Cooperative Games

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Simple games

Definition (Simple games)

A game (N, v) is a **Simple game** when

the valuation function takes two values

- 1 for a winning coalitions
- 0 for the losing coalitions

v satisfies *unanimity*: $v(N) = 1$

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One can represent the game by stating all the wining coalitions. Thanks to monotonicity, it is sufficient to only write down the minimal winning coalitions defined as follows:

Definition (Minimal winning coalition)

Let (N, v) be a TU game. A coalition $\mathcal{C}$ is a **minimal winning coalition** iff $v(\mathcal{C}) = 1$ and $\forall i \in \mathcal{C}$, $v(\mathcal{C} \setminus \{i\}) = 0$.

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$N = \{1, 2, 3, 4\}$.

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Formal definition of common terms in voting

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Let (N, v) be a simple game. A player $i \in N$ is a **dictator** iff $\{i\}$ is a winning coalition.

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Definition (Veto Player)

Let (N, v) be a simple game. A player $i \in N$ is a **veto** player if $N \setminus \{i\}$ is a losing coalition. Alternatively, i is a **veto** player iff for all winning coalition $\mathcal{C}$, $i \in \mathcal{C}$.

It also follows that a veto player is member of every minimal winning coalitions.

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Definition (blocking coalition)

A coalition $\mathcal{C} \subseteq N$ is a **blocking coalition** iff $\mathcal{C}$ is a losing coalition and $\forall S \subseteq N \setminus \mathcal{C}$, $S \setminus \mathcal{C}$ is a losing coalition.

Definition (weighted voting games)

A game $(N, w_{i \in N}, q)$ is a **weighted voting game** when v satisfies unanimity, monotonicity and the valuation function is defined as

$$v(S) = \begin{cases} 1 \text{ when } \sum_{i \in S} w_i \geqslant q \\ 0 \text{ otherwise} \end{cases}$$

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Unanimity requires that $\sum_{i\in N} w_i \geqslant q$.
If we assume that $\forall i \in N\ w_i \geqslant 0$, monotonicity is guaranteed. For the rest of the lecture, we will assume $w_i \geqslant 0$.

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We will note a weighted voting game $(N, w_{i\in N}, q)$ as $[q;\, w_1, \ldots, w_n]$.

A weighted voting game is a **succint** representation, as we only need to define a weight for each agent and a threshold.

Weighted voting game is a strict subclass of voting games.
i.e., all voting games are **not** weighted voting games.

Examples

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Theorem

Let (N, v) be a simple game. Then

$$Core(N,v) = \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{l} x \text{ is an imputation} \\ x_i = 0 \text{ for each non-veto player } i \end{array} \right\}$$

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Proof

- $\subseteq$ Let $x \in Core(N,v)$. By definition $x(N) = 1$. Let i be a non-veto player. $x(N\setminus\{i\}) \geqslant v(N\setminus\{i\}) = 1$. Hence $x(N\setminus\{i\}) = 1$ and $x_i = 0$.
- $\supseteq$ Let x be an imputation and $x_i = 0$ for every non-veto player i. Since $x(N) = 1$, the set V of veto players is non-empty and $x(V) = 1$.

  Let $\mathcal{C} \subseteq N$. If $\mathcal{C}$ is a winning coalition then $V \subseteq \mathcal{C}$, hence $x(\mathcal{C}) \geqslant v(\mathcal{C})$. Otherwise, $v(\mathcal{C})$ is a losing coalition (which may contain veto players), and $x(\mathcal{C}) \geqslant v(\mathcal{C})$. Hence, x is group rational.

□

Definition (Pivotal or swing player)

Let (N, v) be a simple game. A agent i is **pivotal** or **a swing agent** for a coalition $\mathcal{C} \subseteq N \setminus \{i\}$ if agent i turns the coalition $\mathcal{C}$ from a losing to a winning coalition by joining $\mathcal{C}$, i.e., $v(\mathcal{C}) = 0$ and $v(\mathcal{C} \cup \{i\}) = 1$.

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Given a **permutation σ** on N, there is a single pivotal agent.

The Shapley-Shubik index of an agent i is the percentage of permutation in which i is pivotal, i.e.

$$I_{SS}(N, v, i) = \sum_{\mathcal{C} \subseteq N \setminus \{i\}} \frac{|\mathcal{C}|!(|N| - |C| - 1)!}{|N|!} (v(\mathcal{C} \cup \{i\}) - v(\mathcal{C})).$$

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"For each permutation, the pivotal player gets a point."

The Shapley-Shubik power index is the Shapley value.
The index corresponds to the expected marginal utility assuming all join orders to form the grand coalitions are equally likely.

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- For a simple game (N, v), $v(N) = 1$ and $v(\emptyset) = 0$, at least one player i has a power index $\beta_i \neq 0$. Hence, $B = \sum_{j \in N} \beta_j > 0$.

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- The **normalized Banzhaff index** of player i for a simple game (N,v) is defined as $I_B(N,v,i) = \frac{\beta_i}{B}$.

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The index corresponds to the expected marginal utility assuming all coalitions are equally likely.

Examples: [7; 4,3,2,1]

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{2,1,3,4}
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	1	2	3	4
Sh	$\frac{7}{12}$	$\frac{1}{4}$	$\frac{1}{12}$	$\frac{1}{12}$

winning coalitions:
{1,2}
{1,2,3}
{1,2,4}
{1,3,4}
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	1	2	3	4
β	$\frac{5}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
$I_B(N,v,i)$	$\frac{1}{2}$	$\frac{3}{10}$	$\frac{1}{10}$	$\frac{1}{10}$

The Shapley-Shubik index and Banzhaff index may be different.

Representation and Complexitity issues

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- In general, the more succint a representation is, the harder it is to compute, hence we look for a balance between succinctness and tractability.

Weighted graph games

A weighted graph game is a coalitional game defined by an undirected weighted graph $\mathcal{G} = (V, W)$ where V is the set of vertices and $W : V \to V$ is the set of edges weights. For $(i,j) \in V^2$, w_{ij} is the weight of the edge between i and j.

- $N = V$, i.e., each agent is a node in the graph.
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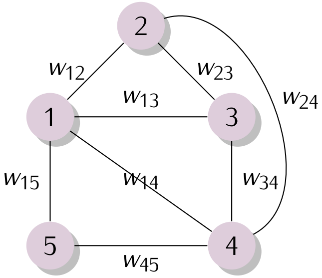

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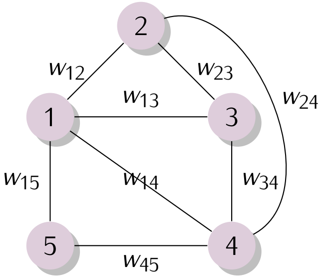

It is a **succint** representation: using an adjacency matrix, we need to provide n^2 entries.
However, it is **not complete**. Some TU games cannot be represented by a weighted graph game (e.g., a majority voting game).

Proposition

Let (V, W) be a weighted graph game. If all the weights are nonnegative then the game is convex.

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Let (V, W) be a weighted graph game. If all the weights are nonnegative then membership of a payoff vector in the core can be tested in polynomial time.

Theorem

Let (V,W) a weighted graph game. The Shapley value of an agent $i \in V$ is $Sh_i(N,v) = \frac{1}{2} \sum_{(i,j) \in N^2 \mid i \neq j} w_{ij}$.

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Proof

Let (V,W) a weighted graph game. We can view this game as the sum of the following $|W|$ games (i.e., one game per edge): $G_{ij} = (V, \{w_{ij}\})$, $(i,j) \in V^2$.
For a game G_{ij}, i and j are substitutes, and all other agents $k \neq i,j$ are dummy agents. Using the symmetry axiom, $Sh_i(G_{ij}) = Sh_j(G_{ij})$. Using the dummy axiom, $Sh_k(G_{ij}) = 0$. Hence, $Sh_i(G_{ij}) = \frac{1}{2} w_{ij}$.
Since (V,W) is the sum of all two-player games, by the additivity axiom, $Sh_k = \sum_{i,j} Sh_k(G_{ij}) = \sum_{k,i} w_{ij}$ □

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Theorem

Let (V,W) be a weighted graph game. Testing the nonemptyness of the core is NP-complete.

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Instead of storing a value for each coalition, we can store the positive synergies between agents.

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Let (N,v) be a superadditive game and (N,s) its **synergy representation**. Then for a coalition $\mathcal{C} \subseteq N$,

$$v(\mathcal{C}) = \left(\max_{\{\mathcal{C}_1,\mathcal{C}_2,\ldots,\mathcal{C}_k\} \in \mathscr{S}_{\mathcal{C}}} \sum_{i=1}^{k} v(\mathcal{C}_i) \right),$$

where $\mathscr{S}_{\mathcal{C}}$ is the set of all partition of $\mathcal{C}$.

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where $\mathscr{S}_{\mathcal{C}}$ is the set of all partition of $\mathcal{C}$.

Example: $N = \{1,2,3\}$, $v(\{i\}) = 1$, $v(\{1,2\}) = 3$, $v(\{1,3\}) = 2$, $v(\{2,3\}) = 2$, $v(\{1,2,3\}) = 4$.
We can represent this game by $v(\{i\}) = 1$, $v(\{1,2\}) = 3$.

This representation may still require a space exponential in the number of agents, but for many games, the space required is much less.

Theorem

It is NP-complete to determine the value of some coalitions for a coalitional game specified with the synergy representation. In particular, it is NP-complete to determine the value of the grand coalition.

V. Conitzer and T. Sandholm, **Complexity of constructing solutions in the core based on synergies among coalitions**, *Artificial Intelligence*, 2006.

Theorem

It is NP-complete to determine the value of some coalitions for a coalitional game specified with the synergy representation. In particular, it is NP-complete to determine the value of the grand coalition.

Theorem

Let (N, v) a TU game specified with the synergy representation and the value of the grand coalition.
Then we can determine in polynomial time whether the core of the game is empty.

V. Conitzer and T. Sandholm, **Complexity of constructing solutions in the core based on synergies among coalitions**, *Artificial Intelligence*, 2006.

Multi-issue representation

Some coalitions may form to solve problems requiring distinct competences. For example, solving a set of tasks requiring different expertises.

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Definition (Decomposition)

The vector of characteristic functions $\langle v_1, v_2, \ldots, v_T \rangle$, with each $v_i : 2^N \rightarrow \mathbb{R}$, is a **decomposition** over T issues of characteristic function $v : 2^N \rightarrow \mathbb{R}$
if for any $S \subseteq N$, $v(S) = \sum_{i=1}^{T} v_i(S)$.

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It is a fully expressive representation (can use 1 issue).

Multi-issue representation

Theorem

The Shapley value of a coalitional game represented with multi-issue representation can be computed in linear time.

Theorem

Checking whether a given value division is in the core is coNP-complete.

V. Conitzer and T. Sandholm. **Computing shapley values, manipulating value division schemes, and checking core membership in multi-issue domains**. In *Proc. of the 19th Nat. Conf. on Artificial Intelligence (AAAI-04)*

A logical approach: Marginal contribution nets (MC-nets)

The idea:

- represent each player by a boolean variable,
- treat the characteristic vector of a coalition as a truth assignment.
- the truth assignment can be used to check whether a formula is satisfied and to compute the value of a coalition.

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Let N be a collection of atomic variables.

Definition (Rule)

A **rule** has a syntactic form (ϕ, w) where ϕ is called the pattern and is a boolean formula containing variables in N and w is called the weight, and is a real number.

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examples:

$(a \wedge b, 5)$: each coalition containing both agents a and b increase its value by 5 units.

$(b, 2)$: each coalition containing b increase its value by 2.

A logical approach: Marginal contribution nets (MC-nets)

Definition (Marginal contribution nets)

An MC-net consists of a set of rules $\{(p_1, w_1), \dots (p_k, w_k)\}$ where the valuation function is given by

$$v(\mathcal{C}) = \sum_{i=1}^{k} p_i(e^{\mathcal{C}}) w_i,$$

where $p_i(e^{\mathcal{C}})$ evaluates to 1 if the boolean formula p_i evaluates to true for the truth assignment $e^{\mathcal{C}}$ and 0 otherwise.

S. Ieong and Y. Shoham, **Marginal contribution nets: a compact representation scheme for coalitional games**, in *Proceedings of the 6th ACM conference on Electronic commerce*, 2005.

Examples

Let us consider an MC-net with the following two rules:

$$(a \wedge b, 5) \text{ and } (b, 2)$$

The coalitional game represented has two agents a and b and the valuation function is defined as follows:

$$\begin{array}{cc} v(\emptyset) = 0 & v(\{b\}) = 2 \\ v(\{a\}) = 0 & v(\{a,b\}) = 5+2 = 7 \end{array}$$

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We can use negations in rules, and negative weights. Let consider the following example:

$$(a \wedge b, 5), (b, 2), (c, 4) \text{ and } (b \wedge \neg c, -2)$$

$$\begin{array}{ccc} v(\emptyset) = 0 & v(\{b\}) = 2-2 = 0 & v(\{a,c\}) = 4 \\ v(\{a\}) = 0 & v(\{a,b\}) = 5+2-2 = 5 & v(\{b,c\}) = 4+2 = 6 \end{array}$$

Theorem (Expressivity)

- MC-nets can represent **any game** when negative literals are allowed in the patterns, or when the weights can be negative.
- When the patterns are limited to conjunctive formula over postivie literals and weights are nonnegative, MC-nets can represent all and only **convex games**.

Proposition

MC-nets generalize Weighted Graph game representation (strict generalization) and the multi-issue representation.

Theorem

Given a TU game represented by an MC-net limited to conjunctive patterns, the **Shapley value** can be computed in time **linear** in the size of the input.

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Proof sketch: we can treat each rule as a game, compute the Shapley value for the rule, and use ADD to sum all the values for the overall game. For a rule, we cannot distinguish the contribution of each agent, by SYM, they must have the same value. It is a bit more complicated when negation occurs (see Ieong and Shoham, 2005).

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Proposition

Determining whether the core is empty or checking whether an imputation lies in the core are coNP-hard.

Proof sketch: due to the fact that MC-nets generalize over weighted graph games.

Hedonic Games and NTU games

Hedonic games

Agents have preferences over coalitions, i.e. agent only cares about the other members of the coalition: "enjoying the pleasure of each other's company".

Let N be a set of agents and $\mathcal{N}_i$ be the set of coalitions that contain agent i, i.e., $\mathcal{N}_i = \{\mathcal{C} \cup \{i\} \mid \mathcal{C} \subseteq N \setminus \{i\}\}$.

Definition (Hedonic games)

An **Hedonic game** is a tuple $(N, (\succeq_i)_{i \in N})$ where

- N is the set of agents
- $\succeq_i \subseteq 2^{\mathcal{N}_i} \times 2^{\mathcal{N}_i}$ is a complete, reflexive and transitive preference relation for agent i, with the interpretation that if $S \succeq_i T$, agent i prefers coalition T at most as much as coalition S.

A. Bogomolnaia and M.O. Jackson, *The stability of hedonic coalition structure*. **Games and Economic Behavior**, 2002.

Stability concepts of Hedonic Games

Let $\Pi \in \mathscr{S}_N$ be a coalition structure, and Π_i denotes the coalition in Π that contains i.

- A coalition structure Π is **core stable** iff $\nexists C \subseteq N \mid \forall i \in C, C \succ_i \Pi_i$.

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- A coalition structure s is **Nash stable**
  $(\forall i \in N)\ (\forall C \in \Pi \cup \{\emptyset\})\ \Pi_i \succsim_i C \cup \{i\}$.
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- A coalition structure Π is **individually stable** iff
  $\nexists i \in N\ \nexists C \in \Pi \cup \emptyset \mid \big((C \cup \{i\} \succ_i \Pi_i) \wedge (\forall j \in C, C \cup \{i\} \succsim_j C)\big)$.
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  No player can move to another coalition that it prefers without making some members of that coalition unhappy.
- A coalition structure Π is **contractually individually stable** iff $\nexists i \in N\ \nexists C \subseteq N \mid$
  $(C \cup \{i\} \succ_i \Pi_i) \wedge (\forall j \in C, C \cup \{i\} \succsim_j C) \wedge (\forall j \in \Pi_i \setminus \{i\}, \Pi_i \setminus \{i\} \succsim_j \Pi_i)$
  No player can move to a coalition it prefers so that the members of the coalition it leaves and it joins are better off

Example

$$\{1,2\} \succ_1 \{1\} \succ_1 \{1,2,3\} \succ_1 \{1,3\}$$

$$\{1,2\} \succ_2 \{2\} \succ_2 \{1,2,3\} \succ_2 \{2,3\}$$

$$\{1,2,3\} \succ_3 \{2,3\} \succ_3 \{1,3\} \succ_3 \{3\}$$

$\{\{1,2\},\{3\}\}$ is in the core and is individually stable.
There is no Nash stable partitions.

$\{\{1\},\{2\},\{3\}\}$	$\{1,2\}$ is preferred by both agent 1 and 2, hence not NS, not IS.
$\{\{1,2\},\{3\}\}$	$\{1,2,3\}$ is preferred by agent 3, so it is not NS, as agents 1 and 3 are worse off, it is not a possible move for IS. no other move is possible for IS.
$\{\{1,3\},\{2\}\}$	agent 1 prefers to be on its own (not NS, then, not IS).
$\{\{2,3\},\{1\}\}$	agent 2 prefers to join agent 1, and agent 1 is better off, hence not NS, not IS.
$\{\{1,2,3\}\}$	agents 1 and 2 have an incentive to form a singleton, hence not NS, not IS.

Nash stability $\Rightarrow$ Individual stability $\Rightarrow$ contractual individual stability

Core stability $\not\Rightarrow$ Nash stability $\not\Rightarrow$ Core stability

Core stability $\not\Rightarrow$ Individual stability

Some classes of games have a non-empty core,
other classes have Nash stable coalition structures.

A. Bogomolnaia and M.O. Jackson, *The stability of hedonic coalition structure*. **Games and Economic Behavior**, 2002.

A representation for hedonic games have been proposed, and is based on MC-nets.

E. Elkind and M. Wooldridge, **Hedonic Coalition Nets**, in *Proc. of 8th Int. Conf. on Autonomous Agents and Multiagent Systems (AAMAS)*, 2009

A general model for NTU games (Non-transferable utility games)

It is **not** always possible to compare the utility of two agents or to transfer utility.

Definition (NTU game)

A NTU game is a tuple $(N, X, V, (\succeq_i)_{i \in N})$ where

- X set of outcomes
- $\succeq_i$ a preference relation (transitive and complete) for agent i over the set of outcomes.
- $V(\mathcal{C})$ a set of outcomes that a coalition $\mathcal{C}$ can bring about

- **Example 1:** hedonic games as a special class of NTU games.
  Let $(N,(\succeq_i^H)_{i\in N})$ be a hedonic game.
  • For each coalition $\mathcal{C}\subseteq N$, create a unique outcome $x_{\mathcal{C}}$.
  • For any two outcomes x_S and x_T corresponding to coalitions S and T that contains agent i, We define $\succeq_i$ as follows: $x_S \succeq_i x_T$ iff $S \succeq_i^H T$.
  • For each coalition $\mathcal{C}\subseteq N$, we define $V(\mathcal{C})$ as $V(\mathcal{C}) = \{x_C\}$

- **Example 1:** hedonic games as a special class of NTU games.
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  • For each coalition $\mathcal{C} \subseteq N$, we define $V(\mathcal{C})$ as $V(\mathcal{C}) = \{x_C\}$
- **Example 2:** a TU game can be viewed as an NTU game.
  Let (N, v) be a TU game.
  • We define X to be the set of all allocations, i.e., $X = \mathbb{R}^n$.
  • For any two allocations $(x, y) \in X^2$, we define $\succeq_i$ as follows: $x \succeq_i y$ iff $x_i \geqslant y_i$.
  • For each coalition $\mathcal{C} \subseteq N$, we define $V(\mathcal{C})$ as $V(\mathcal{C}) = \{x \in \mathbb{R}^n \mid \sum_{i \in N} x_i \leqslant v(\mathcal{C})\}$. $V(\mathcal{C})$ lists all the feasible allocation for the coalition $\mathcal{C}$.

Core

An outcome $x \in X$ is **blocked** by a coalition $\mathcal{C}$ if there is some outcome $y \in V(\mathcal{C})$ such that all members i of $\mathcal{C}$ strictly prefer y to x, i.e., $\exists \mathcal{C} \subseteq N$, $\exists y \in V(\mathcal{C})$ s.t. $\forall i \in \mathcal{C}$, $y \succ_i x$.

The **core** of an NTU game $(N, X, V, (\succeq_i)_{i \in N})$ is defined as:
$Core(N, X, V, (\succeq)) = \{x \in V(N) \mid \nexists \mathcal{C} \subseteq N, \nexists y \in V(\mathcal{C}), \forall i \in \mathcal{C} : y \succ_i x\}$

Games with externalities

One of the purpose of Game theory is to *"determine everything that can be said about coalitions between players, compensations between partners in every coalition, mergers or fights between coalitions "...*

von Neumann and Morgenstern,
Theory of games and economic behaviour, 1944.

1- Which coalition will be formed?
2- How will the coalitional worth be shared between members?
3- How does the presence of other coalitions affect the incentives to cooperate?

Cooperative game theory has focused mainly on point 2.

Coalitional Games with externalities

- In a TU game (N, v), the valuation of a coalition depends only on the members, **not** on the other coalition present in the population.
- The value **can** depend on the other coalitions in the population
  - competitive firms
  - teams in sport

➪ valuation function for a coalition given a coalition structure (in a competitive setting) $v : 2^N \times \mathscr{S} \to \mathbb{R}$
Games **in partition function form**.

➪ valuation function for each agent given a coalition structure (ex: competitive supply chains) $v : N \times \mathscr{S} \to \mathbb{R}$.
Games with **Valuations**.

Definition (Positive and negative spillovers)

A partition function v exhibits

- **positive spillovers** if for any partition π and any two coalitions S and T in π $v(\mathcal{C},\pi\setminus\{S,T\}\cup\{S\cup T\}) \geqslant v(C,\pi)$ for all coalitions $\mathcal{C} \neq S,T$ in π.
- **negative spillovers** if for any partition π and any two coalitions S and T in π $v(\mathcal{C},\pi\setminus\{S,T\}\cup\{S\cup T\}) \leqslant v(C,\pi)$ for all coalitions $\mathcal{C} \neq S,T$ in π.

Valuations

Assumption: Fixed rules of division appear naturally in many economic situations and in theoretical studies based on a two-stage procedure:

1- formation of the coalitions
2- payoff distribution

Definition (Valuation)

A **valuation** v is a mapping which associates to each coalition structure a payoff of individual payoff in $\mathbb{R}^n$.

Definition (Positive and negative spillovers)

A valuation v exhibits

- **positive spillovers** if for any partition π and any two coalitions S and T in π $v_i(\pi \setminus \{S,T\} \cup \{S \cup T\}) \geqslant v_i(\pi)$ for all players $i \notin S \cup T$.
- **negative spillovers** if for any partition π and any two coalitions S and T in π $v_i(\pi \setminus \{S,T\} \cup \{S \cup T\}) \leqslant v_i(\pi)$ for all players $i \notin S \cup T$.

Definition (Core stability)

A coalition structure π is **core stable** if there does not exist a group $\mathcal{C}$ of players a coalition structure π' that contains $\mathcal{C}$ such that $\forall i \in \mathcal{C}$, $v_i(\pi') > v_i(\pi)$.

Definition (α-core Stability)

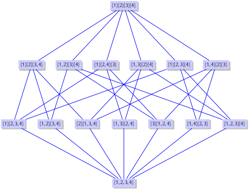
A coalition structure π is **α-core stable** if there does not exist a group $\mathcal{C}$ of players and a partition $\pi'_{\mathcal{C}}$ such that, for all partition $\pi_{N\setminus\mathcal{C}}$ formed by external players, $\forall i \in \mathcal{C}$, $v_i(\pi'_{\mathcal{C}} \cup \pi_{N\setminus\mathcal{C}}) > v_i(\pi)$.

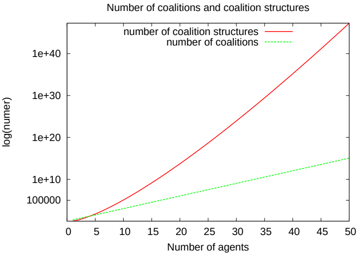
Definition (β-core Stability)

A coalition structure π is **β-core stable** if there does not exist a group $\mathcal{C}$ of players such that for all partitions $\pi_{N\setminus\mathcal{C}}$ of external players, there exists a partition $\pi_{\mathcal{C}}$ of $\mathcal{C}$ such that $\forall i \in \mathcal{C}$, $v_i(\pi_{\mathcal{C}} \cup \pi_{N\setminus\mathcal{C}}) > v_i(\pi)$.

Issues studied in multiagent systems

Search of an optimal coalition structure





The difficulty of searching for the optimal CS is the large search space.

How to distribute the computation of all the coalition values?

- ✪ goal is to minimize computational time
  Computing the value of a coalition can be hard: ex solving a TSP
- ➪ load balancing: distribute coalitions of every size equally among the agents coalitions.

but agents may have different computational speed

A naive approach does not avoid redundancy and may have a high communication complexity.
The current best algorithm works by sharing the computation of coalition of the same size between all the agents.

O. Shehory and S. Kraus. **Methods for task allocation via agent coalition formation**. *Artificial Intelligence*, 1998

T. Rahwan and N. Jennings. **An algorithm for distributing coalitional value calculations among cooperating agents**, *Artificial Intelligence*, 2007

Search of the Optimal Coalition Structure

- First algorithm that guarantees a bound from the optimal $\frac{v(s)}{V(\mathfrak{s}^\star)} \leqslant \mathcal{K}$. It is necessary to visit a least 2^{n-1} CSs, which corresponds to the first two levels of the lattice.
- Best current algorithm is called IP for Interger Partition:
  - Integer Partition: ex $[1,1,2] \rightarrow$ space of coalition structures containing two singletons and a coalition of size 2.
  - Finding bounds for each subspace is easy. Ex: $\max_{S \in [1,1,2]} v(S) \leqslant \max_{\mathcal{C} \in 2^N, |\mathcal{C}|=1} v(\mathcal{C}) + \max_{\mathcal{C} \in 2^N, |\mathcal{C}|=1} v(\mathcal{C}) + \max_{\mathcal{C} \in 2^N, |\mathcal{C}|=2} v(\mathcal{C})$
  - IP uses the representation to efficiently prune part of the space and search the most promising subspaces.

T. Rahwan, S.D. Ramchurn, N. Jennings, and A. Giovannucci. **An anytime algorithm for optimal coalition structure generation**, *Journal of Artificial Intelligence Research*, 2009.

Environments, Safety and Robustness, Communication

- Agents can enter and leave the environment at any time
- The characteristics of the agents may change with time
- Communication links may fail during the negotiation phase

Extending some concepts to Open Environments.

➩ how to avoid recomputing from scratch?

Additional goals of the coalition formation: decreasing the time and the number of messages required to reach an agreement.

- ➪ learning may be used to decrease negotiation time.
- ➪ communication costs are represented in the characteristic function.
- ➪ analysis of the communication complexity of computing the payoff of a player with different stability concepts: they find that it is $\Theta(n)$ when the Shapley value, the nucleolus, or the core is used.

Uncertainty about Knowledge and Task

- Agents may not know some tasks.
- Agents may not know the valuation function, and may use Fuzzy sets to represent the coalition value.
- Expected values of coalitions are used instead of the valuation function.
- Approximation of valuation function: e.g., computing a value for a coalition requires solving a version of the traveling salesman problem and approximations are used to solve that problem.
- Agent do not know the cost incurred by other agents and may only estimate these costs.

Manipulation

A protocol may require that they disclose some private information.

- ➪ Avoid information asymmetry that can be exploited by some agents by using cryptographic techniques.
- ➪ Use computational complexity to protect a protocol.

Other types of manipulations:

- hiding skills
- using false names (anonymous environments)
- colluding

The traditional solution concepts can be vulnerable to false names and to collusion.
Study for some TU games and for weighted voting games.

Long Term Vs Short Term

In general, a coalition is a short-lived entity that is *"formed with a purpose in mind and dissolve when that need no longer exists, the coalition ceases to suit its designed purpose, or critical mass is lost as agents depart".*

- Long term coalitions, and in particular the importance of trust in this content.
- Repeated coalition formation under uncertainty using learning.

Overlapping Coalitions

Agents may simultaneously belong to more than one coalition

- ➪ Fuzzy approach
  - agents can be member of a coalition with a certain degree that represents the risk associated with being in that coalition.
  - agents have different degree of membership, and their payoff depends on this degree.
- ➪ Heuristic algorithms
- ➪ Game theoretical approach (overlapping core)

Conclusion

- Game theory proposes many solution concepts (some of which were not introduced: bargaining set, ϵ-core, least-core, Owen value). Each solution concept has pros and cons.
- Work in AI has dealt with representation issues, and practical coalition formation protocols.
- Many issues are left unexaplored.