Strategic Games

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- How can we formally model such interactions?
- How should **rational** agents behave?

  Game theory is one way.

Outline

- **Today:** non-cooperative games
  - A central topic in Game theory: Strategic Games and Nash equilibrium.
  - Additional topics to provide a broader view of the field.
- **Tomorrow:** cooperative games

Prisoner's dilemma

Two partners in crime, Row (**R**) and Column (**C**), are arrested by the police and are being interrogated in separate rooms. From Row's point of view, four different outcomes can occur:

- Only R confesses ⇒R gets 1 year.
- Only C confesses ⇒R spends 4 years in jail.
- Both confess ⇒Both spend 3 years in prison.
- Neither one confesses ⇒both get 2 years in prison.

The utility of an agent is (5 - number of years in prison).

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	Column confesses	Column does not
Row confesses	2,2	4,1
Row does not	1,4	3,3

We can abstract this game and provide a generic game representation as follows:

Definition (Normal form game)

A **normal form game (NFG)** is $(N, (S_i)_{i \in N}, (u)_{i \in N})$ where

- N is the set of n players.
- S_i is the set of strategies available to agent i.
- $u_i : S_1 \times \cdots \times S_n \to \mathbb{R}^n$ is the **payoff function** of agent i. It maps a **strategy profile** to a **utility**.

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Terminology:

- an element $s = \langle s_1, \ldots, s_n \rangle$ of $S_1 \times \cdots \times S_n$ is called a **strategy profile** or a **joint-strategy**.
- Let $s \in S_1 \times \cdots \times S_n$ and $s'_i \in S_i$. We write (s'_i, s_{-i}) the joint-strategy which is the same as s except for agent i which plays strategy s'_i, i.e., $(s'_i, s_{-i}) = \langle s_1, \ldots, s_{i-1}, s'_i, s_{i+1}, \ldots, s_n \rangle$

What would you do?

- $N = \{Row, Column\}$
- $S_{Row} = S_{Column} = \{cooperate,\ defect\}$
- u_{Row} and u_{Column} are defined by the following bi-matrix.

Row \ *Column*	defect	cooperate
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1. Wait to know the other action?
2. Not confess?
3. Confess?
4. Toss a coin?

Can you use some general principles to explain your choice?

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Definition (strong dominance)

A strategy $x \in S_i$ for player i **(strongly) dominates** another strategy $y \in S_i$ if independently of the strategy played by the opponents, agent i (strictly) prefers x to y, i.e. $\forall s \in S_1 \times \cdots \times S_n$, $u_i(x, s_{-i}) > u_i(y, s_{-i})$

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	C confesses	C does not
R confesses	2,2	4,1
R does not	1,4	3,3

Both players have a dominant strategy: to confess! From Row's point of view:

- if C confesses: R is better off confessing as well.
- if C does not: R can exploit and confess.

Battle of the sexes

	L	R
T	2,2	4,3
B	3,4	1,1

- **Problem:** Where to go on a date: Soccer or Opera?
- **Requirements:**
  - have a date!
  - be at your favourite place!

Do players have a dominant strategy?

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Battle of the sexes possesses two Nash equilibria $\langle O, S \rangle$ and $\langle S, O \rangle$.

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Unique Nash equilibrium: both players confess!

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⇨There is a conflict: the **stable** solution (i.e., the Nash equilibrium) is not **efficient**, as the outcome is not Pareto optimal.

Chicken game

In *Rebel Without a Cause*, James Dean's character's, Jim, is challenged to a "Chickie Run" with Buzz, racing stolen cars towards an abyss.
The one who first jumps out of the car loses and is deemed a "chicken" (coward).

	Jim drives on	Jim turns
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Nash equilibrium: a summary

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- Any combinations of dominant strategies is a Nash equilibrium.
- A Nash equilibrium may not be Pareto optimal.
- Two Nash equilibria may not have the same payoffs

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Assume that there are three strategies: $S_i = \{1,2,3\}$. Player i may decide to play strategy 1 with a probability of $\frac{1}{3}$, strategy 2 with a probability of $\frac{1}{2}$ and strategy 3 with a probability of $\frac{1}{6}$. The mixed strategy is then denoted as $\left\langle \frac{1}{3}, \frac{1}{2}, \frac{1}{6} \right\rangle$.

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Given a mixed strategy profile $p = \langle p_1, \dots, p_n \rangle$, the expected utility for agent i is computed as follows:

$$E_i(p) = \sum_{s \in S_1 \times \dots \times S_n} \left(\left(\prod_{j \in N} p_j(s_j) \right) \times u_i(s) \right)$$

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Battle of the sexes

		y L	$1-y$ R
x	T	2,2	4,3
$1-x$	B	3,4	1,1

The expected utility for the Row player is:
$xy \cdot 2 + x(1-y) \cdot 4 + (1-x)y \cdot 3 + (1-x)(1-y) \cdot 1$
$= -4xy + 3x + 2y + 1$

Given a mixed strategy profile $p = \langle p_1, \ldots, p_n \rangle$, we write (p'_i, p_{-i}) the mixed strategy profile which is the same as p except for player i which plays mixed strategy p'_i, i.e., $(p'_i, p_{-i}) = \langle p_1, \ldots, p_{i-1}, p'_i, p_{i+1}, \ldots, p_n \rangle$.

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T	2,2	4,3
B	3,4	1,1

Let us consider that each player plays the mixed strategy $\langle \frac{3}{4}, \frac{1}{4} \rangle$.
None of the players have an incentive to deviate:

$$E_{row}(T) = \frac{3}{4} \cdot 2 + \frac{1}{4} \cdot 4 = \frac{5}{2} \qquad E_{row}(B) = \frac{3}{4} \cdot 3 + \frac{1}{4} \cdot 1 = \frac{5}{2}$$

(players are indifferent)

Theorem (J. Nash, 195))

Every finite strategic game has got at least one mixed Nash equilibrium.

note: The proofs are non-constructive and use Brouwer's or Kakutani's fixed point theorems.

J.F. Nash. Equilibrium points in n-person games. in *Proc. National Academy of Sciences of the United States of America*, 36:48-49, 1950.

Computing a Nash equilibrium

Complexity: In general, it is a hard problem. It is a PPAD-complete problem.

Daskalakis, Goldberg, Papadimitriou: **The complexity of computing a Nash equilibrium**, in *Proc. 38th Ann. ACM Symp. Theory of Computing (STOC)*, 2006

There are complexity results and algorithms for different classes of games. We will not treat then in this tutorial.

Y. Shoham & K. Leyton-Brown: **Multiagent Systems**, Cambridge University Press, 2009. (Chapter 4)

Nisan, Roughgarden, Tardos & Vazirani: **Algorithmic Game Theory**, Cambridge University Press, 2007. (chapters 2, 3)

Other types of solution concepts for NFGs

Safety strategy

With Nash equilibrium, we assumed that the opponents were **rational agents**. What if the opponents are potentially **malicious**, i.e., their goal could be to minimize the payoff of the player?

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Definition (Maxmin)

For player i,
the **maxmin strategy** is $\operatorname{argmax}_{s_i \in S_i} \min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i})$,
and its **maxmin value** or **safety level** is $\max_{s_i \in S_i} \min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i})$.

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2) the opponents $-i$ choose a (possible mixed) strategy that *minimize* i's payoff.

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⇨the maxmin strategy *maximizes* i's **worst case** payoff.

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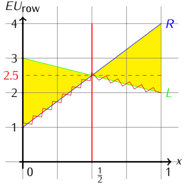

Payoff of Row
when Column plays pure strategy (T or R)
or any mixed strategy (yellow area)

Whatever Column does, Row can guarantee itself a payoff of 2.5 by playing the mixed strategy $\langle \frac{1}{2}, \frac{1}{2} \rangle$.

Punish

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For player i in a 2-player game,
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Player i's strategy against player $-i$ in a 2-player game is a strategy that minimizes $-i$'s best-case payoff

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Proposition

For a player i,

$$\max_{s_i\in S_i} \min_{s_{-i}\in S_{-i}} u_i(s_i, s_{-i}) \leqslant \min_{s_{-i}\in S_{-i}} \max_{s_i\in S_i} u_i(s_i, s_{-i})$$

Minimax theorem

Theorem

Minimax theorem (von Neumann, 1928)

In any finite two-player zero-sum game, for each player i, the maxmin strategy and minmax strategies are the same and are a Nash equilibrium of the game.

Minimax regret

Instead of assuming the opponents are rational (Nash equilibrium) or malicious (minimax), one can assume the **opponent is unpredictable**
⇨avoid **costly mistakes**/minimize their worst-case losses.

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	L	R
T	100,100	0,0
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(T, L) is preferred by both agents.
However, (B, R) is also a NE.
There is no dominance.
How to explain that (T, L) should be preferred?

Minimax regret

Instead of assuming the opponents are rational (Nash equilibrium) or malicious (minimax), one can assume the **opponent is unpredictable**
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One can build a **regret-recording** game where the payoff function r_i is defined by $r_i(s_i,s_{-i}) = u_i(s_i^\star,s_{-i}) - u_i(s_i,s_{-i})$, where $s_i^\star$ is i's best response to s_{-i}, i.e., $r_i(s_i,s_{-i})$ is i's **regret to have chosen s_i instead of $s_i^\star$**.

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$r_i \backslash r_j$	L	R
T	0,0	1,100
B	100,1	0,0

We define $regret_i(s_i)$ as the maximal regret i can have from choosing s_i.
A **regret minimization strategy** is one that **minimizes the $regret_i$ function**.

Correlated equilibrium

Battle of the sexes

	L	R
T	2,2	4,3
B	3,4	1,1

How to avoid the bad outcomes in which the agents fail to coordinate?

💡 **idea:** using a public random variable.

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idea: using a public random variable.

Example: the night before, the couple may condition their strategies based on weather (in the Netherlands, it is raining with a probability of 50%) as follows:
if it rains at 5pm, we go to opera, otherwise, we go to football.

- ⇨ both players increase their expected utility
- ⇨ maybe a fairer solution

Definition (Correlated equilibrium)

Given an n-agent game $G = (N, (S)_{i\in N}, (u_i)_{i\in N})$, a correlated equilibrium is a tuple (v, π, σ), where

- v is a tuple of random variables $v = \langle v_1, \ldots, v_n\rangle$ with respective domains $D = \langle D_1, \ldots, D_n\rangle$,
- π is a joint-distribution over v,
- $\sigma = \langle \sigma_1, \ldots, \sigma_n\rangle$ is a vector of mappings $\sigma_i : D_i \to S_i$,
- and for each agent i and every mapping $\sigma_i' : D_i \to S_i$ it is the case that
  $\sum_{d\in D} \pi(d) u_i(\sigma_1(d_1), \ldots, \sigma_i(d_i), \ldots, \sigma_n(d_n)) \geqslant$
  $\sum_{d\in D} \pi(d) u_i(\sigma_1(d_1), \ldots, \sigma_i'(d_i), \ldots, \sigma_n(d_n))$.

Theorem

For every Nash equilibrium, there exists a corresponding correlated equilibrium.

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Proof

Let $s^\star$ be a Nash equilibrium. We define

- $D_i = S_i$: strategy space and the domains of the random variables are the same.
- $\pi(d) = \prod_{i \in N} s^\star(d_i)$
- $\sigma_i : D_i \to S_i$, $d_i \mapsto s_i$.

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□

- Since a Nash equilibrium always exists, a correlated equilibrium always exists as well.
- However, a correlated equilibrium may not be a Nash equilibrium

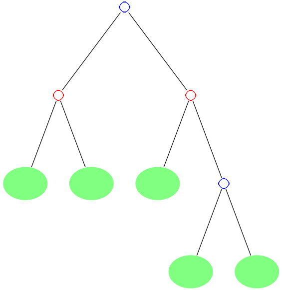
⇒ correlated equilibrium is a generalization of Nash equilibrium.

- We have considered games where each player choose their action **simultaneously**, and we have studied the normal-form representation.
- They are many games which rely on turn-taking, e.g., chess, card games, etc. Game theory has something to say about these games as well.

➩ We now introduce the **extended-form games (EFGs)**, in which a game is represented using a **tree** structure

Extended Form Games (EFGs)

Perfect-information game



A game is described by a **game tree**.

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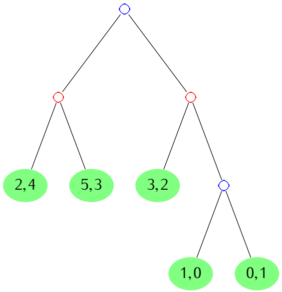

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- the leaf nodes contain the payoff to the agents.

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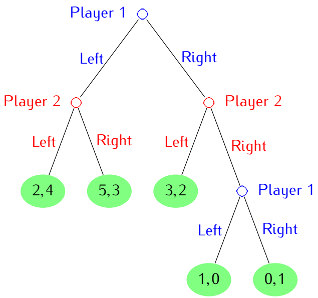

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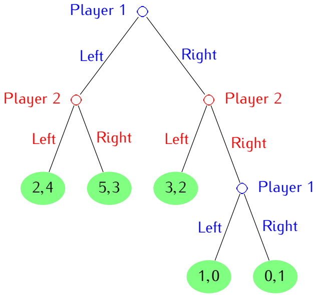

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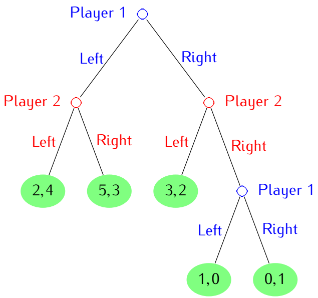

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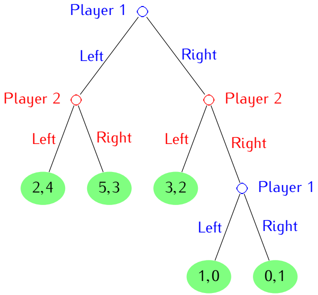

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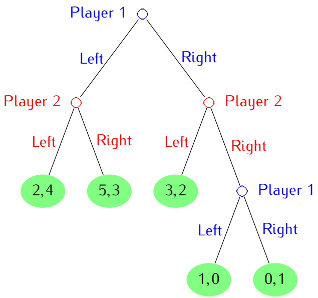

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ex: Player 1 decides for two nodes and has four strategies: (Left, Left), (Left, Right), (Right, Left) and (Right, Right).

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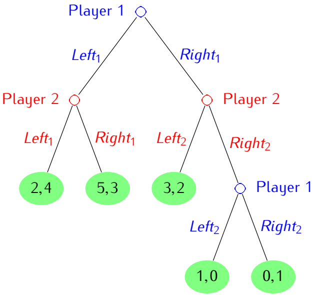

Backward induction: when an agent knows the payoff at each of a node's children, it can decide the best action of the player making the decision for this node.
If there are ties, then how they are broken affects what happens higher up in the tree
⇒Multiple equilibria...

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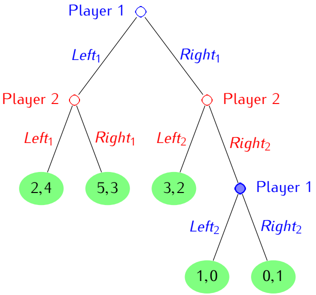

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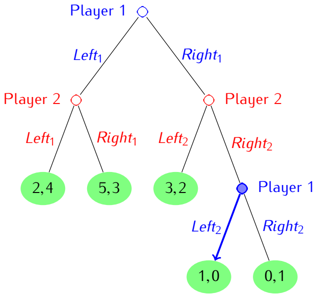

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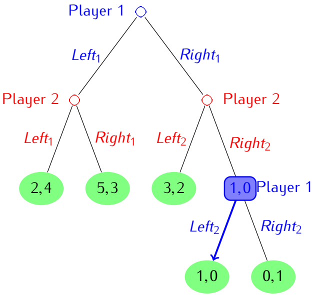

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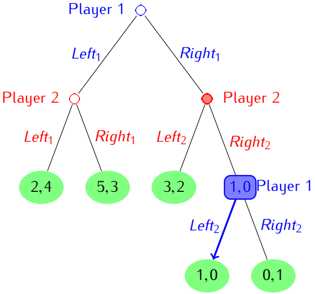

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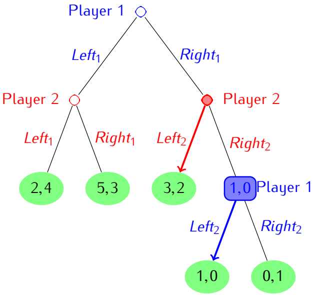

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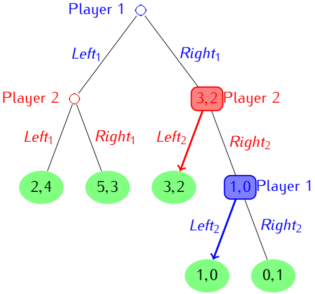

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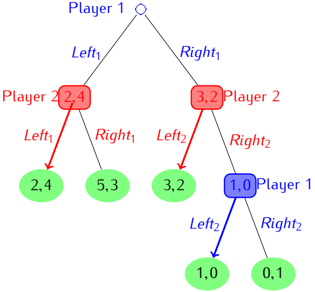

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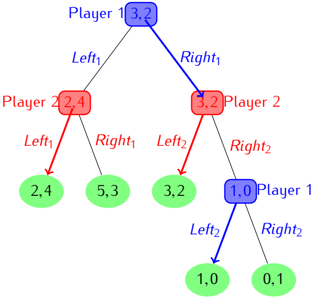

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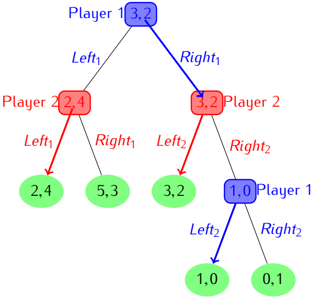

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From an EFG to a NFG

	L_1L_2	L_1R_1	R_1L_2	R_1R_2
L_1L_2	2,4	2,4	5,3	5,3
L_1R_2	2,4	2,4	5,3	5,3
R_1L_2	3,2	1,0	3,2	1,0
R_1R_2	3,2	0,1	3,2	0,1

- There can be an exponential number of pure strategies.

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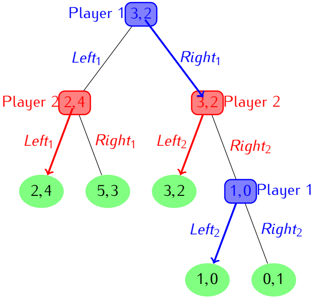

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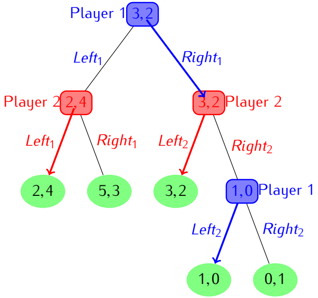

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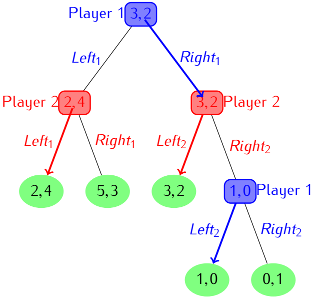

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- Nash equilibrium may be too weak for EFGs.

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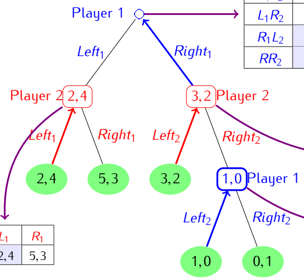

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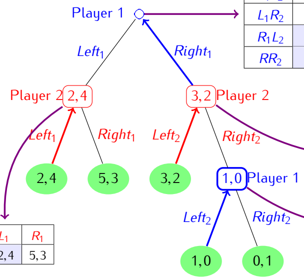

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(L_1L_2, L_1R_2) ✘

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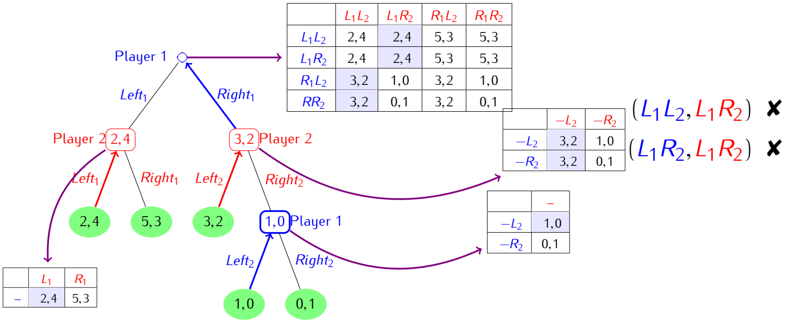

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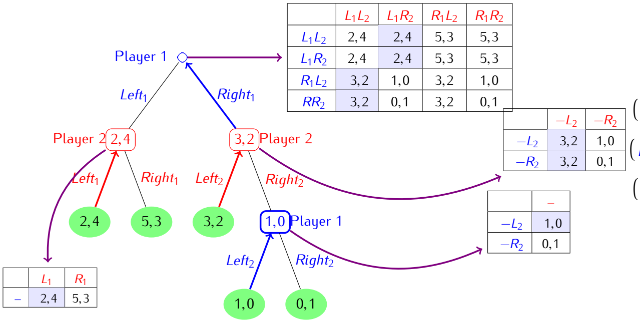

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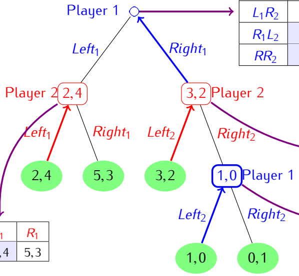

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(L_1R_2, L_1R_2) ✘

(R_1L_2, L_1L_2) ✔

(R_1R_2, L_1L_2) ✘

Other models of games

- **Congestion games:** a special game which always possess a **pure** strategy Nash equilibrium
- **Repeated games:** a NFG is played repeatedly (finitely/infinitely many times).
- **Stochastic games:** uncertainty about the next game to play
- **Bayesian games:** uncertainty about the current game

A **congestion game** is a tuple $(N, R, (S_i)_{i\in N}, (c_r)_{r\in R})$ where:

- $N = \{1,\dots,n\}$ is the set of **players**
- $R = \{1,\dots,m\}$ is the set of facilities or **resources**
- $S_i \subseteq M \setminus \emptyset$ denotes the set of **strategies** of player $i \in N$.
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Every finite congestion game has a pure strategy Nash equilibrium.

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Theorem

Every congestion game is a potential game and every finite potential game is isomorphic to a congestion game

D. Monderer and L. S. Shapley **Potential Games**, in *Games and economic behavior*, 1996.

Repeated games

Prisoner's dilemma

	Defect	Cooperate
Defect	2,2	4,1
Cooperate	1,4	3,3

When players are **rational**, both players confess!
If they trusted each other, they could both not confess and obtain $\langle 3, 3 \rangle$.
If the same players have to repeatedly play the game, then it could be rational not to confess.

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  - infinitely repeated games: the game tree would be infinite, use different techniques.

Infinitely repeated games

What is a strategy? In a repeated game, a **pure strategy** depends also on the **history** of play thus far.

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- **Discounted-sum criterion:** Discounted sum of the payoff received throughout the game by player i: $\sum_{t=0}^{\infty} \gamma^t u_i(s^t)$, where γ is the discount factor (γ models how much the agent cares about the near term compared to long term).

Theorem (A Folk theorem)

Using the average criterion, any payoff vector v such that

- v is **feasible**, i.e., $\exists\lambda \in [0,1]^{\prod_{j\in N}|S_j|}$ s.t. $v_i = \sum_{s\in\prod_{j\in N} S_j} \lambda_s v_i(s)$
- v is **enforceable** $v_i \geqslant \max_{s_i\in S_i} \min_{s_{-i}\in S_{-i}} u_i(s_i, s_{-i})$

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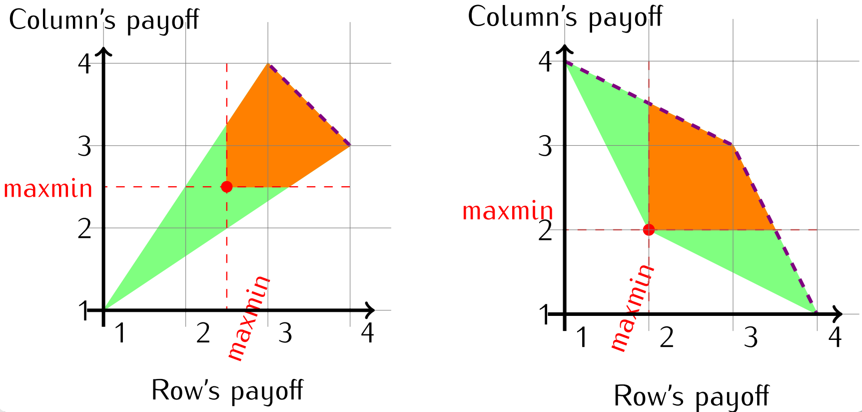

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Definition (Stochastic games)

A stochastic game is tuple $(N,(S_i)_{i\in N},Q,P,(u_i)_{i\in N})$ where

- N is the set of players
- S_i is the strategy space of player i
- Q is a set of NFGs $q=(N,(S_i)_{i\in N},(v_i^q)_{i\in N})$
- $P: Q\times\prod_{i\in N} S_i \times Q \to [0,1]$ is the **transition function**.
  $P(q,s,q')$ is the probability that game q' is played after game q when the joint-strategy s was played in game q.
- $u_i: Q\times\prod_{i\in N} S_i$ is the **payoff function**
  $u_i(q,s)$ is the payoff obtained by agent i when the joint-strategy s was played in game q.

In the definition, for ease of presentation, we assume that all the games have the same strategy space, which is not required.

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Definition (Bayesian game)

A **Bayesian game** is a tuple $(N, (S_i)_{i \in N}, G, P, (I_i)_{i \in N})$:

- N is the set of players.
- S_i is the set of strategies for agent i.
- G is a set of NFGs $g = (N, (S_i)_{i \in N}, (u_i^g)_{i \in N})$.
- P is a **common prior** over all games in G.
- I_i is the information set of agent i (a partition of G). A player knows the set which includes the current game, she does not know, however, which game it is in the set. ex: G is composed of six games, $I_2 = \{\{g_1, g_3, g_4\}, \{g_2, g_5\}\}$. Agent 2 knows the current game is in $\{g_1, g_3, g_4\}$, but she does not know whether the game is g_1, g_3, or g_4.

Evolutionary game theory

- Models organisms in a large population (supposed infinite)
- **two** organisms are drawn randomly and play a 2-player game.
- the payoffs are linked to the fitness of the agents, and then, to their ability to reproduce.
- when an organism reproduces, a child adopt the same strategy as its parent.
- **Goal:** Are the strategies used by the organisms resilient to small mutant invasions? I.e, Is a strategy robust to evolutionary pressures?
  ⇨evolutionary stability.

J. W. Weibull, Evolutionary game theory, the MIT press, 1997

Summary and Concluding remarks

When does an agent play?

- Agents play **simultaneously** (Rock/Paper/Scissors) ➫NFGs
- Agent play **sequentially** (chess, card games) ➫EFGs

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What is known?

- **Complete information** games: the structure of the game and the preference of the agents are common knowledge.
- **Incomplete information games**
  - does a player know the preference of its opponents?
    ⇒uncertainty, learning in games.
  - What kind of opponents? Rational? Malicious?
    ⇒Nash equilibrium, minmax, maxmin, regret.

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What can be observed? Are the agents able to observe the actions of the opponents (**perfect/imperfect information**)

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How does the game develop?

- Is it a one stage game?
- Are there multiple stages? (repeated games) Does the structure of the game change? ➩Stochastic, Bayesian games
- Is the game played forever? ➩Infinitely repeated games

Nobel Laureates		
1972	Arrow	Social choice
1994	Nash, Selten and Harsanyi	Game theory
1996	Vickrey	Mechanism design
1998	Sen	Social choice
2005	Schelling and Aumann	Game theory
2007	Hurwicz, Maskin and Myerson	Mechanism design

- **Game theory:** mathematical study of interaction among independent, self-interested agents. (Two sessions at AAMAS-10)
  - non-cooperative games
  - cooperative games
  - games with sequential actions
  - evolutionary game theory
- **Mechanism design:** study of protocol design for strategic agents (one session at AAMAS-09)
- **Social choice:** study of preference aggregation / collective decision making. (One session at AAMAS-10)

Resources

- Martin J. Osborne and Ariel Rubinstein. **A course in Game Theory**, the MIT Press, 1994. (freely available online)
- Yoav Shoham and Kevin Leyton-Brown. **Multiagent Systems**, Cambridge University Press, 2009
- Michael Wooldridge. **An Introduction to Multiagent Systems**, Wiley, 2009
- Noam Nisan, Tim Roughgarden, Éva Tardos & Vijay V. Vazirani. **Algorithmic Game Theory**, Cambridge University Press, 2007.
- `gametheory.net`

Tomorrow

Cooperative games

When agents work together, the group of agents, as a whole, gets a payoff.

- What groups of agents to form?
- How to distribute the payoff to the individual agents?