A Conjoint Measurement Approach to the Discrete Sugeno Integral

Denis Bouyssou, Thierry Marchant, and Marc Pirlot

1 Introduction and Motivation

In the area of decision-making under uncertainty, the use of fuzzy integrals, most notably the Choquet integral and its variants, has attracted much attention in recent years. It is a powerful and elegant way to extend the traditional model of (subjective) expected utility (on this model, see Fishburn, 1970, 1982). Indeed, integrating with respect to a non-necessarily additive measure allows to weaken the independence hypotheses embodied in the additive representation of preferences underlying the expected utility model that have often been shown to be violated in experiments (see the pioneering experimental findings of Allais, 1953; Ellsberg, 1961). Models based on Choquet integrals have been axiomatized in a variety of ways (see Gilboa, 1987; Schmeidler, 1989; or Wakker, 1989, Chap. 6. For related works in the area of decision-making under risk, see Quiggin, 1982; and Yaari, 1987). Recent reviews of this research trend can be found in Chateauneuf and Cohen (2000), Schmidt (2004), Starmer (2000) and Sugden (2004).

More recently, still in the area of decision-making under uncertainty, Dubois, Prade, and Sabbadin (2000b) have suggested to replace the Choquet integral by a Sugeno integral (see Sugeno, 1974, 1977), the latter being a kind of "ordinal counterpart" of the former, and provided an axiomatic analysis of this model (special cases of the Sugeno integral are analyzed in Dubois, Prade, & Sabbadin 2001b. For a related analysis in the area of decision-making under risk, see Hougaard & Keiding, 1996). Dubois, Marichal, Prade, Roubens, and Sabbadin (2001a) offer a lucid survey of these developments.

Unsurprisingly, people working in the area of multiple criteria decision making (henceforth, MCDM) have considered following a similar path to build models weakening the independence hypotheses embodied in the additive value function

M. Pirlot (✉)
Faculté Polytechnique de Mons, 9 rue de Houdain, 7000 Mons, Belgium
e-mail: marc.pirlot@fpms.ac.be

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model that underlies most of existing MCDM techniques. This offers an alternative to the decomposable and polynomial models studied in Krantz, Luce, Suppes, and Tversky (1971, Chap. 7). The work of Grabisch (1995, 1996) has widely popularized the use of Choquet and Sugeno integrals in MCDM. Since then, there has been many developments in this area. They are surveyed in Grabisch and Roubens (2000) and Grabisch and Labreuche (2004) (an alternative approach to weaken the independence hypotheses of the traditional model that does not use fuzzy integrals is suggested in Gonzales & Perny, 2005).

It is well known that decision-making under uncertainty and MCDM are related areas. When there is only a finite number of states of nature, acts may indeed be viewed as elements of a homogeneous Cartesian product in which the underlying set is the set of all consequences (this is the approach advocated and developped in Wakker, 1989, Chap. 4). In the area of MCDM, a Cartesian product structure is also used to model alternatives. However, in MCDM the product set is generally not homogeneous: alternatives are evaluated on several attributes that do not have to be expressed on the same scale.

The recent development of the use of fuzzy integrals in the area of MCDM should not obscure the fact that there is a major difficulty involved in the transposition of techniques coming from decision-making under uncertainty to the area of MCDM. In the former area, any two consequences can easily be compared: considering constant acts gives a straightforward way to transfer a preference relation on the set of acts to the set of consequences. The situation is vastly different in the area of MCDM. The fact that the underlying product set is not homogeneous invalidates the idea to consider "constant acts". Therefore, there is no obvious way to compare consequences on different attributes. Yet, such comparisons seem to be prerequisite for the application of models based on fuzzy integrals.

Traditional conjoint measurement models (see, e.g., Krantz et al., 1971, Chap. 6; or Wakker, 1989, Chap. 3) lead to compare *preference differences* between consequences. It is indeed easy to give a meaning to a statement like "the preference difference between consequences x_i and y_i on attribute i is equal to the preference difference between consequences x_j and y_j on attribute j" (e.g., because they exactly compensate the same preference difference expressed on a third attribute). These models do *not* lead to comparing in terms of preference consequences expressed on distinct attributes. Indeed, in the additive value function model a statement like "x_i is better than x_j" is easily seen to be meaningless (this is reflected in the fact that, in this model, the origin of the value function on each attribute may be changed independently on each attribute).

In order to bypass this difficulty, most studies involving fuzzy integrals in the area of MCDM postulate that the attributes are somehow "commensurate", while the precise content of this hypothesis is difficult to analyze and test (see, e.g., Dubois, Grabisch, Modave, & Prade, 2000a). Less frequently, researchers have tried to build attributes so that this commensurability hypothesis is adequate. This is the path followed in Grabisch, Labreuche, and Vansnick (2003) who use the MACBETH technique (see Bana e Costa & Vansnick, 1994, 1997, 1999) to build such scales. Such an analysis requires the assessment of a neutral level on each attribute that is supposed to be "equally attractive". In practice, the assessment of such levels does not seem to be an easy task. On a more theoretical level, the precise properties of these commensurate neutral levels are not easy to devise.

A major breakthrough for the application of fuzzy integrals in MCDM has recently been done in Greco, Matarazzo, and Słowiński (2004) who give conditions characterizing binary relations on non-homogeneous product sets that can be represented using a discrete Sugeno integral, using this binary relation as the only primitive. This is an important result that paves the way to a measurement-theoretic analysis of fuzzy integrals in the area of MCDM (Greco et al., 2004, also relate the discrete Sugeno integral model to models based on decision rules that they have advocated in Greco, Matarazzo, & Slowinski, 1999, 2001). It allows to analyze the discrete Sugeno integral model without any commensurateness hypothesis, which is of direct interest to MCDM.

In the present paper, we will present a new model for the representation of preferences, inspired from the work of Bouyssou and Marchant (2007). This non-numerical model, called non-compensatory model, is slightly more general than the discrete Sugeno integral but, when the preference relation is a weak order that has a numerical representation, we will show that both models are equivalent. The analysis of this new model will thus help us to better understand the discrete Sugeno integral and, eventually, to answer some open questions. In particular, we will address the following issues:

- Besides the standard completeness, transitivity and order density conditions, Greco et al. (2004) used only one condition. We will show that it is possible to factorize this condition into two more elementary ones. This helps us to better understand the behavioural content of the conditions. It can also be useful for empirically testing the conditions. Finally, this will permit us to show that the discrete Sugeno integral model can be viewed as a particular case of a general decomposable representation, investigated in Bouyssou and Pirlot (2004) and Greco et al. (2004).
- The correspondence established between weak orders that are representable in the noncompensatory model and those representable by the discrete Sugeno integral model has an interesting byproduct. Starting from any (bounded) numerical representation of a weak order in the noncompensatory model, we provide formulae that allow to build a representation of the weak order by a Sugeno integral.
- Greco et al. (2004) used four conditions in their characterization of the discrete Sugeno integral. We will prove that they are independent.
- In the standard characterizations of the additive model for multi-attributed preferences (e.g., Wakker, 1989), no commensurateness hypothesis is made. Yet, it is well-known that the difference between two levels on attribute i can be compared to the difference between two levels on attribute j. So, in this model, differences

are commensurate and this can be derived from the axioms. This plays an important role in most elicitation techniques.

In their characterization, Greco et al. (2004) did not make any commensurateness hypothesis either. Yet, when we compute a discrete Sugeno integral, we compare levels on different attributes. So, just as with the additive model, it seems that commensurateness must be implied by the axioms and that this could be used in the elicitation. Unfortunately, we will show that the picture is more complex with the discrete Sugeno integral than with the additive model.

- Greco et al. (2004) have shown that, under some conditions, there exists utility functions (one per attribute) that can be used to represent the preferences by means of a discrete Sugeno integral. These utility functions are of course not unique; but to what extent? We will provide a partial answer to this question.

By the way, since the non-compensatory model and the discrete Sugeno integral are equivalent under some conditions, our proof of the characterization of the non-compensatory model can be used as a proof of the characterization of the discrete Sugeno integral. This can prove useful since no proof of it has been published so far.[1]

This paper is organized as follows. The result of Greco et al. (2004) is presented in Sect. 2. We there show how to factorize their main condition into two simpler conditions. Section 3 introduces and characterizes what we will call the noncompensatory model for weak orders. Section 4 analyzes the links between the noncompensatory model for weak orders and the discrete Sugeno integral model. Section 5 presents examples showing that the conditions used in the main result are independent. Section 6 discusses the uniqueness of the representation in the discrete Sugeno integral model and further investigates the commensurateness issue. Section 7 briefly concludes with the mention of some directions for future research.

2 The Discrete Sugeno Integral

2.1 Background on the Discrete Sugeno Integral

Let $\beta = (\beta_1, \beta_2, \dots, \beta_p) \in [0,1]^p$. Let $(\cdot)_\beta$ be a permutation on $P = \{1, 2, \dots, p\}$ such that $\beta_{(1)_\beta} \leq \beta_{(2)_\beta} \leq \cdots \leq \beta_{(p)_\beta}$.

A capacity (see Choquet, 1953) on P is a function $\nu : 2^P \to [0,1]$ such that:

[1] It should be mentioned that a related result for the case of ordered categories is presented without proof in Słowiński, Greco, and Matarazzo (2002). This result is a particular case of the one presented in Greco et al. (2004) for weak orders with a finite number of distinct equivalence classes. A complete and quite simple proof for this particular case was proposed in Bouyssou and Marchant (2007), using comments made on an early version of the latter paper by Greco, Matarazzo, and Słowiński.

- $\nu(\varnothing) = 0$.
- $[A, B \in 2^P$ and $A \subseteq B] \Rightarrow \nu(A) \leq \nu(B)$.

The capacity ν is said to be normalized if, furthermore, $\nu(P) = 1$.

The discrete Sugeno integral of the vector $(\beta_1, \beta_2, \ldots, \beta_p) \in [0,1]^p$ w.r.t. the normalized capacity ν is defined by

$$S_\nu[\beta] = \bigvee_{i=1}^{p} \left[\beta_{(i)_\beta} \wedge \nu(A_{(i)_\beta})\right],$$

where $A_{(i)_\beta}$ is the element of 2^P equal to $\{(i)_\beta, (i+1)_\beta, \ldots, (p)_\beta\}$.

We refer the reader to Dubois, et al. (2001a) and Marichal (2000a, 2000b) for excellent surveys of the properties of the discrete Sugeno integral and its several possible equivalent definitions. Let us simply mention here that the reordering of the components of β in order to compute its Sugeno integral can be avoided noting that we may equivalently write

$$S_\nu[\beta] = \bigvee_{T \subseteq P} \left[\nu(T) \wedge \left(\bigwedge_{i \in T} \beta_i\right)\right]. \tag{1}$$

We will mainly use this presentation of the discrete Sugeno integral below.

2.2 *The Model*

Let $\succsim$ be a binary relation on a set $X = \prod_{i=1}^{n} X_i$ with $n \geq 2$. Elements of X will be interpreted as alternatives evaluated on a set $N = \{1, 2, \ldots, n\}$ of attributes. The relations $\succ$ and $\sim$ are defined as usual. We denote by X_{-i} the set $\prod_{j \in N \setminus \{i\}} X_j$. We abbreviate $Not[x \succsim y]$ as $x \not\succsim y$.

We say that $\succsim$ has a representation in the *discrete Sugeno integral model* if there are a normalized capacity μ on N and functions $u_i : X_i \to [0,1]$ such that, for all $x, y \in X$,

$$x \succsim y \Leftrightarrow S_{\langle \mu, u \rangle}(x) \geq S_{\langle \mu, u \rangle}(y),$$

where $S_{\langle \mu, u \rangle}(x) = S_\mu[(u_1(x_1), u_2(x_2), \ldots, u_n(x_n))]$.

2.3 *Axioms and Result*

A *weak order* is a complete and transitive binary relation. The set $Y \subseteq X$ is said to be dense in X for the weak order $\succsim$ if for all $x, y \in X$, $x \succ y$ implies $x \succsim z$ and $z \succsim y$, for some $z \in Y$. We say that the weak order $\succsim$ on X satisfies the *order-denseness condition* (condition *OD*) if there is a finite or countably infinite set $Y \subseteq X$ that is dense in X for $\succsim$. It is well-known (see Fishburn, 1970, p. 27; or Krantz et al., 1971, p. 40) that there is a real-valued function v on X such that, for all $x, y \in X$,

$$x \succsim y \Leftrightarrow v(x) \geq v(y),$$

if and only if $\succsim$ is a weak order on X satisfying the order-denseness condition.

Remark 1. Let $\succsim$ be a weak order on X. It is clear that $\sim$ is an equivalence and that the elements of $X/\sim$ are linearly ordered. We often abuse terminology and speak of equivalence classes of $\succsim$ to mean the elements of $X/\sim$. When $X/\sim$ is finite, we speak of the first equivalence class of $\succsim$ to mean the elements of $X/\sim$ that precede all others in the induced linear order.

The following condition was introduced in Greco et al. (2004). The relation $\succsim$ on X is said to be strongly 2-graded on attribute $i \in N$ (condition 2^*-graded$_i$) if, for all $x, y, z, w \in X$ and all $a_i \in X_i$,

$$\left.\begin{array}{c} x \succsim z \\ \text{and} \\ y \succsim w \\ \text{and} \\ z \succsim w \end{array}\right\} \Rightarrow \left\{\begin{array}{c} (a_i, x_{-i}) \succsim z \\ \text{or} \\ (x_i, y_{-i}) \succsim w, \end{array}\right.$$

where (a_i, x_{-i}) denotes the element of X obtained from $x \in X$ by replacing its ith coordinate by $a_i \in X_i$. The binary relation will be said to be *strongly 2-graded* (condition 2^*-graded) if it is strongly 2-graded on all attributes $i \in N$.

Although the above condition may look complex, it has a simple interpretation. Consider the particular case of condition 2^*-graded$_i$ in which $z = w$. Suppose that $(x_i, y_{-i}) \not\succsim w$. Since $(y_i, y_{-i}) \succsim w$ and $(x_i, y_{-i}) \not\succsim w$, this suggests that the level x_i is worse than y_i with respect to the alternative w. In this case, $(x_i, x_{-i}) \succsim w$ implies that $(a_i, x_{-i}) \succsim w$, for all $a_i \in X_i$. This means that, once we know that some level y_i is better than x_i w.r.t. to $w \in X$, there does not exist an element in X_i that could be worse than x_i, so that, if $(x_i, x_{-i}) \succsim w$, the same will be true replacing x_i by any element in X_i. This roughly implies that, for each $w \in X$, we can partition the elements of X_i into at most two categories of levels: the "satisfactory" ones and the "unsatisfactory" ones with respect to w. Condition 2^*-graded$_i$ implies these twofold partitions are not unrelated when considering distinct elements z and w in X.

Greco et al. (2004) state the following:

Theorem 1 (Greco et al. (2004, Theorem 3, p. 284)). *Let $\succsim$ be a binary relation on X. This relation has a representation in the discrete Sugeno integral model if and only if (iff) it is a weak order satisfying the order-denseness condition and being strongly 2-graded.*

The necessity of the conditions in this theorem is easy to establish. It is indeed clear that if $\succsim$ has a representation in the discrete Sugeno integral model, then it must be a weak order satisfying *OD*. It is not difficult to show that it must also satisfy 2^*-graded. Indeed, suppose that condition 2^*-graded$_i$ is violated, so that, for some $x, y, z, w \in X$ and some $a_i \in X_i$, we have $x \succsim z$, $y \succsim w$, $z \succsim w$, $(a_i, x_{-i}) \not\succsim z$ and $(x_i, y_{-i}) \not\succsim w$. Using $y \succsim w$ and $(x_i, y_{-i}) \not\succsim w$, we obtain $u_i(x_i) < S_{\langle\mu,u\rangle}(w)$. Because $z \succsim w$, we know that $S_{\langle\mu,u\rangle}(z) \geq S_{\langle\mu,u\rangle}(w)$, so that $S_{\langle\mu,u\rangle}(z) > u_i(x_i)$. Since $x \succsim z$

and $S_{\langle\mu,u\rangle}(z) > u_i(x_i)$, there is some $I \in 2^N$ such that $i \notin I$, $\mu(I) \geq S_{\langle\mu,u\rangle}(z)$ and $u_j(x_j) \geq S_{\langle\mu,u\rangle}(z)$, for all $j \in I$. This implies $S_{\langle\mu,u\rangle}((a_i,x_{-i})) \geq S_{\langle\mu,u\rangle}(z)$, so that $(a_i,x_{-i}) \succsim z$, a contradiction.

In Sect. 4, we give a proof of the sufficiency of the conditions, which links the discrete Sugeno integral model with the noncompensatory model studied in Sect. 3.

2.4 Factorization of 2^**-Graded*$_i$

We say that the relation $\succsim$ satisfies condition $AC1_i$ if, for all $x,y,z,w \in X$,

$$\left.\begin{array}{c} x \succsim y \\ \text{and} \\ z \succsim w \end{array}\right\} \Rightarrow \left\{\begin{array}{c} (z_i,x_{-i}) \succsim y, \\ \text{or} \\ (x_i,z_{-i}) \succsim w. \end{array}\right.$$

We say that $\succsim$ satisfies $AC1$ if it satisfies $AC1_i$ for all $i \in N$.

Condition $AC1$ was proposed and studied in Bouyssou and Pirlot (2004). It plays a central role in the characterization of binary relations (that may be incomplete or intransitive) admitting a decomposable representation of the type:

$$x \succsim y \Leftrightarrow G[u_1(x_1),\ldots,u_n(x_n),u_1(y_1),\ldots,u_n(y_n)] \geq 0,$$

with G being nondecreasing (resp. nonincreasing) in its first (resp. last) n arguments (see Bouyssou & Pirlot, 2004, Theorem 2). We refer to Bouyssou and Pirlot (2004) for a detailed interpretation of this condition. Let us simply mention here that condition $AC1_i$, independently of any transitivity or completeness properties of $\succsim$, allows to order the elements of X_i in such a way that this ordering is compatible with $\succsim$ (see Lemma 3 below).

We say that $\succsim$ is 2-graded on attribute $i \in N$ (condition 2-graded$_i$) if, for all $x,y,z,w \in X$ and all $a_i \in X_i$,

$$\left.\begin{array}{c} x \succsim z \\ \text{and} \\ (y_i,x_{-i}) \succsim z \\ \text{and} \\ y \succsim w \\ \text{and} \\ z \succsim w \end{array}\right\} \Rightarrow \left\{\begin{array}{c} (a_i,x_{-i}) \succsim z \\ \text{or} \\ (x_i,y_{-i}) \succsim w. \end{array}\right.$$

We say that $\succsim$ is *2-graded* (condition 2-graded) if it is 2-graded on all attributes $i \in N$. Condition 2-graded weakens condition 2^*-graded adjoining it the additional premise $(y_i,x_{-i}) \succsim z$. It has a similar interpretation. We have:

Lemma 1. *Let* $\succsim$ *be a weak order on the set* X*. Then* $\succsim$ *satisfies* $AC1_i$ *and* 2*-graded*$_i$ *iff it satisfies* 2^**-graded*$_i$*.*

Proof. [$AC1_i$ & 2-graded$_i \Rightarrow 2^*$-graded$_i$]. Suppose that $x \succsim z$, $y \succsim w$ $z \succsim w$. Using $AC1_i$, $x \succsim z$ and $y \succsim w$ implies either $(y_i, x_{-i}) \succsim z$ or $(x_i, y_{-i}) \succsim w$. In the latter case, one of the two conclusions of 2^*-graded$_i$ holds. In the former case, we have $x \succsim z$, $(y_i, x_{-i}) \succsim z$, $y \succsim w$ and $z \succsim w$, so that 2-graded$_i$ implies either $(a_i, x_{-i}) \succsim z$, for all $a_i \in X_i$ or $(x_i, y_{-i}) \succsim w$, which is the desired conclusion.

[2^*-graded$_i \Rightarrow AC1_i$ & 2-graded$_i$]. It is clear that 2^*-graded$_i$ implies 2-graded$_i$ since 2-graded$_i$ is obtained from 2^*-graded$_i$ by adding to it an additional premise. Suppose that $x \succsim y$ and $z \succsim w$. Since $\succsim$ is complete, we have either $y \succsim w$ or $w \succsim y$. If $y \succsim w$, we have $x \succsim y$, $z \succsim w$ and $y \succsim w$, so that 2^*-graded$_i$ implies $(x_i, z_{-i}) \succsim w$ or $(a_i, x_{-i}) \succsim y$, for all $a_i \in X_i$. Taking $a_i = z_i$ shows that $AC1_i$ holds in this case. The proof is similar if it is supposed that $w \succsim y$. □

Why is this factorization interesting? First, it makes clear that the condition used by Greco et al. (2004) combines two distinct properties: (1) the elements of X_i can be ordered and (2) for each $w \in X$, we can partition the elements of X_i into at most two categories with respect to w. This helps us better understand the behavioural content of the conditions. It can also be useful for empirically testing the validity of the discrete Sugeno integral model. Indeed, if we run an experiment for testing whether a complex condition (like 2^*-graded) is satisfied by subjects, it is likely that it will be rejected. This does not mean that the condition is completely wrong. It can happen that only part of it is wrong. Therefore, testing more elementary conditions can help identify what is wrong with a model. Finally, this factorization permit us to show that the discrete Sugeno integral model can be viewed as a particular case of a general decomposable representation, investigated and characterized in Bouyssou and Pirlot (2004) and Greco et al. (2004). Furthermore, thanks to the factorization, we know exactly what has to be imposed on the decomposable model in order to obtain the discrete Sugeno integral model.

3 The Noncompensatory Model for Weak Orders

This section presents and characterizes the noncompensatory model for weak orders. It will turn out to have intimate connections with the discrete Sugeno integral model.

The following non-numerical model is inspired from the work of Słowiński et al. (2002) and Bouyssou and Marchant (2007) who analyze ordered partitions of a Cartesian product using similar models. A similar model was first suggested in Fishburn (1978).

Definition 1. A weak order $\succsim$ on X has a representation in the *noncompensatory model* if for all $x \in X$, there are sets:

1. $A_i^x \subseteq X_i$, for all $i \in N$.
2. $F^x \subseteq 2^N$ such that

$$[I \in F^x \text{ and } I \subseteq J \in 2^N] \Rightarrow J \in F^x, \tag{2}$$

that are such that, for all $x, y \in X$,

$$x \succsim y \Rightarrow \begin{cases} A_i^x \subseteq A_i^y \\ \text{and} \\ F^x \subseteq F^y \end{cases} \tag{3}$$

and

$$x \succsim y \Leftrightarrow \{i \in N : x_i \in A_i^y\} \in F^y. \tag{4}$$

We often write $A(x,y)$ instead of $\{i \in N : x_i \in A_i^y\}$.

The noncompensatory model[2] can be interpreted as follows. For each $x \in X$ we isolate on each attribute a subset $A_i^x \subseteq X_i$ containing the levels on attribute i that are satisfactory for x. In order for an alternative to be at least as good as x, it must have evaluations that are satisfactory for x on a subset of attributes belonging to F^x. The subsets of attributes belonging to F^x are interpreted as subsets that are "sufficiently important" to warrant preference on x.

With this interpretation in mind, the constraint (3) means that if x is at least as good as y then every level that is satisfactory for x must be satisfactory for y. Furthermore, subsets of attributes that are "sufficiently important" to warrant preference on x must also be "sufficiently important" to warrant preference on y. Given the above interpretation of F^x, the constraint (2) simply says that any superset of a set that is "sufficiently important" to warrant preference on x must have the same property.

Suppose that $x \not\succsim y$ and that $x_i \in A_i^y$, for some $i \in N$. In the noncompensatory model, we have $(z_i, x_{-i}) \not\succsim y$, for all $z_i \in X_i$. It is therefore impossible, starting from x, to obtain an alternative that would be at least as good as y by modifying the evaluation of x on the ith attribute. In other terms, the fact that $A(x,y) \notin F^y$ cannot be compensated by improving the evaluation of x on an attribute in $A(x,y)$. Hence, our name for this model.

We first observe that a weak order having a representation in the noncompensatory model must satisfy $AC1$ and 2-graded.

Lemma 2. *If weak order $\succsim$ on X has a representation in the noncompensatory model, then it satisfies $AC1$ and 2-graded.*

Proof. $[AC1_i]$. Suppose that $x \succsim y$, $z \succsim w$, $(z_i, x_{-i}) \not\succsim y$ and $(x_i, z_{-i}) \not\succsim w$. It is easy to see that $x \succsim y$ and $(z_i, x_{-i}) \not\succsim y$ imply $x_i \in A_i^y$ and $z_i \notin A_i^y$. Similarly, $z \succsim w$ and $(x_i, z_{-i}) \not\succsim w$ imply $z_i \in A_i^w$ and $x_i \notin A_i^w$. Because $\succsim$ is complete, we have either $y \succsim w$ or $w \succsim y$. Hence, we have either $A_i^y \subseteq A_i^w$ or $A_i^w \subseteq A_i^y$, a contradiction.

[2-graded$_i$]. Suppose that 2-graded$_i$ is violated, so that, for some $x, y, z, w \in X$ and some $a_i \in X_i$, $(x_i, x_{-i}) \succsim z$, $(y_i, x_{-i}) \succsim z$, $(y_i, y_{-i}) \succsim w$, $z \succsim w$, $(a_i, x_{-i}) \not\succsim z$ and

[2] The noncompensatory model for weak orders must not be confused with "noncompensatory preferences" as introduced in Fishburn (1976). Noncompensatory preferences in the sense of Fishburn (1976) are preferences that result from an "ordinal aggregation" in the context of MCDM that is quite close from the type of aggregation studied in social choice theory in the vein of Arrow (1963) (for a recent analysis of such preferences, see Bouyssou and Pirlot (2005)). As first shown in Fishburn (1975), noncompensatory preferences that are weak orders are, except in degenerate cases, lexicographic.

$(x_i, y_{-i}) \not\succsim w$. Using the definition of the noncompensatory model, $(y_i, y_{-i}) \succsim w$ and $(x_i, y_{-i}) \not\succsim w$ imply $y_i \in A_i^w$ and $x_i \notin A_i^w$. Similarly, $(x_i, x_{-i}) \succsim z$ and $(a_i, x_{-i}) \not\succsim z$ imply $x_i \in A_i^z$ and $a_i \notin A_i^z$. Since $z \succsim w$, we have $A_i^z \subseteq A_i^w$, a contradiction. □

The main result of this section says that, for weak orders, the noncompensatory model is fully characterized by condition 2^*-graded or, equivalently, by the conjunction of $AC1$ and 2-graded.

Proposition 1. *If a weak order on X satisfies $AC1$ and 2-graded then it has a representation in the noncompensatory model.*

Before proving Proposition 1, we will have to go through a few definitions and lemmas.

Consider an attribute $i \in N$. We define the *left marginal trace* on attribute $i \in N$ letting, for all $x_i, y_i \in X_i$, all $a_{-i} \in X_{-i}$ and all $z \in X$,

$$x_i \succsim_i^+ y_i \Leftrightarrow [(y_i, a_{-i}) \succsim z \Rightarrow (x_i, a_{-i}) \succsim z].$$

Similarly, given $a \in X$, we define the left marginal trace on attribute $i \in N$ with respect to $a \in X$, letting, for all $x_i, y_i \in X_i$ and all $z_{-i} \in X_{-i}$,

$$x_i \succsim_i^{+(a)} y_i \Leftrightarrow [(y_i, z_{-i}) \succsim a \Rightarrow (x_i, z_{-i}) \succsim a].$$

The symmetric and asymmetric parts of $\succsim_i^+$ (resp. $\succsim_i^{+(a)}$) are denoted $\sim_i^+$ and $\succ_i^+$ (resp. $\sim_i^{+(a)}$ and $\succ_i^{+(a)}$). It is clear that $\succsim_i^+$ and $\succsim_i^{+(a)}$ are always reflexive and transitive. They may be incomplete however.

We note a few useful obvious connections between $\succsim_i^{+(a)}$, $\succsim_i^+$ and $\succsim$ in the following lemma.

Lemma 3. *We have, for all $i \in N$, all $z, w \in X$ and all $x_i, y_i \in X_i$:*

1. *$x_i \succsim_i^+ y_i \Leftrightarrow [x_i \succsim_i^{+(a)} y_i$, for all $a \in X]$.*
2. *$[z \succsim w, x_i \succsim_i^+ z_i] \Rightarrow (x_i, z_{-i}) \succsim w$.*
3. *Furthermore, if $\succsim$ is reflexive then, $[z_j \sim_j^+ w_j$, for all $j \in N] \Rightarrow z \sim w$.*
4. *The relation $\succsim_i^+$ is complete iff $AC1_i$ holds.*

Proof. Parts 1 and 2 easily follow from the definitions. Part 3 follows from Part 2 and the fact that $w \succsim w$. It is obvious that negating the completeness of $\succsim_i^+$ is equivalent to negating $AC1_i$. □

Remark 2. When $\succsim$ is a weak order, condition $AC1_i$ is equivalent to supposing that, for all $x_i, y_i \in X_i$ and all $z_{-i}, w_{-i} \in X_{-i}$ $(x_i, z_{-i}) \succ (y_i, z_{-i}) \Rightarrow (x_i, w_{-i}) \succsim (y_i, w_{-i})$, i.e., that attribute i is weakly separable, using the terminology of Bouyssou and Pirlot (2004).

Indeed suppose that $\succsim$ satisfies $AC1_i$ and is such that attribute i is not weakly separable. Therefore there are $x_i, y_i \in X_i$ and $z_{-i}, w_{-i} \in X_{-i}$ such that $(x_i, z_{-i}) \succ (y_i, z_{-i})$ and $(y_i, w_{-i}) \succ (x_i, w_{-i})$. Since $\succsim$ is reflexive, we have $(x_i, z_{-i}) \succsim (x_i, z_{-i})$

and $(y_i, w_{-i}) \succsim (y_i, w_{-i})$. Using $AC1_i$, we have either $y_i \succsim_i^+ x_i$ or $x_i \succsim_i^+ y_i$, so that either $(y_i, z_{-i}) \succsim (x_i, z_{-i})$ or $(x_i, w_{-i}) \succsim (y_i, w_{-i})$, a contradiction.

Conversely, suppose that $\succsim$ is complete and transitive and that attribute i is weakly separable. Suppose that $AC1_i$ is violated so that, since $\succsim$ is complete, $(x_i, x_{-i}) \succsim y$, $(z_i, z_{-i}) \succsim w$, $y \succ (z_i, x_{-i})$ and $w \succ (x_i, z_{-i})$, for some $x, y, z, w \in X$. Since $\succsim$ is a weak order, we obtain $(x_i, x_{-i}) \succ (z_i, x_{-i})$ and $(z_i, z_{-i}) \succ (x_i, z_{-i})$, which violates the weak separability of attribute i.

We say that a weak order $\succsim$ is *weakly separable* if, for all $i \in N$, it is weakly separable for attribute i.

Hence, combining Lemma 1 with Theorem 1 shows that a relation has a representation in the discrete Sugeno integral model iff it is a weakly separable weak order satisfying *OD* and 2-graded.

Bouyssou and Pirlot (2004, Propositions 8 and B.3) have shown that, for weak orders satisfying *OD*, weak separability is a necessary and sufficient condition to obtain a general decomposable representation in which, for all $x, y \in X$,

$$x \succsim y \Leftrightarrow F[u_1(x_1), \dots, u_n(x_n)] \geq F[u_1(y_1), \dots, u_n(y_n)],$$

with F being nondecreasing in all its arguments (see also Greco et al., 2004, Theorem 1). Hence, condition 2-graded is exactly what must be added to go from this general decomposable representation to a representation in the discrete Sugeno integral model.

The following lemma makes precise the structure of the relations $\succsim_i^{+(a)}$ when $\succsim$ is a weak order satisfying $AC1_i$ and 2-graded$_i$.

Lemma 4. *Let $\succsim$ be a weak order on X satisfying $AC1_i$ and 2-graded$_i$. Then:*

1. *$\succsim_i^{+(a)}$ is complete for all $a \in X$.*
2. *$x_i \succ_i^{+(a)} y_i \Rightarrow [x_i \succsim_i^{+(b)} y_i$ for all $b \in X]$.*
3. *$\succsim_i^{+(a)}$ has at most two distinct equivalence classes, for all $a \in X$.*
4. *$[x_i \sim_i^{+(a)} z_i$ and $x_i \succ_i^{+(a)} y_i] \Rightarrow x_i \sim_i^{+(b)} z_i$, for all $b \in X$ such that $a \succsim b$.*
5. *If $a \succsim b$ and both $\succsim_i^{+(a)}$ and $\succsim_i^{+(b)}$ are nontrivial then the first equivalence class of $\succsim_i^{+(a)}$ is included in the first equivalence class of $\succsim_i^{+(b)}$.*

Proof. Parts 1 and 2 follow from Lemma 3 since $AC1_i$ implies that $\succsim_i^+$ is complete.

Part 3. Suppose that $\succsim_i^{+(a)}$ has at least three distinct equivalence classes. This implies that $(x_i, c_{-i}) \succsim a$, $(y_i, c_{-i}) \not\succsim a$, $(y_i, d_{-i}) \succsim a$ and $(z_i, d_{-i}) \not\succsim a$, for some $x_i, y_i, z_i \in X_i$, some $c_{-i}, d_{-i} \in X_{-i}$ and some $a \in X$. Using $AC1_i$, $(x_i, c_{-i}) \succsim a$, $(y_i, d_{-i}) \succsim a$ and $(y_i, c_{-i}) \not\succsim a$ imply $(x_i, d_{-i}) \succsim a$. Using 2-graded$_i$, $(y_i, d_{-i}) \succsim a$, $(x_i, d_{-i}) \succsim a$, $(x_i, c_{-i}) \succsim a$ and $a \succsim a$ imply $(y_i, c_{-i}) \succsim a$ or $(z_i, d_{-i}) \succsim a$, a contradiction.

Part 4. Suppose that $x_i \sim_i^{+(a)} z_i$, $x_i \succ_i^{+(a)} y_i$, $a \succsim b$ and $x_i \succ_i^{+(b)} z_i$ (the proof for the case $z_i \succ_i^{+(b)} x_i$ being similar). By construction, we have $(x_i, w_{-i}) \succsim b$, $(z_i, w_{-i}) \not\succsim b$, $(x_i, t_{-i}) \succsim a$ and $(y_i, t_{-i}) \not\succsim a$. Since $x_i \sim_i^{+(a)} z_i$, we must have $(z_i, t_{-i}) \succsim a$. Using $AC1_i$, $(x_i, w_{-i}) \succsim b$, $(z_i, t_{-i}) \succsim a$ and $(z_i, w_{-i}) \not\succsim b$ imply $(x_i, t_{-i}) \succsim a$. Using

2-graded$_i$, $(z_i,t_{-i}) \succsim a$, $(x_i,t_{-i}) \succsim a$, $(x_i,w_{-i}) \succsim b$ and $a \succsim b$ imply $(z_i,w_{-i}) \succsim b$ or $(y_i,t_{-i}) \succsim a$, a contradiction.

Part 5. Suppose that $a \succsim b$, $x_i \succ_i^{+(a)} y_i$ and $z_i \succ_i^{+(b)} x_i$. Using Part 2, we know that $z_i \succsim_i^{+(a)} x_i$. Because we know from Part 3 that $\succsim_i^{+(a)}$ has at most two equivalence classes, we must have $z_i \sim_i^{+(a)} x_i$. Using Part 4, $a \succsim b$, $z_i \sim_i^{+(a)} x_i$ and $x_i \succ_i^{+(a)} y_i$ imply $z_i \sim_i^{+(b)} x_i$, a contradiction. □

Let $\succsim$ be a weak order on X satisfying $AC1_i$ and 2-graded$_i$. Let $i \in N$. For all $a \in X$, we know that either $\succsim_i^{+(a)}$ is trivial or $\succsim_i^{+(a)}$ has two distinct equivalence classes. Define $B_i^a \subset X_i$ as the empty set in the first case and as the elements in the first equivalence class in the second case. Define C_i^a letting:

$$C_i^a = \bigcup_{\{x \in X : x \succsim a\}} B_i^x.$$

The following lemma studies the properties of the sets C_i^a.

Lemma 5. *Let $\succsim$ be a weak order on X satisfying $AC1$ and 2-graded. For all $x,y,z,w \in X$ and all $i \in N$:*

1. $z \succsim w \Rightarrow C_i^z \subseteq C_i^w$.
2. $\{j \in N : y_j \in C_j^z\} \subseteq \{j \in N : x_j \in C_j^z\} \Rightarrow [x_i \succsim_i^{+(z)} y_i$ for all $i \in N]$.
3. $C_i^x \subsetneq X_i$.

Proof. Part 1. We have $x_i \in C_i^z$ iff $x_i \in B_i^a$, for some $a \succsim z$. Because $z \succsim w$ and $\succsim$ is a weak order, we have $a \succsim z$. Hence, $x_i \in B_i^a$, for some $a \succsim w$, so that $x_i \in C_i^w$.

Part 2. If $\succsim_i^{+(z)}$ is trivial, we have by definition $x_i \sim_i^{+(z)} y_i$. If $\succsim_i^{+(z)}$ is not trivial, it follows from Part 5 of Lemma 4 that C_i^z is equal to the first equivalence class of $\succsim_i^{+(z)}$. If $y_i \in C_i^z$, we have $x_i \in C_i^z$, so that $x_i \sim_i^{+(z)} y_i$. If $y_i \notin C_i^z$, then we have $z_i \succsim_i^{+(z)} y_i$.

Part 3. By construction, B_i^y is strictly included in X_i. As the set C_i^x is obtained by taking the union of sets B_i^y, the conclusion follows. □

Lemma 6. *Let $\succsim$ be a weak order on X satisfying $AC1_i$ and 2-graded$_i$. Define, for all $x \in X$, the set $G^x \subseteq 2^N$ letting $I \in G^x$ whenever we have $\{i \in N : z_i \in C_i^x\} \subseteq I$, for some $z \in X$ such that $z \succsim x$. We have, for all $x,y \in X$:*

1. $x \succsim y \Leftrightarrow \{i \in N : x_i \in C_i^y\} \in G^y$.
2. $[I \in G^x$ and $I \subseteq J] \Rightarrow J \in G^x$.
3. $x \succsim y \Rightarrow G^x \subseteq G^y$.

Proof. Part 1. By construction, if $x \succsim y$ then $\{i \in N : x_i \in C_i^y\} \in G^y$. Let us show that the reverse implication is true. Suppose that $\{i \in N : x_i \in C_i^y\} \in G^y$. This implies that $\{i \in N : z_i \in C_i^y\} \subseteq \{i \in N : x_i \in C_i^y\}$, for some $z \in X$ such that $z \succsim y$. Using Part 2 of Lemma 5, $\{i \in N : z_i \in C_i^y\} \subseteq \{i \in N : x_i \in C_i^y\}$ implies $x_i \succsim_i^{+(y)} z_i$, for all $i \in N$. Hence, $z \succsim y$ implies $x \succsim y$.

Part 2 follows from the definition of the sets G^x.

Part 3. Suppose that $x \succsim y$ and let $I \in G^x$. Let us show that we must have $I \in G^y$. By construction, $I \in G^x$ implies that $\{i \in N : z_i \in C_i^x\} \subseteq I$, for some $z \in X$ such that $z \succsim x$. Consider the alternative $w \in X$ defined in the following way:

- If $z_i \in C_i^x$, let $w_i = z_i$. We have $w_i \in C_i^x$. Using Part 1 of Lemma 5, we know that this implies $w_i \in C_i^y$.
- If $z_i \notin C_i^x$. Using Part 3 of Lemma 5, we know that $C_i^y \subsetneq X_i$. We take w_i to be any element in $X_i \setminus C_i^y$. Because, we know that $C_i^x \subseteq C_i^y$, we have $w_i \notin C_i^x$.

By construction we have, for all $i \in N$, $z_i \in C_i^x \Leftrightarrow w_i \in C_i^x \Leftrightarrow w_i \in C_i^y$. Hence, we have $\{i \in N : z_i \in C_i^x\} = \{i \in N : w_i \in C_i^x\} = \{i \in N : w_i \in C_i^y\}$. The first equality implies $w \succsim x$. Using the fact that $\succsim$ is a weak order, we obtain $w \succsim y$. Hence, we have $\{i \in N : w_i \in C_i^y\} \subseteq I$ and $w \succsim y$. This implies $I \in G^y$. □

Defining $A_i^x = C_i^x$ and $F^x = G^x$, the sufficiency proof of Proposition 1 follows from combining Lemmas 5 and 6.

4 The Noncompensatory Model and the Discrete Sugeno Integral Model

The main result in this section says that if a weak order has a representation in the noncompensatory model and has a numerical representation, then it has a representation in the discrete Sugeno integral model. This will help to complete the proof of Theorem 1.

Proposition 2. *Let $\succsim$ be a weak order on X. Suppose that $\succsim$ can be represented in the noncompensatory model and that there is a real function v on X such that, for all $x, y \in X$,*

$$x \succsim y \Leftrightarrow v(x) \geq v(y). \tag{5}$$

Then $\succsim$ has a representation in the discrete Sugeno integral model.

Proof. Let $\succsim$ be a weak order representable in the noncompensatory model and such that there is a real-valued function v satisfying (5). We may assume w.l.o.g. that, for all $x \in X$, $v(x) \in [0,1]$. Furthermore, if there are minimal elements in X for $\succsim$, we may assume w.l.o.g. that v gives the value 0 to these elements. We consider now any such function v. For all $i \in N$, define u_i letting, for all $x_i \in X_i$,

$$u_i(x_i) = \begin{cases} \sup_{\{w \in X : x_i \in A_i^w\}} v(w) & \text{if } \exists w : x_i \in A_i^w, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Define μ on 2^N letting, for all $I \in 2^N$,

$$\mu(I) = \begin{cases} \sup_{\{w \in X : I \in F^w\}} v(w) & \text{if } \exists w : I \in F^w, \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

Since $I \in F^w$ and $J \supseteq I$ entails $J \in F^w$, we have that $\mu(J) \geq \mu(I)$. Hence, μ is a nondecreasing set function.

Let us show that $\mu(\varnothing) = 0$. If there is no $w \in X$ such that $\varnothing \in F^w$, then we have, by construction, $\mu(\varnothing) = 0$. Suppose that $X_\varnothing = \{w \in X : \varnothing \in F^w\} \neq \varnothing$. From the definition of the noncompensatory model, it follows that, for all $x \in X$ and all $w \in X_\varnothing$, we have $x \succsim w$. Hence, for all $w \in X_\varnothing$, w is minimal for $\succsim$. We therefore have $v(w) = 0$, for all $w \in X_\varnothing$ and, hence, $\mu(\varnothing) = 0$. This shows that μ defined by (7) is a capacity on 2^N. It is not necessarily normalized, i.e., we may not have that $\mu(N) = 1$.

Independently of the normalization of μ, we can compute, for all $x \in X$, $S_{\mu,\mathbf{u}}(x)$ letting:

$$S_{\langle \mu,u \rangle}(x) = \bigvee_{I \subseteq N} \left[\mu(I) \wedge \left(\bigwedge_{i \in I} u_i(x_i) \right) \right]. \tag{8}$$

It is clear that, for all $y \in X$, $S_{\langle \mu,u \rangle}(y) \in [0,1]$. Let us show that, for all $y \in X$, $S_{\langle \mu,u \rangle}(y) = v(y)$, which will complete the proof if μ happens to be normalized.

Let $x, y \in X$ be such that $x \succsim y$. This implies $A(x,y) = \{i \in N : x_i \in A_i^y\} \in F^y$. Hence, for all $i \in A(x,y)$, $y \in \{w \in X : x_i \in A_i^w\}$, so that $u_i(x_i) \geq v(y)$. Similarly, $y \in \{w \in X : A(x,y) \in F^w\}$, so that $\mu(A(x,y)) \geq v(y)$. Hence, for $I = A(x,y)$, we have

$$\mu(I) \wedge \left(\bigwedge_{i \in I} u_i(x_i) \right) \geq v(y).$$

In view of (8), this implies $S_{\langle \mu,u \rangle}(x) \geq v(y)$. Since $\succsim$ is reflexive, this shows that, for all $y \in X$, $S_{\langle \mu,u \rangle}(y) \geq v(y)$.

We now prove that, for all $y \in X$, $S_{\langle \mu,u \rangle}(y) \leq v(y)$. If y is maximal for $\succsim$ (i.e., $y \succsim x$, for all $x \in X$), we have $v(y) \geq v(x)$, for all $x \in X$. The definition of u_i and μ obviously implies that they cannot exceed the maximal value of v on X. Hence, in this case, we have $S_{\langle \mu,u \rangle}(y) \leq v(y)$.

Suppose henceforth that $y \in X$ is not maximal for $\succsim$, so that $x \succ y$, for some $x \in X$. This implies that $A(y,x) = \{i \in N : y_i \in A_i^x\} \notin F^x$. Define $A_y = \bigcup_{z \succ y} A(y,z)$. Because $A(y,z) \subseteq N$, N is a finite set, and $z' \succsim z$ implies $A(y,z') \subseteq A(y,z)$, there is an element $z_0 \in X$ with $z_0 \succ y$ that is such that $A(y,z_0) = A_y$ and $A(y,z) = A_y$, for all $z \in X$ such that $z_0 \succsim z \succ y$.

We claim the following:

Claim 1: for all $j \notin A_y$, $u_j(y_j) \leq v(y)$.
Claim 2: for all $I \subseteq A_y$, $\mu(I) \leq v(y)$.

Proof of Claim 1. Let $j \notin A_y$, so that $y_j \notin A_j^{z_0}$. If the set $\{w \in X : y_j \in A_j^w\}$ is empty, we have $u_j(y_j) = 0$ and the claim trivially holds. Otherwise, let $w \in X$ such that $y_j \in A_j^w$. If $w \succ z_0$, we have $A_j^w \subseteq A_j^{z_0}$, so that $y_j \in A_j^w$ implies $y_j \in A_j^{z_0}$, a contradiction. If $z_0 \succsim w \succ y$, we know that $A(y,w) = A(y,z_0)$. This is contradictory since $y_j \in A_j^w$ and $y_j \notin A_j^{z_0}$. Hence, when $j \notin A_y$, we must have $y \succsim w$, for all $w \in X$ such that $y_j \in A_j^w$. This implies that $u_j(y_j) = \sup_{\{w \in X : y_j \in A_j^w\}} v(w) \leq v(y)$, for all $j \notin A_y$.

Proof of Claim 2. Let $I \subseteq A_y$. If the set $\{w \in X : I \in F^w\}$ is empty, we have $\mu(I) = 0$ and the claim follows. Otherwise, let $w \in X$ such that $I \in F^w$. Suppose that $w \succ z_0$. This implies $F^w \subseteq F^{z_0}$, so that $I \in F^{z_0}$. Because $I \subseteq A_y$, we obtain $A_y \in F^{z_0}$. This is contradictory since $z_0 \succ y$ implies that $A_y = A(y, z_0) \notin F^{z_0}$. Suppose now that $z_0 \succsim w \succ y$. We have $A(y,w) = A_y \notin F^w$. But, since $I \in F^w$ and $I \subseteq A_y$, we obtain $A_y \in F^w$, a contradiction. Hence, for all $w \in X$ such that $I \in F^w$, we have $y \succsim w$. This implies $\mu(I) = \sup_{\{w \in X : I \in F^w\}} v(w) \leq v(y)$.

Using Claims 1 and 2, we establish that $S_{\langle \mu,u \rangle}(y) \leq v(y)$ for any $y \in X$ that is not maximal. Let $I \subseteq N$. We distinguish two cases in order to compute

$$\mu(I) \wedge \left(\bigwedge_{i \in I} u_i(x_i) \right).$$

1. If I is not included in A_y, we know that there is $j \in I$ such that $j \notin A_y$. Hence, using Claim 1, $u_j(y_j) \leq v(y)$ so that $\mu(I) \wedge (\bigwedge_{i \in I} u_i(y_i)) \leq v(y)$.
2. If I is included in A_y, using Claim 2, we have $\mu(I) \leq v(y)$. Hence, we know that $\mu(I) \wedge (\bigwedge_{i \in I} u_i(y_i)) \leq v(y)$.

Hence, for all $I \subseteq N$, we have $\mu(I) \wedge (\bigwedge_{i \in I} u_i(y_i)) \leq v(y)$, so that $S_{\langle \mu,u \rangle}(y) \leq v(y)$. This proves that, for all $y \in X$, $S_{\langle \mu,u \rangle}(y) = v(y)$.

It remains to show that we may always build a representation in the discrete Sugeno integral model using a *normalized* capacity, i.e., a capacity ν such that $\nu(N) = 1$.

Using the above construction, the value of $\mu(N)$ is obtained using (7). We have $\mu(N) = \sup_{w \in X} v(w)$, since for all $w \in X$, $N \in F^w$. If the weak order $\succsim$ is not trivial, we have $\mu(N) > 0$. In order to obtain a representation leading to a normalized capacity, it suffices to apply the above construction to the function u obtained by dividing v by $\mu(N)$. If the weak order $\succsim$ is trivial, it is easy to see that it has a representation in the noncompensatory model such that, for all $x \in X$ and all $i \in N$, $A_i^x = X_i$ and $F^x = \{N\}$. Defining, for all $i \in N$ and all $x_i \in X_i$, $u_i(x_i) = 1$, $\mu(N) = 1$ and $\mu(A) = 0$, for all $A \subsetneq N$, leads to a representation of this trivial weak order in the discrete Sugeno integral model. □

The sufficiency proof of Theorem 1 follows from combining Lemma 1 with Propositions 1 and 2. This amounts to characterizing the discrete Sugeno integral model by the conjunction of any of the following three equivalent sets of conditions:

- Completeness, transitivity, *OD*, *AC*1 and 2-graded
- Completeness, transitivity, *OD*, weak separability and 2-graded
- Completeness, transitivity, *OD* and 2^*-graded

The examples in the following section show no condition in the first set is redundant.

Remark 3. Consider a nontrivial weak order $\succsim$ on X that satisfies the hypotheses of Proposition 2. The proof of this proposition establishes that *any* function $v : X \to [0,1]$ satisfying (5) and giving a value 0 to the minimal elements in X for $\succsim$ (if any) can be used to define a representation in the Sugeno integral model. The functions u_i and the (non-necessarily normalized) capacity μ used in this representation can be defined on the basis of v using (6) and (7).

In other words, any (bounded) numerical representation v of a weak order representable in the noncompensatory model is essentially a Sugeno integral. By “essentially”, we mean that a positive affine transformation may have to be applied first to the numerical representation v in order that the minimal elements in X (if any) receive the value 0 and that the supremum of v is 1. This transformation is only needed to ensure that $\mu(\varnothing) = 0$ and μ is a normalized capacity. Note that applying (6) and (7) to any bounded numerical representation of the preference would yield u_i's and μ such that formula (8) would restate the value of $v(x)$, even if μ does not satisfy $\mu(\varnothing) = 0$ or is not normalized.

Furthermore, as shown in this proof, (6) and (7) can be viewed as *inversion formulas* for the discrete Sugeno integral model in the following sense. If we know the value of $S_{\langle \mu,u \rangle}(x)$, for all $x \in X$, without knowing the functions μ and u_i, it is possible to use (6) and (7) to build functions u_j and a capacity μ that allow to reconstruct all these values using the discrete Sugeno integral formula (8).

5 Independence of Conditions

When strong 2-gradedness is factorized using $AC1$ and 2-gradedness, Theorem 1 uses five conditions: completeness, transitivity, $AC1$, 2-gradedness and order-denseness. The five examples below show that none of these conditions can be dispensed with.

Example 1. Let $X = \{x_1, y_1\} \times \{x_2, y_2\}$. Let $\succsim$ be identical to the weak order

$$(y_1, y_2) \succ [(x_1, y_2), (y_1, x_2)] \succ (x_1, x_2),$$

except that we have removed two arcs from $\succsim$, so as to have $(x_1, y_2) \not\succsim (y_1, x_2)$ and $(y_1, x_2) \not\succsim (x_1, y_2)$. It is clear that $\succsim$ is transitive but is not complete. Since X_1 and X_2 have only two elements, condition 2-graded trivially holds. It is not difficult to check that we have $y_1 \succ_1^+ x_1$ and $y_2 \succ_2^+ x_2$, so that $AC1$ holds.

Example 2. Let $X = \{x_1, y_1\} \times \{x_2, y_2\}$. Let $\succsim$ be identical to the trivial weak order except that we have removed one arc from $\succsim$, so as to have $(x_1, x_2) \not\succsim (y_1, y_2)$. It is not difficult to see that the resulting relation is complete but not transitive (it is a semi-order). Since X_1 and X_2 have only two elements, condition 2-graded trivially holds. It is not difficult to check that we have $y_1 \succ_1^+ x_1$ and $y_2 \succ_2^+ x_2$, so that $AC1$ holds.

Example 3. Let $X = \{x_1, y_1, z_1\} \times \{x_2, y_2\} \times \{x_3, y_3\}$. Let $\succsim$ be the weak order such that:

$$[(x_1, x_2, x_3), (y_1, x_2, x_3)]$$
$$\succ$$

$$[(x_1,x_2,y_3),(x_1,y_2,x_3),(y_1,x_2,y_3),(y_1,y_2,x_3),\\ (y_1,y_2,y_3),(z_1,x_2,x_3),(z_1,x_2,y_3),(z_1,y_2,x_3)]\\ \succ\\ [(z_1,y_2,y_3),(x_1,y_2,y_3)].$$

We have $y_1 \succ_1^+ x_1 \succ_1^+ z_1$, $x_2 \succ_2^+ y_2$ and $x_3 \succ_3^+ y_3$, which shows that $AC1$ holds.

Conditions 2-graded$_2$ and 2-graded$_3$ are trivially satisfied. Condition 2-graded$_1$ is violated since $(x_1,x_2,x_3) \succsim (y_1,x_2,x_3)$, $(y_1,x_2,x_3) \succsim (y_1,x_2,x_3)$, $(y_1,y_2,y_3) \succsim (x_1,x_2,y_3)$ and $(y_1,x_2,x_3) \succsim (x_1,x_2,y_3)$ but $(z_1,x_2,x_3) \not\succsim (y_1,x_2,x_3)$ and $(x_1,y_2,y_3) \not\succsim (x_1,x_2,y_3)$.

Example 4. Let $X = \{x_1,y_1\} \times \{x_2,y_2\} \times \{x_3,y_3\}$. Let $\succsim$ be the weak order such that:

$$[(x_1,x_2,x_3),(x_1,y_2,x_3),(y_1,y_2,x_3)]\\ \succ\\ [(y_1,y_2,y_3),(y_1,x_2,x_3)]\\ \succ\\ [(x_1,x_2,y_3),(x_1,y_2,y_3),(y_1,x_2,y_3)].$$

Condition 2-graded trivially holds. We have $y_2 \succ_2^+ x_2$ and $x_3 \succ_3^+ y_3$, so that conditions $AC1_2$ and $AC1_3$ hold. Since $(x_1,x_2,x_3) \succsim (y_1,y_2,x_3)$ and $(y_1,y_2,y_3) \succsim (y_1,x_2,x_3)$ but $(y_1,x_2,x_3) \not\succsim (y_1,y_2,x_3)$ and $(x_1,y_2,y_3) \not\succsim (y_1,x_2,\ x_3)$, condition $AC1_1$ is violated.

Remark 4. It is easy to check that the weak order in Example 4 satisfies the following condition

$$\left.\begin{array}{c} x \succsim y \\ \text{and} \\ z \succsim y \end{array}\right\} \Rightarrow \left\{\begin{array}{c} (z_i,x_{-i}) \succsim y, \\ \text{or} \\ (x_i,z_{-i}) \succsim y, \end{array}\right.$$

for all $x,y,z \in X$. This condition is a weakening of $AC1_i$ obtained by requiring that $y = w$ in the expression of $AC1_i$ (it is equivalent to requiring that all relations $\succsim_i^{+(a)}$ are complete). It is therefore not possible to weaken $AC1_i$ in this way.

Similarly, it is easy to check that the weak order in Example 3 satisfies the weakening of 2-graded$_i$ obtained by requiring that $z = w$ in the expression of 2-graded$_i$ (and, hence, removing the last redundant premise), i.e., for all $x,y,z \in X$ and all $a_i \in X_i$,

$$\left.\begin{array}{c} x \succsim z \\ \text{and} \\ (y_i,x_{-i}) \succsim z \\ \text{and} \\ y \succsim z \end{array}\right\} \Rightarrow \left\{\begin{array}{c} (a_i,x_{-i}) \succsim z \\ \text{or} \\ (x_i,y_{-i}) \succsim z, \end{array}\right.$$

Hence, condition 2-graded$_i$ cannot be weakened in this way.

Example 5. Let $X = 2^{\mathbb{R}} \times \{0,1\}$. We consider the weak order on X such that $(x_1,x_2) \succsim (y_1,y_2)$ if $[x_2 = 1]$ or $[x_2 = 0,\ y_2 = 0$ and $x_1 \geq^* y_1]$, where $\geq^*$ is any linear order on $2^{\mathbb{R}}$. It is easy to see that $\succsim$ is a weak order. It violates OD since the restriction of $\succsim$ to $2^{\mathbb{R}} \times \{0\}$ is isomorphic to $\geq^*$ on $2^{\mathbb{R}}$ and $\geq^*$ violates *OD*. The relation $\succsim$ has a representation in the noncompensatory model. Indeed, for all $x = (x_1, 1)$, take $A_1^x = \varnothing$, $A_2^x = \{1\}$ and $F^x = \{\{2\}, \{1,2\}\}$. For all $x = (x_1, 0)$, take $A_1^x = \{y_1 \in 2^{\mathbb{R}} : y_1 \geq^* x_1\}$, $A_2^x = \{1\}$ and $F^x = \{\{1\}, \{2\}, \{1,2\}\}$. It is easy to check that this defines a representation of the weak order $\succsim$ in the noncompensatory model. Using Lemma 2, this implies that $\succsim$ satisfies *AC*1 and 2-graded.

6 Uniqueness

This section briefly discusses the uniqueness of the representation in the noncompensatory model and the discrete Sugeno integral model. The "ordinal" character of these models makes them especially attractive to deal with finite sets of alternatives. We therefore restrict our attention to this case in what follows. When X is finite, combining Propositions 1 and 2 with Theorem 1, shows that a binary relation has a representation in the noncompensatory model iff it has a representation in the discrete Sugeno integral model.

6.1 Links Between Representations in the Noncompensatory Model and the Discrete Sugeno Integral Model

Let $\succsim$ be a non-degenerate weak order on a finite set X with $r > 1$ distinct equivalence classes. Suppose that $\succsim$ has a representation in the noncompensatory model using sets A_i^x and F^x. It is easy to deduce from this representation a representation of $\succsim$ in the discrete Sugeno integral model.

It follows from the definition of the noncompensatory model that, if x and y belong to the same equivalence class, we have $A_i^x = A_i^y$, for all $i \in N$, and $F^x = F^y$. Let $A_i^{(k)} = A_i^x$ and $F^{(k)} = F^x$, for some $x \in X$ belonging to the kth equivalence class of $\succsim$.

Take any numbers λ_k such that

$$\lambda_1 = 1 > \lambda_2 > \cdots > \lambda_{r-1} > \lambda_r = 0. \tag{9}$$

For all $i \in N$, define u_i letting, for all $x_i \in X_i$,

$$\begin{cases} u_i(x_i) = \lambda_1 & \text{if } x_i \in A_i^{(1)}, \\ u_i(x_i) = \lambda_2 & \text{if } x_i \in A_i^{(2)} \setminus A_i^{(1)}, \\ u_i(x_i) = \lambda_3 & \text{if } x_i \in A_i^{(3)} \setminus A_i^{(2)}, \\ \quad\vdots & \\ u_i(x_i) = \lambda_{r-1} & \text{if } x_i \in A_i^{(r-1)} \setminus A_i^{(r-2)}, \\ u_i(x_i) = \lambda_r & \text{otherwise}, \end{cases} \tag{10}$$

and μ on 2^N letting, for all $A \in 2^N$,

$$\begin{cases} \mu(A) = \lambda_1 & \text{if } A \in F^{(1)}, \\ \mu(A) = \lambda_2 & \text{if } A \in F^{(2)} \setminus F^{(1)}, \\ \mu(A) = \lambda_3 & \text{if } A \in F^{(3)} \setminus F^{(2)}, \\ \quad \vdots & \\ \mu(A) = \lambda_{r-1} & \text{if } A \in F^{(r-1)} \setminus F^{(r-2)}, \\ \mu(A) = \lambda_r & \text{otherwise.} \end{cases} \tag{11}$$

With such definitions, for all $x \in X$, the value $S_{\langle \mu,u \rangle}(x)$ belongs to $\{\lambda_1, \lambda_2, \ldots, \lambda_r\}$. It is easy to see that $x \in X$ belongs to the kth equivalence class of $\succsim$ iff $\{i \in N : x_i \in A_i^{(k)}\} \in F^{(k)}$ iff $S_{\langle \mu,u \rangle}(x) = \lambda_k$.

The above formulas therefore give a systematic way to build a representation in the discrete Sugeno integral model on the basis of a representation in the noncompensatory model.

Clearly, the real numbers λ_k may be chosen arbitrarily, provided that they satisfy (9). Given a particular choice of λ_k, the representation built above is "minimal" in the sense that it uses as few real numbers as possible in order to build the representation in the Sugeno integral model.

The minimal representation, given a particular choice of λ_k compatible with (9), envisaged above is not the only possible one. Given the numbers λ_k, we can, for instance, use them to define the values of μ through (11). When this is done, it is clear that for each distinct $x_i \in A_i^{(k)} \setminus A_i^{(k-1)}$ we can define $u_i(x_i)$ to take an arbitrary value in the interval $[\lambda_k, \lambda_{k-1})$. Other choices are clearly possible.

6.2 Uniqueness of Representations

It is easy to deduce from the results in Bouyssou and Marchant (2007) the uniqueness of the representation in the noncompensatory model. Consider the kth equivalence class of $\succsim$. We say that attribute $i \in N$ is influent for this equivalence class if there are $x_i, y_i \in X_i$ and $a_{-i} \in X_i$ such that (x_i, a_{-i}) belongs at least to the kth equivalence class of $\succsim$ and (y_i, a_{-i}) belongs to a strictly lower equivalence class. Using the results in Bouyssou and Marchant (2007), it is easy to show that, when each attribute $i \in N$ is influent for the kth equivalence class of $\succsim$, the sets $A_i^{(k)}$ and $F^{(k)}$ are uniquely determined. This condition is not necessary for such a uniqueness however. This is illustrated in the example below adapted from Bouyssou and Marchant (2007).

Example 6. Let $n = 3$, $X = \{x_1, y_1\} \times \{x_2, y_2\} \times \{x_3, y_3\}$. Let $\succsim$ be such that:

$$\begin{aligned} (x_1, x_2, x_3) \succ (y_1, x_2, x_3) \succ [(x_1, x_2, y_3), (x_1, y_2, x_3)] \\ \succ [(x_1, y_2, y_3), (y_1, x_2, y_3), (y_1, y_2, x_3), (y_1, y_2, y_3)]. \end{aligned}$$

It is easy to check that all attributes are influent for the first equivalence class of $\succsim$. We must have $A_1^{(1)} = \{x_1\}$, $A_2^{(1)} = \{x_2\}$, $A_3^{(1)} = \{x_3\}$ and $F^{(1)} = \{\{1,2,3\}\}$. Similarly, all attributes are influent for the third equivalence class. We must have $A_1^{(3)} = \{x_1\}, A_2^{(3)} = \{x_2\}, A_3^{(3)} = \{x_3\}$ and $F^{(3)} = \{\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}\}$.

Attributes 2 and 3 are influent for the second equivalence class of $\succsim$ while attribute 1 is not. In order to satisfy the constraints of the noncompensatory model, we must take $A_1^{(2)} = \{x_1\}$, $A_2^{(2)} = \{x_2\}$, $A_3^{(2)} = \{x_3\}$ and $F^{(2)} = \{\{2,3\},\{1,2,3\}\}$. The conditions ensuring the uniqueness of the representation in the noncompensatory model are investigated in Bouyssou and Marchant (2007). Whenever this representation is not unique, we may use each of these representations as a basis for the analysis in Sect. 6.1.

In order to analyze the uniqueness of the representation in the discrete Sugeno integral model, two points should therefore be kept in mind. First, given a representation in the noncompensatory model, it is possible to deduce several distinct representations in the discrete Sugeno integral model. Second, the representation in the noncompensatory model may not be unique. Combining these two effects, it is clear that the uniqueness of the representation in the discrete Sugeno integral model is quite weak. Since its precise analysis does not seem to be informative, we do not develop this point.

6.3 Commensurateness

When we compute a Sugeno integral, we compare levels on different attributes. This seems to indicate that the axioms of the discrete Sugeno integral model imply the existence of a relation $\succ^c$ defined on $\bigcup_{i\in N} X_i$, with the following interpretation: $x_i \succ^c x_j$ iff x_i is better than x_j. Given a preference relation $\succsim$, there can exist several representations in the discrete Sugeno integral model and it can happen that $u_i(x_i) > u_j(x_j)$ in one representation while $u'_i(x_i) < u'_j(x_j)$ in another one. Hence, stating $x_i \succ^c x_j$ (or the converse) does not make sense for such a pair. So, let us define $\succ^c$ by $x_i \succ^c x_j$ iff $u_i(x_i) > u_j(x_j)$ in *all* representations. In the following proposition, we characterize this relation in terms of the primitive relation $\succsim$.

Proposition 3. *Let $\succsim$ be a weak order representable by means of a Sugeno integral. We have $z_i \succ^c z_j$ if and only if, for some $c,d \in X$, $w_i \in X_i$, $w_j \in X_j$, $a_{-i} \in X_{-i}$ and $b_{-j} \in X_{-j}$, we have*

$$\begin{cases} c \succsim d, \\ (z_i, a_{-i}) \succsim c, \quad (w_i, a_{-i}) \not\succsim c, \\ (w_j, b_{-j}) \succsim d, \quad (z_j, b_{-j}) \not\succsim d. \end{cases} \tag{12}$$

Proof. If (12) holds, then, in any representation, $u_i(z_i) \geq S_{\langle\mu,u\rangle}(c) \geq S_{\langle\mu,u\rangle}(d) > u_j(z_j)$. So, in any representation, $u_i(z_i) > u_j(z_j)$ and, therefore, $z_i \succ^c z_j$.

Suppose now $z_i \succ^c z_j$ and let $(u_i^*)_{i\in N}$ be one of the representations constructed by means of (9), (10) and (11). We therefore know that $u_i^*(z_i) > u_j^*(z_j)$. There is thus k and l with $k < l$ such that $u_j^*(z_j) = \lambda^l$ and $u_i^*(z_i) = \lambda^k$ (this follows from (10)). Hence, $z_i \in A_i^{(k)}$ and $z_j \notin A_j^{(k)}$. So, (12) holds for some $c = d$ belonging to the kth equivalence class of $\succsim$. □

From the definition of $\succ^c$, it is clear that this relation is transitive and asymmetric, i.e., $z_i \succ^c z_j$ implies $z_j \not\succ^c z_i$. We now show that it is also negatively transitive, i.e., $x_i \not\succ^c y_j$ and $y_j \not\succ^c z_l$ implies $x_i \not\succ^c z_l$. Hence, $\succ^c$ is the asymmetric part of a weak order on the set $\bigcup_{i\in N} X_i$. This is in line with the intuitive notion of commensurateness.

Proposition 4. *Let $\succsim$ be a weak order representable by means of a Sugeno integral. Then $\succ^c$ is negatively transitive.*

Proof. Let $(u_i^*)_{i\in N}$ be one of the representations constructed by means of (9), (10) and (11). Suppose $x_i \not\succ^c y_j$ and $y_j \not\succ^c z_l$. If $u_i^*(x_i) > u_j^*(y_j)$, then, as shown in the proof of Proposition 3, (12) holds and, by Proposition 3, $x_i \succ^c y_j$. This contradiction implies $u_i^*(x_i) \leq u_j^*(y_j)$. The same reasoning yields $u_j^*(y_j) \leq u_j^*(z_l)$. By transitivity, $u_j^*(x_i) \leq u_j^*(z_l)$. Suppose now, contrary to negative transitivity, that $x_i \succ^c z_l$. This implies $u_i^*(x_i) > u_l^*(z_l)$, a contradiction. □

To conclude this section, note that the "derived commensurateness", i.e., the relation $\succ^c$, is not easy to interpret and analyze however. Indeed, the way the above relation combines with $\succsim$ remains complex. As shown in the example below, it is quite possible to have $(x_i, x_j, x_{-ij}) \succsim y$, $z_j \succ^c x_i$ and $z_i \succ^c x_j$, while $(z_i, z_j, x_{-ij}) \not\succsim y$. This calls for further analysis.

Example 7. Let $n = 4$ and $X_1 = X_2 = X_3 = X_4 = \{0, 0.01, 0.02, \ldots, 0.99, 1\}$. For all $i \in N$, let $u_i(x_i) = x_i$. Define a normalized capacity μ on N such that: $\mu(\varnothing) = 0$, $\mu(A) = 0.1$, for all $A \subseteq N$ such that $|A| = 1$, $\mu(\{1,2\}) = 0.1$, $\mu(\{1,3\}) = 0.2$, $\mu(\{1,4\}) = 0.301$, $\mu(\{2,3\}) = 0.31$, $\mu(\{2,4\}) = 0.2$, $\mu(\{3,4\}) = 0.3$, $\mu(\{1,2,3\}) = 0.55$, $\mu(\{1,2,4\}) = 0.39$, $\mu(\{1,3,4\}) = 1$, $\mu(\{2,3,4\}) = 0.31$, $\mu(N) = 1$. Define $\succsim$ on X as the relation obtained through the comparison of the values $S_{\langle\mu,u\rangle}(x) = S_\mu[x]$ using the utility functions and the capacity defined above.

We have

$$S_\mu[(0.2, 0, 0.5, 0)] = 0.2 > S_\mu[(0.1, 0, 0.5, 0)] = 0.1,$$

$$S_\mu[(0, 0.2, 0, 0.5)] = 0.2 > S_\mu[(0, 0.15, 0, 0.5)] = 0.15.$$

Since it is clear that $S_\mu[(0.2, 0.2, 0.2, 0.2)] = 0.2$ we thus have

$$\begin{aligned}
(0.2, 0, 0.5, 0) &\succsim (0.2, 0.2, 0.2, 0.2) = c,\\
(0.1, 0, 0.5, 0) &\not\succsim (0.2, 0.2, 0.2, 0.2) = c,\\
(0, 0.2, 0, 0.5) &\succsim (0.2, 0.2, 0.2, 0.2) = d,\\
(0, 0.15, 0, 0.5) &\not\succsim (0.2, 0.2, 0.2, 0.2) = d,\\
c = (0.2, 0.2, 0.2, 0.2) &\succsim (0.2, 0.2, 0.2, 0.2) = d,
\end{aligned}$$

so that the level 0.2 on X_1 is better than the level 0.15 on X_2

Similarly, we have

$$S_\mu[(0,0.46,0.5,0)] = 0.31 > S_\mu[(0,0.3,0.5,0)] = 0.3,$$
$$S_\mu[(0.5,0,0,0.5)] = 0.301 > S_\mu[(0.3,0,0,0.5)] = 0.3.$$

Since we have $S_\mu[(0.31,0.31,0.31,0.31)] = 0.31$ and $S_\mu[(0.301,0.301,0.301,0.301)] = 0.301$, we obtain

$$\begin{aligned}(0,0.46,0.5,0) &\succsim (0.31,0.31,0.31,0.31) = c',\\ (0,0.3,0.5,0) &\not\succsim (0.31,0.31,0.31,0.31) = c',\\ (0.5,0,0,0.5) &\succsim (0.301,0.301,0.301,0.301) = d',\\ (0.3,0,0,0.5) &\not\succsim (0.301,0.301,0.301,0.301) = d',\\ c' &\succsim d',\end{aligned}$$

so that the level 0.46 on X_2 is better than the level 0.3 on X_1.

We have $S_\mu[(0.3,0.15,0.29,0.4)] = 0.3$. Since the level 0.2 on X_1 is better than the level 0.15 on X_2 and 0.46 on X_2 is better than the level 0.3 on X_1, we should obtain that $S_\mu[(0.2,0.46,0.29,0.4)] \geq 0.3$, whereas it is equal to 0.29.

7 Discussion

In this paper, we have analyzed the relations between the discrete Sugeno integral model and the noncompensatory model as well as proposed a factorization of the main condition used in Greco et al. (2004, Theorem 3). By the same token, we have presented a proof of Greco et al. (2004, Theorem 3). We have also discussed the uniqueness of the representation in the discrete Sugeno integral model and shown that the conditions used in Greco et al. (2004, Theorem 3) are independent. Besides, we have analyzed the commensurateness that is implied by the discrete Sugeno integral model and shown that it is more complex than what is usually thought in the literature. Many questions are nevertheless left open. Let us briefly mention here what seems to us the most important ones.

The result in Greco et al. (2004) is a first step in the systematic study of models using fuzzy integrals in MCDM. A first and major open problem is to derive a similar result for the discrete Choquet integral. This appears very difficult and we have no satisfactory answer at this time.

A second open problem is to use the above result as a building block to study particular cases of the discrete Sugeno integral. This was started in Greco et al. (2004) who showed how to characterize ordered weighted minimum and maximum. There are nevertheless many other particular cases of the discrete Sugeno integral that would be worth investigating.

A third problem is to investigate assessment protocols of the various parameters of the discrete Sugeno integral model using the above result and conditions. This will clearly require a deeper investigation of the commensurateness at work in our models.

Finally, it should be mentioned that we have mainly used here the noncompensatory model for weak orders as a tool for analyzing the discrete Sugeno integral model. The noncompensatory model that we introduced can be extended in many possible directions. This will be the subject of a subsequent paper.

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