proof we can find a maximum-weight b'-matching x' in G in strongly polynomial time. Now b arises from b' by at most $|V|$ resettings of b' to $b' + \chi^u$ for some $u \in V$. So it suffices to give a strongly polynomial-time method to obtain a maximum-weight b'-matching from a maximum-weight b-matching x, where $b' = b + \chi^u$ for some $u \in U$.

To this end, define

$$(31.32) \qquad z := \max\{\mathbf{0}, x - \mathbf{2}\} \text{ and } b'' := \min\{b' - Mz, M\mathbf{4}\}$$

(taking the maximum componentwise), where M is the $V \times E$ incidence matrix of G. ($\mathbf{0}$, $\mathbf{2}$, and $\mathbf{4}$ denote the all-0, all-2, and all-4 vector.)

Now we can find a maximum-weight b''-matching x'' in strongly polynomial time. This follows from the fact that $b''(v) \leq 4\deg(v)$ for each vertex v. So we can consider the graph $G_{b''}$ obtained by splitting each vertex v of G into $b''(v)$ copies, and replacing any edge uv by $b''(u)b''(v)$ edges connecting the $b''(u)$ copies of u by the $b''(v)$ copies of v. Then a maximum-weight matching in $G_{b''}$ gives a maximum-weight b''-matching x'' in G''.

Then $x''+z$ is a b'-matching, since $x''+z \geq \mathbf{0}$ and $M(x''+z) \leq b''+Mz \leq b'$. Moreover, $x''+z$ is a maximum-weight b'-matching, since by Lemma 31.4β, there exists a maximum-weight b'-matching x' satisfying $x - \mathbf{2} \leq x' \leq x + \mathbf{2}$. Then $x'-z$ is a b''-matching (since $x'-z \leq \mathbf{4}$), and hence $w^{\mathsf{T}}x'' \geq w^{\mathsf{T}}(x'-z)$. Therefore $w^{\mathsf{T}}(x''+z) \geq w^{\mathsf{T}}x'$. ∎

Elaboration of this method gives an $O(n^2m(n^2+m\log n))$-time algorithm. A similar approach of Anstee [1987] gives $O((m+n\log n)n\log\|b\|_\infty + n^2m)$- and $O(n^2\log n(m+n\log n))$-time algorithms.

For weighted *perfect* b-matching, a similar result follows:

Corollary 31.4a. *Given a graph $G = (V,E)$, $b \in \mathbb{Z}_+^V$, and a weight function $w \in \mathbb{Q}^E$, a minimum-weight perfect b-matching can be found in strongly polynomial time.*

Proof. By flipping signs, it suffices to describe a method finding a maximum-weight perfect b-matching in strongly polynomial time.

We can increase each weight by a constant $C := BW + W$, where $W := \|w\|_\infty + 1$ and $B := \|b\|_1$. So each weight becomes $\geq C - W$ and $\leq C + W$. Then each perfect b-matching has weight at least $\frac{1}{2}B(C-W) = \frac{1}{2}B^2W$, while each nonperfect b-matching has weight at most

$$(31.33) \qquad \begin{aligned}(\tfrac{1}{2}B - 1)(C + W) &= \tfrac{1}{2}BC + \tfrac{1}{2}BW - C - W \\ &= \tfrac{1}{2}B^2W + \tfrac{1}{2}BW + \tfrac{1}{2}BW - BW - W - W < \tfrac{1}{2}B^2W.\end{aligned}$$

So each maximum-weight b-matching is perfect. Therefore, Theorem 31.4 applies. (Alternatively, we could repeat the above reduction process.) ∎

31.5. If b is even

The results on b-matchings can be simplified if b is even. In that case, the proofs can be reduced to the bipartite case. The maximum size of a $2b$-

matching is equal to the minimum weight of a 2-vertex cover, taking b as weight:

Theorem 31.5. *Let $G = (V, E)$ be a graph and let $b \in \mathbb{Z}_+^V$. Then the maximum size of a $2b$-matching is equal to the minimum value of $y^\mathsf{T} b$ taken over 2-vertex covers y; equivalently, the minimum value of*

(31.34) $\quad b(V) + b(N(S)) - b(S),$

taken over stable sets S.

Proof. Make a bipartite graph H as follows. Make a new vertex v' for each $v \in V$, and let $V' := \{v' \mid v \in V\}$. H has vertex set $V \cup V'$ and edges all $u'v$ and uv' for $uv \in E$.

Define $b' : V \cup V' \to \mathbb{Z}_+$ by $b'(v) := b'(v') := b(v)$ for all $v \in V$. Then the maximum size of a $2b$-matching in G is equal to the maximum size of a b'-matching in H. By Corollary 21.1a, this is equal to the minimum b'-weight of a vertex cover in H, which is equal to the minimum of $y^\mathsf{T} b$ over 2-vertex covers y. ∎

It implies the following characterization of the existence of perfect b-matchings for even b:

Corollary 31.5a. *Let $G = (V, E)$ be a graph and let $b \in \mathbb{Z}_+^V$ with b even. Then there exists a perfect b-matching if and only if $b(N(S)) \geq b(S)$ for each stable set S of G.*

Proof. Directly from Theorem 31.5. ∎

This can also be derived directly from Corollary 31.1a. The following two theorems can be derived from the bipartite case in a way similar to the proof of Theorem 31.5, but they also are special cases of results in this chapter.

First we have a characterization of the $2b$-matching polytope:

Theorem 31.6. *Let $G = (V, E)$ be a graph and let $b \in \mathbb{Z}_+^V$. Then the $2b$-matching polytope is determined by*

(31.35)
(i) $x_e \geq 0$ for each $e \in E$,
(ii) $x(\delta(v)) \leq 2b(v)$ for each $v \in V$.

Proof. This is a special case of Theorem 31.2. ∎

Second, we mention a result of Gallai [1957,1958a,1958b]. For a graph $G = (V, E)$ and $w : E \to \mathbb{Z}_+$, a *w-vertex cover* is a function $y : V \to \mathbb{Z}_+$ satisfying $y_u + y_v \geq w(uv)$ for each edge uv.

Theorem 31.7. *Let $G = (V, E)$ be a graph and let $w \in \mathbb{Z}_+^E$ and $b \in \mathbb{Z}_+^V$. Then the maximum weight $w^\mathsf{T}x$ of a $2b$-matching x is equal to the minimum value of $y^\mathsf{T}b$ taken over $2w$-vertex covers y.*

Proof. This follows from Theorem 31.3. ∎

31.6. If b is constant

The results on b-matchings can be specialized to 'k-matchings'. Let $G = (V, E)$ be a graph and let $k \in \mathbb{Z}_+$. A *k-matching* is a function $x \in \mathbb{Z}_+^E$ with $x(\delta(v)) \leq k$ for each vertex v. Thus if we identify k with the all-k vector in $\mathbb{Z}_+^V$, we have a k-matching as before. Therefore, Theorem 31.1 gives a min-max relation for maximum-size k-matching:

Theorem 31.8. *Let $G = (V, E)$ be a graph and let $k \in \mathbb{Z}_+$. Then the maximum size of a k-matching is equal to the minimum value of*

$$k|U| + \sum_K \lfloor \tfrac{1}{2}k|K| \rfloor, \tag{31.36}$$

taken over $U \subseteq V$, where K ranges over the components of $G - U$ spanning at least one edge.

Proof. Directly from Theorem 31.1. ∎

Note that it follows that if k is even, we need not round, and hence the maximum size of a k-matching is equal to $\frac{1}{2}k$ times the maximum-size of a 2-matching. This maximum size is described in Theorem 30.1.

Again, a k-matching x is *perfect* if $x_v = k$ for each vertex v. In characterizing the existence, it is convenient to distinguish between the cases of k odd and k even. Let I_U denote the set of isolated (hence loopless) vertices of $G - U$.

Corollary 31.8a. *Let $G = (V, E)$ be a graph and let $k \in \mathbb{Z}_+$ be odd. Then G has a perfect k-matching if and only if for each $U \subseteq V$, $G - U - I_U$ has at most $k(|U| - |I_U|)$ odd components K.*

Proof. Directly from Corollary 31.1a. ∎

For even k, there is the following result due to Tutte [1952]:

Corollary 31.8b. *Let $G = (V, E)$ be a graph and let $k \in \mathbb{Z}_+$ be even. Then G has a perfect k-matching if and only if $|N(S)| \geq |S|$ for each stable set S.*

Proof. Directly from Corollary 31.5a. ∎

So if k is even, there exists a perfect k-matching if and only if there exists a perfect 2-matching.

We also give the characterization of the *k-matching polytope* (the convex hull of k-matchings):

Theorem 31.9. *Let $G = (V, E)$ be a graph and let $k \in \mathbb{Z}_+$. Then the k-matching polytope is determined by*

(31.37)
(i) $x_e \geq 0$ for each $e \in E$,
(ii) $x(\delta(v)) \leq k$ for each $v \in V$,
(iii) $x(E[U]) \leq \lfloor \frac{1}{2} k|U| \rfloor$ for each $U \subseteq V$ with $k|U|$ odd.

Proof. This is a special case of Theorem 31.2. ∎

31.7. Further results and notes

31.7a. Complexity survey for the b-matching problem

Complexity survey for the maximum-weight b-matching problem:

$*$	$O(n^2 B)$	Pulleyblank [1973]
	$O(n^2 m \log B)$	W.H. Cunningham and A.B. Marsh, III (cf. Marsh [1979])
$*$	$O(m^2 \log n \log B)$	Gabow [1983a]
$*$	$O(n^2 m + n \log B(m + n \log n))$	Anstee [1987]
$*$	$O(n^2 \log n(m + n \log n))$	Anstee [1987]

Here $B := \|b\|_\infty$, and $*$ indicates an asymptotically best bound in the table.

Johnson [1965] extended Edmonds' matching algorithm to an algorithm (not based on splitting vertices) finding a maximum-size b-matching, with running time polynomially bounded in n, m, and B. Gabow [1983a] gave an $O(nm \log n)$-time algorithm to find a maximum-size b-matching.

31.7b. Facets and minimal systems for the b-matching polytope

Edmonds and Pulleyblank (see Pulleyblank [1973]) described the facets of the b-matching polytope. Let $G = (V, E)$ be a graph and let $b \in \mathbb{Z}_+^V$. Call G *b-critical* if for each $u \in V$ there exists a b-matching x such that $x(\delta(u)) = b(u) - 1$ and $x(\delta(v)) = b(v)$ for each $v \neq u$.

Let G be simple and connected with at least three vertices and let $b > \mathbf{0}$. Then an inequality $x(\delta(v)) \leq b(v)$ determines a facet of the b-matching polytope if and only if $b(N(v)) > b(v)$, and if $b(N(v)) = b(v) + 1$, then $E[N(v)] \neq \emptyset$.

Moreover, an inequality $x(E[U]) \leq \lfloor \frac{1}{2} b(U) \rfloor$ determines a facet if and only if $G[U]$ is b-critical and has no cut vertex v with $b(v) = 1$.

Unlike in the matching case, the facet-inducing inequalities do not form a totally dual integral system. The minimal TDI-system for the b-matching polytope was characterized by Cook [1983a] and Pulleyblank [1981]. To describe this, call a graph $G = (V, E)$ *b-bicritical* if G is connected and for each $u \in V$ there is a b-matching x with $x(\delta(u)) = b(u) - 2$ and $x(\delta(v)) = b(v)$ for each $v \neq u$. Then a minimal TDI-system for the b-matching polytope (if G is simple and connected and has at least three vertices and if $b > \mathbf{0}$) is obtained by adding the following to the facet-inducing inequalities:

(31.38) $x(E[U]) \leq \frac{1}{2}b(U)$ for each $U \subseteq V$ with $|U| \geq 3$, $G[U]$ b-bicritical and $b(v) \geq 2$ for each $v \in N(U)$.

(The facets of the 2-matching polytope of a complete graph were also given by Grötschel [1977b].)

The vertices of the 2-matching polytope are characterized by:

Theorem 31.10. *Let $G = (V, E)$ be a graph. Then a 2-matching x is a vertex of the 2-matching polytope P if and only if the edges e with $x_e = 1$ form vertex-disjoint odd circuits.*

Proof. Let x be a 2-matching. Define $F := \{e \in E \mid x_e = 1\}$. Clearly, $\deg_F(v) \leq 2$ for each $v \in V$. So F forms a vertex-disjoint set of paths and circuits.

To see necessity in the theorem, let x be a vertex of P. Suppose that K is a component of F that forms a path or an even circuit. Then we can split K into matchings M and N. Then both $x + \chi^M - \chi^N$ and $x - \chi^M + \chi^N$ belong to P, contradicting the fact that x is a vertex of P.

To see sufficiency, suppose that x is not a vertex of P. Then there exists a nonzero vector y such that $x + y$ and $x - y$ belong to P. If $x_e = 0$ or $x_e = 2$, then $y_e = 0$, as $0 \leq x_e \pm y_e \leq 2$. If e and f are two edges in F incident with a vertex v, then $y_e = -y_f$, since $(x_e + x_f) \pm (y_e + y_f) \leq 2$. Hence, if each component of F is an odd circuit, we have $y = \mathbf{0}$, contradicting our assumption. ∎

31.7c. Regularizable graphs

A graph $G = (V, E)$ is called *regularizable* if there exists a k and a perfect k-matching x with $x \geq \mathbf{1}$. So we obtain a k-regular graph by replacing each edge e by x_e parallel edges. Berge [1978c] characterized regularizability as follows:

Theorem 31.11. *Let $G = (V, E)$ be connected and nonbipartite. Then G is regularizable if and only if $|N(U)| > |U|$ for each nonempty stable set U.*

Proof. Necessity being easy, we show sufficiency. Make a bipartite graph H by making for each vertex v a copy v', and replacing any edge uv by two edges uv' and $u'v$. Then every edge of H belongs to some perfect matching of H. To see this, suppose that edge uv' belongs to no perfect matching. Then by Frobenius' theorem (Corollary 16.2a), there exists a subset X of $V \setminus \{u\}$ such that X has less than $|X|$ neighbours in $V' \setminus \{v'\}$ (in the graph H; here $V' := \{v' \mid v \in V\}$). That is, defining $N'(X) := \cup_{u \in X} N_G(u)$,

(31.39) $|N'(X) \setminus \{v\}| < |X|$.

Let $U := X \setminus N'(X)$. Then U is a stable set. Moreover, $N(U) \subseteq N'(X) \setminus X$. By (31.39), $|N'(X)| \leq |X|$, and therefore $|N(U)| \leq |U|$. So by the condition given in the theorem, $U = \emptyset$; that is, $X \subseteq N'(X)$, and so, by (31.39), $X = N'(X)$. However, as G is connected and nonbipartite, H is connected. This contradicts the fact that $X = N'(X)$ and $X \neq V$.

So each edge of H belongs to a perfect matching. Hence each edge of G belongs to a perfect 2-matching. Adding up these perfect 2-matchings gives a perfect k-matching $x \geq \mathbf{1}$ for some k. ∎

Berge [1978b] remarked that this theorem is equivalent to: a connected nonbipartite graph G is regularizable if and only if the only 2-vertex cover of size $\tau_2(G)$ is the all-1 vector (this follows with (30.2)).

With the help of b-matchings, one can also characterize k-regularizable graphs — graphs that can become k-regular by adding edges parallel to existing edges. Let I_U denote the set of isolated (hence loopless) vertices of $G - U$.

Theorem 31.12. *Let $G = (V, E)$ be a graph and let $k \in \mathbb{Z}_+$. Then G is k-regularizable if and only if for each $U \subseteq V$, $G - U - I_U$ has at most*

(31.40) $$k|U| - k|I_U| - 2|E[U]| - |\delta(U \cup I_U)|$$

components K with $k|K| + |\delta(K)|$ odd.

Proof. From Corollary 31.1a applied to $b : V \to \mathbb{Z}_+$ defined by $b(v) := k - \deg(v)$ for $v \in V$.

Note that the condition implies $b(v) \geq 0$ for each vertex v. For suppose $\deg(v) > k$. If $k = 0$, then (31.40) is negative for $U := V$, a contradiction. So $k > 0$. Taking $U := \{v\}$, the condition implies that $k - k|I_U| - 2|E[U]| - |\delta(\{v\} \cup I_U)| \geq 0$. As $|\delta(v)| > k$, it follows that $I_U \neq \emptyset$, hence $|I_U| = 1$, say $I_U = \{w\}$. So $|E[U]| = 0$, that is, v is loopless. Moreover, $\delta(U \cup I_U) = \emptyset$, that is, $\{v, w\}$ is a component of G. But then the nonnegativity of (31.40) for $U' := \{v, w\}$ implies $2k \geq 2|E[U']| \geq 2\deg(v)$ (as v is loopless), a contradiction. ∎

See also Berge [1978b,1978d,1981].

31.7d. Further notes

Hoffman and Oppenheim [1978] showed that system (31.17) is 'locally strongly modular'; that is, each vertex of the b-matching polytope is determined by a linearly independent set of inequalities among (31.17) (set to equality), where the matrix in the system has determinant ± 1.

Johnson [1965] characterized the vertices of the fractional b-matching polytope. Koch [1979] studied bases (in the sense of the simplex method) for the linear programming problem of finding a maximum-weight b-matching.

Padberg and Wolsey [1984] described a strongly polynomial-time algorithm to find for any vector x the largest λ such that $\lambda \cdot x$ belongs to the b-matching polytope, and to describe $\lambda \cdot x$ as a convex combination of b-matchings.

b-matching algorithms are studied in the books by Gondran and Minoux [1984] and Derigs [1988a].

Chapter 32

Capacitated *b*-matchings

In the previous chapter we studied b-matchings, without upper bound given on the values of the edges. In this chapter we refine the results to the case where each edge has a prescribed 'capacity' that bounds the value on the edge. This can be reduced to uncapacitated b-matching.

32.1. Capacitated *b*-matchings

The capacitated b-matching problem considers b-matchings x satisfying a prescribed capacity constraint $x \leq c$. By a construction of Tutte [1954b], results on capacitated b-matchings can be derived from the results for the uncapacitated case as follows. Denote

(32.1) $$E[X,Y] := \{e \in E \mid \exists x \in X, y \in Y : e = \{x,y\}\}.$$

Theorem 32.1. *Let $G = (V,E)$ be a graph and let $b \in \mathbb{Z}_+^V$ and $c \in \mathbb{Z}_+^E$ with $c > \mathbf{0}$. Then the maximum size of a b-matching $x \leq c$ is equal to the minimum value of*

(32.2) $$b(U) + c(E[W]) + \sum_K \lfloor \tfrac{1}{2}(b(K) + c(E[K,W])) \rfloor,$$

taken over disjoint subsets U, W of V, where K ranges over the components of $G - U - W$.

Proof. To see that the maximum is not more than the minimum, let x be a b-matching with $x \leq c$ and let U, W be disjoint subsets of V. Then $x(E[U] \cup \delta(U)) \leq b(U)$ and $x(E[W]) \leq c(E[W])$. Consider next a component K of $G - U - W$. Then $2x(E[K]) + x(E[K,W]) \leq b(K)$ and $x(E[K,W]) \leq c(E[K,W])$. Hence $x(E[K] \cup E[K,W]) \leq \lfloor \frac{1}{2}(b(K) + c(E[K,W])) \rfloor$, and the inequality follows.

The reverse inequality is proved by reduction to Theorem 31.1. Make a new graph $G' = (V', E')$ by replacing each edge of G by a path of length three. That is, for each edge $e = uv$ introduce two new vertices $p_{e,u}$ and $p_{e,v}$ and three edges: $up_{e,u}$, $p_{e,u}p_{e,v}$, and $p_{e,v}v$.

Define $b' \in \mathbb{Z}_+^{V'}$ by $b'(v) := b(v)$ if $v \in V$ and $b'(p_{e,v}) := c(e)$ for any new vertex $p_{e,v}$. Then the maximum size of a b-matching x in G with $x \leq c$ is equal to the maximum size of a b'-matching in G', minus $c(E)$. By Theorem 31.1, there exists a subset U' of V' such that the maximum size of a b'-matching in G' equals

(32.3) $$b'(U') + \sum_{K'} \lfloor \tfrac{1}{2} b'(K') \rfloor,$$

where K' ranges over the components of $G' - U'$ with $|K'| \geq 2$. (Note that G' has no loops.) We choose U' with $|U'|$ as small as possible.

Let $U := V \cap U'$ and let W be the set of isolated vertices of $G' - U'$ that belong to V. We show that (32.2) is at most (32.3) minus $c(E)$, which proves the theorem.

First observe that

(32.4) if $p_{e,v} \in U'$, then $v \in W$.

Otherwise, deleting $p_{e,v}$ from U' does not increase (32.3), contradicting the minimality of $|U'|$. (Here we use that $p_{e,v}$ has degree 2 and that $b'(p_{e,v}) > 0$, that is, $c(e) > 0$. Then $b'(U')$ decreases by $c(e)$ while the sum in (32.3) increases by at most $\lfloor \frac{1}{2} c(e) + 1 \rfloor$, which is at most $c(e)$.)

Hence

(32.5) $$\begin{aligned} b'(U') &= b(U) + b'(U' \setminus V) = b(U) + \sum_{v \in W} c(\delta(v)) \\ &= b(U) + 2c(E[W]) + c(\delta(W)). \end{aligned}$$

Consider a component K' of $G' - U'$ with $|K'| \geq 2$. If K' does not intersect V, then it is equal to $\{p_{e,u}, p_{e,v}\}$ for some edge $e = uv$ of G with $u, v \in U$. So $b'(K') = 2c(e)$. If K' intersect V, let $K := K' \cap V$. Then K is a component of $G - U - W$. Indeed, any edge spanned by K gives a path of length 3 in K' (by (32.4)), and any path in K' between vertices in K gives a path in K. Any edge of G leaving K gives a path of length 3 in G' connecting K' and $U \cup W$. So

(32.6) $$K' = K \cup \{p_{e,u} \mid e = uv \in E, u \in K\} \cup \{p_{e,v} \mid e = uv \in E, u \in K, v \in U\}.$$

Hence

(32.7) $$b'(K') = b(K) + c(E[K,W]) + 2c(E[K]) + 2c(E[K,U]).$$

Therefore, (32.3) is equal to

(32.8) $$\begin{aligned} &b(U) + 2c(E[W]) + c(\delta(W)) + c(E[U]) \\ &+ \sum_{K} \big(\lfloor \tfrac{1}{2}(b(K) + c(E[K,W])) \rfloor + c(E[K]) + c(E[K,U]) \big), \end{aligned}$$

where K ranges over the components of $G - U - W$. Since

(32.9) $$c(E) = c(E[W]) + c(\delta(W)) + c(E[U]) + \sum_K (c(E[K]) + c(E[K, U])),$$

(32.3) minus $c(E)$ is equal to (32.2). ∎

This implies for *perfect* b-matchings:

Corollary 32.1a. *Let $G = (V, E)$ be a graph and let $b \in \mathbb{Z}_+^V$ and $c \in \mathbb{Z}_+^E$ with $c > \mathbf{0}$. Then G has a perfect b-matching $x \leq c$ if and only if for each partition T, U, W of V, $G[T]$ has at most*

(32.10) $$b(U) - b(W) + 2c(E[W]) + c(E[T, W])$$

components K with $b(K) + c(E[K, W])$ odd.

Proof. Directly from Theorem 32.1. ∎

32.2. The capacitated b-matching polytope

Let $G = (V, E)$ be a graph and let $b \in \mathbb{Z}_+^V$ and $c \in \mathbb{Z}_+^E$. The *c-capacitated b-matching polytope* is the convex hull of the b-matchings x satisfying $x \leq c$. A description of this polytope follows again from that for the uncapacitated b-matching polytope.

Theorem 32.2. *Let $G = (V, E)$ be a graph and let $b \in \mathbb{Z}_+^V$ and $c \in \mathbb{Z}_+^E$. The c-capacitated b-matching polytope is determined by*

(32.11)
$$\begin{array}{lll}
\text{(i)} & 0 \leq x_e \leq c(e) & (e \in E),\\
\text{(ii)} & x(\delta(v)) \leq b(v) & (v \in V),\\
\text{(iii)} & x(E[U]) + x(F) \leq \lfloor \tfrac{1}{2}(b(U) + c(F)) \rfloor & (U \subseteq V, F \subseteq \delta(U),\\
 & & b(U) + c(F) \text{ odd}).
\end{array}$$

Proof. It is easy to show that each b-matching $x \leq c$ satisfies (32.11). To show that the inequalities (32.11) completely determine the c-capacitated b-matching polytope, let $x \in \mathbb{R}^E$ satisfy (32.11). Let $G' = (V', E')$ and $b' \in \mathbb{Z}_+^{V'}$ be as in the proof of Theorem 32.1. Define $x' \in \mathbb{R}^{E'}$ by $x'(up_{e,u}) := x'(vp_{e,v}) := x_e$ and $x'(p_{e,u}p_{e,v}) := c(e) - x_e$, for any edge $e = uv$ of G. We show that x' belongs to the b'-matching polytope with respect to G'.

By Theorem 31.2, it suffices to check the constraints (31.6) for x' with respect to G' and b'. That is, we should check (where $\delta' := \delta_{G'}$ and $E'[U']$ is the set of edges in E' spanned by U'):

(32.12)
$$\begin{array}{lll}
\text{(i)} & x'(e') \geq 0 & (e' \in E'),\\
\text{(ii)} & x'(\delta'(v')) \leq b'(v') & (v' \in V'),\\
\text{(iii)} & x'(E'[U']) \leq \lfloor \tfrac{1}{2} b'(U') \rfloor & (U' \subseteq V' \text{ with } b'(U') \text{ odd}).
\end{array}$$

Trivially we have (32.12)(i) by (32.11)(i). Moreover, for each vertex $v \in V$ one has $x'(\delta'(v)) \leq b'(v)$ by (32.11)(ii). For any vertex $p_{e,u}$ of G', with $e = uv \in E$, one has $x'(\delta'(p_{e,u})) = c(e) = b'(p_{e,u})$.

To prove (32.12)(iii), we first show that it suffices to prove it for those $U' \subseteq V'$ satisfying for each edge $e = uv \in E$:

(32.13) (i) if $u, v \in U'$, then $p_{e,u} \in U'$ and $p_{e,v} \in U'$,
(ii) if $p_{e,u} \in U'$, then $u \in U'$.

To see (32.13)(i), first let $u, v \in U'$ and $p_{e,u} \notin U'$. Define $U'' := U' \cup \{p_{e,u}, p_{e,v}\}$. Then

(32.14)
$$x'(E'[U']) \leq x'(E'[U'']) - x'(\delta'(p_{e,u})) \leq \lfloor \tfrac{1}{2} b'(U'') \rfloor - b'(p_{e,u}) \leq \lfloor \tfrac{1}{2} b'(U') \rfloor.$$

To see (32.13)(ii), let $p_{e,u} \in U'$ and $u \notin U'$. Define $U'' := U' \setminus \{p_{e,u}, p_{e,v}\}$. If $p_{e,v} \notin U'$, then

(32.15)
$$x'(E'[U']) = x'(E'[U'']) \leq \lfloor \tfrac{1}{2} b'(U'') \rfloor \leq \lfloor \tfrac{1}{2} b'(U') \rfloor.$$

If $p_{e,v} \in U'$, then

(32.16)
$$x'(E'[U']) = x'(E'[U'']) + x'(\delta'(p_{e,v})) \leq \lfloor \tfrac{1}{2} b'(U'') \rfloor + b'(p_{e,v}) = \lfloor \tfrac{1}{2} b'(U') \rfloor.$$

This proves that we may assume (32.13) (as repeated application of these modifications gives finally (32.13)). Let $U := U' \cap V$ and let F be the set of those edges $e = uv$ in $\delta(U)$ with $u \in U$, $v \notin U$, and $p_{e,u} \in U'$. Then $x'(E'[U']) = x(E[U]) + c(E[U]) + x(F)$ and $b'(U') = b(U) + 2c(E[U]) + c(F)$. Hence (32.11)(iii) implies (32.12)(iii).

So x' is a convex combination of b'-matchings in G'. Each such b'-matching y satisfies $y(\delta'(v')) = b'(v')$ for each 'new' vertex $v' = p_{e,u}$ (as x' satisfies this equality). Hence each such b'-matching corresponds to a b-matching subject to c in G, and we obtain x as convex combination of b-matchings subject to c in G. ∎

Similarly, the *c-capacitated perfect b-matching polytope* is the convex hull of the perfect b-matchings x satisfying $x \leq c$. Theorem 32.2 implies the following (announced by Edmonds and Johnson [1970] (cf. Green-Krótki [1980], Aráoz, Cunningham, Edmonds, and Green-Krótki [1983])):

Corollary 32.2a. *The c-capacitated perfect b-matching polytope is determined by*

(32.17)
$$\begin{array}{rll} \text{(i)} & 0 \leq x_e \leq c(e) & (e \in E), \\ \text{(ii)} & x(\delta(v)) = b(v) & (v \in V), \\ \text{(iii)} & x(\delta(U) \setminus F) - x(F) \geq 1 - c(F) & (U \subseteq V, F \subseteq \delta(U), \\ & & b(U) + c(F) \text{ odd}). \end{array}$$

Proof. Directly from Theorem 32.2, as (32.17)(ii) implies that $x(E[U]) = \frac{1}{2} b(U) - \frac{1}{2} x(\delta(U))$. ∎

32.3. Total dual integrality

System (32.11) generally is not TDI (cf. the example in Section 30.5). To obtain a TDI-system, one should delete the restriction in (32.11)(iii) that $b(U) + c(F)$ is odd. Thus we obtain:

(32.18)
$$\begin{array}{lll} \text{(i)} & 0 \leq x_e \leq c(e) & (e \in E), \\ \text{(ii)} & x(\delta(v)) \leq b(v) & (v \in V), \\ \text{(iii)} & x(E[U]) + x(F) \leq \lfloor \frac{1}{2}(b(U) + c(F)) \rfloor & (U \subseteq V, F \subseteq \delta(U)). \end{array}$$

Theorem 32.3. *System* (32.18) *is TDI.*

Proof. Let $G' = (V', E')$ and $b' \in \mathbb{Z}_+^{V'}$ be as in the proof of Theorem 32.1. By Corollary 31.3a, the following system is TDI:

(32.19)
$$\begin{array}{lll} \text{(i)} & x'(e') \geq 0 & (e' \in E'), \\ \text{(ii)} & x'(\delta'(v')) \leq b'(v') & (v' \in V'), \\ \text{(iii)} & x'(E'[U']) \leq \lfloor \frac{1}{2} b'(U') \rfloor & (U' \subseteq V'). \end{array}$$

Since setting inequalities to equalities maintains total dual integrality (Theorem 5.25), the following system is TDI:

(32.20)
$$\begin{array}{lll} \text{(i)} & x'(e') \geq 0 & (e' \in E'), \\ \text{(ii)} & x'(\delta'(v)) \leq b(v) & (v \in V), \\ \text{(iii)} & x'(up_{e,u}) + x'(p_{e,u}p_{e,v}) = c(e) & (u \in e = uv \in E), \\ \text{(iv)} & x'(E'[U']) \leq \lfloor \frac{1}{2} b'(U') \rfloor & (U' \subseteq V'). \end{array}$$

The inequalities (32.14), (32.15), and (32.16) show that in (32.20)(iv) we may restrict the U' to those satisfying (32.13). So U' is determined by $U := U' \cap V$ and $F := \{e = uv \in E \mid u, p_{e,u} \in U', v \notin U'\}$.

Moreover, with Theorem 5.27 we can eliminate the variables $x'(up_{e,u})$ for $e \in E$ and $u \in e$ with the equalities (32.20)(iii). That is, we replace $x'(up_{e,u})$ by $c(e) - y_e$, where we set $y_e := x'(p_{e,u}p_{e,v})$ for $e = uv \in E$. Then:

(32.21) $x'(E'[U']) = y(E[U]) + 2(c(E[U]) - y(E[U])) + c(F) - y(F)$ and $b'(U') = b(U) + 2c(E[U]) + c(F)$.

Hence the system becomes:

(32.22)
$$\begin{array}{lll} \text{(i)} & y_e \geq 0 & (e \in E), \\ \text{(ii)} & y_e \leq c(e) & (e \in E), \\ \text{(iii)} & -y(\delta(v)) \leq b(v) - c(\delta(v)) & (v \in V), \\ \text{(iv)} & -y(E[U]) - y(F) \leq \lfloor \frac{1}{2} b(U) + c(F) \rfloor - c(E[U]) - c(F) & \\ & & (U \subseteq V, F \subseteq \delta(U)). \end{array}$$

Setting y_e to $c(e) - x_e$, the system becomes (32.18) and remains TDI. ∎

32.4. The weighted capacitated b-matching problem

By the construction given in the proof of Theorem 32.1, the weighted capacitated b-matching problem can easily be reduced to the uncapacitated variant:

Theorem 32.4. *Given a graph $G = (V, E)$, $b \in \mathbb{Z}_+^V$, $c \in \mathbb{Z}_+^E$, and a weight function $w \in \mathbb{Q}^E$, a maximum-weight b-matching $x \leq c$ can be found in strongly polynomial time.*

Proof. We may assume that $w \geq \mathbf{0}$. Make $G' = (V', E')$ and $b' \in \mathbb{Z}_+^{V'}$ as in the proof of Theorem 32.1. Moreover, define a weight function w' on the edges of G' by $w'(up_{e,u}) := w'(p_{e,u}p_{e,v}) := w'(p_{e,v}v) := w(e)$ for any edge $e = uv$ of G.

Let x' be a maximum-weight b'-matching in G'. Then we may assume that for each edge $e = uv$ of G one has $x'(up_{e,u}) = c(e) - x'(p_{e,u}p_{e,v}) = x'(p_{e,v}v)$. (This follows from the fact that we can assume that $x'(up_{e,u}) = x'(p_{e,v}v)$, since if say $x'(up_{e,u}) = x'(p_{e,v}v) + \tau$ with $\tau > 0$, we can decrease $x'(up_{e,u})$ by τ and increase $x'(p_{e,u}p_{e,v})$ by τ. Next we can reset $x'(p_{e,u}p_{e,v}) := c(e) - x'(up_{e,u})$.)

Now define $x_e := x'(up_{e,u})$ for each edge $e = uv$ of G. Then x is a maximum-weight b-matching with $x \leq c$. ∎

Similarly, for the weighted capacitated perfect b-matching problem:

Theorem 32.5. *Given a graph $G = (V, E)$, $b \in \mathbb{Z}_+^V$, $c \in \mathbb{Z}_+^E$, and a weight function $w \in \mathbb{Q}^E$, a minimum-weight perfect b-matching $x \leq c$ can be found in strongly polynomial time.*

Proof. As in the previous proof replace each edge by a path of length three, yielding the graph G', and define b', and w' similarly. Let x' be a maximum-weight perfect b'-matching in G'. Then for each edge $e = uv$ of G one has $x'(up_{e,u}) = c(e) - x'(p_{e,u}p_{e,v}) = x'(p_{e,v}v)$. Defining $x_e := x'(up_{e,u})$ for each edge $e = uv$ of G, gives a maximum-weight b-matching $x \leq c$. ∎

32.4a. Further notes

Cook [1983b] and Cook and Pulleyblank [1987] determined the facets and the minimal TDI-system for the capacitated b-matching polytope.

Johnson [1965] extended Edmonds' matching algorithm to an algorithm (not based on reduction to matching) that finds a maximum-size capacitated b-matching, with running time bounded by a polynomial in n, m, and $\|b\|_\infty$. Gabow [1983a] gave an $O(nm \log n)$-time algorithm for this.

Cunningham and Green-Krótki [1991] showed the following. Let $G = (V, E)$ be a graph, let $b \in \mathbb{Z}_+^V$ and $c \in \mathbb{Z}_+^E$. Then the convex hull of the integer vectors $y \leq b$ for which there is a perfect y-matching $x \leq c$ is determined by the inequalities

$$\text{(32.23)} \qquad \begin{aligned} &\mathbf{0} \leq y \leq b, \\ &y\Big(\bigcup_{i=0}^{k} A_i\Big) - y(B) \leq \sum_{i=1}^{k}(b(A_i) - 1) + c(E[A_0]) + c(E[A_0, V \setminus B]), \end{aligned}$$

where A_0 and B are disjoint subsets of V and where $A_1, \ldots, A_k$ are some of the components of $G - A_0 - B$ such that $b(A_i) + c(E[A_0, A_i])$ is odd for each $i = 1, \ldots, k$.

This characterizes the convex hull of degree-sequences of capacitated b-matchings, where the *degree-sequence* of $x \in \mathbb{Z}^E$ is the vector $y \in \mathbb{Z}^E$ defined by $y_v = x(\delta(v))$ for $v \in V$.

This generalizes the results of Balas and Pulleyblank [1989] on the matchable set polytope (Section 25.5d) and of Koren [1973] on the convex hull of degree-sequences of simple graphs (Section 33.6c below). See also Cunningham and Green-Krótki [1994] and Cunningham and Zhang [1992].

Chapter 33

Simple *b*-matchings and *b*-factors

A special case of capacitated b-matchings is obtained when we take capacity 1 on every edge. So the b-matching takes values 0 and 1 only. Such a b-matching is called *simple*. A simple b-matching is the incidence vector of some set of edges. If the b-matching is simple and perfect it is called a *b-factor*.
In this chapter we derive results on simple b-matchings and b-factors in a straightforward way from those on capacitated b-matchings obtained in the previous chapter.

33.1. Simple *b*-matchings and *b*-factors

Call a b-matching x *simple* if x is a 0,1 vector. We can identify simple b-matchings with subsets F of E with $\deg_F(v) \leq b(v)$ for each $v \in V$.

Simple b-matchings are special cases of capacitated b-matchings, namely by taking capacity function $c = \mathbf{1}$. Hence a min-max relation for maximum-size simple b-matching follows from the more general capacitated version:

Theorem 33.1. *Let $G = (V, E)$ be a graph and let $b \in \mathbb{Z}_+^V$. Then the maximum size of a simple b-matching is equal to the minimum value of*

$$b(U) + |E[W]| + \sum_K \lfloor \tfrac{1}{2}(b(K) + |E[K, W]|) \rfloor, \tag{33.1}$$

taken over all disjoint subsets U, W of V, where K ranges over the components of $G - U - W$.

Proof. The theorem is the special case $c = \mathbf{1}$ of Theorem 32.1. ∎

A *b-factor* is a simple perfect b-matching. In other words, it is a subset F of E with $\deg_F(v) = b(v)$ for each $v \in V$. The existence of a b-factor was characterized by Tutte [1952,1974] (cf. Ore [1957]):

Corollary 33.1a. *Let $G = (V, E)$ be a graph and let $b \in \mathbb{Z}_+^V$. Then G has a b-factor if and only if for each partition T, U, W of V, the graph $G[T]$ has at most*

(33.2) $$b(U) - b(W) + 2|E[W]| + |E[T, W]|$$

components K with $b(K) + |E[K, W]|$ odd.

Proof. Directly from Theorem 33.1 (or Corollary 32.1a). ∎

(An algorithmic proof was given by Anstee [1985], yielding an $O(n^3)$-time algorithm to find a b-factor. Tutte [1981] gave another proof and a sharpening.)

33.2. The simple b-matching polytope and the b-factor polytope

Given a graph $G = (V, E)$ and a vector $b \in \mathbb{Z}_+^V$, the *simple b-matching polytope* is the convex hull of the simple b-matchings in G. It can be characterized by (Edmonds [1965b]):

Theorem 33.2. *The simple b-matching polytope is determined by*

(33.3)
$$\begin{array}{rll} \text{(i)} & 0 \leq x_e \leq 1 & (e \in E), \\ \text{(ii)} & x(\delta(v)) \leq b(v) & (v \in V), \\ \text{(iii)} & x(E[U]) + x(F) \leq \lfloor \frac{1}{2}(b(U) + |F|) \rfloor & (U \subseteq V, F \subseteq \delta(U), \\ & & b(U) + |F| \text{ odd}). \end{array}$$

Proof. The theorem is a special case of Theorem 32.2. ∎

Given a graph $G = (V, E)$ and a vector $b \in \mathbb{Z}_+^V$, the *b-factor polytope* is the convex hull of (the incidence vectors of) b-factors in G. As it is a face of the simple b-matching polytope (if nonempty), we have:

Corollary 33.2a. *The b-factor polytope is determined by*

(33.4)
$$\begin{array}{rll} \text{(i)} & 0 \leq x_e \leq 1 & (e \in E), \\ \text{(ii)} & x(\delta(v)) = b(v) & (v \in V), \\ \text{(iii)} & x(\delta(U) \setminus F) - x(F) \geq 1 - |F| & (U \subseteq V, F \subseteq \delta(U), \\ & & b(U) + |F| \text{ odd}). \end{array}$$

Proof. Directly from Theorem 33.2. ∎

33.3. Total dual integrality

Consider the system (extending (33.3)):

(33.5)

$$\begin{array}{rlr} \text{(i)} & 0 \leq x_e \leq 1 & (e \in E), \\ \text{(ii)} & x(\delta(v)) \leq b(v) & (v \in V), \\ \text{(iii)} & x(E[U]) + x(F) \leq \lfloor \frac{1}{2}(b(U) + |F|) \rfloor & (U \subseteq V, F \subseteq \delta(U)). \end{array}$$

A special case of Theorem 32.3 is (cf. Cook [1983b]):

Theorem 33.3. *System* (33.5) *is TDI.*

Proof. Directly from Theorem 32.3. ∎

It implies for the b-factor polytope:

Corollary 33.3a. *System* (33.4) *is totally dual half-integral.*

Proof. By Theorems 33.3 and 5.25, the system obtained from (33.5) by setting (33.5)(ii) to equality, is TDI. Then each inequality (33.5) is a half-integer sum of inequalities (33.4), and the theorem follows. ∎

This can be extended to:

Corollary 33.3b. *Let $w \in \mathbb{Z}^E$ with $w(C)$ even for each circuit C. Then the problem of minimizing $w^{\mathsf{T}}x$ subject to* (33.4) *has an integer optimum dual solution.*

Proof. If $w(C)$ is even for each circuit, there is a subset U of V with $\{e \in E \mid w(e) \text{ odd}\} = \delta(U)$. Now replace w by $w' := w + \sum_{v \in U} \chi^{\delta(v)}$. Then $w'(e)$ is an even integer for each edge e. Hence by Corollary 33.3a there is an integer optimum dual solution y'_v $(v \in V)$, z_U $(U \subseteq V$, $b(U)$ odd) for the problem of minimizing ${w'}^{\mathsf{T}}x$ subject to (33.4). Now setting $y_v := y'_v - 1$ if $v \in U$ and $y_v := y'_v$ if $v \notin U$ gives an integer optimum dual solution for w. ∎

33.4. The weighted simple b-matching and b-factor problem

Also algorithmic results can be derived from the general capacity case, but some arguments can be simplified. While finding a minimum-weight b-factor can be reduced to finding a minimum-weight perfect b-matching, there is a more direct construction, since we can assume that b is not too large. We give the precise arguments in the proofs below.

Theorem 33.4. *Given a graph $G = (V, E)$, $b \in \mathbb{Z}_+^V$, and a weight function $w \in \mathbb{Q}^E$, a maximum-weight simple b-matching can be found in strongly polynomial time.*

Proof. We may assume that $b(v) \leq \deg_G(v)$ for each $v \in V$, since replacing $b(v)$ by $\min\{b(v), \deg_G(v)\}$ for each v does not change the problem.

Now the techniques described in Chapters 31 and 32 (replacing each vertex by $b(v)$ vertices, and next each edge by a path of length three), yield a strongly polynomial reduction to the maximum-weight matching problem. ∎

So a maximum-size simple b-matching and a b-factor (if any) can be found in polynomial time.

A similar construction applies to the weighted b-factor problem:

Theorem 33.5. *Given a graph $G = (V, E)$, $b \in \mathbb{Z}_+^V$, and a weight function $w \in \mathbb{Q}^E$, a minimum-weight b-factor can be found in strongly polynomial time.*

Proof. We may assume that $b(v) \leq \deg_G(v)$ for each $v \in V$, since otherwise there is no b-factor. Now the reduction techniques described in Chapters 31 and 32 yield a strongly polynomial reduction to the minimum-weight perfect matching problem. ∎

33.5. If b is constant

Again we can specialize the results above to k-matchings and k-factors, for $k \in \mathbb{Z}_+$. First we have for the maximum size of a simple k-matching:

Theorem 33.6. *Let $G = (V, E)$ be a graph and let $k \in \mathbb{Z}_+$. The maximum size of a simple k-matching is equal to the minimum value of*

$$k|U| + |E[W]| + \sum_K \lfloor \tfrac{1}{2}(k|K| + |E[K, W]|) \rfloor, \tag{33.6}$$

taken over all disjoint subsets U, W of V, where K ranges over the components of $G - U - W$.

Proof. Directly from Theorem 33.1. ∎

A *k-factor* is a simple perfect k-matching. In other words, it is a subset F of E with (V, F) k-regular. Theorem 33.6 implies a classical theorem of Belck [1950]:

Corollary 33.6a. *A graph $G = (V, E)$ has a k-factor if and only if for each partition T, U, W of V, $G[T]$ has at most*

$$k(|U| - |W|) + 2|E[W]| + |E[T, W]| \tag{33.7}$$

components K with $k|K| + |E[K, W]|$ odd.

Proof. Directly from Theorem 33.6. ∎

Petersen [1891] showed that the following is easy:

Theorem 33.7. *Each connected $2k$-regular graph G with an even number of edges has a k-factor.*

Proof. Make an Eulerian tour in G, and colour the edges alternatingly red and blue. Then the red edges form a k-factor. ∎

33.6. Further results and notes

33.6a. Complexity results

Urquhart [1967] gave an $O(b(V)n^3)$-time algorithm for finding a maximum-weight simple b-matching. This was improved by Gabow [1983a] to $O(b(V)m\log n)$ (by reduction to the $O(nm\log n)$-time algorithm of Galil, Micali, and Gabow [1982, 1986] for maximum-weight matching) and to $O(b(V)n^2)$. For maximum-*size* simple b-matching, Gabow [1983a] gave algorithms of running time $O(\sqrt{b(V)}\,m)$ (by reduction to Micali and Vazirani [1980]) and to $O(nm\log n)$.

33.6b. Degree-sequences

A sequence $d_1,\ldots,d_n$ is called a *degree-sequence* of a graph $G=(V,E)$ if we can order the vertices as $v_1,\ldots,v_n$ such that $\deg_G(v_i)=d_i$ for $i=1,\ldots,n$.

From Corollary 33.1a one can derive the characterization of degree-sequences of simple graphs due to Erdős and Gallai [1960]: there exists a simple graph with degrees $d_1\geq d_2\geq\cdots\geq d_n\geq 0$ if and only if $\sum_{i=1}^n d_i$ is even and

$$\sum_{i=1}^{k} d_i \leq k(k-1)+\sum_{i=k+1}^{n}\min\{k,d_i\} \tag{33.8}$$

for $k=1,\ldots,n$.

Havel [1955] gave the following recursive algorithm to decide if a sequence is the degree-sequence of a simple graph. A sequence $d_1\geq d_2\geq\cdots\geq d_n$ is the degree-sequence of a simple graph if and only if $0\leq d_n\leq n-1$ and $d_1-1,d_2-1,\ldots,d_{d_n}-1,d_{d_n+1},\ldots,d_{n-1}$ is the degree sequence of a simple graph.

Koren [1973] showed that the convex hull of degree-sequences of simple graphs on a finite vertex set V is determined by:

$$\begin{array}{lll}\text{(i)} & x_v\geq 0 & \text{for each } v\in V,\\ \text{(ii)} & x(U)-x(W)\leq |U|(|V|-|W|-1) & \text{for disjoint } U,W\subseteq V.\end{array} \tag{33.9}$$

If the graph need not be simple (but yet is loopless), condition (33.8) can be replaced by $\sum_{i=2}^n d_i\geq d_1$, as can be shown easily (cf. Hakimi [1962a]). Related work was done by Peled and Srinivasan [1989], who showed that system (33.9) is totally dual integral and characterized vertices, facets, and adjacency on the polytope determined by (33.9).

Kundu [1973] showed that if both sequences $d_1\geq\cdots\geq d_n\geq k$ and $d_1-k\geq\cdots\geq d_n-k\geq 0$ are realizable (as degree-sequence of a simple graph), then the first sequence is realizable by a graph with a k-factor (answering a question of Grünbaum [1970]). See also Edmonds [1964] and Cai, Deng, and Zang [2000].

33.6c. Further notes

Cook [1983b] and Cook and Pulleyblank [1987] determined the facets and the minimal TDI-system for the simple b-matching polytope. Hausmann [1978a,1981] characterized adjacency on the simple b-matching polytope.

Lovász [1972f] extended the Edmonds-Gallai decomposition to b-factors (cf. Lovász [1972e] and Graver and Jurkat [1980]). For a sharpening of Corollary 33.1a by specializing T, U, W, see Tutte [1974,1978].

Fulkerson, Hoffman, and McAndrew [1965] showed the following. Let $G = (V, E)$ be a graph such that any two odd circuits have a vertex in common or are connected by an edge. Let $b \in \mathbb{Z}_+^V$. Then G has a b-factor if and only if $b(V)$ is even and

$$b(U) + 2|E[W]| + |E[T, W]| \geq b(W) \tag{33.10}$$

for each partition T, U, W of V (cf. Mahmoodian [1977]).

Baebler [1937] showed that any k-regular l-connected graph has an l-factor if k is odd and l is even. Era [1985] proved the following conjecture of Akiyama [1982]: for each k there exists a t such that for each r-regular graph $G = (V, E)$ with $r \geq t$, E can be partitioned into $E_1, \ldots, E_s$ with for each $i = 1, \ldots, s$ one has $k \leq \deg_{E_i}(v) \leq k + 1$ for each vertex v.

Katerinis [1985] showed that if k', k, k'' are odd natural numbers with $k' \leq k \leq k''$, then any graph G having a k'-factor and a k''-factor, also has a k-factor. Related results are reported in Enomoto, Jackson, Katerinis, and Saito [1985].

Goldman [1964] studied augmenting paths for simple b-matchings by reduction to $\mathbf{1}$-matchings. More on b-matchings and b-factors can be found in Bollobás [1978], Tutte [1984], and Bollobás, Saito, and Wormald [1985].

Chapter 34

b-edge covers

The covering analogue of a b-matching is the b-edge cover. It is not difficult to derive min-max relations, polyhedral characterizations, and algorithms for b-edge covers from those for b-matchings.

34.1. *b*-edge covers

Let $G = (V, E)$ be a graph and let $b \in \mathbb{Z}_+^V$. A *b-edge cover* is a function $x \in \mathbb{Z}_+^E$ satisfying

(34.1) $\qquad x(\delta(v)) \geq b(v)$

for each $v \in V$.

There is a direct analogue of Gallai's theorem (Theorem 19.1), also given in Gallai [1959a], relating maximum-size b-matchings and minimum-size b-edge covers:

Theorem 34.1. *Let $G = (V, E)$ be a graph without isolated vertices and let $b \in \mathbb{Z}_+^V$. Then the maximum size of a b-matching plus the minimum size of a b-edge cover is equal to $b(V)$.*

Proof. Let x be a minimum-size b-edge cover. For any $v \in V$, reduce $x(\delta(v))$ by $x(\delta(v)) - b(v)$, by reducing x_e on edges $e \in \delta(v)$. We obtain a b-matching y of size

(34.2) $$y(E) \geq x(E) - \sum_{v \in V} (x(\delta(v)) - b(v)) = b(V) - x(E).$$

Hence the maximum-size of a b-matching is at least $b(V) - x(E)$.

Conversely, let y be a maximum-size b-matching. For any $v \in V$, increase $y(\delta(v))$ by $b(v) - y(\delta(v))$, by increasing y_e on edges $e \in \delta(v)$. We obtain a b-edge cover x of size

(34.3) $$x(E) \leq y(E) + \sum_{v \in V} (b(v) - y(\delta(v))) = b(V) - y(E).$$

Hence the minimum-size of a b-edge cover is at most $b(V) - y(E)$. ∎

(An alternative way of proving this is by applying Gallai's theorem for the case $b = \mathbf{1}$ directly to the graph G_b described in (31.2), obtained from G by splitting any vertex v into $b(v)$ vertices.)

With Theorem 34.1, we can derive a min-max relation for minimum-size b-edge cover from that for maximum-size b-matching. Let I_U denote the set of isolated (hence loopless) vertices of $G - U$.

Corollary 34.1a. *Let $G = (V, E)$ be a graph and let $b \in \mathbb{Z}_+^V$. Then the minimum size of a b-edge cover is equal to the maximum value of*

(34.4) $$b(I_U) + \sum_K \lceil \tfrac{1}{2} b(K) \rceil,$$

taken over $U \subseteq V$, where K ranges over the components of $G - U - I_U$.

Proof. Directly from Theorems 34.1 and 31.1. ∎

The construction in the proof of Theorem 34.1 also implies that a minimum-size b-edge cover can be found in polynomial time.

34.2. The b-edge cover polyhedron

Given a graph $G = (V, E)$ and $b \in \mathbb{Z}_+^V$, the *b-edge cover polyhedron* is the convex hull of the b-edge covers. The inequalities describing the b-edge cover polyhedron can be easily derived from the description of the edge cover polytope, similar to Theorem 31.2.

Theorem 34.2. *Let $G = (V, E)$ be a graph and let $b \in \mathbb{Z}_+^V$. Then the b-edge cover polyhedron is determined by the inequalities*

(34.5)
$$\begin{array}{rll} \text{(i)} & x_e \geq 0 & (e \in E), \\ \text{(ii)} & x(\delta(v)) \geq b(v) & (v \in V), \\ \text{(iii)} & x(E[U] \cup \delta(U)) \geq \lceil \tfrac{1}{2} b(U) \rceil & (U \subseteq V, b(U) \text{ odd}). \end{array}$$

Proof. Similar to the proof of Theorem 31.2, by construction of G_b and reduction to the description of the edge cover polytope (Corollary 27.3a). The theorem also follows from Theorem 34.3 below. ∎

34.3. Total dual integrality

The constraints (34.5) are totally dual integral if we delete the parity condition in (34.5)(iii):

$$\text{(34.6)}\qquad \begin{array}{rlr} \text{(i)} & x_e \geq 0 & (e \in E), \\ \text{(ii)} & x(\delta(v)) \geq b(v) & (v \in V), \\ \text{(iii)} & x(E[U] \cup \delta(U)) \geq \lceil \frac{1}{2} b(U) \rceil & (U \subseteq V). \end{array}$$

It is equivalent to the following:

Theorem 34.3. *Let $G = (V, E)$ be a graph, $b \in \mathbb{Z}_+^V$, and $w \in \mathbb{Z}_+^E$. Then the minimum weight $w^\mathsf{T} x$ of a b-edge cover x is equal to the maximum value of*

$$\text{(34.7)}\qquad \sum_{v \in V} y_v b(v) + \sum_{U \subseteq V} z_U \lceil \tfrac{1}{2} b(U) \rceil,$$

where $y \in \mathbb{Z}_+^V$ and $z \in \mathbb{Z}_+^{\mathcal{P}(V)}$ satisfy

$$\text{(34.8)}\qquad \sum_{v \in V} y_v \chi^{\delta(v)} + \sum_{U \subseteq V} z_U \chi^{E[U] \cup \delta(U)} \leq w.$$

Proof. We derive this from Theorem 32.3. Define $B := b(V) + 1$. Then the minimum is attained by a b-edge cover $x < B \cdot \mathbf{1}$. So adding $x_e \leq B$ for $e \in E$ as inequalities to (34.6) does not make it TDI if it wasn't. Let $\tilde{b}(v) := B \cdot \deg(v) - b(v)$ for each $v \in V$. Then by Theorem 32.3, the following system is TDI:

$$\text{(34.9)}\qquad \begin{array}{lr} 0 \leq \tilde{x}_e \leq B & (e \in E), \\ \tilde{x}(\delta(v)) \leq \tilde{b}(v) & (v \in V), \\ \tilde{x}(E[U] \cup F) \leq \lfloor \frac{1}{2}(\tilde{b}(U) + B|F|) \rfloor & (U \subseteq V, F \subseteq \delta(U)). \end{array}$$

Hence also the following system is TDI (by resetting $x_e = B - \tilde{x}_e$ for each $e \in E$):

$$\text{(34.10)}\qquad \begin{array}{lr} 0 \leq x_e \leq B & (e \in E), \\ x(\delta(v)) \geq b(v) & (v \in V), \\ x(E[U] \cup F) \geq \lceil \frac{1}{2}(b(U) - B|\delta(U) \setminus F|) \rceil & (U \subseteq V, F \subseteq \delta(U)). \end{array}$$

Now we can restrict ourselves in the last set of inequalities to those with $F = \delta(U)$, as otherwise the right-hand side is negative. So we have system (34.6) added with the superfluous inequalities $x_e \leq B$ for $e \in E$. ∎

Equivalently, in TDI terms:

Corollary 34.3a. *System* (34.6) *is totally dual integral.*

Proof. Directly from Theorem 34.3. ∎

34.4. The weighted *b*-edge cover problem

A minimum-weight b-edge cover can be found in strongly polynomial time, by reduction to maximum-weight b-matching:

Theorem 34.4. *For any graph $G = (V, E)$, $b \in \mathbb{Z}_+^V$, and weight function $w \in \mathbb{Q}^E$, a minimum-weight b-edge cover can be found in strongly polynomial time.*

Proof. Define $B := \|b\|_\infty$. Then we can assume that a minimum-weight b-edge cover x satisfies $x_e \leq B$ for each $e \in E$. Define $\tilde{b}(v) := B \cdot \deg(v) - b(v)$ for each $v \in V$. By Theorem 32.4, we can find a maximum-weight $\tilde{b}$-matching x in strongly polynomial time. Defining $x_e := B - \tilde{x}_e$ for each e then gives a minimum-weight b-edge cover. ∎

34.5. If *b* is even

The results can be simplified if b is even. In that case, the proofs can be reduced to the bipartite case.

Minimum-size $2b$-edge cover relates to maximum-weight 2-stable set, taking b as weight. Here a 2-*stable set* is a function $y \in \mathbb{Z}_+^V$ with $y_u + y_v \leq 2$ for each edge uv.

Theorem 34.5. *Let $G = (V, E)$ be a graph and let $b \in \mathbb{Z}_+^V$. Then the minimum size of a $2b$-edge cover is equal to the maximum value of $y^{\mathsf{T}} b$ where y is a 2-stable set; equivalently, to the maximum value of*

$$b(V) + b(S) - b(N(S)), \tag{34.11}$$

taken over stable sets S.

Proof. Similar to the proof of Theorem 31.5. (Alternatively, the present theorem can be derived with Theorem 34.1 from Theorem 31.7.) ∎

For a graph $G = (V, E)$ and $w : E \to \mathbb{Z}_+$, a w-*stable set* is a function $y : V \to \mathbb{Z}_+$ satisfying $y_u + y_v \leq w(uv)$ for each edge uv. Gallai [1957,1958a, 1958b] showed:

Theorem 34.6. *Let $G = (V, E)$ be a graph and let $b \in \mathbb{Z}_+^V$ and $w \in \mathbb{Z}_+^E$. Then the minimum weight $w^{\mathsf{T}} x$ of a $2b$-edge cover is equal to the maximum value of $y^{\mathsf{T}} b$ where y is a $2w$-stable set.*

Proof. This follows from Theorem 34.3. ∎

34.6. If *b* is constant

The above results can also be specialized to k-edge covers, for $k \in \mathbb{Z}_+$. That is, b is constant.

Let $G = (V, E)$ be a graph and let $k \in \mathbb{Z}_+$. A k-*edge cover* is a function $x \in \mathbb{Z}_+^E$ with $x(\delta(v)) \geq k$ for each vertex v. Thus if we identify k with the all-k

vector in $\mathbb{Z}_+^V$, we have a k-edge cover as before. Therefore, Corollary 34.1a gives the following, where I_U denotes the set of isolated (hence loopless) vertices of $G - U$:

Theorem 34.7. *Let $G = (V, E)$ be a graph and let $k \in \mathbb{Z}_+$. Then the minimum size of a k-edge cover is equal to the maximum value of*

$$k|I_U| + \sum_K \lceil \tfrac{1}{2} k |K| \rceil, \tag{34.12}$$

over $U \subseteq V$, where K ranges over the components of $G - U - I_U$.

Proof. Directly from Corollary 34.1a. ∎

Note that it follows that if k is even, we need not round, and hence the minimum size of a k-edge cover is equal to $\frac{1}{2}k$ times the minimum-size of a 2-edge cover.

34.7. Capacitated *b*-edge covers

The capacitated b-edge cover problem considers b-edge covers x satisfying a prescribed capacity constraint $x \leq c$. Results on capacitated b-edge covers can be easily derived from the results on capacitated b-matchings.

For minimum-size capacitated b-edge cover, one has:

Theorem 34.8. *Let $G = (V, E)$ be a graph and let $b \in \mathbb{Z}_+^V$ and $c \in \mathbb{Z}_+^E$. Then the minimum size of a b-edge cover $x \leq c$ is equal to the maximum value of*

$$b(U) - c(E[U]) + \sum_K \lceil \tfrac{1}{2}(b(K) - c(E[K, U])) \rceil, \tag{34.13}$$

taken over all pairs T, U of disjoint subsets of V, where K ranges over the components of $G[T]$.

Proof. Define $b'(v) := c(\delta(v)) - b(v)$ for each $v \in V$. Then by Theorem 32.1,

$$\begin{aligned}
&\text{minimum size of a } b\text{-edge cover } x \leq c \\
&= c(E) - \text{maximum size of a } b'\text{-matching } x' \leq c \\
&= c(E) - \min_{T,U,W} \big(b'(U) + c(E[W]) \\
&\quad + \sum_K \lfloor \tfrac{1}{2}(b'(K) + c(E[K, W])) \rfloor\big) \\
&= \max_{T,U,W} c(E) - 2c(E[U]) - c(\delta(U)) + b(U) - c(E[W]) \\
&\quad - \sum_K \lfloor \tfrac{1}{2}(2c(E[K]) + c(\delta(K)) - b(K) + c(E[K, W])) \rfloor \\
&= \max_{T,U,W} b(U) - c(E[U]) + \sum_K \lceil \tfrac{1}{2}(b(K) - c(E[K, U])) \rceil
\end{aligned} \tag{34.14}$$

(since $c(E) = c(E[U]) + c(\delta(U)) + c(E[W]) + c(E[T, W])$), where T, U, W range over partitions of V and where K ranges over the components of $G[T]$. ∎

This reduction also implies that a minimum-size b-edge cover $x \leq c$ can be found in strongly polynomial time.

Let $G = (V, E)$ be a graph, let $b \in \mathbb{Z}_+^V$ and $c \in \mathbb{Z}_+^E$. The *c-capacitated b-edge cover polytope* is the convex hull of the b-edge covers x satisfying $x \leq c$. The description of the inequalities follows again from that for the capacitated b-matching polytope.

Theorem 34.9. *The c-capacitated b-edge cover polytope is determined by*

(34.15)
$$\begin{array}{rll} \text{(i)} & 0 \leq x_e \leq c(e) & (e \in E), \\ \text{(ii)} & x(\delta(v)) \geq b(v) & (v \in V), \\ \text{(iii)} & x(E[U]) + x(\delta(U) \setminus F) \geq \lceil \frac{1}{2}(b(U) - c(F)) \rceil & \\ & & (U \subseteq V, F \subseteq \delta(U), b(U) - c(F) \text{ odd}). \end{array}$$

Proof. From Theorem 32.2, by setting $\tilde{b}(v) := c(\delta(v)) - b(v)$ and $\tilde{x}_e := c(e) - x_e$. ∎

By deleting the parity condition in (34.15)(iii), the system becomes totally dual integral:

Theorem 34.10. *The following system is TDI:*

(34.16)
$$\begin{array}{rll} \text{(i)} & 0 \leq x_e \leq c(e) & (e \in E), \\ \text{(ii)} & x(\delta(v)) \geq b(v) & (v \in V), \\ \text{(iii)} & x(E[U]) + x(\delta(U) \setminus F) \geq \lceil \frac{1}{2}(b(U) - c(F)) \rceil & \\ & & (U \subseteq V, F \subseteq \delta(U)). \end{array}$$

Proof. From Theorem 32.3, with the substitutions as given in the proof of the previous theorem. ∎

The weighted capacitated b-edge cover problem can easily be reduced to the uncapacitated variant:

Theorem 34.11. *Given a graph $G = (V, E)$, $b \in \mathbb{Z}_+^V$, $c \in \mathbb{Z}_+^E$, and a weight function $w \in \mathbb{Q}^E$, a minimum-weight b-edge cover $x \leq c$ can be found in strongly polynomial time.*

Proof. From Theorem 32.4, with the construction given in the proof of Theorem 34.8. ∎

Agarwal, Sharma, and Mittal [1982] showed that a minimum-weight b-edge cover $x \leq c$ can be obtained from a minimum-weight 'fractional' b-edge cover $x' \leq c$ with the help of a minimum-weight **1**-edge cover algorithm.

34.8. Simple b-edge covers

Call a b-edge cover x *simple* if x is a 0,1 vector. Thus we can identify simple b-edge covers with subsets F of E such that $\deg_F(v) \geq b(v)$ for each $v \in V$.

So defining $\tilde{b}(v) := \deg_G(v) - b(v)$ for $v \in V$, a vector x is a simple b-edge cover if and only if $\mathbf{1} - x$ is a simple $\tilde{b}$-matching. This reduces simple b-edge cover problems to simple $\tilde{b}$-matching problems. With this reduction, Theorem 33.1 gives:

Theorem 34.12. *Let $G = (V, E)$ be a graph and let $b \in \mathbb{Z}_+^V$ with $b(v) \leq \deg(v)$ for each $v \in V$. Then the minimum size of a simple b-edge cover is equal to the maximum value of*

$$(34.17) \qquad b(U) - |E[U]| + \sum_K \lceil \tfrac{1}{2}(b(K) - |E[K, U]|) \rceil,$$

taken over all pairs T, U of disjoint subsets of V, where K ranges over the components of $G[T]$.

Proof. From Theorem 33.1 applied to $\tilde{b}$. ∎

The *simple b-edge cover polytope* is the convex hull of the simple b-edge covers in G.

Theorem 34.13. *The simple b-edge cover polytope is determined by*

$$(34.18) \qquad \begin{array}{rll} \text{(i)} & 0 \leq x_e \leq 1 & (e \in E), \\ \text{(ii)} & x(\delta(v)) \geq b(v) & (v \in V), \\ \text{(iii)} & x(E[U]) + x(\delta(U) \setminus F) \geq \lceil \tfrac{1}{2}(b(U) - |F|) \rceil & \\ & & (U \subseteq V, F \subseteq \delta(U),\ b(U) - |F| \text{ odd}). \end{array}$$

Proof. This is a special case of Theorem 34.9. ∎

Again the system is TDI:

Theorem 34.14. *System* (34.18) *is totally dual integral after deleting the parity condition in* (iii).

Proof. The theorem is a special case of Theorem 34.10. ∎

Simple b-matchings are special cases of capacitated b-matchings, namely by taking the capacity function $c = \mathbf{1}$. Hence a minimum-weight simple b-edge cover can be found in strongly polynomial time:

Theorem 34.15. *Given a graph $G = (V, E)$, $b \in \mathbb{Z}_+^V$, and a weight function $w \in \mathbb{Q}^E$, a minimum-weight simple b-edge cover can be found in strongly polynomial time.*

Proof. The theorem is a special case of Theorem 34.11. ∎

We can specialize these results to k-edge covers, for $k \in \mathbb{Z}_+$. A *simple k-edge cover* is a set of edges covering each vertex at least k times. Thus it corresponds to subgraphs of minimum degree at least k. A min-max relation for minimum-size simple k-edge cover reads:

Theorem 34.16. *Let $G = (V, E)$ be a graph and let $k \in \mathbb{Z}_+$. Then the minimum size of a simple k-edge cover is equal to the maximum value of*

(34.19) $$k|U| - |E[U]| + \sum_K \lceil \tfrac{1}{2}(k|K| - |E[K,U]|) \rceil,$$

taken over all pairs T, U of disjoint subsets of V, where K ranges over the components of $G[T]$.

Proof. This is a special case of Theorem 34.12. ∎

34.8a. Simple b-edge covers and b-matchings

Let $G = (V, E)$ be a graph and let $b \in \mathbb{Z}_+^V$ with $b(v) \leq \deg_G(v)$ for each $v \in V$. Define

(34.20) $\nu^{\mathrm{s}}(b) :=$ the maximum size of a simple b-matching,
$\rho^{\mathrm{s}}(b) :=$ the minimum size of a simple b-edge cover.

Similar to Theorem 34.1, there is a relation between $\nu^{\mathrm{s}}(b)$ and $\rho^{\mathrm{s}}(b)$, generalizing Gallai's theorem (Theorem 19.1):

(34.21) $$\nu^{\mathrm{s}}(b) + \rho^{\mathrm{s}}(b) = b(V).$$

To see this, let M be a maximum-size simple b-matching. For each $v \in V$, add to M $b(v) - \deg_M(v)$ edges incident with v. We can do this in such a way that we obtain a simple b-edge cover F with $|F| \leq |M| + \sum_{v \in V}(b(v) - \deg_M(v)) = b(V) - |M|$. So $\rho^{\mathrm{s}}(b) \leq b(V) - |M| = b(V) - \nu^{\mathrm{s}}(b)$.

To see the reverse inequality, let F be a minimum-size simple b-edge cover. For each $v \in V$, delete from F $\deg_F(v) - b(v)$ edges incident with v. We obtain a simple b-matching M with $|M| \geq |F| - \sum_{v \in V}(\deg_F(v) - b(v)) = b(V) - |F|$. So $\nu^{\mathrm{s}}(b) \geq b(V) - |F| = b(V) - \rho^{\mathrm{s}}(b)$, which shows (34.21).

There is a second relation between simple b-matchings and simple b-edge covers. Define $\tilde{b}(v) := \deg_G(v) - b(v)$ for each $v \in V$. Then trivially (by complementing),

(34.22) $$\nu^{\mathrm{s}}(b) + \rho^{\mathrm{s}}(\tilde{b}) = |E|.$$

(34.21) implies

(34.23) $$b(V) - 2\nu^{\mathrm{s}}(b) = \rho^{\mathrm{s}}(b) - \nu^{\mathrm{s}}(b) = 2\rho^{\mathrm{s}}(b) - b(V),$$

and (34.22) implies

(34.24) $$\rho^{\mathrm{s}}(b) - \nu^{\mathrm{s}}(b) = \rho^{\mathrm{s}}(\tilde{b}) - \nu^{\mathrm{s}}(\tilde{b}).$$

Hence

(34.25) $$b(V) - 2\nu^{\mathrm{s}}(b) = \tilde{b}(V) - 2\nu^{\mathrm{s}}(\tilde{b}) = 2\rho^{\mathrm{s}}(b) - b(V) = 2\rho^{\mathrm{s}}(\tilde{b}) - \tilde{b}(V).$$

So the 'deficiency' of a maximum-size b-matching is equal to the 'surplus' of a minimum-size b-edge cover, and this parameter is invariant under replacing b by $\tilde{b} = \deg_G - b$.

34.8b. Capacitated *b*-edge covers and *b*-matchings

The results of the previous section hold more generally for capacitated b-matchings. Let $G = (V, E)$ be a graph, let $b \in \mathbb{Z}_+^V$ and let $c \in \mathbb{Z}_+^E$ with $b(v) \leq c(\delta(v))$ for each $v \in V$. Define

(34.26) $\nu^c(b) :=$ the maximum size of a b-matching $x \leq c$,
$\rho^c(b) :=$ the minimum size of a b-edge cover $x \leq c$.

Then:

(34.27) $\nu^c(b) + \rho^c(b) = b(V)$.

To see this, consider a maximum-size b-matching $x \leq c$. We can increase x to obtain a b-edge cover $y \leq c$, in such a way that $y(E) \leq x(E) + \sum_{v \in V}(b(v) - x(\delta(v))) = b(V) - x(E)$. So $\rho^c(b) \leq b(V) - x(E) = b(V) - \nu^c(b)$.

To see the reverse inequality, consider a minimum-size b-edge cover $y \leq c$. We can decrease y to obtain a b-matching $x \leq y$ such that $x(E) \geq y(E) - \sum_{v \in V}(y(\delta(v)) - b(v)) = b(V) - y(E)$. So $\nu^c(b) \geq b(V) - y(E) = b(V) - \rho^c(b)$, which shows (34.27).

Again, there is a second relation between capacitated b-matchings and capacitated b-edge covers. Define $\tilde{b}(v) := c(\delta(v)) - b(v)$ for each $v \in V$. Then trivially (by replacing x by $c - x$),

(34.28) $\nu^c(b) + \rho^c(\tilde{b}) = c(E)$.

Combining (34.27) and (34.28) gives as in (34.25):

(34.29) $b(V) - 2\nu^c(b) = \tilde{b}(V) - 2\nu^c(\tilde{b}) = 2\rho^c(b) - b(V) = 2\rho^c(\tilde{b}) - \tilde{b}(V)$.

So the 'deficiency' of a maximum-size b-matching $x \leq c$ is equal to the 'surplus' of a minimum-size b-edge cover $y \leq c$, and this parameter is invariant under replacing b by $\tilde{b} := c \circ \delta - b$.

Chapter 35

Upper and lower bounds

In the previous chapters we considered nonnegative integer functions satisfying certain lower *or* upper bounds. We now turn over to the more general case where we put both upper *and* lower bounds. We also relax the condition that the functions be nonnegative. Again, the results can be proved by refining the results of previous chapters — thus all results are obtained essentially by reduction to the fundamental results of Tutte and Edmonds.

35.1. Upper and lower bounds

Let $G = (V, E)$ be a graph and let $a, b \in \mathbb{Z}^V$ and $d, c \in \mathbb{Z}^E$. We will consider functions $x \in \mathbb{Z}^E$ satisfying

$$\text{(35.1)} \qquad \begin{array}{rll} \text{(i)} & d(e) \leq x_e \leq c(e) & \text{for all } e \in E, \\ \text{(ii)} & a(v) \leq x(\delta(v)) \leq b(v) & \text{for all } v \in V. \end{array}$$

The existence of such a function is characterized in the following theorem. (As usual, $E[X, Y]$ denotes the set of edges xy in E with $x \in X$ and $y \in Y$.)

Theorem 35.1. *Let $G = (V, E)$ be a graph and let $a, b \in \mathbb{Z}^V$ with $a \leq b$ and $d, c \in \mathbb{Z}^E$ with $d < c$. Then there exists an $x \in \mathbb{Z}^E$ satisfying* (35.1) *if and only if for each partition T, U, W of V, the number of components K of $G[T]$ with $b(K) = a(K)$ and*

$$\text{(35.2)} \qquad b(K) + c(E[K, W]) + d(E[K, U])$$

odd is at most

$$\text{(35.3)} \qquad b(U) - 2d(E[U]) - d(E[T, U]) - a(W) + 2c(E[W]) + c(E[T, W]).$$

Proof. To see necessity, consider a component K of $G[T]$ with $b(K) = a(K)$. Then

$$\text{(35.4)} \qquad 2x(E[K]) = b(K) - x(\delta(K)) = b(K) - x(E[K, U]) - x(E[K, W]).$$

Hence, if (35.2) is odd, we have $x(E[K, U]) \geq d(E[K, U]) + 1$ or $x(E[K, W]) \leq c(E[K, W]) - 1$. So $x(E[T, U]) - d(E[T, U]) + c(E[T, W]) - x(E[T, W])$ is at least the number of such components. On the other hand,

$$x(E[T,U]) - x(E[T,W]) = x(\delta(U)) - x(\delta(W)) \leq b(U) - 2d(E[U]) - a(W) + 2c(E[W]). \tag{35.5}$$

This proves necessity.

To see sufficiency, we may assume that $d = \mathbf{0}$, since the theorem is invariant under replacing $a(v)$ by $a(v) - d(\delta(v))$ and $b(v)$ by $b(v) - d(\delta(v))$ for each v, and c by $c - d$ and d by $\mathbf{0}$. (It does not change the parity of (35.2) and does not change (35.3).)

We show sufficiency by application of Corollary 32.1a. Define

$$R := \{v \in V \mid a(v) < b(v)\}. \tag{35.6}$$

Extend G to a graph $G' = (V', E')$, and define $b' \in \mathbb{Z}_+^{V'}$ and $c' \in \mathbb{Z}_+^{E'}$, as follows. For each $v \in V$, let $b'(v) := b(v)$ and for each $e \in E$, let $c'(e) := c(e)$. Introduce a new vertex v_0, with $b'(v_0) := b(V)$, and a loop v_0v_0 at v_0, with $c'(v_0v_0) := \infty$. Moreover, for each $v \in R$ introduce an edge vv_0 with $c'(vv_0) := b(v) - a(v)$.

Now a function x as required exists if and only if there exists a perfect b'-matching $x' \leq c'$ in G'. So it suffices to test the constraints given by Corollary 32.1a for G', b', and c'. Assuming x' does *not* exist, we can partition V' into T', U', and W' such that $G'[T']$ has more than $b'(U') - b'(W') + 2c'(E'[W']) + c'(E'[T', W'])$ components K' with $b'(K') + c'(E'[K', W'])$ odd. By parity, the excess is at least 2. (This follows from the fact that $b'(V') = 2b(V)$ is even.)

Let $T := T' \setminus \{v_0\}$, $U := U' \setminus \{v_0\}$, and $W := W' \setminus \{v_0\}$.

First assume that $v_0 \in U'$; so $T' = T$ and $W' = W$. Then the number of components K of $G'[T'] = G[T]$ with $b(K) + c(E[K, W])$ odd is trivially at most $b(T) + c(E[T, W])$, and hence at most

$$\begin{aligned} &b(U) + b(V) - b(W) + 2c(E[W]) + c(E[T, W]) \\ &= b'(U') - b'(W') + 2c'(E'[W']) + c'(E'[T', W']), \end{aligned} \tag{35.7}$$

a contradiction.

Second assume that $v_0 \in W'$. Then $c'(E'W') = \infty$, which is again a contradiction.

Hence we may assume that $v_0 \in T'$; so $U' = U$ and $W' = W$. Then $G'[T']$ has exactly one component containing v_0. All other components K are components of $G[T]$ that are disjoint from R (since no vertex in K is adjacent to v_0). So $G[T]$ has more than

$$\begin{aligned} &b'(U') - b'(W') + 2c'(E'[W']) + c'(E'[T', W']) \\ &= b(U) - a(W) + 2c(E[W]) + c(E[T, W]) \end{aligned} \tag{35.8}$$

components K contained in $V \setminus R$ with $b(K) + c(E[K, W])$ odd. This contradicts the condition of the theorem. ∎

By taking $d = \mathbf{0}$ and $c = \boldsymbol{\infty}$ we obtain as special case (where again I_U denotes the set of isolated (hence loopless) vertices of $G - U$):

Corollary 35.1a. *Let $G = (V, E)$ be a graph and let $a, b \in \mathbb{Z}_+^V$ with $a \leq b$. Then there exists a function $x \in \mathbb{Z}_+^E$ satisfying*

$$a(v) \leq x(\delta(v)) \leq b(v) \tag{35.9}$$

for each $v \in V$ if and only if for each $U \subseteq V$, $G - U - I_U$ has at most $b(U) - a(I_U)$ components K with $b(K)$ odd and $a(K) = b(K)$.

Proof. We show sufficiency. Suppose that no such x exists. By Theorem 35.1 (for $d = \mathbf{0}$, $c = \mathbf{\infty}$), there exists a partition T, U, W of V with $E[W] = \emptyset$ and $E[T, W] = \emptyset$ such that the number of components K of $G[T]$ with $b(K) = a(K)$ and $b(K)$ odd, is more than $b(U) - a(W)$. We may assume that each component K of $G[T]$ spans at least one edge: otherwise, if $K = \{v\}$, moving v from T to W, decreases the number of such components by at most 1, while $b(U) - a(W)$ decreases by at least 1 (since $b(v) = a(v)$ and $b(v)$ is odd).

So we can assume that $W = I_U$, in which case we have a contradiction with the condition in the present corollary. ∎

Another special case, for $d = \mathbf{0}$, and $c = \mathbf{1}$, is the characterization of Lovász [1970c] of the existence of subgraphs with prescribed degrees:

Corollary 35.1b. *Let $G = (V, E)$ be a graph and let $a, b \in \mathbb{Z}_+^V$ with $a \leq b$. Then E has a subset F such that*

$$a(v) \leq \deg_F(v) \leq b(v) \tag{35.10}$$

for each $v \in V$ if and only if for each partition T, U, W of V, the number of components K of $G[T]$ with $b(K) = a(K)$ and $b(K) + |E[K, W]|$ odd is at most $b(U) - a(W) + 2|E[W]| + |E[T, W]|$.

Proof. This is the case $d = \mathbf{0}$, $c = \mathbf{1}$ of Theorem 35.1. ∎

The construction described in the proof of Theorem 35.1 also implies:

Theorem 35.2. *Given a graph $G = (V, E)$, $a, b \in \mathbb{Z}^V$, $d, c \in \mathbb{Z}^E$, and $w \in \mathbb{Q}^E$, a vector $x \in \mathbb{Z}^E$ satisfying $d \leq x \leq c$ and $a(v) \leq x(\delta(v)) \leq b(v)$ for each $v \in V$, and minimizing $w^{\mathsf{T}}x$, can be found in strongly polynomial time.*

Proof. The construction in the proof of Theorem 35.1 reduces this to Theorem 32.4. ∎

35.2. Convex hull

We now characterize the convex hull of the functions $x \in \mathbb{Z}^E$ satisfying (35.1):

Theorem 35.3. *Let $G = (V, E)$ be a graph and let $a, b \in \mathbb{Z}^V$ and $d, c \in \mathbb{Z}^E$ with $a \leq b$ and $d \leq c$. Then the convex hull of the vectors $x \in \mathbb{Z}^E$ satisfying* (35.1) *is determined by* (35.1) *together with the inequalities*

(35.11) $$x(E[U]) - x(E[W]) + x(F \cap \delta(U)) - x(H \cap \delta(W)) \leq \lfloor \tfrac{1}{2}(b(U) - a(W) + c(F) - d(H)) \rfloor,$$

where U and W are two disjoint subsets of V and where F and H partition $\delta(U \cup W)$, with $b(U) - a(W) + c(F) - d(H)$ odd.

Proof. Necessity of (35.11) follows by adding up the following inequalities, each implied by (35.1):

(35.12) $$\begin{aligned} &x(E[U]) + \tfrac{1}{2}x(\delta(U)) \leq \tfrac{1}{2}b(U),\\ &-x(E[W]) - \tfrac{1}{2}x(\delta(W)) \leq -\tfrac{1}{2}a(W),\\ &\tfrac{1}{2}x(F) \leq \tfrac{1}{2}c(F),\\ &-\tfrac{1}{2}x(H) \leq -\tfrac{1}{2}d(H). \end{aligned}$$

The left-hand sides add up to the left-hand side of (35.11), and the right-hand side to the unrounded right-hand side of (35.11).

To see sufficiency of (35.11), we may assume that $d = \mathbf{0}$. Indeed, the theorem is invariant under resetting $a(v) := a(v) - d(\delta(v))$, and $b(v) := b(v) - d(\delta(v))$ for all $v \in V$, and $c := c - d$ and $d := \mathbf{0}$. Then, as above, we can reduce the theorem to Corollary 32.2a characterizing the convex hull of capacitated b-matchings.

Let x satisfy (35.1) and (35.11). Let R, G', b', and c' be as in the proof of Theorem 35.1. Define $x'(e) := x_e$ for each $e \in E$, $x'(vv_0) := b(v) - x(\delta(v))$ for each $v \in R$, and $x'(v_0v_0) := 2x(E)$.

We show that x' belongs to the c'-capacitated perfect b'-matching polytope (with respect to G'). This implies that x belongs to the convex hull of vectors $x \in \mathbb{Z}_+^E$ satisfying (35.1).

By Corollary 32.2a, it suffices to check

(35.13) $$\begin{array}{rll} \text{(i)} & 0 \leq x'(e') \leq c'(e') & (e' \in E'),\\ \text{(ii)} & x'(\delta'(v')) = b'(v') & (v' \in V'),\\ \text{(iii)} & x'(\delta'(U') \setminus F') - x'(F') \geq 1 - c'(F') & \\ & & (U' \subseteq V',\ F' \subseteq \delta'(U') \text{ with}\\ & & b'(U') + c'(F') \text{ odd}). \end{array}$$

(35.13)(i) and (ii) are direct. To see (35.13)(iii), let $U' \subseteq V'$ and $F' \subseteq \delta'(U')$ with $b'(U')+c'(F')$ odd. We may assume that $v_0 \in U'$ (as we can replace U' by its complement, since $b'(V') = 2b(V)$ is even). Let $W := \{v \in V \mid vv_0 \in F\}$ and $U := V \setminus (U' \cup W)$. Let $F := F' \cap E$ and $H := \delta_E(U \cup W) \setminus F$.

Now $b'(U') = b(V) + b(V \setminus (U \cup W))$ and $c'(F') = c(F) + (b - a)(W)$. So $b'(U') + c'(F')$ odd implies that $b(U) - a(W) + c(F)$ is odd. So by (35.11),

(35.14) $$2x(E[U]) - 2x(E[W]) + 2x(F \cap \delta(U)) - 2x(H \cap \delta(W)) \leq b(U) - a(W) + c(F) - 1.$$

Hence

(35.15) $$x'(\delta'(U') \setminus F') - x'(F') = x(H) + \sum_{v \in U}(b(v) - x(\delta(v))) - x(F) - \sum_{v \in W}(b(v) - x(\delta(v)))$$

$$\begin{aligned}&= x(H) + b(U) - 2x(E[U]) - x(\delta(U)) - x(F) - b(W)\\ &+ 2x(E[W]) + x(\delta(W)) = -2x(E[U]) + 2x(E[W]) - 2x(F \cap \delta(U))\\ &+ 2x(H \cap \delta(W)) + (b(U) - b(W))\\ &\geq 1 - b(U) + a(W) - c(F) + (b(U) - b(W))\\ &= 1 - c(F) - b(W) + a(W) = 1 - c'(F'),\end{aligned}$$

proving (35.13)(iii). ∎

The special case $d = \mathbf{0}$, $c = \boldsymbol{\infty}$ was mentioned by Schrijver and Seymour [1977]:

Corollary 35.3a. *Let $G = (V, E)$ be a graph and let $a, b \in \mathbb{Z}_+^V$. Then the convex hull of those $x \in \mathbb{Z}^E$ satisfying*

(35.16) (i) $x_e \geq 0$ for each $e \in E$,
(ii) $a(v) \leq x(\delta(v)) \leq b(v)$ for each $v \in V$,

is determined by (35.16) *together with the inequalities:*

(35.17) $x(E[U]) - x(E[W]) - x(\delta(W) \setminus \delta(U)) \leq \lfloor \frac{1}{2}(b(U) - a(W)) \rfloor,$

where U and W are disjoint subsets of V with $b(U) - a(W)$ odd.

Proof. This is a special case of Theorem 35.3. ∎

Similarly, we can characterize the convex hull of subgraphs with prescribed bounds on the degrees:

Corollary 35.3b. *Let $G = (V, E)$ be a graph and let $a, b \in \mathbb{Z}_+^V$ with $a \leq b$. Then the convex hull of the incidence vectors of subsets F of E satisfying*

(35.18) $a(v) \leq \deg_F(v) \leq b(v)$

for each $v \in V$, is determined by

(35.19) (i) $0 \leq x_e \leq 1$ for each $e \in E$,
(ii) $a(v) \leq x(\delta(v)) \leq b(v)$ for each $v \in V$,

together with the inequalities

(35.20) $x(E[U]) - x(E[W]) + x(F \cap \delta(U)) - x(H \cap \delta(W))$
$\leq \lfloor \frac{1}{2}(b(U) - a(W) + |F|) \rfloor,$

where U and W are disjoint subsets of V and where F and H partition $\delta(U \cup W)$, with $b(U) - a(W) + |F|$ odd.

Proof. Again this is a special case of Theorem 35.3. ∎

We note that for the $V \times E$ incidence matrix M of any graph $G = (V, E)$, any $a, b \in \mathbb{Z}^V$, $d, c \in \mathbb{Z}^E$, and any $k, l \in \mathbb{Z}$ one has:

(35.21)
$$\begin{aligned}&\text{conv.hull}\{x \in \mathbb{Z}^E \mid d \le x \le c, a \le Mx \le b, k \le x(E) \le l\}\\ &= \text{conv.hull}\{x \in \mathbb{Z}^E \mid d \le x \le c, a \le Mx \le b\}\\ &\cap \{x \in \mathbb{R}^E \mid k \le x(E) \le l\}.\end{aligned}$$

This can be proved similarly to Corollary 18.10a.

35.3. Total dual integrality

System (35.1) together with the inequalities (35.11) generally is not TDI (cf. the example in Section 30.5). To obtain a totally dual integral system we should delete the restriction in (35.11) that $b(U) - a(W) + c(F) - d(H)$ be odd. Thus we obtain the system:

(35.22)
$$\begin{array}{rll}\text{(i)} & d(e) \le x_e \le c(e) & (e \in E),\\ \text{(ii)} & a(v) \le x(\delta(v)) \le b(v) & (v \in V),\\ \text{(iii)} & \multicolumn{2}{l}{x(E[U]) - x(E[W]) + x(F \cap \delta(U)) - x(H \cap \delta(W))}\\ & \quad \le \lfloor \frac{1}{2}(b(U) - a(W) + c(F) - d(H))\rfloor & \\ & & (U, W \subseteq V, U \cap W = \emptyset,\ F, H\\ & & \text{partition } \delta(U \cup W)).\end{array}$$

Theorem 35.4. *System* (35.22) *is totally dual integral.*

Proof. Again we may assume $d = \mathbf{0}$. Let R, G', b', and c' be as in the proof of Theorem 35.1. By Theorem 32.3, the following system, in the variable $x' \in \mathbb{R}^{E'}$, is TDI (where $\delta' := \delta_{G'}$ and $E'[U']$ is the set of edges in E' spanned by U'):

(35.23)
$$\begin{array}{rll}\text{(i)} & 0 \le x'(e') \le c'(e') & (e' \in E'),\\ \text{(ii)} & x'(\delta'(v')) = b'(v') & (v' \in V'),\\ \text{(iii)} & x'(E'[U']) + x'(F') \le \lfloor \frac{1}{2}(b'(U') + c'(F'))\rfloor & \\ & & (U' \subseteq V', F' \subseteq \delta'(U')).\end{array}$$

Adding the equality

(35.24)
$$x'(v_0v_0) - x'(E) = 0$$

to (35.23) maintains total dual integrality (since (35.24) is valid for each vector x' satisfying (35.23)).

We can restrict the inequalities (35.23)(iii) to those with $v_0 \notin U'$. To see this, assume $v_0 \in U'$. Define $U := U' \cap V$ and $U'' := V \setminus U'$. Then

(35.25)
$$\begin{aligned}x'(E'[U']) &= x'(v_0v_0) + x'(E'[U]) + \sum_{v \in U \cap R} x'(vv_0)\\ &= x'(E) + x'(E'[U]) + \sum_{v \in U \cap R} x'(vv_0) = x'(E'[U'']) + \sum_{v \in U} x'(\delta'(v))\end{aligned}$$

and

$$\lfloor \tfrac{1}{2}(b'(U') + c'(F')) \rfloor = \lfloor \tfrac{1}{2}(b'(U'') + 2b'(U) + c'(F')) \rfloor = \lfloor \tfrac{1}{2}(b'(U'') + c'(F')) \rfloor + \sum_{v \in U} b'(v), \tag{35.26}$$

since $b'(U') = b'(U) + b'(v_0) = b'(U) + b'(V)$. So the inequality

$$x'(E'[U']) + x'(F') \leq \lfloor \tfrac{1}{2}(b'(U') + c'(F')) \rfloor \tag{35.27}$$

is a sum of

$$x'(E'[U'']) + x'(F') \leq \lfloor \tfrac{1}{2}(b'(U'') + c'(F')) \rfloor \tag{35.28}$$

and of $x'(\delta'(v)) = b'(v)$ for $v \in U$. So we can assume that $v_0 \notin U'$.

Now adding an integer multiple of a valid equality to another constraint, maintains total dual integrality. So using (35.23)(ii) we can replace (35.23)(i) by:

$$\begin{aligned} 0 \leq x'(e) &\leq c(e) && (e \in E), \\ a(v) \leq x'(\delta(v)) &\leq b(v) && (v \in V), \end{aligned} \tag{35.29}$$

since for $v \in R$, subtracting $x'(\delta'(v)) = b(v)$ from $0 \leq x'(vv_0) \leq b(v) - a(v)$ gives $-b(v) \leq -x'(\delta(v)) \leq -a(v)$.

For $U' \subseteq V$ and $F' \subseteq \delta'(U')$, let $W := \{v \in V \mid vv_0 \in F'\}$ and $F := F' \cap E$. As

$$\begin{aligned} & x'(E'[U']) + x'(F') - \sum_{v \in W} x'(\delta'(v)) \\ &= x'(E'[U']) + x'(F') - 2x'(E[W]) - x'(\delta(W)) \\ &= x'(E'[U' \setminus W]) + x'(F') - x'(E[W]) - x'(\delta(W) \setminus \delta(U' \setminus W)) \end{aligned} \tag{35.30}$$

and

$$\lfloor \tfrac{1}{2}(b'(U') + c'(F')) \rfloor - b(W) = \lfloor \tfrac{1}{2}(b(U' \setminus W) - a(W) + c(F)) \rfloor, \tag{35.31}$$

we can replace (35.23)(iii) by (taking $U := U' \setminus W$):

$$x'(E'[U]) + x'(F) - x'(E[W]) - x'(\delta(W) \setminus \delta(U)) \leq \tfrac{1}{2}\lfloor b(U) - a(W) + c(F) \rfloor \text{ for } U, W \subseteq V \text{ with } U \cap W = \emptyset \text{ and for } F \subseteq \delta(U \cup W). \tag{35.32}$$

Each of the variables $x'(vv_0)$ ($v \in R$) and $x'(v_0v_0)$ occurs exactly once in the system, in an equality constraint, with coefficient 1. So we can delete these variables maintaining total dual integrality (Theorem 5.27), and we obtain system (35.22). ∎

As special cases one can derive the total dual integrality of the systems corresponding to $d = \mathbf{0}$, $c = \infty$ and to $d = \mathbf{0}$, $c = \mathbf{1}$ (the subgraph polytope).

35.4. Further results and notes

35.4a. Further results on subgraphs with prescribed degrees

Corollary 35.1b of Lovász [1970c] implies the following. Let $G = (V, E)$ be a graph and let $b, b' \in \mathbb{Z}_+^V$ with $b + b' > \deg_G$. Then E can be partitioned into a simple b-matching and a simple b'-matching if and only if

(35.33) $\quad |E[U, W]| \leq b(U) + b'(W)$

for each pair of disjoint subsets U and W of V.

This corresponds to the case $a < b$ in Corollary 35.1b, by taking $a := \deg_G - b'$. Then there are no components K with $a(K) = b(K)$.

The condition can be equivalently described as:

(35.34) $$\sum_{v \notin U} \max\{0, a(v) - \deg_{G-U}(v)\} \leq b(u)$$

for $U \subseteq V$.

This implies the following result of Lovász [1970c]:

(35.35) Let $G = (V, E)$ be a graph of maximum degree k and let $k_1, k_2 \geq 0$ with $k_1 + k_2 = k + 1$. Then E can be partitioned into a simple k_1-matching and a simple k_2-matching

(since $|E[U, W]| \leq (k_1 + k_2) \min\{|U|, |W|\} \leq k_1|U| + k_2|W|$). A special case is a result noted by Tutte [1978]: for all $0 \leq r \leq k$, each k-regular graph has a subgraph in which each degree belongs to $\{r, r + 1\}$.

Thomassen [1981a] gave the following short direct proof of (35.35). In fact he proved the following extension of (35.35):

(35.36) Let $G = (V, E)$ be a graph in which each vertex has degree k or $k + 1$ and let $1 \leq k' < k$. Then G has a subgraph $G' = (V, E')$ in which each vertex has degree k' or $k' + 1$.

Note that (35.35) follows from this by embedding G into a k-regular graph.

To prove (35.36), it suffices to prove the case $k' = k - 1$. Let U be the set of vertices of degree $k + 1$ in G. We can assume that deleting any edge of G results in a vertex of degree less than k. Hence no two distinct vertices in U are adjacent. There may be loops at the vertices in U; let W be the set of those vertices in U that are not incident with a loop. Since each vertex in W has degree $k+1$ and each vertex in $V \setminus U$ has degree k, by Hall's marriage theorem, G contains a matching M connecting W to $V \setminus U$. Now deleting the edges in M and deleting, for each vertex $v \in U \setminus W$, one of the loops attached at v, gives a graph G' as required.

A 'dual' consequence was noted by Gupta [1978]:

(35.37) Let $G = (V, E)$ be a graph of minimum degree δ and let $\delta_1, \delta_2 \geq 0$ with $\delta_1 + \delta_2 = \delta - 1$. Then E can be partitioned into E_1 and E_2 such that $G_i = (V, E_i)$ has minimum degree at least δ_i for $i = 1, 2$.

Gupta [1978] mentioned that the following direct derivation from Theorem 20.6 was shown to him by C. Berge:

Apply induction on δ_1, the case $\delta_1 = 0$ being trivial. If $\delta_1 > 0$, by the induction hypothesis E can be partitioned into E_1 and E_2 such that $\delta(G_1) \geq \delta_1 - 1$ and

$\delta(G_2) \geq \delta_2 + 1 = \delta - \delta_1$. (Here $G_i = (V, E_i)$ for $i = 1, 2$.) We choose this partition with $|E_2|$ minimal.

Let S be the set of vertices v with $\deg_{E_1}(v) = \delta_1 - 1$. By the minimality of $|E_2|$, S spans no edge of E_2. Let $F := \delta(S) \cap E_2$. So $\deg_F(v) = \deg_{E_2}(v) = \deg_G(v) - \deg_{E_1}(v) \geq \delta - \delta_1 + 1$ for each $v \in S$. Let $p := \delta - \delta_1 + 1 = \delta_2 + 2$. Now by Theorem 20.6, F can be partitioned into $F_1, \ldots, F_p$ such that each vertex v is covered by at least $\min\{p, \deg_F(v)\}$ of the F_i. Then replacing E_1 by $E_1 \cup F_1$ and E_2 by $E_2 \setminus F_1$ gives a partition as required. Indeed, if $\deg_F(v) \geq p$, then $\deg_{F_i}(v) \geq 1$ for each i, implying

(35.38) $$\deg_{E_1 \cup F_1}(v) = \deg_{E_1}(v) + \deg_{F_1}(v) \geq (\delta_1 - 1) + 1 = \delta_1$$

and

(35.39) $$\deg_{E_2 \setminus F_1}(v) \geq \sum_{i=2}^{p} \deg_{F_i}(v) \geq p - 1 = \delta_2 + 1.$$

If $\deg_F(v) < p$, then $v \notin S$ and $\deg_{F_1}(v) \leq 1$, and hence

(35.40) $$\deg_{E_1 \cup F_1}(v) \geq \deg_{E_1}(v) \geq \delta_1$$

and

(35.41) $$\deg_{E_2 \setminus F_1}(v) \geq \deg_{E_2}(v) - 1 \geq (\delta_2 + 1) - 1 = \delta_2.$$

This proves (35.37).

Las Vergnas [1978] showed that if $a \leq \mathbf{1} \leq b$ holds, a simpler condition can be formulated in Corollary 35.1b:

(35.42) for each $U \subseteq V$, the number of odd components K of $G - U$ with $|K| = 1$ and $a(K) = 1$, or with $|K| \geq 3$ and $a(K) = b(K)$ is at most $b(U)$.

Anstee [1985] gave a proof of Lovász's theorem, with an $O(n^3)$-time algorithm to find the subgraph. Heinrich, Hell, Kirkpatrick, and Liu [1990] gave a simplified proof of Lovász's theorem for $a < b$, implying an $O(\sqrt{a(V)}\, m)$-time algorithm.

Lovász [1970c] also characterized the minimum deviation that subsets can have from prescribed lower and upper bounds on the degrees. In fact, he showed the following (where $\alpha_+ := \max\{0, \alpha\}$ for any $\alpha \in \mathbb{R}$): Let $G = (V, E)$ be a graph and let $a, b \in \mathbb{Z}_+^V$ with $a \leq b$. Then the minimum of

(35.43) $$\sum_{v \in V} ((a(v) - \deg_F(v))_+ + (\deg_F(v) - b(v))_+)$$

over $F \subseteq E$ is equal to the maximum value of

(35.44) $a(W) - b(U) - 2|E[W]| - |E[T, W]| +$number of components K of $G[T]$ with $a(K) = b(K)$ and with $a(K) + |E[K, W]|$ odd,

taken over all partitions T, U, W of V.

Let $B : V \to \mathcal{P}(\mathbb{Z}_+)$. The B-matching problem asks for a subgraph H of G such that $\deg_H(v) \in B(v)$ for each $v \in V$. In general, this is NP-complete, even when $B(v) \in \{\{1\}, \{0, 3\}\}$ for each $v \in V$ (Lovász [1972f]).

If $|\mathbb{Z}_+ \setminus B(v)| = 1$ for each vertex v, Lovász [1973a] gave a characterization. Lovász [1972f] investigated the case where $\mathbb{Z}_+ \setminus B(v)$ contains no two consecutive integers, for which Cornuéjols [1988] gave a polynomial-time algorithm, and Sebő [1993b] a good characterization.

For algorithms to find subgraphs of minimum deviation see Hell and Kirkpatrick [1993]. Other work on subgraphs with prescribed degrees includes Berge and Las Vergnas [1978], Shiloach [1981], Kano and Saito [1983], Akiyama and Kano [1985a], Kano [1985,1986], Anstee [1990], Cai [1991], and Li and Cai [1998]. A survey is given by Akiyama and Kano [1985b].

35.4b. Odd walks

Let $G = (V, E)$ be an undirected graph, let $s, t \in V$, and let $l : E \to \mathbb{Q}$. Call a walk *odd* if it has an odd number of edges. Then a shortest odd $s-t$ walk without repeated edges can be found as follows. For each edge e of G, set $d(e) := 0$ and $c(e) := 1$, and add an edge $\tilde{e}$ parallel to e, of length $l(\tilde{e}) := -l(e)$, and define $d(\tilde{e}) := -1$, $c(\tilde{e}) := 0$. Let M be the $V \times E'$ incidence matrix of the extended graph $G' = (V, E')$. Define $b : V \to \mathbb{Z}$ by $b(s) := b(t) := 1$ and $b(v) := 0$ for each $v \in V \setminus \{s, t\}$. Then a shortest odd $s-t$ walk without repeated edges can be found by finding an $x \in \mathbb{Z}^{E'}$ satisfying $d \leq x \leq c$ and $Mx = b$ and minimizing $l^{\mathsf{T}} x$.

So by Theorem 35.2, this can be solved in strongly polynomial time. Better running times were given by Goldberg and Karzanov [1994,1996]: $O(m)$ for finding such an odd $s-t$ walk, $O(nm \log n)$ and $O(nm\sqrt{\log L})$ for finding a shortest such odd $s-t$ walk, strengthened to $O(m \log n)$ and $O(m\sqrt{\log L})$ for nonnegative lengths. (L is the maximum absolute value of the lengths, assuming they are integer.)

Chapter 36

Bidirected graphs

In the previous chapter we considered integer solutions of $d \le x \le c$, $a \le Mx \le b$ where M is the incidence matrix of an undirected graph. Earlier, in Chapter 12, we considered the same problem if M is the incidence matrix of a directed graph.
Edmonds and Johnson [1970] showed that M can more generally be the incidence matrix of a 'bidirected' graph — a structure that comprises both undirected and directed graphs. That is, M has entries 0, ± 1, and ± 2, such that the sum of the absolute values of the entries in any column is equal to 2. The results are obtained by a simple reduction to the undirected case, although the elaboration takes some effort.
The results could be formulated just in terms of matrices, but the graph-theoretic interpretation is helpful in formulating, visualizing, and proving the results.

36.1. Bidirected graphs

A *bidirected graph* is a triple $G = (V, E, \sigma)$, where (V, E) is an undirected graph and where σ assigns to each $e \in E$ and $v \in e$ a 'sign' $\sigma_{e,v} \in \{+1, -1\}$.

If e is a loop, that is, e is a family $\{v, v\}$, we may assign different signs to the two occurrences of v. However, in the problems discussed in this chapter, loops where the signs are different are irrelevant. So we assume that the signs in a loop are the same, either both $+1$, or both -1.

Clearly, undirected graphs and directed graphs can be considered as special cases of bidirected graphs. Graph terminology for the graph (V, E) extends in an obvious way to the bidirected graph (V, E, σ).

Let $G = (V, E, \sigma)$ be a bidirected graph. The edges e for which $\sigma_{e,v} = 1$ for each $v \in e$ are called the *positive edges*, those with $\sigma_{e,v} = -1$ for each $v \in e$ the *negative edges*, and the remaining edges are called the *directed edges*. The $V \times E$ *incidence matrix* of G is the $V \times E$ matrix M defined by:

(36.1) $$M_{v,e} := \sigma_{e,v} \text{ if } e \text{ is not a loop,}$$
$$M_{v,e} := 2\sigma_{e,v} \text{ if } e \text{ is a loop,}$$

setting $\sigma_{e,v} := 0$ if $v \notin e$. It follows that an integer matrix M is the $V \times E$ incidence matrix of a bidirected graph if and only if the sum of the absolute values of the entries in any column of M is equal to 2.

For vectors $a, b \in \mathbb{Z}^V$ and $d, c \in \mathbb{Z}^E$, we consider integer solutions $x \in \mathbb{R}^E$ of

(36.2) (i) $d \leq x \leq c$,
(ii) $a \leq Mx \leq b$.

The related existence and optimization problems can be reduced as follows to the case where G is just an undirected graph. First, we can assume that G has no negative edges, since multiplying the corresponding column of M by -1 gives an equivalent problem. Next, any directed edge $f = su$, with $\sigma_{f,s} = -1$ and $\sigma_{f,u} = +1$, can be handled as follows.

(36.3) Extend G by a new vertex t and replace edge e by two new positive edges st and tu. This makes the bidirected graph $G' = (V', E')$. Define $a', b' \in \mathbb{Z}^{V'}$ by $a'(v) := a(v)$ and $b'(v) := b(v)$ for $v \in V$ and $a'(t) := b'(t) := 0$. Define $d', c' \in \mathbb{Z}^{E'}$ by $d'(e) := d(e)$ and $c'(e) := c(e)$ for $e \in E \setminus \{f\}$, and $d'(st) := -\infty$, $c'(st) := \infty$, $d'(tu) := d(f)$, and $c'(tu) := c(f)$. Let M' be the $V' \times E'$ incidence matrix of G'.

Then there is a one-to-one relation between (integer) solutions of (36.2) and those for the system corresponding to G', M', a', b', c', d': just define $x(tu) := x_f$ and $x(st) := -x_f$.

Algorithmically, this gives a direct reduction to the undirected case:

Theorem 36.1. *For $w \in \mathbb{Q}^E$, an integer vector x maximizing $w^{\mathsf{T}}x$ over* (36.2) *can be found in strongly polynomial time.*

Proof. By multiplying columns of M by -1, we can assume that G has no negative edges. Next, apply (36.3) to each directed edge. This reduces the problem to one on a bidirected graph with all edges positive, that is, on an undirected graph. Hence, the theorem follows from Theorem 35.2. ∎

We next consider characterizations. Let $G = (V, E, \sigma)$ be a bidirected graph. For any $T \subseteq V$, $G[T]$ denotes the bidirected subgraph of G induced by T (that is, $G[T] = (T, E[T], \sigma')$, where σ' is the restriction of σ to pairs e, v with $e \in E[T]$). We set for $U \subseteq V$:

(36.4) $\delta(U) := \delta_E(U)$.

For disjoint $X, Y \subseteq V$, we denote:

(36.5) $E[X, Y^+] := \{e \in \delta(X) \mid \exists v \in Y : \sigma_{e,v} = +1\}$,
$E[X, Y^-] := \{e \in \delta(X) \mid \exists v \in Y : \sigma_{e,v} = -1\}$.

For any vector z, let z_+ be the vector obtained from z by setting each negative entry to 0. Similarly, let z_- be the vector obtained from z by setting each positive entry to 0. So $z = z_+ + z_-$.

In the following theorem the condition that $d < c$ is not really a restriction: if $d_e = c_e$ we know that $x_e = d_e$ and hence we can dispose of e by contracting it appropriately. But if we delete the condition $d < c$, the formulation of the theorem would be more complicated.

Theorem 36.2. *Let $a \leq b$ and $d < c$. Then there exists an integer vector x satisfying* (36.2) *if and only if for each partition T, U, W of V, the number of components K of $G[T]$ with $b(K) = a(K)$ and*

$$b(K)+c(E[K,W^+])+c(E[K,U^-])+d(E[K,U^+])+d(E[K,W^-]) \tag{36.6}$$

odd is at most

$$y_+^\mathsf{T} b + y_-^\mathsf{T} a - (y^\mathsf{T} M)_- c - (y^\mathsf{T} M)_+ d, \tag{36.7}$$

where $y := \chi^U - \chi^W$.

Proof. The validity of the theorem is invariant under multiplying a row v of M by -1 and replacing $b(v)$ by $-a(v)$ and $a(v)$ by $-b(v)$ (then if $v \in U$ we move v to W, and if $v \in W$ we move v to U). Similarly, the validity is invariant under multiplying a column e by -1 and replacing $c(e)$ by $-d(e)$ and $d(e)$ by $-c(e)$.

Hence, to see necessity, we can assume that $W = \emptyset$ and $E[T, U^-] = \emptyset$. So $y_+ = y$ and $y_- = \mathbf{0}$ and $E[T, U^+] = \delta(T)$. Then

$$\begin{aligned}(x-d)(\delta(T)) &= (x-d)(E[T,U^+]) \leq (y^\mathsf{T} M)_+(x-d)\\ &\leq (y^\mathsf{T} M)_+(x-d) - (y^\mathsf{T} M)_-(c-x)\\ &= y^\mathsf{T} Mx - (y^\mathsf{T} M)_+ d - (y^\mathsf{T} M)_- c\\ &= y_+^\mathsf{T} Mx + y_-^\mathsf{T} Mx - (y^\mathsf{T} M)_+ d - (y^\mathsf{T} M)_- c\\ &\leq y_+^\mathsf{T} b + y_-^\mathsf{T} a - (y^\mathsf{T} M)_+ d - (y^\mathsf{T} M)_- c.\end{aligned} \tag{36.8}$$

On the other hand, for each component K of $G[T]$ one has $(x-d)(\delta(K)) \geq 0$. Moreover, if $b(K) = a(K)$ and (36.6) is odd, then $(x-d)(\delta(K))$ is odd, since

$$(x-d)(\delta(K)) \equiv (x-d)(\delta(K)) + 2x(E[K]) \equiv b(K) + d(\delta(K)) \pmod 2 \tag{36.9}$$

So $(x-d)(\delta(T))$ is not less than the number of components K of $G[T]$ with $a(K) = b(K)$ and (36.6) odd, showing necessity of the condition.

To show sufficiency, we can assume that G has no negative edges, since we can multiply columns of M by -1. We show sufficiency by induction on the number of directed edges. If this number is 0, the theorem reduces to Theorem 35.1. So we can assume that there is an edge $f = su$ with $\sigma_{f,s} = -1$ and $\sigma_{f,u} = +1$. Then we apply construction (36.3), to obtain $G' = (V', E'), M', a', b', d', c'$.

Now there exists an integer vector x satisfying $d \leq x \leq c$ and $a \leq Mx \leq b$ if there exists an integer vector x' satisfying $d' \leq x' \leq c'$ and $a' \leq M'x' \leq b'$. So we can assume that no such x' exists. By induction (as G' has fewer directed edges than G), we know that V' can be partitioned into T', U', and W' such that the number of components K' of $G'[T']$ with $b'(K') = a'(K')$ and

$$(36.10)\qquad b'(K') + c'(E'[K', W'^{+}]) + c'(E'[K', U'^{-}]) + d'(E'[K', U'^{+}]) + d'(E'[K', U'^{-}])$$

odd, is more than

$$(36.11)\qquad y'^{\mathsf{T}}_{+}b' + y'^{\mathsf{T}}_{-}a' - (y'^{\mathsf{T}}M')_{-}c' - (y'^{\mathsf{T}}M')_{+}d',$$

where $y' := \chi^{U'} - \chi^{W'}$.

Since $c'(st) = \infty$ we know that $(y'^{\mathsf{T}}M')_{st} \geq 0$, that is, $y'_s + y'_t \geq 0$. Similarly, since $d'(st) = -\infty$ we know that $(y'^{\mathsf{T}}M')_{st} \leq 0$, that is, $y'_s + y'_t \leq 0$. So $y'_s = -y'_t$, and hence either $s \in U'$, $t \in W'$, or $s \in W'$, $t \in U'$, or $s, t \in T'$.

Let $U := U' \cap V$, $W := W' \cap V$, and $T := T' \cap V$. Then for any component K' of $G'[T']$ with $b'(K') = a'(K')$ and (36.10) odd, $K := K' \cap V$ is a component of $G[T]$ with $b(K) = a(K)$ and (36.6) odd. Moreover, (36.11) is equal to (36.7). Hence we have a contradiction with the condition given in the theorem. ∎

36.2. Convex hull

Also the convex hull of the integer solutions of (36.2) can be characterized (where we do not assume $d < c$):[20]

Theorem 36.3. *The convex hull of the integer solutions of* (36.2) *is determined by*

$$(36.12)\qquad \begin{array}{rl} \text{(i)} & d \leq x \leq c, \\ \text{(ii)} & a \leq Mx \leq b, \\ \text{(iii)} & \tfrac{1}{2}((\chi^U - \chi^W)M + \chi^F - \chi^H)x \\ & \quad \leq \lfloor \tfrac{1}{2}(b(U) - a(W) + c(F) - d(H)) \rfloor \\ & \qquad\qquad \text{for } U, W \subseteq V \text{ with } U \cap W = \emptyset, \\ & \qquad\qquad \text{and for partitions } F, H \text{ of } \delta(U \cup W) \\ & \qquad\qquad \text{with } b(U) - a(W) + c(F) - d(H) \text{ odd.} \end{array}$$

Proof. Necessity of (36.12) follows from the facts that $\frac{1}{2}((\chi^U - \chi^W)M + \chi^F - \chi^H)$ is an integer vector and that for each vector x satisfying (36.2) one has $\chi^U Mx \leq \chi^U b = b(U)$, $\chi^W Mx \geq \chi^W a = a(W)$, $\chi^F x \leq \chi^F c = c(F)$, and $\chi^H x \geq \chi^H d = d(H)$.

Again, to show sufficiency, we can assume that G has no negative edges, and we apply induction on the number of directed edges. If this number is 0, the theorem reduces to Theorem 35.3.

So we can assume that there is a directed edge $f = su$. Again, construct $G' = (V', E')$, M', a', b', d', c', as in (36.3). By induction we know that the theorem holds for the new structure.

[20] In order to reduce notation, in this chapter we take incidence vectors χ^U, χ^W, χ^F, and χ^H as *row* vectors.

Let $x \in \mathbb{R}^E$ satisfy (36.12) for G, a, b, c, d. Define $x' \in \mathbb{R}^{E'}$ by $x'(e) := x(e)$ for each $e \in E \setminus \{f\}$, and $x'(st) := -x(f)$ and $x'(tu) := x(f)$. Now it suffices to show that x' satisfies the inequalities for G', a', b', c', d' (since of x' is a convex combination of integer solutions, also x is).

So let U' and W' be disjoint subsets of V' and let F' and H' partition $\delta'(U' \cup W')$, with $b'(U') - a'(W') + c'(F') - d'(H')$ odd. Since $c'(st) = \infty$ and $d'(st) = -\infty$, we know that $st \notin \delta'(U' \cup W')$.

Let $U := U' \cap V$ and $W := W' \cap V$. Moreover, let F and H arise from F' and H' by replacing any occurrence of tu by f. Then

(36.13) $$\tfrac{1}{2}((\chi^{U'} - \chi^{W'})M' + \chi^{F'} - \chi^{H'})x' = \tfrac{1}{2}((\chi^{U} - \chi^{W})M + \chi^{F} - \chi^{H})x$$

since $x'(F') = x(F)$ and $x'(H') = x(H)$, and moreover, $\chi^{U'}M'x' = \chi^U Mx$ and $\chi^{W'}M'x' = \chi^W Mx$ (as $\chi^t M'x' = x'(st) + x'(tu) = 0$).

Also we have

(36.14) $$\begin{aligned}&\lfloor \tfrac{1}{2}(b'(U') - a'(W') + c'(F') - d'(H'))\rfloor\\ &= \lfloor \tfrac{1}{2}(b(U) - a(W) + c(F) - d(H))\rfloor,\end{aligned}$$

as $a'(t) = b'(t) = 0$. Hence (36.12) gives the required inequality for U', W', F', H'. ∎

The special case $a = b$, $d = \mathbf{0}$ was announced by Edmonds and Johnson [1970] and elaborated by Aráoz, Cunningham, Edmonds, and Green-Krótki [1983]. It amounts to, for $b \in \mathbb{Z}_+^V$ and $c \in \mathbb{Z}_+^E$:

(36.15) $$\mathbf{0} \leq x \leq c, Mx = b.$$

Then:

Corollary 36.3a. *The convex hull of the integer solutions of* (36.15) *is determined by* (36.15) *together with the constraints*

(36.16) $$x(\delta(U) \setminus F) - x(F) \geq 1 - c(F)$$

where $U \subseteq V$ *and* $F \subseteq \delta(U)$ *with* $b(U) + c(F)$ *odd.*

Proof. Directly from Theorem 36.3, by replacing Mx by b in (36.12)(iii). ∎

For undirected graphs we obtain a characterization of the capacitated perfect b-matching polytope as special case — cf. Corollary 32.2a.

36.3. Total dual integrality

System (36.12) generally is not totally dual integral (cf. the example in Section 30.5). However, if we delete the parity condition in (36.12)(iii):

(36.17)
$$\begin{array}{rl} \text{(i)} & d \leq x \leq c, \\ \text{(ii)} & a \leq Mx \leq b, \\ \text{(iii)} & \frac{1}{2}((\chi^U - \chi^W)M + \chi^F - \chi^H)x \\ & \quad \leq \lfloor \frac{1}{2}(b(U) - a(W) + c(F) - d(H)) \rfloor \\ & \qquad \text{for } U, W \subseteq V \text{ with } U \cap W = \emptyset \text{ and for partition} \\ & \qquad F, H \text{ of } \delta(U \cup W), \end{array}$$

we obtain a totally dual integral system:

Theorem 36.4. *System* (36.17) *is totally dual integral.*

Proof. Again we can assume that there are no negative edges, and apply induction on the number of directed edges of G. If there is no directed edge, the theorem reduces to Theorem 35.4. If there is a directed edge $f = su$, we again construct $G' = (V', E')$, M', a', b', d', c' as in (36.3).

Let Σ and Σ' be the systems for G, a, b, c, d, and for G', a', b', c', d', respectively. By induction we know that Σ' is totally dual integral. Now the constraint $x'(st) + x'(tu) = 0$ belongs to Σ'. This implies the total dual integrality of the system Σ'' obtained from Σ' by adding an integer multiple of $x'(st)+x'(tu) = 0$ to any other constraint of Σ' so as to make the coefficient of the variable $x'(st)$ equal to 0.

Now deleting the constraints $x'(st) + x'(tu) = 1$ and the variable $x'(st)$ from Σ'', identifying $x'(e) = x(e)$ for all $e \in E \setminus \{f\}$, and identifying $x'(tu) = x(f)$, gives again a totally dual integral system. We show that it is system Σ.

Indeed, $a' \leq M'x' \leq b'$ becomes $a \leq Mx \leq b$. Similarly, for each $e \in E \setminus \{f\}$, $d'(e) \leq x'(e) \leq c'(e)$ becomes $d(e) \leq x_e \leq c(e)$, and $d'(tu) \leq x'(tu) \leq c'(tu)$ becomes $d(f) \leq x(f) \leq c(f)$, while $d'(st) \leq x'(st) \leq c'(st)$ is void (as the bounds are $-\infty$ and $+\infty$).

Consider next the following inequality of Σ':

(36.18)
$$\begin{array}{l} \frac{1}{2}((\chi^{U'} - \chi^{W'})M + \chi^{F'} - \chi^{H'})x' \\ \leq \lfloor \frac{1}{2}(b'(U') - a'(W') + c'(F') - d'(H')) \rfloor, \end{array}$$

where U' and W' are disjoint subsets of V' and where F' and H' partition $\delta'(U' \cup W')$.

Since $c'(st) = \infty$, $d'(s,t) = -\infty$, we know that $st \notin \delta'(U' \cup W')$. Consider the coefficient of $x'(st)$ in (36.18). If this coefficient is 0, (36.18) reduces to (36.17)(iii). If this coefficient is positive, then $s, t \in U'$. Set $U'' := U' \setminus \{t\}$ and $W'' := W' \cup \{t\}$. Then in Σ'', (36.18) becomes (by subtracting $x'(st) + x'(tu) = 0$):

(36.19)
$$\begin{array}{l} \frac{1}{2}((\chi^{U''} - \chi^{W''})M + \chi^{F'} - \chi^{H'})x' \\ \leq \lfloor \frac{1}{2}(b'(U'') - a'(W'') + c'(F') - d'(H')) \rfloor \end{array}$$

(since $b'(t) = a'(t) = 0$). In (36.19), the coefficient of $x'(st)$ is 0, and hence (36.19) reduces to (36.17)(iii).

We proceed similarly if the coefficient of $x'(st)$ in (36.18) is negative. ∎

A consequence is the total dual half-integrality of the original system:

Corollary 36.4a. *System* (36.12) *is totally dual half-integral.*

Proof. This follows from the fact that each inequality in (36.17) is a half-integer nonnegative combination of inequalities in (36.12). ∎

A special case is the total dual half-integrality of

$$\text{(36.20)} \qquad \begin{array}{rll} \text{(i)} & x \geq \mathbf{0}, & \\ \text{(ii)} & Mx = b, & \\ \text{(iii)} & x(\delta(U)) \geq 1 & \text{for each } U \subseteq V \text{ with } b(U) \text{ odd} \end{array}$$

(Edmonds and Johnson [1970]):

Corollary 36.4b. *System* (36.20) *is totally dual half-integral.*

Proof. This is a special case of Corollary 36.4a. ∎

From this one can derive (Barahona and Cunningham [1989]):

Corollary 36.4c. *Let $w \in \mathbb{Z}^E$ with $w(C)$ even for each circuit C. Then the problem of minimizing $w^{\mathsf{T}}x$ subject to* (36.20) *has an integer optimum dual solution.*

Proof. If $w(C)$ is even for each circuit, there is a subset U of V with $\{e \in E \mid w(e) \text{ odd}\} = \delta(U)$. Now replace w by $w' := w + \sum_{v \in U} M_v^{\mathsf{T}}$, where M_v denotes row v of M. Then $w'(e)$ is an even integer for each edge e. Hence by Corollary 36.4b there is an integer optimum dual solution y'_v ($v \in V$), z_U ($U \subseteq V$, $b(U)$ odd) for the problem of minimizing ${w'}^{\mathsf{T}}x$ subject to (36.20). Now setting $y_v := y'_v - 1$ if $v \in U$ and $y_v := y'_v$ if $v \notin U$ gives an integer optimum dual solution for w. ∎

36.4. Including parity conditions

We are not yet at the end of our self-refining trip. As was observed by Edmonds and Johnson [1973], the results can be generalized even further by including parity constraints. This can be reduced to the previous case by adding loops at the vertices at which there is a parity constraint.

Let $G = (V, E, \sigma)$ be a bidirected graph and let M be the $V \times E$ incidence matrix of G. (For definitions and terminology, see Section 36.1.) Let $a, b \in \mathbb{Z}^V$ and $d, c \in \mathbb{Z}^E$ and let S^{odd} and S^{even} be two disjoint subsets of V. We consider integer solutions x of:

(36.21)
$$\begin{array}{rll} \text{(i)} & d \le x \le c, & \\ \text{(ii)} & a \le Mx \le b, & \\ \text{(iii)} & (Mx)_v \text{ is odd} & \text{if } v \in S^{\text{odd}}, \\ \text{(iv)} & (Mx)_v \text{ is even} & \text{if } v \in S^{\text{even}}. \end{array}$$

The problem of finding a maximum-weight integer vector x satisfying (36.21) can be easily reduced to the special case without parity constraints, discussed in the previous chapter:

Theorem 36.5. *For any $w \in \mathbb{Q}^E$, an integer vector x maximizing $w^{\mathsf{T}}x$ over* (36.21) *can be found in strongly polynomial time.*

Proof. The condition $(Mx)_v$ is odd, can be replaced by $1 \le (Mx)_v + 2z_v \le 1$, where z_v is a new integer variable (bounded by $-\infty$ and ∞). Similarly, for the even case. This gives a reduction to the problem of Theorem 36.1, which implies the present theorem. ∎

We next characterize the existence of an integer vector x satisfying (36.21). To this end we make the following assumptions, which can easily be satisfied:

(36.22)
(i) $a(v)$ and $b(v)$ are odd (if finite) for each $v \in S^{\text{odd}}$,
(ii) $a(v)$ and $b(v)$ are even (if finite) for each $v \in S^{\text{even}}$,
(iii) if $a(v) = b(v)$, then $v \in S^{\text{odd}} \cup S^{\text{even}}$.

Define $S := S^{\text{odd}} \cup S^{\text{even}}$. Moreover, for any vector z, again let z_+ arise from z by replacing any negative component by 0, and let z_- arise from z by replacing any positive component by 0. So $z = z_+ + z_-$.

Theorem 36.6. *Assume* (36.22) *and that $d < c$. Then there exists an integer vector $x \in \mathbb{Z}^E$ satisfying* (36.21) *if and only if for each partition T, U, W of V, the number of components K of $G[T]$ contained in $S^{\text{odd}} \cup S^{\text{even}}$ and with*

(36.23)
$$|K \cap S^{\text{odd}}| + c(E[K, W^+]) + c(E[K, U^-]) + d(E[K, U^+]) + d(E[K, W^-])$$

odd is at most

(36.24)
$$y_+^{\mathsf{T}} b + y_-^{\mathsf{T}} a - (y^{\mathsf{T}} M)_- c - (y^{\mathsf{T}} M)_+ d,$$

where $y := \chi^U - \chi^W$.

Proof. Define $L := \{v \in S \mid a(v) < b(v)\}$, $L^{\text{odd}} := L \cap S^{\text{odd}}$, and $L^{\text{even}} := L \cap S^{\text{even}}$.

Extend the bidirected graph G by a loop l at any vertex $v \in L$, where l has two positive ends at v. This makes the bidirected graph $G' = (V, E', \sigma')$, with $V \times E'$ incidence matrix M'. Define $a'(v) := a(v)$ and $b'(v) := b(v)$ for each $v \in V \setminus L$. Moreover, $a'(v) := b'(v) := 1$ for $v \in L^{\text{odd}}$ and $a'(v) := b'(v) := 0$

for $v \in L^{\text{even}}$. Define $d'(e) := d(e)$ and $c'(e) := c(e)$ for each $e \in E$. For each loop l at $v \in L$, define $d'(l) := \frac{1}{2}(b'(v) - b(v))$ and $c'(l) := \frac{1}{2}(a'(v) - a(v))$.

Now there exist an integer vector x satisfying (36.21) if and only if there exists an integer vector $x' \in \mathbb{Z}^{E'}$ satisfying $d' \leq x' \leq c'$ and $a' \leq M'x' \leq b'$. So we should show that the conditions given in the present theorem imply those given in Theorem 36.2 (for the modified structure). (Since in Theorem 36.2 the condition $d < c$ is required, we had to exclude loops at vertices in $S \setminus L$.)

To this end, let T, U, W partition V. Then any component K of $G'[T]$ with $b'(K) = a'(K)$ and

(36.25) $$b'(K) + c'(E'[K, W^+]) + c'(E'[K, U^-]) + d'(E'[K, U^+]) + d'(E[K, W^-])$$

odd, is a component of $G[T]$ contained in S, with $|K \cap S^{\text{odd}}| + c(E[K, W^+]) + c(E[K, U^-]) + d(E[K, U^+]) + d(E[K, W^-])$ odd (note that $a'(v) = b'(v) \iff v \in S$, and that $b'(K) \equiv |K \cap S^{\text{odd}}| \bmod 2$). Moreover, for $y := \chi^U - \chi^W$ one has

(36.26) $$\begin{aligned} & y_+^{\mathsf{T}} b' + y_-^{\mathsf{T}} a' - (y^{\mathsf{T}} M')_- c' - (y^{\mathsf{T}} M')_+ d' \\ & = y_+^{\mathsf{T}} b + y_-^{\mathsf{T}} a - (y^{\mathsf{T}} M)_- c - (y^{\mathsf{T}} M)_+ d, \end{aligned}$$

since

(36.27) $$\begin{aligned} & y_+^{\mathsf{T}} b' = b'(U) = b(U \setminus L) + |U \cap L^{\text{odd}}|, \\ & y_-^{\mathsf{T}} a' = -a'(W) = -a(W \setminus L) - |W \cap L^{\text{odd}}|, \\ & (y^{\mathsf{T}} M')_- c' = (y^{\mathsf{T}} M)_- c - 2(\tfrac{1}{2}(a'(W \cap L) - a(W \cap L))) \\ & = (y^{\mathsf{T}} M)_- c - |W \cap L^{\text{odd}}| + a(W \cap L), \\ & (y^{\mathsf{T}} M')_+ d' = (y^{\mathsf{T}} M)_+ d + 2(\tfrac{1}{2}(b'(U \cap L) - b(U \cap L))) \\ & = (y^{\mathsf{T}} M)_+ d + |U \cap L^{\text{odd}}| - b(U \cap L). \end{aligned}$$ ∎

A special case is the following result on orientations by Frank, Tardos, and Sebő [1984].

Corollary 36.6a. *Let $G = (V, E)$ be an undirected graph and let $l, u \in \mathbb{Z}_+^V$ be such that $l(v) \equiv u(v)$ (mod 2) for each $v \in V$. Then G has an orientation $D = (V, A)$ such that*

(36.28) $$l(v) \leq \deg_D^{\text{out}}(v) \leq u(v) \text{ and } \deg_D^{\text{out}}(v) \equiv u(v) \pmod 2$$

for each $v \in V$ if and only if for each partition T, U, W of V, the number of components K of $G[T]$ with $u(K) + |E[K]| + |E[K, U]|$ odd is at most

(36.29) $$u(U) - l(W) - |E[U]| + |E[W]| + |\delta(W)|.$$

Proof. Let $D' = (V, A')$ be an arbitrary orientation of G. Let $\delta^{\text{out}}(U) := \delta_{D'}^{\text{out}}(U)$ and $\delta^{\text{in}}(U) := \delta_{D'}^{\text{in}}(U)$ for any $U \subseteq V$.

Then G has an orientation as required in the theorem if and only if there exists a vector $x \in \mathbb{Z}^{A'}$ with $\mathbf{0} \leq x \leq \mathbf{1}$ and

$$l(v) \leq x(\delta^{\text{in}}(v)) + |\delta^{\text{out}}(v)| - x(\delta^{\text{out}}(v)) \leq u(v) \tag{36.30}$$

and

$$x(\delta^{\text{in}}(v)) + |\delta^{\text{out}}(v)| - x(\delta^{\text{out}}(v)) \equiv u(v) \pmod 2 \tag{36.31}$$

for each $v \in V$. (This can be seen by reversing the orientation if and only if $x_a = 1$.)

Define for each $v \in V$,

$$a(v) := l(v) - |\delta^{\text{out}}(v)| \text{ and } b(v) := u(v) - |\delta^{\text{out}}(v)|. \tag{36.32}$$

Moreover, let $d, c \in \mathbb{Z}^{A'}$ with $d = \mathbf{0}$ and $c = \mathbf{1}$. Let M be the $V \times A'$ incidence matrix of D' (such that $M_{v,a} = -1$ if a leaves v and $M_{v,a} = +1$ if a enters v). Let S^{odd} and S^{even} be the sets of vertices v with $b(v)$ odd and even, respectively.

Then the existence of an orientation as required is equivalent the existence of an integer vector x satisfying (36.21). Hence, by Theorem 36.6, it is equivalent to the condition that for each partition T, U, W of V the number of components K of $G[T]$ with (for the bidirected graph $G = (V, E, \sigma)$ obtained from M):

$$b(K) + |E[K, W^+]| + |E[K, U^-]| \tag{36.33}$$

odd is at most

$$u(U) - \sum_{v \in U} |\delta^{\text{out}}(v)| - l(W) + \sum_{v \in W} |\delta^{\text{out}}(v)| + |\delta^{\text{out}}(U)| + |\delta^{\text{in}}(W)|. \tag{36.34}$$

Now (36.33) is equal to

$$\begin{aligned} &u(K) - \sum_{v \in K} |\delta^{\text{out}}(v)| + |E[K, W^+]| + |E[K, U^-]| \\ &= u(K) - |E[K]| + |\delta^{\text{out}}(K)| + |E[K, W^+]| + |E[K, U^-]| \\ &\equiv u(K) - |E[K]| + |\delta^{\text{out}}(K)| + |E[K, W^+]| + 2|E[K, U^+]| \\ &+ |E[K, U^-]| \equiv u(K) + |E[K]| + |E[K, U]| \pmod 2, \end{aligned} \tag{36.35}$$

since $|\delta^{\text{out}}(K)| = |E[K, U^+]| + |E[K, W^+]|$ and $|E[K, U]| = |E[K, U^+]| + |E[K, U^-]|$. Moreover, (36.34) is equal to (36.29), proving the corollary. ∎

One can similarly derive the following two further orientation results of Frank, Tardos, and Sebő [1984].

Corollary 36.6b. *Let $G = (V, E)$ be an undirected graph and let $u \in \mathbb{Z}_+^V$. Then G has an orientation $D = (V, A)$ such that*

$$\deg_D^{\text{out}}(v) \leq u(v) \text{ and } \deg_D^{\text{out}}(v) \equiv u(v) \pmod 2 \tag{36.36}$$

for each $v \in V$ if and only if for each $U \subseteq V$ the number of components K of $G - U$ with $u(K) + |E[K]| + |\delta(K)|$ odd is at most $u(U) - |E[U]|$.

Proof. Similar to the proof of Corollary 36.6a. ∎

Corollary 36.6c. *Let $G = (V, E)$ be an undirected graph and let $l \in \mathbb{Z}_+^V$. Then G has an orientation $D = (V, A)$ such that*

$$(36.37) \qquad \deg_D^{\text{out}}(v) \geq l(v) \text{ and } \deg_D^{\text{out}}(v) \equiv l(v) \pmod 2$$

for each $v \in V$ if and only if for each $U \subseteq V$ the number of components K of $G - U$ with $l(K) + |E[K]| + |\delta(K)|$ odd is at most $|E[U]| + |\delta(U)| - l(U)$.

Proof. Similar to the proof of Corollary 36.6a. ∎

36.5. Convex hull

The convex hull of the integer solutions of (36.21) is characterized by:

Theorem 36.7. *Assuming* (36.22)*, the convex hull of the integer solutions of* (36.21) *is determined by* (36.21)(i) *and* (ii)*, together with the constraints*

$$(36.38) \qquad \tfrac{1}{2}((\chi^U - \chi^W)M + \chi^F - \chi^H)x \leq \lfloor \tfrac{1}{2}(b(U) - a(W) + c(F) - d(H)) \rfloor,$$

where U and W are disjoint subsets of $V \setminus S$ and where F and H partition $\delta(U \cup W \cup R)$ for some $R \subseteq S$ with $|R \cap S^{\text{odd}}| + b(U) - a(W) + c(F) - d(H)$ odd.

Proof. To see necessity of (36.38), let x be an integer vector satisfying (36.21), and choose U, W, R, F and H as described in the theorem. As x satisfies $d \leq x \leq c$ and $a \leq Mx \leq b$ one directly has $((\chi^U - \chi^W)M + \chi^F - \chi^H)x \leq b(U) - a(W) + c(F) - d(H)$. So it suffices to show that strict inequality holds. Now $(\chi^U + \chi^W + \chi^R)M + \chi^F + \chi^H$ is an even vector. So (using (36.21)(iii) and (iv))

$$(36.39) \qquad \begin{aligned} &((\chi^U - \chi^W)M + \chi^F - \chi^H)x \equiv \chi^R Mx \equiv |R \cap S^{\text{odd}}| \\ &\not\equiv b(U) - a(W) + c(F) - d(H) \pmod 2 \end{aligned}$$

This shows strict inequality.

We next show that (36.38) determines the convex hull, by reduction to Theorem 36.3. Let L, L^{odd}, L^{even}, $G' = (V, E')$, M', a', b', d', c' be as in the proof of Theorem 36.6. Let $x \in \mathbb{R}^E$ satisfy (36.21)(i) and (ii) and all constraints (36.38). Define $x \in \mathbb{R}^{E'}$ by $x'(e) := x(e)$ for each $e \in E$, and $x'(l) := a'(v) - x(\delta(v))$ for the loop l at any $v \in L$. Then $d' \leq x' \leq c'$ and $a' \leq M'x' \leq b'$. It suffices to show that x' is a convex combination of integer solutions of this system. By Theorem 36.3, it suffices to check condition (36.12)(iii) for G', x'.

Let U' and W' be disjoint subsets of V and let F and H partition $\delta'(U' \cup W')$, with $b'(U') - a'(W') + c'(F) - d'(H)$ odd. Define $U := U' \setminus S$, $W := W' \setminus S$, and $R := (U' \cup W') \cap S$. Then $|R \cap S^{\text{odd}}| + b(U) - a(W) + c(F) - d(H)$ is odd, since $|R \cap S^{\text{odd}}| \equiv b'(U' \cap S) - a'(W' \cap S) \pmod 2$. Moreover,

$$\begin{aligned}(36.40)\qquad &\chi^{U'} M'x' = \chi^{U} Mx + b(U' \cap (S \setminus L)) + |U' \cap L^{\text{odd}}|,\\ &\chi^{W'} M'x' = \chi^{W} Mx + a(W' \cap (S \setminus L)) + |W' \cap L^{\text{odd}}|,\\ &b'(U') = b(U) + b(U' \cap (S \setminus L)) + |U' \cap L^{\text{odd}}|, \text{ and}\\ &a'(W') = a(W) + a(W' \cap (S \setminus L)) + |W' \cap L^{\text{odd}}|.\end{aligned}$$

Hence (36.38) for x implies (36.12)(iii) for x'. ∎

36.5a. Convex hull of vertex-disjoint circuits

Green-Krótki [1980] and Aráoz, Cunningham, Edmonds, and Green-Krótki [1983] showed that the previous theorem implies a characterization of the convex hull of disjoint sets of circuits:

Corollary 36.7a. *Let $G = (V, E)$ be a graph. Then the convex hull of the vectors χ^F where F is the edge set of the union of a number of vertex-disjoint circuits is given by:*

$$\begin{array}{llll}(36.41) & \text{(i)} & 0 \leq x_e \leq 1 & (e \in E),\\ & \text{(ii)} & x(\delta(v)) \leq 2 & (v \in V),\\ & \text{(iii)} & x(\delta(U) \setminus F) - x(F) \geq 1 - |F| & (U \subseteq V,\ F \subseteq \delta(U), |F| \text{ odd}).\end{array}$$

Proof. This follows directly from Theorem 36.7, since x is an incidence vector χ^F of the edge set of a vertex-disjoint union of disjoint circuits if and only if (36.41)(i) and (ii) are satisfied, together with: $x(\delta(v))$ even for each $v \in V$. So we can take $a = \mathbf{0}$, $b = \mathbf{2}$, $d = \mathbf{0}$, $c = \mathbf{1}$, $S^{\text{even}} = V$, and $S^{\text{odd}} = \emptyset$. In particular, U and W are empty in (36.38). ∎

Note that Corollaries 29.2e and 36.7a imply that the polytope described in Corollary 36.7a is obtained from the $\emptyset$-join polytope by adding the constraint (36.41)(ii).

This has as consequence Corollary 29.2f (due to Seymour [1979b]) characterizing the circuit cone. Given a graph $G = (V, E)$, the *circuit cone* is the cone in $\mathbb{R}^E$ generated by the incidence vectors of circuits. This cone is determined by:

$$\begin{array}{llll}(36.42) & \text{(i)} & x_e \geq 0 & \text{for each } e \in E,\\ & \text{(ii)} & x(D) \geq 2x_e & \text{for each cut } D \text{ and } e \in D.\end{array}$$

To prove this, we may assume (by scaling) that $x(E) \leq 1$. Then (36.42)(ii) implies (36.41)(iii), and hence the characterization follows from Corollary 36.7a.

36.6. Total dual integrality

We finally show that the system given by (36.21)(i) and (ii) and (36.38) after deleting the parity constraint on R, is TDI:

Theorem 36.8. *Assuming* (36.22)*, the following system is TDI (setting $T := V \setminus S$):*

(36.43)
(i) $d \leq x \leq c$,
(ii) $\frac{1}{2}a_v \leq \frac{1}{2}(Mx)_v \leq \frac{1}{2}b_v$, for $v \in S$,
(iii) $a_v \leq (Mx)_v \leq b_v$, for $v \in T$,
(iv) $\frac{1}{2}((\chi^U - \chi^W)M + \chi^F - \chi^H)x \leq \frac{1}{2}(b(U) - a(W) + c(F) - d(H) - \varepsilon)$,

where U and W are disjoint subsets of T, where F and H partition $\delta(U \cup W \cup R)$ for some $R \subseteq S$, and where $\varepsilon \in \{0,1\}$ such that $\varepsilon \equiv |R \cap S^{\mathrm{odd}}| + b(U) - a(W) + c(F) - d(H)$ (mod 2).

Proof. The partition of V into S and T induces a partition of M, a, b into M_S, a_S, b_S and M_T, a_T, b_T. By Theorem 36.4, the system

(36.44)
(i) $d \leq x \leq c$,
(ii) $0 \leq z \leq \frac{1}{2}(b_S - a_S)$,
(iii) $M_S x + 2z = b_S$,
(iv) $a_T \leq M_T x \leq b_T$

becomes TDI by adding the inequalities

(36.45)
$$\frac{1}{2}((\chi^U - \chi^W)M + \chi^F - \chi^H)x + z(U \cap S) - z(W \cap S) \leq \lfloor \frac{1}{2}(b(U) - b(W \cap S) - a(W \cap T) + c(F) - d(H)) \rfloor,$$

for disjoint subsets U, W of V and partitions F, H of $\delta(U \cup W)$. (36.44) is equivalent to:

(36.46)
$$\frac{1}{2}((\chi^U - \chi^W)M + \chi^F - \chi^H)x + z(U \cap S) - z(W \cap S) \leq \frac{1}{2}(b(U) - b(W \cap S) - a(W \cap T) + c(F) - d(H) - \varepsilon),$$

where $\varepsilon \in \{0,1\}$ and

(36.47)
$$\varepsilon \equiv b(U) - b(W \cap S) - a(W \cap T) + c(F) - d(H) \text{ (mod 2)}.$$

Substituting $z := \frac{1}{2}(b_S - M_S x)$ in (36.44)(ii) gives (36.43)(ii), and in (36.46) gives

(36.48)
$$\frac{1}{2}((\chi^U - \chi^W)M + \chi^F - \chi^H)x + \frac{1}{2}b(U \cap S) - \frac{1}{2}\chi^{U \cap S} M_S x - \frac{1}{2}b(W \cap S) + \frac{1}{2}\chi^{W \cap S} M_S x \leq \frac{1}{2}(b(U) - b(W \cap S) - a(W \cap T) + c(F) - c(H) - \varepsilon).$$

Equivalently:

(36.49)
$$\frac{1}{2}((\chi^{U \cap T} - \chi^{W \cap T})M + \chi^F - \chi^H)x \leq \frac{1}{2}(b(U \cap T) - a(W \cap T) + c(F) - d(H) - \varepsilon).$$

This is equivalent to (36.43)(iv), and total dual integrality is maintained by Theorem 5.27. Note that

(36.50)
$$\begin{aligned} \varepsilon &\equiv b(U) - b(W \cap S) - a(W \cap T) + c(F) - d(H) \\ &\equiv b(U \cap T) - a(W \cap T) + c(F) - d(H) + b(U \cap S) + b(W \cap S) \\ &\equiv b(U \cap T) - a(W \cap T) + c(F) - d(H) + |R \cap S^{\mathrm{odd}}| \text{ (mod 2)}, \end{aligned}$$

where $R := (U \cup W) \cap S$. ∎

We remark that the coefficients of the inequalities in (36.43) generally are not all integer.

36.7. Further results and notes

36.7a. The Chvátal rank

The results on the convex hull in this chapter (and in previous chapters) can be interpreted in terms of the so-called 'Chvátal rank' of a system of inequalities or of a matrix. (This relates to the cutting planes reviewed in Section 5.21.)

For any polyhedron P, let P_{I} denote the *integer hull* of P, that is, the convex hull of the integer vectors in P. If P is a rational polyhedron, then P_{I} is again a rational polyhedron. This polyhedron can be approached as follows.

Define for any polyhedron P, the set P' by:

(36.51) $$P' := \bigcap_{H \supseteq P} H_{\mathrm{I}},$$

where H ranges over all rational affine halfspaces containing P as a subset. Here an *affine halfspace* is a set of the form

(36.52) $$H = \{x \in \mathbb{R}^n \mid w^{\mathsf{T}} x \leq \alpha\}$$

for some nonzero $w \in \mathbb{R}^n$ and some $\alpha \in \mathbb{R}$. It is *rational* if w and α are rational. So trivially (since $P \subseteq H \Rightarrow P_{\mathrm{I}} \subseteq H_{\mathrm{I}}$):

(36.53) $$P \supseteq P' \supseteq P_{\mathrm{I}}.$$

Note that if H is as in (36.52) and w is integer, with relatively prime components, then

(36.54) $$H_{\mathrm{I}} = \{x \in \mathbb{R}^n \mid w^{\mathsf{T}} x \leq \lfloor \alpha \rfloor\}.$$

So P' arises from P by adding a 'first round of cuts'. Observe that if $P = \{x \mid Mx \leq b\}$ for some rational $m \times n$ matrix M and some vector $b \in \mathbb{Q}^m$, then in (36.51) we can restrict the affine hyperplanes H to those for which there exists a vector $y \in \mathbb{Q}^m_+$ with $y^{\mathsf{T}} M$ integer and nonzero and

(36.55) $$H = \{x \mid (y^{\mathsf{T}} M) x \leq y^{\mathsf{T}} b\}$$

(by Farkas' lemma).

It can be shown that P' is a rational polyhedron again. To P' we can apply this operation again, and obtain $P'' = (P')'$. We thus obtain a series of polyhedra P, P', P'',$\ldots$, $P^{(t)}$, $\ldots$, satisfying

(36.56) $$P \supseteq P' \supseteq P'' \supseteq \cdots \supseteq P^{(t)} \supseteq \cdots P_I.$$

Now Chvátal [1973a] (cf. Schrijver [1980b]) showed that for each polyhedron P there is a finite t with $P^{(t)} = P_{\mathrm{I}}$. The smallest such t is called the *Chvátal rank* of P.

It can be proved more strongly (Cook, Gerards, Schrijver, and Tardos [1986]) that for each rational matrix M there is a finite value t such that the polyhedron $P := \{x \mid Mx \leq b\}$ has Chvátal rank at most t, for each integer vector b (of appropriate dimension). The smallest such t is called the *Chvátal rank* of M. So each totally unimodular matrix has Chvátal rank 0.

The *strong Chvátal rank* of M is, by definition, the Chvátal rank of the matrix

(36.57) $$\begin{pmatrix} I \\ -I \\ M \\ -M \end{pmatrix}.$$

So the strong Chvátal rank of M is the smallest t such that for all integer vectors d, c, a, b the polyhedron $\{x \mid d \leq x \leq c, a \leq Mx \leq b\}$ has Chvátal rank at most t. So M is totally unimodular if and only if M is integer and has strong Chvátal rank 0 (this is the Hoffman-Kruskal theorem).

Theorem 36.3 implies that the $V \times E$ incidence matrix of a bidirected graph has strong Chvátal rank at most 1. (Matrices of strong Chvátal rank at most 1 are said in Gerards and Schrijver [1986] to have the *Edmonds-Johnson property*.)

Theorem 36.9. *The $V \times E$ incidence matrix of a bidirected graph has strong Chvátal rank at most* 1.

Proof. We must show that for each integer d, c, a, b, one has $P' = P_{\mathrm{I}}$ for $P := \{x \mid d \leq x \leq c, a \leq Mx \leq b\}$. This follows from

(36.58) $$P' \subseteq \{x \in P \mid \forall y \in \{0, \tfrac{1}{2}\}^n : y^{\mathsf{T}}M \in \mathbb{Z}^n \Rightarrow y^{\mathsf{T}}Mx \leq \lfloor y^{\mathsf{T}}b \rfloor\} \\ = P_{\mathrm{I}} \subseteq P',$$

where the equality follows from Theorem 36.3. ∎

It is generally not true that also the *transpose* M^{T} of these matrices have Chvátal rank at most 1, as is shown by the incidence matrix M of the complete graph K_4. In Section 68.6c we shall study the Chvátal rank of such matrices M^{T}.

36.7b. Further notes

Gabow [1983a] gave an $O(m^{\frac{3}{2}})$-time algorithm for finding a maximum $s-t$ flow in a bidirected graph with unit capacities. Moreover, he gave $O(m^2 \log n)$- and $O(n^2 m)$-time algorithms for finding a minimum-cost bidirected $s-t$ flow of given value, with unit capacities.

Chapter 37

The dimension of the perfect matching polytope

In this chapter the dimension of the perfect matching polytope is characterized. It implies a characterization of the dimension of the perfect matching space — the linear space spanned by the incidence vectors of perfect matchings.
The basis of determining the dimension is formed by the matching-covered graphs without nontrivial tight cuts. For such graphs, there is an easy formula for the dimension.
Key result (needed in characterizing the perfect matching lattice in the next chapter) is a characterization of Lovász of the matching-covered graphs without nontrivial tight cuts: the 'braces' and the 'bricks'.

37.1. The dimension of the perfect matching polytope

Naddef [1982] gave a min-max formula for the dimension of the perfect matching polytope. By the work of Edmonds, Lovász, and Pulleyblank [1982], it is equivalent to the following.

Let $G = (V, E)$ be a graph and let E_0 be the set of edges covered by at least one perfect matching. Defining $G_0 := (V, E_0)$, one trivially has:

$$\dim(P_{\text{perfect matching}}(G)) = \dim(P_{\text{perfect matching}}(G_0)). \tag{37.1}$$

So when investigating the dimension of the perfect matching polytope, we can confine ourselves to *matching-covered* graphs, that is, to graphs in which each edge is contained in at least one perfect matching.

A further reduction can be obtained by considering tight cuts. A cut C is called *odd* if $C = \delta(U)$ for some $U \subseteq V$ with $|U|$ odd. A cut C is called *tight* if it is odd and each perfect matching intersects C in exactly one edge.

Let $G = (V, E)$ be a graph and let $U \subseteq V$. Recall that G/U denotes the graph obtained from G by contracting U to one vertex, which vertex we will call U. In the obvious way, we will consider the edge set of G/U as a subset of the edge set of G. Hence, for any $x \in \mathbb{R}^E$, we can speak of the *projection* of x to the edges of G/U.

Theorem 37.1. *Let $G = (V, E)$ be a matching-covered graph and let $\delta(U)$ be a tight cut. Define $G_1 := G/U$ and $G_2 := G/\overline{U}$ (where $\overline{U} := V \setminus U$). Then*

(37.2) $$\dim(P_{\text{perfect matching}}(G)) = \dim(P_{\text{perfect matching}}(G_1)) + \dim(P_{\text{perfect matching}}(G_2)) - |\delta(U)| + 1.$$

Proof. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. Then a vector $x \in \mathbb{R}^E$ belongs to the perfect matching polytope of G if and only if its projections to E_1 and E_2 belong to the perfect matching polytopes of G_1 and G_2 respectively. Moreover, since G is matching-covered and since $\delta(U)$ is tight, the projection of $P_{\text{perfect matching}}(G)$ on $\delta(U)$ has dimension equal to $|\delta(U)| - 1$. ∎

This theorem gives a reduction if there exists a nontrivial tight cut. (A cut C is called *nontrivial* if $C = \delta(U)$ for some U with $1 < |U| < |U| - 1$.) Then:

Theorem 37.2. *Let $G = (V, E)$ be a matching-covered graph without any nontrivial tight cut and with at least one perfect matching. Then*

(37.3) $$\dim(P_{\text{perfect matching}}(G)) = |E| - |V| + k,$$

where k is the number of bipartite components of G.

Proof. We may assume that G is connected. If G is bipartite, the result follows from Theorem 18.6. If G is nonbipartite, consider a vector x in the relative interior of the perfect matching polytope of G. Since G is matching-covered, we know that $x_e > 0$ for each edge e, and since G has no nontrivial tight cut, we know that $x(C) > 1$ for each nontrivial odd cut C. Hence the only constraints in (25.2) satisfied by x with equality are the constraints $x(\delta(v)) = 1$ for $v \in V$. So $\dim(P_{\text{perfect matching}}(G)) \geq |E| - |V|$.

To see equality, we show that the constraints $x(\delta(v)) = 1$ are independent. For let $u \in V$, and choose an odd-length $u - u$ walk $(u, e_1, \ldots, e_t, u)$. For each $e \in E$, let x_e be the number of odd i with $e = e_i$, minus the number of even i with $e = e_i$. Then $x(\delta(u)) = 2$ and $x(\delta(v)) = 0$ for all $v \neq u$. ∎

Theorems 37.1 and 37.2 describe the decomposition of the dimension problem. We now aggregate these results.

For any cut C, any set U with $C = \delta(U)$ is called a *shore* of C. Two cuts C and C' are called *cross-free* if they have shores U and U' that are disjoint. A collection $\mathcal{F}$ of cuts is *cross-free* if each two cuts in $\mathcal{F}$ are cross-free.

Let $\mathcal{F}$ be a cross-free collection of nontrivial cuts. An *$\mathcal{F}$-contraction* of G is a graph obtained from G by choosing a $U_0 \subseteq V$ with $\delta(U_0) \in \mathcal{F}$, contracting U_0, and contracting each maximal proper subset U of $V \setminus U_0$ with $\delta(U) \in \mathcal{F}$.

One easily checks that, if G is connected, there exist precisely $|\mathcal{F}| + 1$ $\mathcal{F}$-contractions. Let $\text{nonbip}_G(\mathcal{F})$ denote the number of $\mathcal{F}$-contractions that are nonbipartite.

Corollary 37.2a. *Let $G = (V, E)$ be a connected matching-covered graph with $|V| \geq 2$. Let $\mathcal{F}$ be any inclusionwise maximal cross-free collection of nontrivial tight cuts. Then*

(37.4) $$\dim(P_{\text{perfect matching}}(G)) = |E| - |V| - \text{nonbip}_G(\mathcal{F}) + 1.$$

Proof. The corollary follows directly by induction from Theorems 37.1 and 37.2, as follows.

If $\mathcal{F} = \emptyset$, then (37.4) follows from Theorem 37.2. If $\mathcal{F} \neq \emptyset$, choose a cut $\delta(U) \in \mathcal{F}$. Let $G_1 := G/U$ and $G_2 := G/\overline{U}$ (where $\overline{U} := V \setminus U$). Then G_1 and G_2 are connected and matching-covered again.

Let $\mathcal{F}_1$ be the set of cuts in $\mathcal{F}$ that have a shore properly contained in $V \setminus U$ and let $\mathcal{F}_2$ be the set of cuts in $\mathcal{F}$ that have a shore properly contained in U.

Then $\mathcal{F}_1$ forms an inclusionwise maximal cross-free collection of nontrivial tight cuts in G_1. So inductively

(37.5) $$\dim(P_{\text{perfect matching}}(G_1)) = |EG_1| - |VG_1| - \text{nonbip}_{G_1}(\mathcal{F}_1) + 1.$$

Similarly,

(37.6) $$\dim(P_{\text{perfect matching}}(G_2)) = |EG_2| - |VG_2| - \text{nonbip}_{G_2}(\mathcal{F}_2) + 1.$$

Now each $\mathcal{F}$-contraction of G is an $\mathcal{F}_i$ contraction of G_i for exactly one $i \in \{1, 2\}$. Hence

(37.7) $$\text{nonbip}_G(\mathcal{F}) = \text{nonbip}_{G_1}(\mathcal{F}_1) + \text{nonbip}_{G_2}(\mathcal{F}_2).$$

Since moreover $|EG| = |EG_1| + |EG_2| - |\delta(U)|$ and $|VG_1| + |VG_2| = |VG| + 2$, we obtain (37.4) with Theorem 37.1. ∎

37.2. The perfect matching space

We derive from Corollary 37.2a a characterization of the perfect matching space and its dimension. The *perfect matching space* of a graph $G = (V, E)$ is the linear hull of the incidence vectors of perfect matchings; that is,

(37.8) $$S_{\text{perfect matching}}(G) := \text{lin.hull}\{\chi^M \mid M \text{ perfect matching in } G\}.$$

(Here lin.hull denotes linear hull.)

Corollary 37.2a directly gives for the dimension of the perfect matching space:

Corollary 37.2b. *Let $G = (V, E)$ be a connected matching-covered graph with $|V| \geq 2$. Let $\mathcal{F}$ be any inclusionwise maximal cross-free collection of nontrivial tight cuts. Then*

(37.9) $$\dim(S_{\text{perfect matching}}(G)) = |E| - |V| - \text{nonbip}_G(\mathcal{F}) + 2.$$

Proof. The dimension of the perfect matching space is 1 more than the dimension of the perfect matching polytope. So the corollary follows from Corollary 37.2a. ∎

With the help of the description of the perfect matching polytope we can similarly describe the perfect matching space in terms of equations:

Theorem 37.3. *The perfect matching space of a graph $G = (V, E)$ is equal to the set of vectors $x \in \mathbb{R}^E$ satisfying*

(37.10) (i) $x_e = 0$ if e is contained in no perfect matching,
(ii) $x(C) = x(\delta(v))$ for each tight cut C and each vertex v.

Proof. Condition (37.10) clearly is necessary for each vector x in the perfect matching space. To see sufficiency, let $x \in \mathbb{R}^E$ satisfy (37.10). We can assume that G has at least one perfect matching.

By adding sufficiently many incidence vectors of perfect matchings to x, we can achieve that $x_e \geq 0$ for each edge e, and $x_e > 0$ for at least one edge e, and $x(C) \geq x(\delta(v))$ for each odd cut C and each vertex v. By scaling we can achieve that $x(\delta(v)) = 1$ for each $v \in V$. Then x belongs to the perfect matching polytope of G, and hence to the perfect matching space. ∎

37.3. The brick decomposition

For any inclusionwise maximal cross-free collection $\mathcal{F}$ of nontrivial tight cuts, the family of $\mathcal{F}$-contractions is called a *brick decomposition.* (We note here that it does not mean that each $\mathcal{F}$-contraction is a brick as defined in Section 37.6.)

Lovász [1987] showed that a brick decomposition is a unique family of graphs (up to isomorphism), independently of the maximal cross-free collection of tight cuts chosen:

Theorem 37.4. *All brick decompositions of a matching-covered graph $G = (V, E)$ are the same (up to isomorphism).*

Proof. By induction on $|V|$. Consider two maximal cross-free collection $\mathcal{F}$ and $\mathcal{F}'$ of nontrivial tight cuts.

Case 1: $\mathcal{F}$ and $\mathcal{F}'$ have a common member $\delta(U)$. By induction, the result of two decompositions of $G/\overline{U}$ is the same (where $\overline{U} := V \setminus U$). Similarly, the result of two decompositions of G/U is the same. The theorem follows.

Case 2: There exist $C \in \mathcal{F}$ and $C' \in \mathcal{F}'$ with C and C' cross-free. Let $\mathcal{F}''$ be a maximal cross-free collection of nontrivial tight cuts containing C and C'. By Case 1, the decompositions of G by $\mathcal{F}$ and $\mathcal{F}''$ result in the same family of graphs. Similarly, the decompositions of G by $\mathcal{F}'$ and $\mathcal{F}''$ result in the same family of graphs. The theorem follows.

Case 3: There exist $C = \delta(U) \in \mathcal{F}$ and $C' = \delta(U') \in \mathcal{F}'$ with $|U \cap U'|$ odd and at least 3. Then trivially $C'' := \delta(U \cap U')$ is tight again. Let $\mathcal{F}''$ be a maximal cross-free collection of nontrivial tight cuts containing C''. By Case 2, the decompositions of G by $\mathcal{F}$ and $\mathcal{F}''$ result in the same family of graphs. Similarly, the decompositions of G by $\mathcal{F}'$ and $\mathcal{F}''$ result in the same family of graphs. Again, the theorem follows.

Case 4: None of the previous cases applies. Let $C = \delta(U) \in \mathcal{F}$ and $C' = \delta(U') \in \mathcal{F}'$. Then $\mathcal{F} = \{C\}$ and $\mathcal{F}' = \{C'\}$. For suppose that say $\mathcal{F}$ contains another cut $C'' = \delta(U'')$. We can assume that $U \subseteq U''$ and that $U'' \cap U'$ is odd. So $|U'' \cap U'| = 1$ (as Case 3 does not apply), and therefore $|U \cap U'| = 1$ (as Case 2 does not apply). However, $U \cup U'$ is odd and disjoint from $U'' \setminus U$, implying that $U \cup U'$ is at most $|V| - 2$, and so Case 3 applies, a contradiction.

So $\mathcal{F} = \{C\}$ and $\mathcal{F}' = \{C'\}$. We can now assume that $U \cap U'$ is odd. Since Case 3 does not apply, $|U \cap U'| = 1$ and $|U \cup U'| = |V| - 1$. Let $U \cap U' = \{u\}$ and $U' \cup U = V \setminus \{v\}$.

Now $\{u, v\}$ is a 2-vertex-cut in G, separating $U \setminus \{u\}$ and $U' \setminus \{u\}$. For suppose that there is an edge e connecting $U \setminus \{u\}$ and $U' \setminus \{u\}$. Let M be a perfect matching containing e. Let f be the edge in M covering u. Then f leaves at least one of U and U'. Since e leaves both U and U', this contradicts the fact that U and U' give tight cuts.

As G has no cut vertices (as G is matching-covered), this implies that $G/\overline{U}$ and G/U' are isomorphic graphs, and similarly that G/U and $G/\overline{U}$ are isomorphic. The theorem follows. ∎

37.4. The brick decomposition of a bipartite graph

All graphs in the brick decomposition of a bipartite graph are bipartite:

Theorem 37.5. *Let G be a matching-covered graph and let $\mathcal{F}$ be an inclusionwise maximal cross-free collection of nontrivial tight cuts. Then G is bipartite if and only if each $\mathcal{F}$-contraction is bipartite.*

Proof. It suffices to prove that for any nontrivial tight cut $\delta(U)$:

(37.11) G is bipartite if and only if G/U and $G/\overline{U}$ are bipartite

(where $\overline{U} := VG \setminus U$). Sufficiency in (37.11) is direct (actually, it holds for any cut). To see necessity in (37.11), note that, since G is matching-covered, U has neighbours only in the largest colour class of the bipartite graph $G - U$. So G/U is bipartite, and similarly, $G/\overline{U}$ is bipartite. ∎

37.5. Braces

A bipartite graph $G = (V, E)$, with colour classes U and W, is called a *brace* if G is matching-covered with $|V| \geq 4$ and for all distinct $u, u' \in U$ and $w, w' \in W$, the graph $G - u - u' - w - w'$ has a perfect matching.

By Hall's marriage theorem (Theorem 22.1), a connected bipartite graph $G = (V, E)$ with equal-sized colour classes U and W is a brace if and only if for each subset X of U with $1 \leq |X| \leq |U| - 2$ one has

(37.12) $\qquad |N(X)| \geq |X| + 2.$

Theorem 37.6. *Each tight cut in a brace is trivial.*

Proof. Let $G = (V, E)$ be a brace with colour classes U and W, and suppose that $\delta(T)$ is a nontrivial tight cut. As $|T|$ is odd, by symmetry we can assume that $|U \cap T| < |W \cap T|$.

Then $|U \cap T| = |W \cap T| - 1$, since there exists a perfect matching intersecting $\delta(T)$ in exactly one edge. Since $\delta(T)$ is nontrivial, $1 \leq |U \cap T| \leq |U| - 2$.

Moreover, there is no edge e connecting $U \cap T$ and $W \setminus T$. Otherwise this e would be contained in a perfect matching M. This perfect matching also contains an edge connecting $U \setminus T$ and $W \cap T$, contradicting the tightness of $\delta(T)$.

So $N(U \cap T) \subseteq W \cap T$, and hence $|N(U \cap T)| \leq |U \cap T| + 1$, contradicting (37.12). ∎

37.6. Bricks

A graph G is called a *brick* if G is 3-connected and bicritical, and has at least four vertices. (A graph G is called *bicritical* if $G - u - v$ has a perfect matching for any two distinct vertices u, v.)

The following key result was shown by Edmonds, Lovász, and Pulleyblank [1982]:

Theorem 37.7. *Each tight cut in a brick is trivial.*

Proof. Let $G = (V, E)$ be a brick, and suppose that it has a nontrivial tight cut C_0. Let $\mathcal{C}$ be the collection of odd cuts in G.

For any $b \in \mathbb{Q}^V$, consider the linear program

(37.13)
$$\begin{array}{lll} \text{minimize} & \displaystyle\sum_{e=uv \in E} (b(u) + b(v)) x_e & \\ \text{subject to} & x(C) \geq 1 & (C \in \mathcal{C}), \\ & x_e \geq 0 & (e \in E). \end{array}$$

and its dual

$$\text{(37.14)} \quad \begin{array}{lll} \text{maximize} & \displaystyle\sum_{C \in \mathcal{C}} y(C) & \\ \text{subject to} & \displaystyle\sum_{C \ni e} y(C) \leq b(u) + b(v) & (e = uv \in E), \\ & y(C) \geq 0 & (C \in \mathcal{C}). \end{array}$$

We first show:

(37.15) there exist $y \in \mathbb{Q}_+^{\mathcal{C}}$ and $b \in \mathbb{Q}_+^V$ such that $\sum_{C \ni e} y(C) \leq b(u) + b(v)$ for each edge $e = uv$, and such that $y(\mathcal{C}) = b(V)$ and $y(C_0) > 0$.

To prove this, define $w = \chi^{C_0}$ (the incidence vector of C_0 in $\mathbb{R}^E$). As C_0 is tight, the maximum of $w(M)$ over perfect matchings M is equal to 1. Hence, by Edmonds' perfect matching polytope theorem (Theorem 25.1) and by linear programming duality, there exists a vector $z \in \mathbb{Q}^{\mathcal{C}}$ such that

(37.16)
(i) $\sum_{C \ni e} z(C) \leq -w(e)$ for each edge e,
(ii) $z(C) \geq 0$ if C is nontrivial,
(iii) $z(\mathcal{C}) = -1$.

For $v \in V$, define $b(v) := -z(\delta(v))$ if $z(\delta(v)) < 0$, and $b(v) := 0$ otherwise. For $C \in \mathcal{C}$, define $y(C) := z(C)$ if $z(C) > 0$, and $y(C) := 0$ otherwise. Then (37.16) implies:

(37.17)
(i) $b(u) + b(v) \geq \sum_{C \ni e} y(C) + w(e)$ for each edge $e = uv$,
(ii) $b \geq \mathbf{0}$, $y \geq \mathbf{0}$,
(iii) $b(V) = y(\mathcal{C}) + 1$.

So resetting $y(C_0) := y(C_0) + 1$ gives b and y as required in (37.15), proving (37.15).

This implies:

(37.18) for some vector $b \in \mathbb{Z}_+^V$ there exists an integer optimum solution $y \in \mathbb{Z}_+^{\mathcal{C}}$ of (37.14) such that $y(C_0) \geq 1$.

Indeed, in (37.15) we can assume (by scaling) that b and y are integer. Then by the properties described in (37.15), y is a feasible solution of (37.14). Since the maximum in (37.13) is at least $b(V)$ (as any perfect matching M satisfies $w(M) = b(V)$), and since $y(\mathcal{C}) = b(V)$, we know that y is an optimum solution of (37.14). This proves (37.18).

Now fix a b as in (37.18), with $b(v)$ minimal. Then

(37.19) for any optimum solution y of (37.14) and any $C \in \mathcal{C}$ one has that if $y(C) > 0$, then C is tight.

Indeed, any perfect matching M attains the maximum (37.13) (as the maximum value equals $b(V)$). So if $y(C) > 0$, by complementary slackness, $|M \cap C| = 1$. This shows (37.19).

Call a vector $y \in \mathbb{R}_+^{\mathcal{C}}$ *laminar* if the collection $\{C \in \mathcal{C} \mid y(C) > 0\}$ is laminar. Then:

(37.20) there exists a laminar integer optimum solution of (37.14) such that $y(C) \geq 1$ for at least one nontrivial tight cut C.

To see this, choose an integer optimum solution y of (37.14) such that $y(C) \geq 1$ for at least one nontrivial tight cut C, with

(37.21) $$\sum_{C \in \mathcal{C}} y(C)s(C)$$

minimized, where $s(C)$ denotes the number of pairs of vertices separated by C. We show that y is laminar.

Suppose to the contrary that C and C' cross, with $y(C) > 0$ and $y(C') > 0$. We can choose $U', U'' \subseteq V$ such that $C = \delta(U)$, $C' = \delta(U')$, and $|U \cap U'|$ is odd. Let $D := \delta(U \cap U')$ and $D' := \delta(U \cup U')$. Let $\varepsilon := \min\{y(C), y(C')\}$. Decrease $y(C)$ and $y(C')$ by ε, and increase $y(D)$ and $y(D')$ by ε. Then we obtain again a feasible solution of (37.14), while (37.21) is smaller. So both D and D' are trivial. Hence $U \cap U' = \{u\}$ and $U \cup U' = V \setminus \{v\}$ for some vertices u and v. As G is 3-connected, there is an edge e connecting $U \setminus U'$ and $U' \setminus U$. Since G is matching-covered, there is a perfect matching M containing e. So $e \in C \cap C'$. As C and C' are tight, e is the only edge of M intersecting $C \cup C'$. Hence no edge of M intersects $D = \delta(U \cap U')$, a contradiction. This proves (37.20).

Fix y satisfying (37.20). We note that the first set of constraints in (37.14) gives:

(37.22) if $e = uv \in C$ and $y(C) > 0$, then $b(u) > 0$ or $b(v) > 0$.

Moreover,

(37.23) for each $u \in V$, $b(u) = 0$ or $y(\delta(u)) = 0$.

Otherwise, decreasing $b(u)$ and $y(\delta(u))$ by 1 would give b and y with smaller $b(V)$.

We also show:

(37.24) if $y(\delta(U)) > 0$, then $G[U]$ is connected.

If not, let K be an odd component of $G[U]$ and let e be an edge in $\delta(U)$ not incident with K. Let M be a perfect matching containing e. Then M intersects $\delta(U)$ in more than one edge (since K is odd), while $\delta(U)$ is tight since $y(\delta(U)) > 0$. This contradiction proves (37.24).

Now choose an odd cut $C = \delta(U)$ with $y(C) > 0$, an edge $e_0 = u_0v \in C$ with $u_0 \in U$ and $b(u_0) > 0$, such that $|U|$ is as small as possible. (Such U, e_0, u_0 exist by (37.22).)

By (37.23), $|U| > 1$. Let $U_1, \ldots, U_k$ be the maximal proper subsets of U with $y(\delta(U_i)) > 0$. By (37.20), the U_i are pairwise disjoint. Note that $u_0 \notin U_1 \cup \cdots \cup U_k$, by the minimality of $|U|$.

Define

(37.25) $U' := U \setminus (U_1 \cup \cdots \cup U_k)$, $U_+ := \{u \in U' \mid b(u) > 0\}$, and $U_0 := U' \setminus U_+$.

Then

(37.26) there is no edge joining distinct sets among $U_0, U_1, \ldots, U_k$.

Directly from (37.22) and the minimality of $|U|$.

Moreover,

(37.27) there is no edge $e = uv$ with $u \in U_+$ and $v \in U'$.

For suppose that such an edge e exists. Then there is a perfect matching containing e. Hence, by complementary slackness, we have equality in the corresponding constraint of (37.14). As $b(u)+b(v) > 0$, we know that $y(C) > 0$ for some C with $e \in C$. Then $C = \delta(S)$ for some $S \subseteq U$. This contradicts the definition of the U_i, proving (37.27).

As $G[U]$ is connected (by (37.24)), it follows that $U_0 = \emptyset$. Next

(37.28) $|U_+| = k + 1$.

For consider any perfect matching M containing edge e_0. Then M intersects any $\delta(U_i)$ in exactly one edge (as each $\delta(U_i)$ is tight, by (37.19)) and it also intersects $\delta(U)$ in exactly one edge, namely e_0. Since $|M \cap \delta(U)| = 1$, we know with (37.26) that the edge in $M \cap \delta(U_i)$ connects U_i and U_+. Moreover, no edge in M connects two vertices in U_+ (by (37.27)). Hence we have (37.28).

(37.29) No edge connects any U_i with $V \setminus U$.

Otherwise, the same counting as for proving (37.28) gives $|U_+| = k$, a contradiction.

As $|U| > 1$ we know $k > 0$. Choose $s, t \in U_+$. As G is bicritical, $G - s - t$ has a perfect matching M. Then M intersects each $\delta(U_i)$ at least once, and hence (by (37.29)) $|U_+ \setminus \{s, t\}| \geq k$, a contradiction. ∎

37.7. Matching-covered graphs without nontrivial tight cuts

The foregoing is used in obtaining the following basic result of Lovász [1987]:

Theorem 37.8. *Let $G = (V, E)$ be a connected graph with at least four vertices. Then G is matching-covered without nontrivial tight cuts if and only if G is a brick or a brace.*

Proof. If G is a brick or a brace, then trivially G is matching-covered. Moreover, Theorems 37.6 and 37.7 show that braces and bricks have no nontrivial tight cuts.

Conversely, assume that G is matching-covered and has no nontrivial tight cuts.

Case 1: G is not bicritical. We show that G is a brace. As G is not bicritical, by Tutte's 1-factor theorem (Theorem 24.1a) there exists a subset U of V such that $G - U$ has $|U|$ odd components, with $|U| \geq 2$. As G is matching-covered, U is a stable set, and $G - U$ has no even components. For each component K of $G - U$, $\delta(K)$ is tight, and hence trivial, that is $|K| = 1$. So G is bipartite, and U is one of its colour classes. If G is not a brace, there exists a subset X of U with $1 \leq |X| \leq |U| - 2$ and $|N(X)| \leq |X| + 1$. Let $Y \subseteq V \setminus U$ with $N(X) \subseteq Y$ and $|Y| = |X| + 1$. Then $\delta(X \cup Y)$ is a nontrivial tight cut, a contradiction.

Case 2: G is bicritical. We show that G is a brick. So we must show that G is 3-connected. As G is matching-covered, G is trivially 2-connected. Suppose that $\{u, v\}$ is a 2-vertex-cut. Let K be a component of $G - u - v$. As $G - u - v$ has a perfect matching, $|K|$ is even. Then $\delta(K \cup \{u\})$ is a nontrivial cut which is tight, since the intersection of $\delta(K \cup \{u\})$ with any perfect matching M is odd and at most 2 (as each edge in the intersection is incident with u or v). ∎

Chapter 38

The perfect matching lattice

This chapter is devoted to giving a proof of the deep theorem of Lovász [1987] characterizing the perfect matching lattice of a graph — the lattice generated by the incidence vectors of perfect matchings.

We summarize concepts and results from previous chapters that we need in the proof. Let $G = (V, E)$ be a graph. The following notions will be used:

- A cut C in G is *tight* if each perfect matching intersects C in exactly one edge.
- A cut C is *trivial* if $C = \delta(v)$ for some vertex v.
- G is *matching-covered* if each edge is contained in a perfect matching.
- G is *bicritical* if for each two distinct vertices u and v, the graph $G-u-v$ has a perfect matching.
- G is a *brick* if it is 3-connected and bicritical and has at least 4 vertices.
- A subset B of V is a *barrier* if $G - B$ has at least $|B|$ odd components. A *maximal barrier* is an inclusionwise maximal barrier. A *nontrivial barrier* is a barrier B with $|B| \geq 2$.

Moreover, the following results will be used:

- the perfect matching lattice of a bipartite graph is equal to the set of integer vectors in the perfect matching space (this is an easy consequence of Kőnig's edge-colouring theorem, see Theorem 20.12).
- Any two distinct inclusionwise maximal barriers in a connected matching-covered graph are disjoint (Corollary 24.11a).
- A graph is a brick if and only if it is nonbipartite and matching-covered and has no nontrivial tight cuts (a consequence of Theorem 37.8).
- A graph is bicritical if and only if it has no nontrivial barrier (a consequence of Tutte's 1-factor theorem (Corollary 24.1a)).

Throughout this chapter, $\overline{U}$ denotes the complement of U.

38.1. The perfect matching lattice

The *perfect matching lattice* (usually briefly the *matching lattice*) of a graph $G = (V, E)$ is the lattice generated by the incidence vectors of perfect matchings in G; that is,

(38.1) $$L_{\text{perfect matching}}(G) := \text{lattice}\{\chi^M \mid M \text{ perfect matching in } G\}.$$

So it is a sublattice of $\mathbb{Z}^E$ and is contained in the perfect matching space of G.

In Section 20.8 we saw that the perfect matching lattice of a *bipartite* graph $G = (V, E)$ is equal to the intersection of $\mathbb{Z}^E$ with the perfect matching space of G. This characterization does not hold in general for nonbipartite graphs, as is shown by the Petersen graph. However, as was proved by Lovász [1987], any graph for which the characterization does not hold, contains the Petersen graph in some sense. In particular, for any graph without Petersen graph minor, the characterization remains valid.

In analyzing the perfect matching lattice of G, two initial observations are of interest:

- We can assume that G is matching-covered, since any edge contained in no perfect matching can be deleted;
- If G has a nontrivial tight cut, we can reduce the analysis by considering the two graphs obtained by contracting either of the shores of the cut.

So we can focus the investigations on nonbipartite matching-covered graphs without nontrivial tight cuts; that is, by Theorem 37.8, on bricks.

38.2. The perfect matching lattice of the Petersen graph

We will need a characterization of the perfect matching lattice of the Petersen graph, which is easy to prove:

Theorem 38.1. *Let G be the Petersen graph and let C be a 5-circuit in G. Then the perfect matching lattice consists of all integer vectors x in the perfect matching space with $x(EC)$ even.*

Proof. Inspection of the Petersen graph (cf. Figure 38.1) shows that each edge of G is contained in exactly two perfect matchings, that (hence) G has exactly six perfect matchings, that any two perfect matchings intersect each other in exactly one edge, and that each perfect matching intersects EC in an even number of edges.

Let $M_0 := \delta(VC)$ (the set of edges intersecting VC in one vertex). Then M_0 is a perfect matching of G. Let $M_1, \ldots, M_5$ be the five other perfect matchings of G. So each of the M_i intersects M_0 in one edge.

By adding appropriate integer multiples of $\chi^{M_1}, \ldots, \chi^{M_5}$ to x we can achieve that $x_e = 0$ for each $e \in M_0$. As x is in the perfect matching space, we know that there exists a number t such that $x(\delta(v)) = t$ for each vertex v. Hence, as $|EC|$ is odd, $x_e = \frac{1}{2}t$ for all $e \in EC$; similarly, for each edge e in the 5-circuit vertex-disjoint from C one has $x_e = \frac{1}{2}t$. As $x(EC)$ is even, we know that $\frac{5}{2}t$ is even, hence $\frac{1}{2}t$ is even. Now the vector

$$(38.2) \qquad y := \chi^{M_1} + \cdots \chi^{M_5} - \chi^{M_0}$$

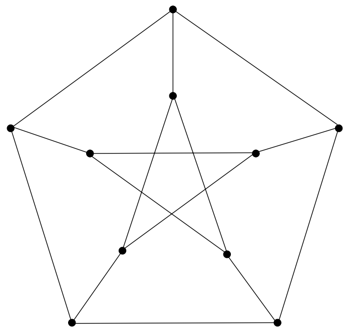


Figure 38.1
The Petersen graph

satisfies $y_e = 0$ for $e \in M_0$ and $y_e = 2$ for $e \notin M_0$. Hence x is an integer multiple of y, proving that x belongs to the perfect matching lattice of G. ▮

38.3. A further fact on the Petersen graph

In the proof of the characterization of the perfect matching lattice, we need a further, technical fact on the Petersen graph.

Let $G = (V, E)$ be a graph and let $b : V \to \mathbb{Z}_+$. Recall that a *b-factor* is a subset F of E with $\deg_F(v) = b(v)$ for each $v \in V$.

Theorem 38.2. *Let $G = (V, E)$ be the Petersen graph and let C be a 5-circuit in G. Let $b : V \to \mathbb{Z}_+$ be such that*

- *either there exists a $u \in V$ with $b(u) = 3$ and $b(v) = 1$ for all $v \neq u$,*
- *or there exist distinct $u, u' \in V$ with $b(u), b(u') \in \{0, 2\}$ and $b(v) = 1$ for all $v \neq u, u'$, such that if $b(u) = b(u') = 0$, then u and u' are nonadjacent.*

Then there exist b-factors F and F' such that $|F \cap EC|$ and $|F' \cap EC|$ have different parities.

Proof. By induction on $b(V)$. If $b(u) = b(u') = 0$ for some distinct $u, u' \in V$, then u and u' are nonadjacent. Let x be the common neighbour of u and u' and let y be the neighbour of x distinct from u and u'. Then $G - x - N(x)$ forms a 6-circuit (by inspection — cf. Figure 38.1), D say. Split ED into two matchings, M and M'. Adding edge xy to M and M' gives b-factors as required since EC intersects ED in an odd number of edges (as $ED = EG \setminus \delta(N(x))$, and $|EC|$ is odd and $|EC \cap \delta(N(x))|$ is even).

If $b(u') = 2$, choose a neighbour u'' of u' with u'' different from and nonadjacent to u. Define $b'(u'') := 0$, $b'(u) := b(u)$, and $b'(v) := 1$ for all other vertices. By induction, there exist b'-factors F and F' such that F and

F' intersect EC in different parities. Adding edge $u'u''$ to F and F' gives b-factors as required.

If $b(u) = 3$ for some $u \in V$, we can choose any neighbour u' of u, define $b'(u) := 2$, $b'(u') := 0$, and $b'(v) := 1$ for all $v \neq u, u'$, and apply induction as above. ∎

38.4. Various useful observations

In this section we prove a few easy facts that turn out to be useful.

Let $G = (V, E)$ be a graph and let $U \subseteq V$. Recall that G/U denotes the graph obtained from G by contracting U to one vertex, which vertex we will call U. In the obvious way, we will consider the edge set of G/U as a subset of the edge set of G. Hence, for any $x \in \mathbb{R}^E$, we can speak of the *projection* of x to the edges of G/U.

We now characterize when G/U is a brick if G is a brick:

Theorem 38.3. *Let $G = (V, E)$ be a brick and let $U \subseteq V$. Then G/U is a brick if and only if $G - U$ is 2-connected and factor-critical.*

Proof. Necessity being easy, we prove sufficiency.

First, let $G - U$ be 2-connected. Then G/U is 3-connected, for suppose that vertices u and u' of G/U form a 2-vertex-cut of G/U. If both u and u' are different from vertex U of G/U, then u, u' would also form a 2-vertex-cut of G, contradicting the 3-connectivity of G. If, say, u' is equal to vertex U of G/U, then u is a cut vertex of $G - U$, contradicting the 2-connectivity of $G - U$.

Second, let $G - U$ be factor-critical. To see that G/U is bicritical, let B be a nontrivial barrier of G/U. If B does not contain vertex U of G/U, then B would also be a nontrivial barrier of G, contradicting the bicriticality of G. If B contains vertex U, then $G - U$ is not factor-critical. ∎

Maximal barriers leave factor-critical components:

Theorem 38.4. *Let $G = (V, E)$ be a graph with a perfect matching and let B be a maximal barrier. Then each component K of $G - B$ is factor-critical.*

Proof. Suppose not. Then K has a nonempty subset B' such that $(G[K]) - B'$ has at least $|B'| + 1$ odd components. Hence $B \cup B'$ is a barrier of G, contradicting the maximality of B. ∎

We note that

(38.3) if $B_1, \ldots, B_k$ are the maximal nontrivial barriers of a graph $G = (V, E)$, having a perfect matching, then for each $u \in V \setminus (B_1 \cup \cdots \cup B_k)$, the graph $G - u$ is factor-critical.

In bicritical graphs, nonempty stable sets have many neighbours (a *neighbour* of S is a vertex not in S adjacent to at least one vertex in S):

Theorem 38.5. *Let $G = (V, E)$ be bicritical with $|V| \geq 4$. Then any nonempty stable set S has at least $|S| + 2$ neighbours.*

Proof. Suppose that $|N(S)| \leq |S| + 1$. Since $|V \setminus S| \geq |S|$ (as G has a perfect matching), we know $|V \setminus S| \geq 2$. Hence we can choose two vertices $v, v' \in V \setminus S$ such that $|N(S) \setminus \{v, v'\}| < |S|$. This however contradicts the fact that $G - v - v'$ has a perfect matching, since each vertex in S should be matched to a vertex in $N(S)$. ∎

It will also be useful to make the following observation:

Theorem 38.6. *Let $G = (V, E)$ be a graph and let U be an odd subset of V, such that for each edge $e \in \delta(U)$ there is a perfect matching M_e with $M_e \cap \delta(U) = \{e\}$. Define $G_1 := G/\overline{U}$ and $G_2 := G/U$, and let $x \in \mathbb{Z}^E$. If, for each $i = 1, 2$, the projection of x to EG_i belongs to the perfect matching lattice of G_i, then x belongs to the perfect matching lattice of G.*

Proof. Let x' and x'' be the projections of x to EG_1 and to EG_2, respectively. Since x' belongs to the perfect matching lattice of G_1, there exist perfect matchings $M'_1, \ldots, M'_{k'}$ and $N'_1, \ldots, N'_{l'}$ of G_1 such that

(38.4) $$x' = \sum_{i=1}^{k'} \chi^{M'_i} - \sum_{j=1}^{l'} \chi^{N'_j}.$$

Similarly, there exist perfect matchings $M''_1, \ldots, M''_{k''}$ and $N''_1, \ldots, N''_{l''}$ of G_2 such that

(38.5) $$x'' = \sum_{i=1}^{k''} \chi^{M''_i} - \sum_{j=1}^{l''} \chi^{N''_j}.$$

Consider any $e \in \delta(U)$. Then $x'_e = x_e = x''_e$. Hence, using the projections of M_e to EG_1 and to EG_2, we can assume that

(38.6) $$|\{i = 1, \ldots, k' \mid e \in M'_i\}| = |\{i = 1, \ldots, k'' \mid e \in M''_i\}| \text{ and}$$
$$|\{j = 1, \ldots, l' \mid e \in N'_j\}| = |\{j = 1, \ldots, l'' \mid e \in N''_j\}|,$$

since we can add the projection of M_e to EG_1 to both sums in (38.4), if the number of i with $e \in M'_i$ is less than the number of i with $e \in M''_i$; similarly, if it would be more.

We can do this for each $e \in \delta(U)$, to obtain (38.6) for each $e \in \delta(U)$. It implies that $k' = k''$ and $l' = l''$. It also implies that we can 'match' the M'_i and M''_i in common edges in $\delta(U)$. That is, by permuting indices, we can assume that M'_i and M''_i have an edge in $\delta(U)$ in common, for each $i = 1, \ldots, k'$. In other words, each $M'_i \cup M''_i$ is a perfect matching of G. Similarly, we can assume that each $N'_j \cup N''_j$ is a perfect matching of G. Then

$$x = \sum_{i=1}^{k'} \chi^{M_i' \cup M_i''} - \sum_{j=1}^{l'} \chi^{N_j' \cup N_j''}. \tag{38.7}$$

So x belongs to the perfect matching lattice of G. ∎

38.5. Simple barriers

In this section, we fix a brick $G = (V, E)$ and an edge e such that $G - e$ is matching-covered, and study barriers of $G - e$. In particular we focus on 'simple' barriers of $G - e$. They play an important role in the proof of the characterization of the perfect matching lattice.

For any $B \subseteq V$, let $I(B)$ denote the set of isolated vertices of $G - e - B$ and let $K(B)$ denote the set of nonisolated vertices of $G - e - B$. Then B is called a *simple barrier* of $G - e$ if $|I(B)| = |B| - 1$. So a simple barrier is a barrier of $G - e$, and hence a stable set (as $G - e$ is matching-covered). Note that each singleton is a simple barrier.

For any simple barrier B of $G-e$, $K(B)$ is an odd component of $G-e-B$, since $G-e$ is matching-covered and connected. (Trivially, $|K(B)|$ is odd, since $|V|$ is even and $|I(B)| = |B| - 1$. If $K(B)$ would not be connected, let L be an odd component of $K(B)$ and let f be an edge connecting $K(B) \setminus L$ and B. Let M be a perfect matching of $G - e$ containing f. Necessarily some edge in M leaves L. But then more that one edge in M connects $K(B)$ and B, and also each vertex in $I(B)$ is matched to B, while $|I(B)| = |B| - 1$, a contradiction.)

Since a barrier B of $G - e$ with $|B| \geq 2$ is not a barrier of G (since G is bicritical), e necessarily connects two odd components of $G - e - B$. If B is a simple barrier of $G - e$ with $|B| \geq 2$, then e connects $K(B)$ with some vertex $v_1 \in I(B)$. (G has a perfect matching M intersecting $\delta(K(B))$ in at least three edges, and hence M contains an edge connecting $K(B)$ and $I(B)$. This edge must be e.)

Then the perfect matchings M of G are of two types:

(38.8) M does not contain e, in which case M matches B with the components of $G - e - B$,
or M contains e, in which case two of the edges in M leaving B are incident with $K(B)$, and the other edges in M leaving B are incident with $I(B) \setminus \{v_1\}$.

We now give some further easy properties of simple barriers. Recall that a subset U of the vertex set V of a graph G is called *matchable* if $G[U]$ has a perfect matching.

Theorem 38.7. *Let $G = (V, E)$ be a brick, let $e \in E$ be such that $G - e$ is matching-covered and let B be a simple barrier of $G - e$. Let $e = v_1v_2$ with $v_1 \in B \cup I(B)$ and $v_2 \in K(B)$. Then:*

(38.9) (i) *if* $|B| \geq 2$, *then* $v_1 \in I(B)$;
(ii) *for any* $u \in B$, *the set* $(B - u) \cup I(B)$ *is matchable;*
(iii) *for any distinct* $u, u' \in B$, *the set* $(B - u - u') \cup (I(B) - v_1)$ *is matchable;*
(iv) $G - e/K(B)$ *is matching-covered;*
(v) $G[B \cup I(B)]$ *is connected;*
(vi) *any cut vertex* v *of* $G[B \cup I(B)]$ *belongs to* $I(B)$;
(vii) *if* $Y \subseteq I(B)$ *and* $G[B \cup I(B)] - Y$ *has at least* $|Y| + 1$ *components, then it contains precisely* $|Y| + 1$ *components and any component of* $G[B \cup I(B)] - Y$ *not containing* v_1 *consists of a singleton vertex in* B.

Proof. Since all assertions are trivial if $|B| = 1$, we can assume that $|B| \geq 2$. We saw above that then $v_1 \in I(B)$, proving (i).

(ii) follows from the fact that $G - u - v_2$ has a perfect matching. Similarly, (iii) follows from the fact that $G - u - u'$ has a perfect matching, necessarily containing e. (iv) is directly implied by the fact that $G - e$ is matching-covered, and (v) follows from (ii).

To see (vi), assume that $v \in B$. Choose a component K of $G[B \cup I(B)] - v$ not containing v_1. Since $(B - v) \cup I(B)$ is matchable by (ii), $|K \cap B| = |K \cap I(B)|$. Choose $v' \in K \cap B$. Then $(B - v - v') \cup (I(B) - v_1)$ is matchable by (iii). However, $|K \cap B \setminus \{v'\}| < |K \cap I(B)|$, a contradiction. This proves (vi).

To prove (vii), let α be the number of components of $G[B \cup I(B)] - Y$ containing v_1, let β be the number of other components intersecting $I(B)$, and let γ be the number of other components (hence each consisting of a singleton vertex in B). So $\alpha + \beta + \gamma \geq |Y| + 1$. Now by Theorem 38.5, each component K satisfies

(38.10) $$|K \cap B| \geq |K \cap I(B)| + 1.$$

Indeed, if $K \cap I(B) = \emptyset$, this is trivial. If $K \cap I(B) \neq \emptyset$, then by Theorem 38.5, $|K \cap I(B)| + 2 \leq N(K \cap I(B))| \leq |K \cap B| + 1$, as $N(K \cap I(B)) \subseteq (K \cap B) \cup \{v_2\}$. This proves (38.10).

Moreover,

(38.11) if $v_1 \notin K$ and $K \cap I(B) \neq \emptyset$, then $|K \cap B| \geq |K \cap I(B)| + 2$,

since then $N(K \cap I(B)) \subseteq K \cap B$.

(38.10) and (38.11) imply

(38.12) $$\alpha + 2\beta + \gamma \leq \sum_K (|K \cap B| - |K \cap I(B)|) = |B| - |I(B) \setminus Y| = |Y| + 1 \leq \alpha + \beta + \gamma,$$

where K ranges over the components of $G[B \cup I(B)] - Y$. Hence $\beta = 0$, and (vii) follows. ∎

We next consider the case where v_2 is a cut vertex of $G[K(B)]$.

Theorem 38.8. *Let $G = (V, E)$ be a brick and let $e = v_1v_2$ be an edge such that $G - e$ is matching-covered. Let B be a simple barrier of $G - e$ with $v_1 \in I(B)$ and $v_2 \in K(B)$. Let Z be a union of components of $G[K(B)] - v_2$, with $Z \neq K(B) - v_2$. Then $G/(Z \cup \{v_2\})$ is matching-covered and has exactly one brick in its brick decomposition.*

Proof. Define $U := Z \cup \{v_2\}$ and $L := K(B) \setminus U$. Note that L is matchable, since $G - v - v'$ has a perfect matching for some $v, v' \in B$ (necessarily containing e and containing no edge connecting $K(B)$ and B). So $|L|$ is even.

We first show that

(38.13) G/U is matching-covered.

Consider first any perfect matching M of $G - e$. Then M has exactly one edge leaving $B \cup I(B)$. Hence M has exactly one edge leaving U (since if there were at least three, then at least two of them should leave $B \cup I(B)$). So M gives a perfect matching in G/U. Since $G - e$ is matching-covered, this implies that each edge of G/U except the image of e is contained in a perfect matching of G/U.

As $L \neq \emptyset$ and $|L|$ is even, G has a perfect matching M with at least three edges leaving $L \cup \{v_2\}$. So it contains at least two edges connecting L and B. Hence M contains e, and all other edges leaving $B \cup I(B)$ connect it with L. So the image of M is a perfect matching in G/U containing the image of e. This shows (38.13).

To see that G/U has only one brick in its brick decomposition, choose a counterexample with $|B|$ as small as possible. This implies:

(38.14) $|N(X) \cap I(B)| > |X|$ for each nonempty subset X of $B \setminus N(L)$.

Assume that this is not the case. Since $|B \cap N(L)| \geq 2$ (as G is 3-connected), we know $|X| \leq |B| - 2$, and so $|N(X) \cap I(B)| \leq |B| - 2 = |I(B)| - 1$, implying $I(B) \not\subseteq N(X)$. Each neighbour of $I(B) \setminus N(X)$ belongs to $(B \setminus X) \cup \{v_2\}$, as there is no edge connecting X and $I(B) \setminus N(X)$. So, using Theorem 38.5,

(38.15)
$$\begin{aligned} |B| - |X| = |B \setminus X| &\geq |N(I(B) \setminus N(X))| - 1 \geq |I(B) \setminus N(X)| + 1 \\ &= |B| - |N(X) \cap I(B)|, \end{aligned}$$

implying $|N(X) \cap I(B)| = |X|$ and $v_1 \notin N(X)$. Define $B' := B \setminus X$. Then B' is a simple barrier of $G - e$ again, with $I(B') = I(B) \setminus N(X)$ and $K(B') = K(B) \cup N(X) \cup X$.

Let S be the union of X, $N(X) \cap I(B)$, and the contracted vertex U of G/U. Then each perfect matching of G/U has exactly one edge leaving S (as X is matched to $(I(B) \cap N(X)) \cup \{U\}$ in G/U, since $X \cap N(L) = \emptyset$). So S determines a tight cut in G/U. As $G/\overline{S}$ is bipartite, it suffices to show that the brick decomposition of $G/U/S$ contains exactly one brick.

Since $X \cap N(L) = \emptyset$, L is a union of components of $G[K(B')] - v_2$. Then

(38.16) $G/U/S = G/(K(B') \setminus L) \cup \{v_2\}$.

Hence, by the minimality of B, G/U has a brick decomposition with exactly one brick. This shows (38.14).

We finally derive from (38.14) that G/U has only one brick in its brick decomposition, in fact, that it *is* a brick — equivalently that $G - U$ is 2-connected and factor-critical (Theorem 38.3).

Assume that $G-U$ is not 2-connected, and let v be a cut vertex of $G-U$. Then each component of $G - U - v$ intersects $B \cup I(B)$ as G is 3-connected. Hence v is a cut vertex of $G[B \cup I(B)]$. So Theorem 38.7(vi) applies. In particular, $v \in I(B)$.

Since each component of $G[L]$ is adjacent to at least two vertices in B (since G is 3-connected), we know by Theorem 38.7(vii) that all vertices in B adjacent to L belong to the same component of $G[B \cup I(B)] - v$ as v_1. Any other component consists of one vertex, w say, in B. But then this contradicts (38.14), taking $X = \{w\}$. So $G - U$ is 2-connected.

To show that $G - U$ is factor-critical, suppose to the contrary that there exists a nonempty subset Y of $\overline{U}$ such that $G - U - Y$ has at least $|Y| + 1$ odd components.

Then $Y \subseteq I(B)$. Otherwise choose $v \in Y \setminus I(B)$. So $v \in L \cup B$. Then $G-U-v$ has no perfect matching. However, as G is bicritical, $G-v-v_2$ has a perfect matching M. Then the restriction of M to $\overline{U}$ is a perfect matching of $G - U - v$, a contradiction. So $Y \subseteq I(B)$.

Each component of $G-U-Y$ containing a component of L has at least two elements in B (since G is 3-connected). So $G[B \cup I(B)] - Y$ has precisely $|Y|+1$ components. Hence it has $|Y|$ singleton components in B, without neighbours in L (by Theorem 38.7(vii)). Let X be the union of these components. Each neighbour y of any $x \in X$ with $y \notin U$ belongs to Y. So $|X| \geq |Y| \geq |N(X) \cap I(B)|$, contradicting (38.14). ∎

We next consider *pairs* of simple barriers B_1, B_2. The following auxiliary theorem is of special interest for disjoint simple barriers B_1 and B_2 of $G - e$ where B_2 intersects $I(B_1)$.

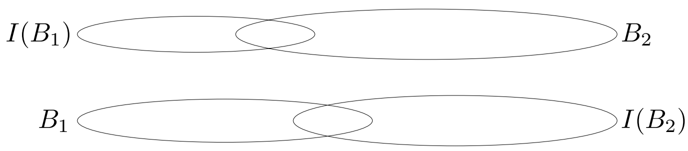


Figure 38.2

Theorem 38.9. *Let $G = (V, E)$ be a brick and let $e = v_1v_2 \in E$ be such that $G-e$ is matching-covered. Let B_1 and B_2 be disjoint simple barriers of $G-e$ with $v_1 \in I(B_1)$ and $v_2 \in I(B_2)$. Then*

(38.17) (i) *$I(B_1) \cap I(B_2) = \emptyset$;*
(ii) *$B_1 \cup I(B_2)$ and $B_2 \cup I(B_1)$ are stable sets;*
(iii) *$|B_1 \cap I(B_2)| = |B_2 \cap I(B_1)|$;*
(iv) *$B_2 \setminus I(B_1)$ is again a simple barrier of $G - e$, with $I(B_2 \setminus I(B_1)) = I(B_2) \setminus B_1$.*

Proof. (i) follows from the fact that all neighbours of any $u \in I(B_1) \cap I(B_2)$ belong to $B_1 \cap B_2 = \emptyset$. Since $N(I(B_2)) \subseteq B_2 \cup \{v_1\}$, which is disjoint from B_1, we have that $B_1 \cup I(B_2)$ is a stable set. Similarly, $B_2 \cup I(B_1)$ is a stable set, implying (ii).

Since $I(B_2) \setminus B_1 \subseteq I(B_2 \setminus I(B_1))$, we know

(38.18)
$$\begin{aligned}&|I(B_2)| - |I(B_2) \cap B_1| = |I(B_2) \setminus B_1| \leq |I(B_2 \setminus I(B_1))|\\ &\leq |B_2 \setminus I(B_1)| - 1 = |B_2| - 1 - |B_2 \cap I(B_1)|\\ &= |I(B_2)| - |B_2 \cap I(B_1)|.\end{aligned}$$

So $|B_2 \cap I(B_1)| \leq |B_1 \cap I(B_2)|$, and hence by symmetry $|B_2 \cap I(B_1)| = |B_1 \cap I(B_2)|$, and we have equality throughout in (38.18). This gives (iii) and (iv). ∎

The last auxiliary theorem in this section reads:

Theorem 38.10. *Let G be a brick and let $e = v_1v_2$ be an edge of G with $G-e$ matching-covered. Let B_1 and B_2 be simple barriers of $G-e$, and define $J_i := B_i \cup I(B_i)$ for $i = 1, 2$, with $v_1 \in J_1$ and $v_2 \in J_2$, and $X := V \setminus (J_1 \cup J_2)$. Assume that $J_1 \cap J_2 = \emptyset$, and that, for each $u \in X$, $G-e-u$ is factor-critical and $G - u/J_1$ and $G - u/J_2$ are 2-connected. Then if $G - e$ has a 2-vertex-cut separating J_1 and J_2, it has a 2-vertex-cut $\{u, u'\}$ separating J_1 and J_2 such that for some component K of $G - e - u - u'$, both $G/(K \cup \{u\})$ and $G/\overline{K \cup \{u\}}$ are bricks.*[21]

Proof. Note that if $\{u, u'\}$ is $J_1 - J_2$ separating in $G-e$ (which by definition implies that $u, u' \notin J_1 \cup J_2$), then $G-e-u-u'$ has a perfect matching (by the assumption in the theorem). Moreover, since G is 3-connected, $G - u - u'$ is connected. Hence $G - e - u - u'$ has exactly two components, one containing J_1 and one containing J_2. We will apply Theorem 38.3.

We first show:

(38.19) Let $\{u, u'\}$ be J_1-J_2 separating in $G-e$ and let K be a component of $G - e - u - u'$. Then the graph $G[K \cup \{u\}]$ is factor-critical.

By symmetry, we may assume that $J_1 \subseteq K$. Define $S := K \cup \{u\}$. Choose a vertex $v \in S$. We prove that $G[S] - v$ has a perfect matching. If $v = u$, then $G[S] - v = G[K]$ has a perfect matching (as K is a component of $G-e-u-u'$). So let $v \neq u$. As $G-e-u'$ is factor-critical by the assumption

[21] It is important to note that it is not concluded that also $G/(K \cup \{u'\})$ and $G/\overline{K \cup \{u'\}}$ are bricks.

in the theorem, $G - e - u' - v$ has a perfect matching M. Since $|K|$ is even, the edge in M incident with u, connects u with K. So M contains a matching spanning $S \setminus \{v\}$. This proves (38.19).

In order to prove that $G[K \cup \{u\}]$ is 2-connected, we need a special kind of 2-vertex-cut and a special order of the components:

(38.20) there exist a pair u, u' separating J_1 and J_2 in $G - e$ and components $K \supseteq J_1$ and $L \supseteq J_2$ of $G - e - u - u'$ such that for each $v \in K \setminus J_1$, $\{u', v\}$ does not separate J_1 and J_2 in $G - e$ and for each $v \in L \setminus J_2$, $\{u, v\}$ does not separate J_1 and J_2 in $G - e$.

To prove this, let $\{u, u'\}$ be a 2-vertex-cut separating J_1 and J_2 in $G - e$. Let K and L be the components of $G-e-u-u'$ containing J_1 and J_2, respectively. We choose u and u' such that L is minimal. Then by the minimality of L, for each $v \in L \setminus J_2$, neither $\{u, v\}$ nor $\{u', v\}$ separates J_1 and J_2 in $G - e$.

If (38.20) does not hold, then there exist $v, v' \in K \setminus J_2$ such that $\{u, v\}$ and $\{u', v'\}$ are vertex-cuts in $G - e$, each separating J_1 and J_2. Let Y be the component of $G-e-u-v$ not containing u'. Since $N_{G-e}(Y) \subseteq \{u, v\}$, we know that $v_1 \in Y$ and hence $J_1 \subseteq Y$. Let Y' be the component of $G - e - u' - v'$ not containing u. Again $J_1 \subseteq Y'$. Hence $J_1 \subseteq Y \cap Y'$. Now $N_{G-e}(Y \cap Y') \subseteq \{v, v'\}$ (since $N_{G-e}(Y \cap Y') \subseteq N_{G-e}(Y) \cup N_{G-e}(Y') \subseteq \{u, u', v, v'\}$; but u' is not a neighbour of $Y \cap Y'$ since u' is not in component Y of $G - e - u - v$; similarly for u). This implies $v \neq v'$. Hence $v' \in Y$.

Let A be the component of $G - e - u - v$ different from Y, and let A' be the component of $G - e - u' - v'$ different from Y'. Then $Y' \cap A = \emptyset$. Indeed, $N(Y' \cap A) \subseteq N(Y') \cup N(A) \subseteq \{u, v, u', v'\}$. Moreover, $u, v' \notin N(Y' \cap A)$, since $u \in A'$ and $v' \in Y$. So $|N(Y' \cap A)| \leq 2$, implying $Y' \cap A = \emptyset$ by the 3-connectivity of G.

Similarly, $K \cap A \cap A' = \emptyset$. Indeed, $N(K \cap A \cap A') \subseteq N(K) \cup N(A) \cup N(A') = \{u, v, u', v'\}$. Moreover, $v, v' \notin N(K \cap A \cap A')$, since $v \in Y'$ and $v' \in Y$. So $|N(K \cap A \cap A')| \leq 2$, implying $K \cap A \cap A' = \emptyset$.

So $K \cap A$ intersects neither Y' nor A', hence $K \cap A \subseteq \{u', v'\}$. However, $u' \notin K$ and $v' \notin A$. So $K \cap A = \emptyset$. Hence $K \subseteq Y \cup \{v\}$. So $Y = K \setminus \{v\}$, implying that $|Y|$ is odd, a contradiction (since $G - e - u - v$ has a perfect matching). This proves (38.20).

Let u, u' be as in (38.20). By symmetry, it suffices to show:

(38.21) $G[K \cup \{u\}]$ is 2-connected.

Let $S := K \cup \{u\}$. Suppose that there exists a $v \in S$ with $G[S \setminus \{v\}]$ disconnected. Let Z be a component of $G[S \setminus \{v\}]$ not containing v_1, and let Y be any other component. If $u \notin Z$, then $N(Z) \subseteq \{v, u'\}$, contradicting the 3-connectivity of G. So $u \in Z$.

So $u \notin Y$, and hence $N(Y) \subseteq \{u', v, v_2\}$, implying by the 3-connectivity of G, that $v_1 \in Y$. So $N_{G-e}(Y) = \{u', v\}$. If $v \notin J_1$, then $J_1 \subseteq Y$ (as $G[J_1]$ is connected), implying that $\{u', v\}$ is $J_1 - J_2$ separating in $G - e$, contradicting the condition in (38.20). So $v \in J_1$.

If $Y \not\subseteq J_1$, then $Y \setminus J_1$ has only two neighbours in G/J_1: J_1 and u', contradicting the fact that $G - u'/J_1$ is 2-connected (by the condition in the theorem). So $Y \subseteq J_1$.

Let M be a perfect matching in $G-u'-v_1$. So M intersects $\delta(J_1)$ in exactly two edges (since $|I(B_1) \setminus \{v_1\}| = |B_1| - 2$ and $u' \in K(B_1)$, as $u' \notin J_1$). If $G[J_1 \setminus \{v\}]$ is connected, then $Y = J_1 \setminus \{v\}$. Then M contains an edge leaving J_1 and not incident with v. This contradicts the fact that $N_{G-e}(Y) \subseteq \{u', v\}$ and that M does not cover u'.

So v is a cut vertex of $G[J_1]$, and hence by Theorem 38.7(vi), v belongs to $I(B_1)$. Now $v \neq v_1$, since $v_1 \in Y$. By Theorem 38.7(vii), $G[J_1 \setminus \{v\}]$ has two components, one containing v_1 and one consisting only of some neighbour, w say, of v. So $Z \cap J_1 = \{w\}$ and $|(Y \setminus \{v_1\}) \cup \{v\}|$ is even. Then M contains a matching with union $(Y \setminus \{v_1\}) \cup \{v\}$. Hence at most one edge in M leaves J_1, a contradiction. This shows (38.21). ∎

38.6. The perfect matching lattice of a brick

We now prove the theorem of Lovász [1987]:

Theorem 38.11. *Let $G = (V, E)$ be a brick different from the Petersen graph. Then the perfect matching lattice of G is equal to the set of integer vectors in the perfect matching space of G.*

Proof. We choose a counterexample with $|V| + |E|$ minimal. Let x be an integer vector in the perfect matching space of G that is not in the perfect matching lattice of G. We can assume that $x(\delta(v)) = 0$ for each vertex v (this can be achieved by adding an appropriate integer multiple of χ^M to x, for some perfect matching M in G).

Claim 1. Let $\delta(U)$ be an odd cut in G such that both G/U and $G/\overline{U}$ are matching-covered and have exactly one brick in their brick decompositions. Then there exist no perfect matchings M and N of G with $|M \cap \delta(U)| - |N \cap \delta(U)| = 2$.

Proof of Claim 1. Suppose to the contrary that such perfect matchings M, N exist. In particular, $|U|, |\overline{U}| \geq 3$. As $x(\delta(U))$ is even (since $x(\delta(v))$ is even for each vertex v), by adding an appropriate integer multiple of $\chi^M - \chi^N$ to x we can achieve that $x(\delta(U)) = 0$.

Let x' and x'' be the projections of x to the edges of $G/\overline{U}$ and G/U, respectively. Let $H := G/\overline{U}$.

Consider any minimal subset W of U, such that $|W| \geq 3$ and such that $\delta(W)$ is a tight cut of H. (Such a set exists, since $\delta(U)$ is tight in H.) Since H has exactly one brick in its brick decomposition, we know that $H/\overline{W}$ or H/W is bipartite and matching-covered. If $H/\overline{W}$ is bipartite and matching-covered, the colour class of $H/\overline{W}$ not containing vertex $\overline{W}$ would be a nontrivial barrier in G. This contradicts the fact that G is a brick.

So H/W is bipartite and matching-covered. Hence the projection of x' to the edges of H/W belongs to the perfect matching lattice of H/W. So (by Theorem 38.6) the projection y of x' to the edges of $I := H/\overline{W}$ is not in the perfect matching lattice of I.

By the minimality of W, I is a brick. Since y is not in the perfect matching lattice of I, by the minimality of $|V| + |E|$, I is the Petersen graph and has a 5-circuit disjoint from vertex $\overline{W}$ of I with $y(EC)$ odd.

As $\delta(W)$ is not tight in G (since G is a brick), G has a perfect matching L satisfying $|L \cap \delta(W)| \geq 3$, and hence $|L \cap \delta(W)| = 3$ (since I is the Petersen graph). Then by Theorem 38.2 (defining $b(\overline{W}) := 3$ and $b(v) := 1$ for each vertex $v \neq \overline{W}$ of I), we can modify L on the edges of I not incident with $\overline{W}$ to obtain a perfect matching L' of G such that the intersections of L and L' with EC have different parities. Resetting $x := x + \chi^L - \chi^{L'}$ we achieve that $x(EC)$, and hence $x'(EC)$, is even.

Hence the projection of the new x on the edges of $G/\overline{U}$ is in the perfect matching lattice of $G/\overline{U}$. We can perform similar resettings to achieve that the projection of the new x on the edges of G/U is in the perfect matching lattice of G/U. Then the new x, and hence also the original x, belongs to the perfect matching lattice of G, by Theorem 38.6. This contradicts our assumption. *End of Proof of Claim 1*

There exists an edge e with $G - e$ matching-covered

To see this, we first show:

Claim 2. There are no edges e and f such that $G - e - f$ is matching-covered and bipartite.

Proof of Claim 2. Suppose that such e and f exist. As $G - e - f$ is matching-covered, the colour classes of $G - e - f$ have the same size, and as G is matching-covered and nonbipartite, e is spanned by one of the colour classes, and f by the other.

Let M be a perfect matching in G containing e and f and let N be a perfect matching in G not containing e and f. By adding an appropriate integer multiple of $\chi^M - \chi^N$ to x we can achieve that $x_e = 0$. Since x is in the perfect matching space of G, this implies that $x_f = 0$. By Corollary 20.12a, the restriction of x to $G - e - f$ is in the perfect matching lattice of $G - e - f$. Hence x belongs to the perfect matching lattice of G, contradicting our assumption. *End of Proof of Claim 2*

This gives:

Claim 3. There is an edge e such that $G - e$ is matching-covered.

Proof of Claim 3. For each edge e, let $\mathcal{M}_e$ denote the collection of perfect matchings of G containing e. Choose any edge e with $\mathcal{M}_e$ inclusionwise minimal. We prove that $G-e$ is matching-covered.

Suppose that $G-e$ is not matching-covered. Hence there is an edge $f \neq e$ such that each perfect matching of G containing f, also contains e; that is, $\mathcal{M}_f \subseteq \mathcal{M}_e$. By the minimality of $\mathcal{M}_e$, $\mathcal{M}_f = \mathcal{M}_e$. Hence there is no perfect matching containing exactly one of e, f. We show that

(38.22) $G-e-f$ is bipartite.

As there is no perfect matching containing e but not containing f, by Tutte's 1-factor theorem, there exists a subset B of V spanning e such that $G-f-B$ has more than $|B|-2$ odd components; hence, by parity, at least $|B|$ odd components. As $|B| \geq 2$ and as G is bicritical, f connects two distinct odd components, K_1 and K_2 say, of $G-f-B$. Moreover, as G is bicritical, each component of $G-f-B$ is odd.

We show that $G-e-f$ is bipartite with colour classes B and $W := V \setminus B$. That is, e is the only edge contained in B, and each component of $G-f-B$ is a singleton.

To see this, first assume that some component K of $G-f-B$ is not a singleton. Then $\delta(K)$ is a nontrivial cut, and hence it is not tight. So there exists a perfect matching M with $|M \cap \delta(K)| \geq 3$. If $f \notin M$, then (adding up over all components of $G-f-B$), $|M \cap \delta(B)| \geq |B|+2$, a contradiction. If $f \in M$, then similarly $|M \cap \delta(B)| \geq |B|$, again a contradiction (since $e \in M$).

Second assume that B spans some edge e' different from e. Let M be a perfect matching containing e'. If $f \notin M$, then $|M \cap \delta(B)| \geq |B|$, contradicting the fact that $e' \in M$. If $f \in M$, then $|M \cap \delta(B)| \geq |B|-2$, contradicting the fact that both e and e' belong to M. This shows (38.22).

In particular, any odd circuit in G contains exactly one of e and f. By Claim 2, $G-e-f$ is not matching-covered. Hence there is an edge g such that each perfect matching containing g contains e or f. Hence $\mathcal{M}_g = \mathcal{M}_e = \mathcal{M}_f$. So, as before, each of $G-e-f$, $G-e-g$, $G-f-g$ is bipartite. Hence each odd circuit in G contains exactly one edge from each pair taken from e, f, g, a contradiction. *End of Proof of Claim 3*

Each maximal barrier of $G-e$ is simple

We fix an edge e with $G-e$ matching-covered. Let e connect vertices v_1 and v_2.

Claim 4. Let B be a maximal barrier of $G-e$. Then B is simple and $G/\overline{K(B)}$ is a brick.

Proof of Claim 4. As the claim is trivial if $|B|=1$, we can assume $|B| \geq 2$; that is, B is nontrivial. Since G has no nontrivial barrier, B is not a barrier of G, and hence e connects two different components of $G-e-B$.

By Theorem 38.4, each component K of $G - e - B$ is factor-critical. So it suffices to show (by Theorem 38.3) that $G[K(B)]$ is 2-connected. In other words, $G - e - B$ has precisely one block[22].

Let $\mathcal{K}$ denote the collection of components of $G - e - B$, and let $\mathcal{L}$ denote the collection of blocks of $G - e - B$. For $K \in \mathcal{K}$, let $\mathcal{L}_K$ denote the set of blocks of $G[K]$.

It is useful to state the following formulas (38.23) and (38.25). For any perfect matching M of G and any $K \in \mathcal{K}$ one has

(38.23) $$\sum_{L \in \mathcal{L}_K} (|M \cap \delta(L)| - 1) = |M \cap \delta(K)| - 1.$$

This can be shown inductively as follows. Consider any subsets U' and U'' of a set U of vertices with $U' \cup U'' = U$, $|U' \cap U''| = 1$, and no edge connecting $U' \setminus U''$ and $U'' \setminus U'$. Then $|M \cap \delta(U)| - 1 = (|M \cap \delta(U')| - 1) + (|M \cap \delta(U'')| - 1)$, since

(38.24) $$\begin{aligned} |M \cap \delta(U')| + |M \cap \delta(U'')| &= |M \cap \delta(U' \cup U'')| + |M \cap \delta(U' \cap U'')| \\ &= |M \cap \delta(U)| + 1. \end{aligned}$$

One also has

(38.25) $$\sum_{K \in \mathcal{K}} (|M \cap \delta(K)| - 1) = 2|M \cap \{e\}|,$$

since

(38.26) $$\begin{aligned} \sum_{K \in \mathcal{K}} |M \cap \delta(K)| &= |M \cap \delta(B)| + 2|M \cap \{e\}| = |B| + 2|M \cap \{e\}| \\ &= |\mathcal{K}| + 2|M \cap \{e\}|. \end{aligned}$$

Suppose now that the claim is not true — that is, $|\mathcal{L}| \geq 2$. We derive:

(38.27) for each $L \in \mathcal{L}$ and for each edge $f \in \delta(L)$, G has a perfect matching M with $M \cap \delta(L) = \{f\}$.

Indeed, if $f \neq e$, let M be a perfect matching of $G - e$ containing f. By (38.23) and (38.25), M intersects $\delta(L)$ in exactly one edge. So $M \cap \delta(L) = \{f\}$.

Suppose next that $f = e$. As $|\mathcal{L}| \geq 2$ by assumption, there exists a block $L' \neq L$. As G has no tight nontrivial cuts, G has a perfect matching M with $|M \cap \delta(L')| \geq 3$, and hence by (38.23) and (38.25), $|M \cap \delta(L)| = 1$, that is, $M \cap \delta(L) = \{e\}$. This proves (38.27).

Now for each $L \in \mathcal{L}$ there exists a perfect matching M with $|M \cap \delta(L)| \geq 3$, and hence, by (38.23) and (38.25), $|M \cap \delta(L)| = 3$ and $|M \cap \delta(L')| = 1$ for all other $L' \in \mathcal{L}$. Moreover, let N be a perfect matching not containing e. Then adding an appropriate integer multiple of $\chi^M - \chi^N$ to x we can achieve that $x(\delta(L)) = 0$, while $x(\delta(L'))$ does not change for any other $L' \in \mathcal{L}$.

As we can do this for all $L \in \mathcal{L}$, we can assume that

[22] A *block* of a graph H is an inclusionwise maximal set L of vertices with $|L| \geq 2$ and with $G[L]$ 2-connected.

(38.28) $x(\delta(L)) = 0$ for all $L \in \mathcal{L}$.

Since x is in the perfect matching space, with (38.23) this gives that

(38.29) $x(\delta(K)) = 0$ for all $K \in \mathcal{K}$.

Moreover, $x_e = 0$, since

(38.30) $$2x_e = \sum_{K \in \mathcal{K}} x(\delta(K)) - x(\delta(B)) = -x(\delta(B)) = -\sum_{v \in B} x(\delta(v)) = 0.$$

Let H be the matching-covered bipartite graph obtained from $G - e$ by contracting each $K \in \mathcal{K}$ to a vertex. Since $x(\delta(K)) = 0$ for each $K \in \mathcal{K}$ and $x(\delta(v)) = 0$ for each $v \in B$, and since $x_e = 0$, we know from Corollary 20.12a that $x|EH$ is in the perfect matching lattice of H. Now for each $K \in \mathcal{K}$ and for each $f \in \delta(K)$ with $f \neq e$, there exists a matching M in $G-e$ containing f, and hence there is a matching with union $K \setminus \{v\}$, where v is the vertex in K incident with f. We therefore can extend each perfect matching of H to a perfect matching of $G-e$ intersecting each $\delta(K)$ in one edge. This implies that we may assume that $x_f = 0$ for each $f \in \delta(B)$.

Hence each edge f with $x_f \neq 0$ is spanned by some $L \in \mathcal{L}$. Let $\mathcal{L}'$ be the collection of those blocks $L \in \mathcal{L}$ spanning at least one edge f with $x_f \neq 0$. We choose x satisfying all previous assumptions and such that $|\mathcal{L}'|$ is as small as possible.

As each $K \in \mathcal{K}$ is factor-critical, each $L \in \mathcal{L}$ is factor-critical. Hence, by Theorem 38.3,

(38.31) $G/\overline{L}$ is a brick for each $L \in \mathcal{L}$.

Moreover,

(38.32) we can assume that, for each $L \in \mathcal{L}$ with $G/\overline{L}$ the Petersen graph, there is a 5-circuit C in $G[L]$ with $x(EC)$ even.

Indeed, choose any 5-circuit C in $G[L]$, and suppose that $x(EC)$ is odd. Let M be a perfect matching in G with $|M \cap \delta(L)| = 3$. By Theorem 38.2, we can modify M on the edges spanned by L so as to obtain a perfect matching N with $|N \cap EC|$ having parity different from $|M \cap EC|$, and such that M and N coincide for all edges not spanned by L. Now adding $\chi^M - \chi^N$ to x makes $x(EC)$ even, and does not invalidate our previous assumptions. This shows (38.32).

We show next:

(38.33) for each $\mathcal{L}_0 \subseteq \mathcal{L}$ with $x_f = 0$ for each $f \in \delta(\bigcup \mathcal{L}_0)$, one has $\mathcal{L}_0 \subseteq \mathcal{L}'$.

We show this by induction on $|\mathcal{L}_0|$. If $\mathcal{L}_0 = \emptyset$, this is trivial. If $\mathcal{L}_0 \neq \emptyset$, we can choose an $L \in \mathcal{L}_0$ such that L has a vertex v such that each $L' \in \mathcal{L}_0$ with $L' \neq L$ is disjoint from $L \setminus \{v\}$. Hence each $f \in \delta(L)$ with $x_f \neq 0$ is incident with v. By (38.31) and (38.32), $x|E(G/\overline{L})$ is in the perfect matching lattice of $G/\overline{L}$. So

(38.34) $$x|E(G/\overline{L}) = \sum_{M} \lambda_M \chi^M,$$

where M ranges over perfect matchings of $G/\overline{L}$ and where $\lambda_M \in \mathbb{Z}$. Let $\mathcal{M}$ denote the collection of perfect matchings of $G/\overline{L}$ not containing an edge leaving L at v. So if $f \in M \in \mathcal{M}$ and f is incident with $\overline{L}$, then $x_f = 0$. By (38.27), for each $f \in \delta(L)$ we can choose a perfect matching N_f of G/L containing f. Then for each perfect matching M of $G/\overline{L}$, let $\widetilde{M} := M \cup N_f$ where f is the edge of M leaving L. Then, by replacing x by

(38.35) $$x - \sum_{M \in \mathcal{M}} \lambda_M \chi^{\widetilde{M}},$$

x changes only on edges spanned by L, and we achieve that $x_f = 0$ for each edge $f \in \delta(v)$ spanned by L. Hence for $\mathcal{L}_0' := \mathcal{L}_0 \setminus \{L\}$ we have $x_f = 0$ for each $f \in \delta(\bigcup \mathcal{L}_0')$. Therefore, by the induction hypothesis, $\mathcal{L}_0' \subseteq \mathcal{L}'$. So $x_f = 0$ for each $f \in \delta(L)$. Hence, taking the λ_M as above, by replacing x by

(38.36) $$x - \sum_{M} \lambda_M \chi^{\widetilde{M}},$$

where M ranges over all perfect matchings of $G/\overline{L}$, we achieve that $x|E(G/\overline{L}) = \mathbf{0}$. This proves (38.33).

Applying (38.33) to $\mathcal{L}_0 := \mathcal{L}$, we derive that $x = \mathbf{0}$, a contradiction.

End of Proof of Claim 4

We remind that for each maximal nontrivial barrier B of $G - e$ one has $e \in \delta(K(B))$ and:

(38.37) for each perfect matching M of G: $e \in M \iff |M \cap \delta(K(B))| = 3$.

Pairs of simple barriers of $G - e$

Claim 5. Let B_1 and B_2 be simple barriers of $G - e$ and let $J_i := B_i \cup I(B_i)$ (for $i = 1, 2$), with $J_1 \cap J_2 = \emptyset$ and $v_i \in J_i$ (for $i = 1, 2$). Then $H := G - e/J_1/J_2$ is not a brick.

Proof of Claim 5. Suppose that H is a brick. By adding an appropriate integer multiple of $\chi^M - \chi^N$ to x, where M and N are perfect matchings in G containing e and not containing e, respectively, we can achieve $x_e = 0$. Then, since $x(\delta(v)) = 0$ for each vertex v, we have that $x(\delta(J_1)) = x(\delta(J_2)) = 0$. As $G - e/\overline{J_1}$ and $G - e/\overline{J_2}$ are bipartite and as H is a brick, it follows that the respective projections of x belong to the perfect matching space of $G - e/\overline{J_1}$, $G - e/\overline{J_2}$, and H.

As x is not in the perfect matching lattice of G, by Theorem 38.6 at least one of these projections is not in the corresponding perfect matching lattice. As $G - e/\overline{J_1}$ and $G - e/\overline{J_2}$ are bipartite, it follows (as G is a minimal counterexample to Theorem 38.11) that H is the Petersen graph and that $x(EC)$ is odd for some 5-circuit C in H disjoint from vertices J_1 and J_2 of H. Then it suffices to show:

(38.38) G has perfect matchings M and N, each containing e, such that M and N intersect EC in different parities,

since then adding $\chi^M - \chi^N$ to x turns the parity of $x(EC)$.

To prove (38.38), let

(38.39) $X := VG \setminus (J_1 \cup J_2) = K(B_1) \cap K(B_2)$.

So $VH = X \cup \{J_1, J_2\}$. We first show that for $i = 1, 2$:

(38.40) if $|J_i| \geq 3$, and a and b are distinct neighbours of vertex J_i of H with $a, b \in X$, then $\{aJ_i, bJ_i\}$ is the image of a matching in G.

To see this, we can assume that $i = 1$.

If J_1 and J_2 are adjacent vertices of H, then a and b are the only neighbours of J_1 in X. Choose $z \in B_2$. As G is bicritical, $G - v_1 - z$ has a perfect matching M. Then M matches up all vertices in $J_2 \setminus \{z\}$. Moreover, all but two vertices in B_1 are matched with vertices in $I(B_1) \setminus \{v_1\}$. Hence two edges of M connect B_1 and K. So M contains edges connecting a and b with B_1.

If J_1 and J_2 are nonadjacent vertices of H, let z be the vertex distinct from a, b adjacent in H to J_1. Since G is bicritical, $G - v_1 - z$ has a perfect matching M. All but two vertices in B_1 are matched with vertices in $I(B_1) \setminus \{v_1\}$. Since M misses z, M contains edges connecting a and b with B_1. This shows (38.40).

Moreover, we have:

(38.41) if J_1 and J_2 are adjacent vertices in H, and $|J_1| \geq 3$ and $|J_2| \geq 3$, then J_1 has a neighbour a_1 in X, and J_2 has a neighbour a_2 in X, such that $\{a_1 J_1, J_1 J_2, J_2 a_2\}$ is the image of a matching in G.

Let f be an edge of $G - e$ connecting J_1 and J_2. By (38.40), J_1 has a neighbour a_1 in X such that there exists an edge connecting a_1 and J_1 disjoint from f. Similarly, J_2 has a neighbour a_2 in X such that there exists an edge connecting a_2 and J_2 disjoint from f. This gives the a_1 and a_2 required in (38.41).

By Theorem 38.2, we can find subsets F_1 and F_2 of the edge set of H such that for each $j = 1, 2$,

(38.42) (i) each vertex in X is incident with exactly one edge in F_j,
(ii) for each $i = 1, 2$, if $|J_i| = 1$, then J_i is incident with none of the edges in F_j, and, if $|J_i| \geq 3$, then J_i is incident with exactly two edges in F_j,
(iii) $|F_1 \cap EC|$ and $|F_2 \cap EC|$ have different parities.

(Note that if $|J_1| = |J_2| = 1$, then J_1 and J_2 are not adjacent, as then $J_1 = \{v_1\}$ and $J_2 = \{v_2\}$, $e = v_1 v_2$, and $H = G - e$.)

If J_1 and J_2 are adjacent vertices of H and $|J_1| \geq 3$, $|J_2| \geq 3$, we can choose the F_j such that moreover

(38.43) $a_1 J_1$, $a_2 J_2$ belong to both F_1 and F_2,

where a_1 and a_2 are as in (38.41). To see this, note that a_1 and a_2 are nonadjacent (as the Petersen graph has no 4-circuit). Then there exist by Theorem 38.2 subsets F'_1 and F'_2 of the edge set of H such that for each $j = 1, 2$, each vertex of H different from a_1 and a_2 is incident with exactly one edge in F'_j, while a_1 and a_2 are not covered by F'_j, and such that $|F'_1 \cap EC|$ and $|F'_2 \cap EC|$ have different parities. Extending the F'_j with the edges $a_1 J_1$ and $a_2 J_2$ gives F_j as required.

By Theorem 38.7(iii), (38.40) and (38.41), F_1 and F_2 are projections of perfect matchings M and N of G containing e, as required in (38.38).

End of Proof of Claim 5

This claim can be sharpened as follows:

Claim 6. Let B_1 and B_2 be simple barriers of $G - e$ and let $J_i := B_i \cup I(B_i)$ (for $i = 1, 2$), with $J_1 \cap J_2 = \emptyset$ and $v_i \in J_i$ (for $i = 1, 2$). Define $X := V \setminus (J_1 \cup J_2)$. If $G - e - u$ is factor-critical for each $u \in X$ and $H := G - e/J_1/J_2$ is bicritical, then $G/J_1/J_2$ has a 2-vertex-cut intersecting $\{J_1, J_2\}$.

Proof of Claim 6. If $G - u/J_1$ is not 2-connected for some $u \in X$, then $\{u, J_1\}$ is a 2-vertex-cut in G/J_1 (since G is 3-connected), hence in $G/J_1/J_2$, as required. So we may assume that $G - u/J_1$ and $G - u/J_2$ are 2-connected for each $u \in X$.

Let H be bicritical. By Claim 5, H is not a brick. Hence H is not 3-connected. Let $\{u, u'\}$ be a 2-vertex-cut of H. If $\{u, u'\}$ intersects $\{J_1, J_2\}$ we are done. So suppose that $\{u, u'\}$ is disjoint from $\{J_1, J_2\}$. Since G is 3-connected and e connects J_1 and J_2, we know that $\{u, u'\}$ separates J_1 and J_2. Hence, by Theorem 38.10, we may assume that the components K and L of $G - e - u - u'$ are such that $G/(K \cup \{u\})$ and $G/\overline{K \cup \{u\}}$ are bricks.

Define $U := K \cup \{u\}$. Then G has a perfect matching M with $|M \cap \delta(U)| \geq 3$, since G has no nontrivial tight cuts. As each edge in $\delta(U) \setminus \{e\}$ is incident with u or u', we know $|M \cap \delta(U)| = 3$. Let $f \in \delta(U) \setminus \{e\}$ and let N be a perfect matching in $G - e$ containing f. Then $|N \cap \delta(U)| = 1$, contradicting Claim 1. *End of Proof of Claim 6*

$G - e$ has exactly two maximal nontrivial barriers

By Corollary 24.11a, we know:

(38.44) any two distinct maximal barriers of $G - e$ are disjoint.

Since each maximal nontrivial barrier B contains $N(v_1)\setminus\{v_2\}$ or $N(v_2)\setminus\{v_1\}$ (as e connects $I(B)$ and $K(B)$), we know that $G-e$ has at most two maximal nontrivial barriers. In fact:

Claim 7. $G-e$ has exactly two maximal nontrivial barriers B_1 and B_2.

Proof of Claim 7. First assume that $G-e$ has no nontrivial barriers; that is, $G-e$ is bicritical. This contradicts Claim 6 for $B_1 := \{v_1\}$ and $B_2 := \{v_2\}$. ($G-e-u$ is factor-critical for each $u \in V$ by (38.3).) So $G-e$ has at least one maximal nontrivial barrier, B_1 say. Let $J_1 := B_1 \cup I(B_1)$, and assume without loss of generality that $v_1 \in I(B_1)$.

Assume that there is exactly one maximal nontrivial barrier. Then $G-e/J_1$ has no nontrivial barrier; that is, it is bicritical. By Claim 4, G/J_1 is a brick, and hence is 3-connected. This contradicts Claim 6, taking $B_2 := \{v_2\}$. ($G-e-u$ is factor-critical for each $u \in V \setminus J_1$ by (38.3).)

End of Proof of Claim 7

Decomposition of G

Having the two maximal nontrivial barriers B_1 and B_2, assuming $v_1 \in I(B_1)$ and $v_2 \in I(B_2)$, we define

(38.45) $\quad J_1 := B_1 \cup I(B_1)$ and $J_2 := B_2 \cup I(B_2)$.

Note that J_1 and J_2 might intersect. Define $J_1' := J_1 \setminus J_2$, $J_2' := J_2 \setminus J_1$, $B_1' := B_1 \setminus I(B_2)$, and $B_2' := B_2' \setminus I(B_1)$. By Theorem 38.9, B_1' and B_2' are simple barriers again, with $I(B_1') = I(B_1) \setminus B_2$ and $I(B_2') = I(B_2) \setminus B_1$.

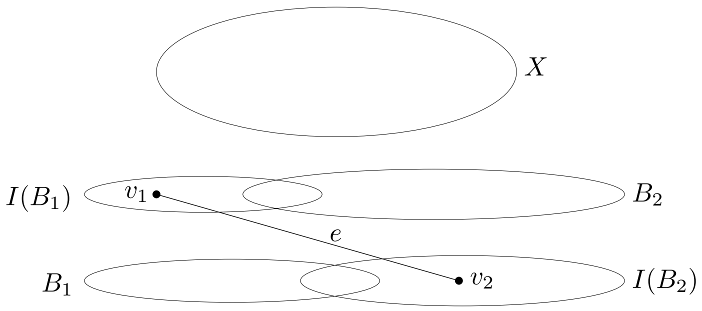


Figure 38.3

Thus we obtain a decomposition of V into

(38.46) $\quad B_1', B_2', I(B_1'), I(B_2'), B_1 \cap I(B_2), B_2 \cap I(B_1),$
$X := K(B_1) \cap K(B_2),$

where e connects $I(B'_1)$ and $I(B'_2)$.

By Theorem 38.9, $G - X$ is bipartite, with colour classes $B_1 \cup I(B_2)$ and $B_2 \cup I(B_1)$.

$G[X]$ has exactly two components

Claim 8. $G[X]$ is disconnected.

Proof of Claim 8. Consider $H := G - e/J_1/J'_2$. Note that H is isomorphic to $G - e/J'_1/J_2$, since, if $J_1 \cap J_2 \neq \emptyset$, then $J_1 \cap J_2$ has neighbours both in J'_1 and J'_2, and nowhere else (by Theorems 38.5 and 38.9).

By Claim 5, H is not a brick. However,

(38.47) H is bicritical.

To see this, choose two distinct vertices v, v' of H. We can assume that $v \neq J'_2$ and $v' \neq J_1$. (If $v = J'_2$ or $v' = J_1$ then exchange v and v'.) Let w be equal to v if $v \neq J_1$ and let w be any vertex in B_1 if $v = J_1$. Similarly, let w' be equal to v' if $v' \neq J'_2$ and let w' be any vertex in B'_2 if $v' = J'_2$. Then $G - e - w - w'$ has a perfect matching, since $\{w, w'\}$ is neither contained in B_1 nor in B_2. As B_1 is a simple barrier in $G - e$, each vertex in $I(B_1)$ is matched to a vertex in B_1. Similarly, each vertex in $I(B'_2)$ is matched to a vertex in B'_2. Hence this perfect matching gives a perfect matching of $H - v - v'$. This proves (38.47).

By Claim 6, $G/J_1/J'_2$ has a 2-vertex-cut $\{u, u'\}$ intersecting $\{J_1, J'_2\}$. ($G - e - u$ is factor-critical for each $u \in X$ by (38.3).) If $\{u, u'\} = \{J_1, J'_2\}$ we are done. So we can assume that $u' \notin \{J_1, J'_2\}$. If $u = J_1$, then u' is a cut vertex of $G - J_1$, contradicting Claim 4. If $u = J'_2$, observe that $G/J_1/J'_2$ is isomorphic to $G/J'_1/J_2$, where the isomorphism brings vertex J_1 to vertex J'_1, and vertex J'_2 to vertex J_2. So u' is a cut vertex of $G - J_2$, again contradicting Claim 4.

End of Proof of Claim 8

We have that

(38.48) each component of $G[X]$ is even,

as for any $u \in B'_1$, $G[K(B_2)] - u$ has a perfect matching M. Then trivially no edge in M connects $K(B_2)$ and J_2. Moreover, no edge in M connects $K(B_1)$ and J_1, since $e \notin M$ (as e is not contained in $K(B_2)$) and since each vertex in $I(B'_1)$ is matched to a vertex in $B'_1 \setminus \{u\}$ (note that $J'_1 \subseteq K(B_2)$).

For any subset L of X, any perfect matching M of G, and any $i \in \{1, 2\}$, define

(38.49) $\lambda_i(M, L) :=$ the number of edges in M connecting L and B_i.

Claim 9. For any component L of $G[X]$ and any perfect matching M of G containing e one has $\{\lambda_1(M, L), \lambda_2(M, L)\} = \{0, 2\}$.

Proof of Claim 9. Since $\lambda_1(M,L)+\lambda_2(M,L)=|M\cap\delta(L)|$ is even (as $|L|$ is even by (38.48)) and since $\lambda_i(M,L)\leq 2$ for $i=1,2$ (since M has two edges connecting $K(B_i)$ and B_i), it suffices to show that $\lambda_1(M,L)\neq\lambda_2(M,L)$.

Suppose that $\lambda_1(M,L)=\lambda_2(M,L)$. Since $e\in M$, $|M\cap\delta(J_1')|=3$. As no edge connects L and $I(B_i)$ (since e is the only edge connecting $K(B_i)$ and $I(B_i)$, but $v_1,v_2\notin L$), we have that M has $\lambda_1(M,L)$ edges connecting L and J_1'. Hence for $U:=J_1'\cup L$,

$$\begin{aligned}(38.50)\qquad |M\cap\delta(U)|&=|M\cap\delta(J_1')|+|M\cap\delta(L)|-2\lambda_1(M,L)\\&=3+\lambda_1(M,L)+\lambda_2(M,L)-2\lambda_1(M,L)=3.\end{aligned}$$

Moreover, any perfect matching N of $G-e$ satisfies $|N\cap\delta(U)|=1$. Indeed, $|N\cap\delta(J_1')|=1$ and $|N\cap\delta(J_2)|=1$. So $|N\cap\delta(X)|\leq 2$. Hence if $|N\cap\delta(U)|\geq 3$, then $N\cap\delta(U)$ contains an edge leaving neither J_1' nor J_2. Hence N has an edge connecting L and $X\setminus L$, a contradiction. So $|N\cap\delta(U)|=1$.

We show that both $G/\overline{U}$ and G/U are matching-covered, AND THat each has a unique brick in its brick decomposition, contradicting Claim 1.

Consider $G':=G/J_2$. Then G' is a brick by Claim 4, and L is a nonempty union of components of $G'-J_1'-\{J_2\}$. Moreover, $G'-e$ is matching-covered (since each perfect matching of $G-e$ has exactly one edge in $\delta(J_2)$) and B_1' is a simple barrier of $G'-e$. So by Theorem 38.8 (taking $Z:=X\setminus L$ and $v_2=J_2$), $G'/\overline{U}=G/\overline{U}$ is matching-covered and has a unique brick in its brick decomposition.

Let $U':=J_2'\cup(X\setminus L)$. Similarly, $G/\overline{U'}$ is matching-covered and has a unique brick in its brick decomposition. Since $\overline{U'}=U\cup(J_1\cap J_2)$, we have $\overline{U\cup U'}=J_1\cap J_2$. So $G/U/U'$ is matching-covered and bipartite. As U' gives a tight cut in G/U, also G/U is matching-covered and has a unique brick in its brick decomposition. *End of Proof of Claim 9*

Claim 10. $G[X]$ has exactly two components.

Proof of Claim 10. Let M be any perfect matching of G containing e. Then $\lambda_i(M,X)\leq 2$ for $i=1,2$, and hence by Claim 9, $G[X]$ has exactly two components. *End of Proof of Claim 10*

Conclusion

Let L_1 and L_2 be the components of $G[X]$. For $j=1,2$, let Z_j be the set of pairs $\{b,b'\}$ with $b\in B_1$, $b'\in B_2'$ such that $L_j\cup\{b,b'\}$ is matchable. In particular, if $b\in B_1$ and $b'\in B_2'$ are adjacent, then $\{b,b'\}\in Z_1\cap Z_2$. Then

Claim 11. For each $j=1,2$, any $b\in N(L_j)$ belongs to some pair in Z_j.

Proof of Claim 11. As $b\in N(L_j)$, there is an edge f joining b and L_j. Let M be a perfect matching of $G-e$ containing f. Then $\lambda_1(M,L_j)=\lambda_2(M,L_j)=1$, and hence $\{b,b'\}\in Z_j$ for some b'. *End of Proof of Claim 11*

Note that if $b \in N(L_j)$ for some j, then $b \in B_1' \cup B_2'$ (since X has no neighbour in $I(B_1) \cup I(B_2)$).

Claim 12. Each pair in Z_1 intersects each pair in Z_2.

Proof of Claim 12. Suppose to the contrary that there exist disjoint pairs $\{b, b'\} \in Z_1$ and $\{c, c'\} \in Z_2$, taking $b, c \in B_1$ and $b', c' \in B_2'$. By definition of Z_j, $L_1 \cup \{b, b'\}$ and $L_2 \cup \{c, c'\}$ are matchable. Moreover, by Theorem 38.7, also $J_1 \setminus \{b, c, v_1\}$ and $J_2' \setminus \{b', c', v_2\}$ are matchable. Together with e, this gives a perfect matching M of G containing e with $\lambda_1(M, L_1) \leq 1$ and $\lambda_2(M, L_1) \leq 1$. This contradicts Claim 9. *End of Proof of Claim 12*

Claim 13. $Z_1 \cap Z_2 = \emptyset$, $|B_1| = |B_2| = 2$, $I(B_1) \cap B_2 = I(B_2) \cap B_1 = \emptyset$, $B_1 \cup B_2$ is a stable set, and Z_1 and Z_2 are perfect matchings on $B_1 \cup B_2$.

Proof of Claim 13. We have $|N(L_j) \cap B_i| \geq 2$ for $j = 1, 2$ and $i = 1, 2$, since (for $j = 1$, $i = 1$, say) L_1 has at least two neighbours in $K(B_2)$ (as $G[K(B_2)]$ is 2-connected), which must belong to B_1.

Assume that $Z_1 \cap Z_2 \neq \emptyset$. Let $\{c, c'\} \in Z_1 \cap Z_2$ with $c \in B_1$ and $c' \in B_2'$. We can choose $b \in N(L_1) \cap B_1$ with $b \neq c$. Then $\{b, c'\} \in Z_1$ (by Claims 11 and 12). We can choose $b' \in N(L_2) \cap B_2'$ with $b' \neq c'$. Again, $\{b', c\} \in Z_2$. As $\{b, c'\}$ and $\{b', c\}$ are disjoint, this contradicts Claim 12. So $Z_1 \cap Z_2 = \emptyset$.

Then $B_1 \cup B_2'$ is a stable set, since if there is an edge connecting $b \in B_1$ and $b' \in B_2'$, then $L_1 \cup \{b, b'\}$ and $L_2 \cup \{b, b'\}$ are matchable, and hence $\{b, b'\} \in Z_1 \cap Z_2$, a contradiction.

This implies $B_1 \cap I(B_2) = \emptyset$, since otherwise there is an edge connecting $b \in B_1 \cap I(B_2)$ and $b' \in B_2' = B_2 \setminus I(B_1)$ (since $B_1 \cap I(B_2)$ has more than $|B_1 \cap I(B_2)| = |B_2 \cap I(B_1)|$ neighbours in B_2, by Theorem 38.5). Hence, by (38.17)(iii), $B_2 \cap I(B_1) = \emptyset$. So $B_2' = B_2$.

Next, for each $j = 1, 2$, no two pairs in Z_j intersect. For assume that $\{b, b'\}$, $\{b, c'\}$ belong to Z_1 with b', c' different vertices in B_2. As $|N(L_2) \cap B_1| \geq 2$, we can choose (by Claim 11) $\{d, d'\} \in Z_2$, with $d \in B_1$ and $d \neq b$. However, then $d' = b'$ and $d' = c'$ by Claim 12, a contradiction, as $b' \neq c'$.

So Z_j consists of disjoint pairs. As each pair in Z_1 intersects each pair in Z_2, we have that each Z_j consists of two disjoint pairs, that Z_1 and Z_2 cover the same set of vertices, and that $Z_1 \cap Z_2 = \emptyset$. In particular,

(38.51) $$|N(X) \cap B_1| = |N(X) \cap B_2| = 2.$$

Finally we show that $|B_i| = 2$ for $i = 1, 2$. Suppose that (say) $|B_1| \geq 3$. Then $|I(B_1)| \geq 2$. Choose $v \in I(B_1) \setminus \{v_1\}$. As G is bicritical, $G - v - v_1$ has a perfect matching M. Necessarily, at least three edges of M connect B_1 and $K(B_1)$, hence (as $B_1 \cup B_2$ is stable) M has at least three edges connecting X and B_1. So $|N(X) \cap B_1| \geq 3$, contradicting (38.51).

End of Proof of Claim 13

This claim in particular implies that

(38.52) v_1 and v_2 have degree 3

(since all neighbours of v_1 belong to $B_1 \cup \{v_2\}$). We can set

(38.53) $B_1 = \{b_1, b_1'\}$, $B_2 = \{b_2, b_2'\}$,
$Z_1 = \{\{b_1, b_2'\}, \{b_1', b_2\}\}$, $Z_2 = \{\{b_1, b_2\}, \{b_1', b_2'\}\}$.

Claim 14. $L_j \cup B_i$ is matchable for all $i, j \in \{1, 2\}$.

Proof of Claim 14. We may assume $i = 2$, $j = 1$. Let M and N be matchings spanning $L_1 \cup \{b_1', b_2\}$ and $L_1 \cup \{b_1, b_2'\}$, respectively. The path P in $M \cup N$ starting at b_1' ends at b_2', as if P would end at b_1, then $L_1 \cup \{b_1, b_2\}$ is matchable (while $\{b_1, b_2\} \notin Z_1$), and if it would end at b_2, then $L_1 \cup \{b_1, b_1', b_2, b_2'\}$ is matchable, implying that G has a perfect matching M' containing e with $\lambda_1(M', L_1) = \lambda_2(M', L_1) = 2$, contradicting Claim 9. So $M \triangle EP$ is a perfect matching on $L_1 \cup \{b_2, b_2'\}$. *End of Proof of Claim 14*

Claim 15. $G - e'$ is matching-covered for each edge e' of G.

Proof of Claim 15. Since G is connected and e is chosen arbitrarily under the condition that $G - e$ is matching-covered, we can assume that e' is incident with e. In particular, we can assume that e' connects v_1 and b_1. Suppose that $G - e'$ is not matching-covered. Then there exists an edge $f \neq e'$ such that each perfect matching of G containing f also contains e'. So f is disjoint from e'.

First assume that f is incident with v_2. We may assume that f connects v_2 with vertex b_2. By definition of Z_1, $L_1 \cup \{b_1, b_2'\}$ is matchable. Since also L_2 is matchable, we can find a perfect matching of G containing f but not e', contradicting our assumption.

So we may assume that f is incident with L_1. Let M' be a perfect matching of G containing f. If M' does not intersect $\delta(L_1)$, we can extend $M'[L_1] \cup \{v_1 b_1', v_2 b_2'\}$ by a matching spanning $L_2 \cup \{b_1, b_2\}$ to obtain a perfect matching containing f but not e', a contradiction. So M' intersects $\delta(L_1)$. Hence, necessarily, it contains an edge joining L_1 with b_1' (as $e' \in M'$). So also it contains an edge joining L_1 and b_2. Therefore, M' contains a matching M spanning $L_1 \cup \{b_1', b_2\}$. Let N be a matching spanning $L_1 \cup \{b_1, b_2'\}$.

Like in Claim 14, the path P in $M \cup N$ starting at b_1' ends at b_2'. Similarly, the path Q in $M \cup N$ starting at b_2 ends at b_1. At least one of $M \triangle EP$ and $M \triangle EQ$ contains f (since f is in M and on at most one of P, Q). As $L_2 \cup \{b_1, b_1'\}$ and $L_2 \cup \{b_2, b_2'\}$ are matchable (by Claim 14), there is a perfect matching containing f and not e', a contradiction. *End of Proof of Claim 15*

This gives with (38.52) that

(38.54) G is 3-regular,

since by Claim 15 we can take for e any edge of G.

Claim 16. $|L_1| = |L_2| = 2$.

Proof of Claim 16. Since G is 3-regular, each $b \in B_1 \cup B_2$ has a unique neighbour in L_j, for each $j = 1, 2$. In fact, for any $j = 1, 2$,

(38.55) if $b \in B_1$, $b' \in B_2$, and $\{b, b'\} \notin Z_j$, then the neighbours of b and b' in L_j coincide.

For assume that the neighbour c of b in L_j differs from the neighbour c' of b' in L_j. As G is bicritical, $G - c - c'$ has a perfect matching M. Let M' be the set of edges in M intersecting L_j. As $|L_j|$ is even, M' spans either $L_j - c - c'$ or $(L_j - c - c') \cup (B_1 - b) \cup (B_2 - b')$. Extending M' with the edges bc and $b'c'$, we obtain a matching spanning $L_j \cup \{b, b'\}$, contradicting $\{b, b'\} \notin Z_j$, or spanning $L_j \cup B_1 \cup B_2$, contradicting Claim 9. This shows (38.55).

Now (38.55) implies that $N(B_1) \cap L_1 = N(B_2) \cap L_1$. As this set is not a 2-vertex-cut of G, we have $|L_1| = 2$. Similarly, $|L_2| = 2$.

End of Proof of Claim 16

So both L_1 and L_2 consist of a single edge. Therefore, G is the Petersen graph, contradicting our assumption. ∎

38.7. Synthesis and further consequences of the previous results

The previous results imply a characterization of the matching lattice for matching-covered graphs (Lovász [1987]):

Corollary 38.11a. *Let $G = (V, E)$ be a matching-covered graph and let $x \in \mathbb{Z}^E$. Then x belongs to the perfect matching lattice of G if and only if for some maximal cross-free collection $\mathcal{F}$ of nontrivial tight cuts:*

(38.56) (i) *$x(D) = x(\delta(v))$ for each $D \in \mathcal{F}$ and each $v \in V$;*
(ii) *for every Petersen brick resulting from the given tight cut decomposition, and for some 5-circuit C in that brick, the sum of the x_e over edges e mapping to EC, is even.*

Proof. Directly from Theorems 38.6, 38.1, and 38.11. ∎

Corollary 38.11a implies the following (conjectured by Lovász [1985]):

Corollary 38.11b. *Let $G = (V, E)$ be a matching-covered graph and let $x \in 2\mathbb{Z}^E$ be such that $x(C) = x(C')$ for any two tight cuts C and C'. Then x belongs to the perfect matching lattice of G.*

Proof. Directly from Corollary 38.11a. ∎

Moreover, there is the following corollary for regular graphs (recall that a *k-graph* is a k-regular graph with $|C| \geq k$ for each odd cut):

Corollary 38.11c. *Let $G = (V, E)$ be a k-graph. Then the all-2 vector **2** belongs to the perfect matching lattice of G. If G has no subgraph homeomorphic to the Petersen graph, then the all-1 vector belongs to the perfect matching lattice of G.*

Proof. Directly from Corollary 38.11a. ∎

A special case is the following result of Seymour [1979a], which also follows from the conjecture of Tutte [1966], proved by Robertson, Seymour, and Thomas [1997], Sanders, Seymour, and Thomas [2000], and Sanders and Thomas [2000], that each bridgeless cubic graph without Petersen graph minor, is 3-edge-colourable.

Corollary 38.11d. *Let $G = (V, E)$ be a bridgeless cubic graph without Petersen graph minor. Then the all-1 vector **1** belongs to the perfect matching lattice of G.*

Proof. This is a special case of Corollary 38.11c. ∎

Similarly, the following consequence, a theorem of Seymour [1979a], supports a positive answer to the question of Fulkerson [1971a] whether each cubic graph G satisfies $\chi'(G_2) = 6$:

Corollary 38.11e. *Let $G = (V, E)$ be a bridgeless cubic graph. Then the all-2 vector **2** in $\mathbb{R}^E$ belongs to the perfect matching lattice of G.*

Proof. Again, this is a special case of Corollary 38.11c. ∎

38.8. What further might (not) be true

The conjecture that the perfect matchings in any graph would constitute a Hilbert base, is too bold: Let G be the graph obtained from the Petersen graph by adding one additional edge (connecting nonadjacent vertices of the Petersen graph). Let $x_e := 1$ if e is an edge of the Petersen graph, and $x_e := 0$ if e is the new edge. Then x belongs to the perfect matching cone[23] and to the perfect matching lattice (since G is a brick). However, x is not a nonnegative integer combination of perfect matchings, since the Petersen graph is not 3-edge-colourable. (This example was given by Goddyn [1993].)

Two weaker conjectures might yet hold true. The first one is due to L. Lovász (cf. Goddyn [1993]):

[23] The *perfect matching cone* is the cone generated by the incidence vectors of the perfect matchings.

(38.57) (?) for any graph without Petersen graph minor, the incidence vectors of the perfect matchings form a Hilbert base. (?)

The second one was given in Section 28.6 above ((28.28)), and is due to Seymour [1979a] (the *generalized Fulkerson conjecture*):

(38.58) (?) each k-graph contains $2k$ perfect matchings, covering each edge exactly twice. (?)

(A *k-graph* is a k-regular graph $G = (V, E)$ with $d_G(U) \geq k$ for each odd $U \subseteq V$.) For $k = 3$, (38.58) was asked by Fulkerson [1971a]:

(38.59) (?) each bridgeless cubic graph has 6 perfect matchings covering each edge precisely twice. (?)

What has been proved by Robertson, Seymour, and Thomas [1997], Sanders, Seymour, and Thomas [2000], and Sanders and Thomas [2000] is:

(38.60) each bridgeless cubic graph without Petersen graph minor is 3-edge-colourable.

This is a special case of conjecture (38.57), and of the 4-*flow conjecture* of Tutte [1966]:

(38.61) (?) each bridgeless graph without Petersen graph minor has three cycles covering each edge precisely twice. (?)

(A *cycle* is an edge-disjoint union of circuits.) Related is the following theorem of Alspach, Goddyn, and Zhang [1994]:

(38.62) the circuits of a graph G form a Hilbert base $\iff$ G has no Petersen graph minor.

It implies that the *circuit double cover conjecture* (asked by Szekeres [1973], conjectured by Seymour [1979b]):

(38.63) (?) each bridgeless graph has a family of circuits covering each edge precisely twice, (?)

is true for graphs without Petersen graph minor:

(38.64) each bridgeless graph without Petersen graph minor has a family of circuits covering each edge precisely twice.

(For cubic graphs this was shown by Alspach and Zhang [1993].) This is also a special case of the 4-flow conjecture (38.61).

Seymour [1979b] conjectures that

(38.65) (?) each *even* integer vector x in the circuit cone is a nonnegative integer combination of incidence vectors of circuits. (?)

This is more general than the circuit double cover conjecture.

Bermond, Jackson, and Jaeger [1983] have proved that

(38.66) each bridgeless graph has a family of circuits covering each edge precisely four times.

Tarsi [1986] mentioned the following strengthening of the circuit double cover conjecture:

(38.67) (?) in each bridgeless graph there exists a family of at most 5 cycles covering each edge precisely twice. (?)

Finally, the 5-*flow conjecture* of Tutte [1954a]:

(38.68) (?) each bridgeless graph has a nowhere-zero 5-flow, (?)

can be formulated in terms of circuits as follows (by Theorem 28.4):

(38.69) (?) each bridgeless graph can be oriented such that there exist directed circuits, covering each edge at least once and at most four times. (?)

Seymour [1981b] showed that each bridgeless graph has a nowhere-zero 6-flow; equivalently:

(38.70) each bridgeless graph can be oriented such that there exist directed circuits, covering each edge at least once and at most five times.

It improves an earlier result of Jaeger [1976,1979] that each bridgeless graph has a nowhere-zero 8-flow. This is equivalent to: each bridgeless graph contains three cycles covering all edges.

Notes. More on nowhere-zero flows and circuit covers can be found in Itai, Lipton, Papadimitriou, and Rodeh [1981], Bermond, Jackson, and Jaeger [1983], Bouchet [1983], Steinberg [1984], Alon and Tarsi [1985], Fraisse [1985], Jaeger, Khelladi, and Mollard [1985], Tarsi [1986], Khelladi [1987], Möller, Carstens, and Brinkmann [1988], Catlin [1989], Goddyn [1989], Jamshy and Tarsi [1989,1992], Fan [1990,1993, 1995,1998], Jackson [1990], Zhang [1990,1993c], Raspaud [1991], Alspach and Zhang [1993], Fan and Raspaud [1994], Huck and Kochol [1995], Lai [1995], Steffen [1996], and Galluccio and Goddyn [2002]. Surveys were given by Jaeger [1979,1985,1988], Zhang [1993a,1993b], and Seymour [1995a], and a book was devoted to it by Zhang [1997b]. The extension to matroids is discussed in Section 81.10.

38.9. Further results and notes

38.9a. The perfect 2-matching space and lattice

Let $G = (V, E)$ be a graph. The *perfect* 2-*matching space* of G is the linear hull of the perfect 2-matchings in G. This space is easily characterized with the help of Corollary 30.2b:

Theorem 38.12. *The perfect 2-matching space of G consists of all vectors $x \in \mathbb{R}^E$ such that $x_e = 0$ if e is not in the support of any perfect 2-matching and such that $x(\delta(v)) = x(\delta(u))$ for all $u, v \in V$.*

Proof. Clearly each vector x in the perfect 2-matching space satisfies the condition. To see the reverse, let x satisfy the condition. By adding appropriate multiples of perfect 2-matchings, we can assume that $x \geq \mathbf{0}$. If $x = \mathbf{0}$ we are done, so we can assume $x \neq \mathbf{0}$. Then, by scaling, we can assume that $x(\delta(v)) = 2$ for each vertex v. Hence, by Corollary 30.2b, x belongs to the perfect 2-matching polytope of F, and therefore to the perfect 2-matching space. ∎

The *perfect* 2-*matching lattice* of G is the lattice generated by the perfect 2-matchings in G. Jungnickel and Leclerc [1989] showed that a characterization of the perfect 2-matching lattice can be easily derived from the theorem of Petersen that the edges of any $2k$-regular graph can be decomposed into k 2-factors (Corollary 30.7b):

Theorem 38.13. *The perfect 2-matching lattice of G consists of all integer vectors x in the perfect 2-matching space of G with $x(\delta(v))$ even for one (hence for each) vertex v.*

Proof. Trivially, each vector x in the perfect 2-matching lattice satisfies the condition. To see the reverse, let x satisfy the condition. By adding integer multiples of perfect 2-matchings, we can assume that $x \geq \mathbf{0}$. Replace each edge e by x_e parallel edges, yielding graph G', of degree $2k$ for some integer $k > 0$. Now by Corollary 30.7b, the edges of G' can be partitioned into k 2-factors. This gives a decomposition of x as a sum of k perfect 2-matchings in G. ∎

38.9b. Further notes

De Carvalho, Lucchesi, and Murty [2002a,2002b] showed that each brick G different from K_4, the prism $\overline{C_6}$, and the Petersen graph, has an edge e such that $G - e$ is a matching-covered graph with precisely one brick in its brick decomposition (conjectured by L. Lovász in 1987). Having this, the proof of Theorem 38.11 can be shortened considerably (de Carvalho, Lucchesi, and Murty [2002c]). (Earlier related work was done by de Carvalho and Lucchesi [1996].)

Naddef and Pulleyblank [1982] study the relation between ear-decompositions and the GF(2)-rank of the incidence vectors of the perfect matchings.

Kilakos [1996] characterized the lattice generated by the matchings M that have a positive coefficient in at least one fractional $\chi'^*(G)$-edge-colouring (these matchings form a face of the matching polytope of G).

Alexander Schrijver

Combinatorial Optimization

Polyhedra and Efficiency

Volume B: Matroids, Trees, Stable Sets

October 21, 2002

Springer
Berlin Heidelberg New York
Barcelona Hong Kong
London Milan Paris
Tokyo

Table of Contents

Volume B

Part VI: Cliques, Stable Sets, and Colouring 1081

Volume A

Volume C