Cooperative Games

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Why study coalitional games?

Coalitional (or Cooperative) games are a branch of game theory in which **cooperation** or collaboration between agents can be modeled. Coalitional games can also be studied from a computational point of view (e.g., the problem of succint reprensentation and reasoning).

A coalition may represent a set of:

- persons or group of persons (labor unions, towns)
- objectives of an economic project
- artificial agents

We have a population N of n agents.

Definition (Coalition)

A **coalition** $\mathcal{C}$ is a set of agents: $\mathcal{C} \in 2^N$.

Two main classes of games

1- Games with Transferable Utility (TU games)

- Two agents can **compare** their utility
- Two agents can **transfer** some utility

Definition (valuation or characteristic function)

A *valuation function* v associates a real number $v(S)$ to any subset S, i.e., $v : 2^N \to \mathbb{R}$

Definition (TU game)

A TU game is a pair (N, v) where N is a set of agents and where v is a valuation function.

2- Games with Non Transferable Utility (NTU games)

It is **not** always possible to compare the utility of two agents or to transfer utility (e.g., no price tags). Agents have preference over coalitions.

Informal example: a task allocation problem

- A set of tasks requiring different expertises needs to be performed, tasks may be decomposed.
- Agents do not have enough resource on their own to perform a task.
- Find complementary agents to perform the tasks
  - robots have the ability to move objects in a plant, but multiple robots are required to move a heavy box.
  - transportation domain: agents are trucks, trains, airplanes, ships... a task is a good to be transported.
- **Issues:**
  - What coalition to form?
  - How to reward each each member when a task is completed?

Some types of TU games

$\forall \mathcal{C}_1, \mathcal{C}_2 \subseteq N \mid \mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$, $i \in N$, $i \notin \mathcal{C}_1$

- **additive (or inessential):** $v(\mathcal{C}_1 \cup \mathcal{C}_2) = v(\mathcal{C}_1) + v(\mathcal{C}_2)$ trivial from the game theoretic point of view
- **superadditive:** $v(\mathcal{C}_1 \cup \mathcal{C}_2) \geqslant v(\mathcal{C}_1) + v(\mathcal{C}_2)$ satisfied in many applications: it is better to form larger coalitions.
- **weakly superadditive:** $v(\mathcal{C}_1 \cup \{i\}) \geqslant v(\mathcal{C}_1) + v(\{i\})$
- **subadditive:** $v(\mathcal{C}_1 \cup \mathcal{C}_2) \leqslant v(\mathcal{C}_1) + v(\mathcal{C}_2)$
- **convex:** $\forall \mathcal{C} \subseteq \mathcal{T}$ and $i \notin \mathcal{T}$,
  $v(\mathcal{C} \cup \{i\}) - v(\mathcal{C}) \leqslant v(\mathcal{T} \cup \{i\}) - v(\mathcal{T})$.
  Convex game appears in some applications in game theory and have nice properties.
- **monotonic:** $\forall \mathcal{C} \subseteq \mathcal{T} \subseteq N$ $v(\mathcal{C}) \leqslant v(\mathcal{T})$.

The main problem

In the game (N, v) we want to form the **grand coalition**.

Each agent i will get a **personal payoff** x_i.

What are the interesting **properties** that x should satisfy?

How to **determine** the payoff vector x?

problem: a game (N, v) in which v is a worth of a coalition
solution: a payoff vector $x \in \mathbb{R}^n$

An example

$$N = \{1,2,3\}$$
$$v(\{1\}) = 0,\ v(\{2\}) = 0,\ v(\{3\}) = 0$$
$$v(\{1,2\}) = 90$$
$$v(\{1,3\}) = 80$$
$$v(\{2,3\}) = 70$$
$$v(\{1,2,3\}) = 105$$

What should we do?

- form $\{1,2,3\}$ and share equally $\langle 35,35,35\rangle$?
- 3 can say to 1 "let's form $\{1,3\}$ and share $\langle 40,0,40\rangle$".
- 2 can say to 1 "let's form $\{1,2\}$ and share $\langle 45,45,0\rangle$".
- 3 can say to 2 "OK, let's form $\{2,3\}$ and share $\langle 0,46,24\rangle$".
- 1 can say to 2 and 3, "fine! $\{1,2,3\}$ and $\langle 33,47,25\rangle$
- … is there a "good" solution?

Some properties

Let $x \in \mathbb{R}^n$ be a solution of the TU game (N, v)

Feasible solution: $\sum_{i \in N} x(i) \leqslant v(N)$.

Anonymity: a solution is independent of the names of the player.

Definition (marginal contribution)

The **marginal contribution** of agent i for a coalition $\mathcal{C} \subseteq N \setminus \{i\}$ is $mc_i(\mathcal{C}) = v(\mathcal{C} \cup \{i\}) - v(\mathcal{C})$.

Let mc_i^{min} and mc_i^{max} denote the minimal and maximal marginal contribution.

x is **reasonable from above** if $\forall i \in N \ x^i < mc_i^{max}$

- ➪ mc_i^{max} is the strongest **threat** that an agent can use against a coalition.

x is **reasonable from below** if $\forall i \in N \ x^i > mc_i^{min}$

- ➪ mc_i^{min} is a minimum acceptable reward.

Some properties

Let x, y be two solutions of a TU-game (N, v).

Efficiency: $x(N) = v(N)$

- the payoff distribution is an allocation of the entire worth of the grand coalition to all agents.

Individual rationality: $\forall i \in N, x(i) \geqslant v(\{i\})$

- agent obtains at least its self-value as payoff.

Group rationality: $\forall \mathcal{C} \subseteq N$, $\sum_{i \in \mathcal{C}} x(i) = v(\mathcal{C})$

- if $\sum_{i \in \mathcal{C}} x(i) < v(\mathcal{C})$ some utility is lost.
- if $\sum_{i \in \mathcal{C}} x(i) > v(\mathcal{C})$ is not possible.

Pareto Optimal: $\sum_{i \in N} x(i) = v(N)$

- no agent can improve its payoff without lowering the payoff of another agent.

An **imputation** is a payoff distribution x that is efficient and individually rational.

The core

D Gillies, **Some theorems in *n*-person games**. *PhD thesis, Department of Mathematics, Princeton, N.J.*, 1953.

- A condition for a coalition to form:

  all agents prefer to be in it.

  i.e., none of the participants wishes she were in a different coalition or by herself ➾ **Stability**.
- Stability is a necessary but not sufficient condition, (e.g., there may be multiple stable coalitions).
- The **core** is a stability concepts where no agents prefer to deviate to form a different coalition.
- For simplicity, we will only consider the problem of the stability of the grand coalition:

➾ Is the grand coalition stable? $\Leftrightarrow$ Is the core non-empty?

The core relates to the stability of the grand coalition:
No group of agents has any incentive to change coalition.

Definition (*core* of a game (N,v))

Let (N,v) be a TU game, and assume we form the grand coalition N.
The **core** of (N,v) is the set:

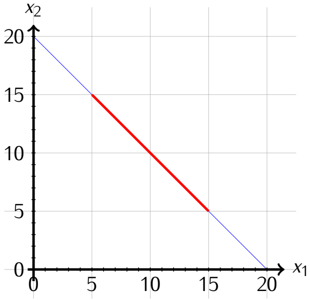
$$Core(N,v) = \{x \in \mathbb{R}^n \mid x \text{ is a group rational imputation}\}$$

Equivalently,

$$Core(N,v) = \{x \in \mathbb{R}^n \mid x(N) \leqslant v(N) \wedge x(\mathcal{C}) \geqslant v(\mathcal{C})\ \forall \mathcal{C} \subseteq N\}$$

$$N = \{1,2\}$$
$$v(\{1\}) = 5,\ v(\{2\}) = 5$$
$$v(\{1,2\}) = 20$$

$$core(N,v) = \{(x_1,x_2) \in \mathbb{R}^2 \mid x_1 \geqslant 5, x_2 \geqslant 5, x_1 + x_2 = 20\}$$



The core may not be fair: the core only considers stability.

Issues with the core

- The core may not always be non-empty.
- When the core is not empty, it may not be 'fair'.
- It may not be easy to compute.

➥ Are there classes of games that have a non-empty core?

➥ Is it possible to characterize the games with non-empty core?

Definition (Convex games)

A game (N, v) is **convex** iff
$\forall \mathcal{C} \subseteq \mathcal{T}$ and $i \notin \mathcal{T}$, $v(\mathcal{C} \cup \{i\}) - v(\mathcal{C}) \leqslant v(\mathcal{T} \cup \{i\}) - v(\mathcal{T})$.

TU-game is convex if the marginal contribution of each player increases with the size of the coalition he joins.

Theorem

A TU game (N, v) is convex iff for all coalition S and T
$v(S) + v(T) \leqslant v(S \cup T) + v(S \cap T)$

Theorem

A convex game has a non-empty core

Games with Coalition structures

Coalition Structure

Definition (Coalition Structure)

A **coalition structure (CS)** is a partition of the grand coalition into coalitions.
$\mathcal{S} = \{\mathcal{C}_1, \dots, \mathcal{C}_k\}$ where $\cup_{i \in \{1..k\}} \mathcal{C}_i = N$ and $i \neq j \Rightarrow \mathcal{C}_i \cap \mathcal{C}_j = \emptyset$.
We note $\mathscr{S}_N$ the set of all coalition structures over the set N.

ex: $\{\{1,3,4\}\{2,7\}\{5\}\{6,8\}\}$ is a coalition structure for $n = 8$ agents.

We start by defining a game with coalition structure, and see how we can define the core of such a game.

Game with Coalition Structure

Definition (TU game)

A TU game is a pair (N, v) where N is a set of agents and where v is a valuation function.

Definition (Game with Coalition Structures)

A **TU-game with coalition structure** $(N, v, \mathcal{S})$ consists of a TU game (N, v) and a CS $\mathcal{S} \in \mathscr{S}_N$.

- We assume that the players agreed upon the formation of $\mathcal{S}$ and only the payoff distribution choice is left open.
- The CS may model affinities among agents, may be due to external causes (e.g. affinities based on locations).
- The agents may refer to the value of coalitions with agents outside their coalition (i.e. opportunities they would have outside of their coalition).
- (N, v) and $(N, v, \{N\})$ represent the same game.

The set of **feasible** payoff vectors for $(N, v, \mathcal{S})$ is
$X_{(N,v,\mathcal{S})} = \{x \in \mathbb{R}^n \mid \text{ for every } \mathcal{C} \in \mathcal{S} \;\; x(\mathcal{C}) \leqslant v(\mathcal{C})\}$.

Definition (Core of a game with CS)

The **core** $Core(N, v, \mathcal{S})$ of $(N, v, \mathcal{S})$ is defined by
$\{x \in \mathbb{R}^n \mid (\forall \mathcal{C} \in \mathcal{S}, x(\mathcal{C}) \leqslant v(\mathcal{C})) \text{ and } (\forall \mathcal{C} \subseteq N, x(\mathcal{C}) \geqslant v(\mathcal{C}))\}$

We have $Core(N, v, \{N\}) = Core(N, v)$.

The next theorem is due to Aumann and Drèze.

R.J. Aumann and J.H. Drèze. **Cooperative games with coalition structures**, *International Journal of Game Theory*, 1974

Definition (Substitutes)

Let (N,v) be a game and $(i,j) \in N^2$. Agents i and j are **substitutes** iff $\forall \mathcal{C} \subseteq N \setminus \{i,j\}$, $v(\mathcal{C} \cup \{i\}) = v(\mathcal{C} \cup \{j\})$.

A nice property of the core related to fairness:

Theorem

Let $(N,v,\mathcal{S})$ be a game with coalition structure,
let i and j be substitutes, and let $x \in Core(N,v,\mathcal{S})$.
If i and j belong to different members of $\mathcal{S}$, then $x_i = x_j$.

The nucleolus

D. Schmeidler, **The nucleolus of a characteristic function game**. *SIAM Journal of applied mathematics*, 1969.

Excess of a coalition

Definition (Excess of a coalition)

Let (N,v) be a TU game, $\mathcal{C} \subseteq N$ be a coalition, and x be a payoff distribution over N. The **excess** $e(\mathcal{C},x)$ of coalition $\mathcal{C}$ at x is the quantity $e(\mathcal{C},x) = v(\mathcal{C}) - x(\mathcal{C})$.

An example: let $N = \{1,2,3\}$, $\mathcal{C} = \{1,2\}$, $v(\{1,2\}) = 8$, $x = \langle 3,2,5 \rangle$, $e(\mathcal{C},x) = v(\{1,2\}) - (x_1 + x_2) = 8 - (3+2) = 3$.

We can interpret a positive excess ($e(\mathcal{C},x) \geqslant 0$) as the amount of **dissatisfaction** or **complaint** of the members of $\mathcal{C}$ from the allocation x.

We can use the excess to define the core:
$Core(N,v) = \{x \in \mathbb{R}^n \mid x \text{ is an imputation and } \forall \mathcal{C} \subseteq N, e(\mathcal{C},x) \leqslant 0\}$

This definition shows that no coalition has any complaint: each coalition's demand can be granted.

$$N = \{1,2,3\},\ v(\{i\}) = 0 \text{ for } i \in \{1,2,3\}$$
$$v(\{1,2\}) = 5,\ v(\{1,3\}) = 6,\ v(\{2,3\}) = 6$$
$$v(N) = 8$$

Let us consider two payoff vectors $x = \langle 3,3,2 \rangle$ and $y = \langle 2,3,3 \rangle$.
Let $e(x)$ denote the sequence of **excesses** of all coalitions at x.

$x = \langle 3,3,2 \rangle$

coalition $\mathcal{C}$	$e(\mathcal{C}, x)$
$\{1\}$	-3
$\{2\}$	-3
$\{3\}$	-2
$\{1,2\}$	-1
$\{1,3\}$	1
$\{2,3\}$	1
$\{1,2,3\}$	0

$y = \langle 2,3,3 \rangle$

coalition $\mathcal{C}$	$e(\mathcal{C}, y)$
$\{1\}$	-2
$\{2\}$	-3
$\{3\}$	-3
$\{1,2\}$	0
$\{1,3\}$	1
$\{2,3\}$	0
$\{1,2,3\}$	0

Which payoff should we prefer? x or y? Let us write the excess in the decreasing order (from the greatest excess to the smallest)

$\langle 1,1,0,-1,-2,-3,-3 \rangle$ $\langle 1,0,0,0,-2,-3,-3 \rangle$

Definition (lexicographic order of $\mathbb{R}^m$ $\geqslant_{lex}$)

Let $\geqslant_{lex}$ denote the **lexicographical** ordering of $\mathbb{R}^m$,
i.e., $\forall (x,y) \in \mathbb{R}^m$, $x \geqslant_{lex} y$ iff

$$\begin{cases} \text{x=y or} \\ \exists t \text{ s. t. } 1 \leqslant t \leqslant m \text{ s. t. } \forall i \text{ s. t. } 1 \leqslant i \leqslant t\ x_i = y_i \text{ and } x_t > y_t \end{cases}$$

example: $\langle 1,1,0,-1,-2,-3,-3 \rangle \geqslant_{lex} \langle 1,0,0,0,-2,-3,-3 \rangle$

Let l be a sequence of m reals. We denote by $l^{\blacktriangleright}$ the **reordering** of l in **decreasing** order.

In the example, $e(x) = \langle -3,-3,-2,-1,1,1,0 \rangle$,
then $e(x)^{\blacktriangleright} = \langle 1,1,0,-1,-2,-3,-3 \rangle$.

Hence, we can say that y is better than x by writing

$$e(x)^{\blacktriangleright} \geqslant_{lex} e(y)^{\blacktriangleright}.$$

Definition (Nucleolus)

Let (N, v) be a TU game.
Let $\mathcal{I}mp$ be the set of all imputations.
The **nucleolus** $Nu(N, v)$ is the set
$Nu(N, v) = \left\{x \in \mathcal{I}mp \mid \forall y \in \mathcal{I}mp \;\; e(y)^{\blacktriangleright} \geqslant_{lex} e(x)^{\blacktriangleright}\right\}$

Theorem (Non-emptyness of the nucleolus)

Let (N, v) be a TU game, if $\mathcal{I}mp \neq \emptyset$,
then the nucleolus $Nu(N, v)$ is **non-empty**.

For a TU game (N, v) the nucleolus $Nu(N, v)$ is non-empty when $\mathcal{I}mp \neq \emptyset$, which is a great property as agents will always find an agreement. But there is more!

Theorem

The nucleolus has **at most one** element

In other words, there is **one** agreement which is stable according to the nucleolus.

Theorem

Let (N, v) be a superadditive game and $\mathcal{I}mp$ be its set of imputations. Then, $\mathcal{I}mp \neq \emptyset$.

Proof

Let (N, v) be a superadditive game.
Let x be a payoff distribution defined as follows:
$x_i = v(\{i\}) + \frac{1}{|N|}\left(v(N) - \sum_{j\in N} v(\{j\})\right)$.

- $v(N) - \sum_{j\in N} v(\{j\}) > 0$ since (N, v) is superadditive.
- It is clear x is individually rational ✔
- It is clear x is efficient ✔

Hence, $x \in \mathcal{I}mp$. □

Theorem

Let (N, v) be a TU game with a non-empty core. Then $Nu(N, v) \subseteq Core(N, v)$

The kernel.

M. Davis. and M. Maschler, **The kernel of a cooperative game**. *Naval Research Logistics Quarterly*, 1965.

Definition (Excess)

For a TU game (N,v), the excess of coalition $\mathcal{C}$ for a payoff distribution x is defined as $e(\mathcal{C},x) = v(\mathcal{C}) - x(\mathcal{C})$.

We saw that a positive excess can be interpreted as an amount of complaint for a coalition.
We can also interpret the excess as a potential to generate more utility.

Definition (Maximum surplus)

For a TU game (N, v), the **maximum surplus** $s_{k,l}(x)$ of **agent k over agent l** with respect to a payoff distribution x is the **maximum excess** from a coalition that **includes k** but does **exclude** l, i.e.,
$s_{k,l}(x) = \max_{\mathcal{C} \subseteq N \mid k \in \mathcal{C},\, l \notin \mathcal{C}} e(\mathcal{C}, x)$.

Definition (Kernel)

Let $(N, v, \mathcal{S})$ be a TU game with coalition structure. The **kernel** is the set of imputations $x \in X_{(N,v,\mathcal{S})}$ such that for every coalition $\mathcal{C} \in CS$, if $(k, l) \in \mathcal{C}^2$, $k \neq l$, then we have either $s_{kl}(x) \geqslant s_{lk}(x)$ or $x_k = v(\{k\})$.

$s_{kl}(x) < s_{lk}(x)$ calls for a transfer of utility from k to l unless it is prevented by individual rationality, i.e., by the fact that $x_k = v(\{k\})$.

Properties

Theorem

Let $(N, v, \mathcal{S})$ a game with coalition structure, and let $\mathcal{I}mp \neq \emptyset$. Then we have $Nu(N, v, \mathcal{S}) \subseteq K(N, v, \mathcal{S})$

Theorem

Let $(N, v, \mathcal{S})$ a game with coalition structure, and let $\mathcal{I}mp \neq \emptyset$. The kernel $K(N, v, \mathcal{S})$ of the game is non-empty.

Proof

Since the Nucleolus is non-empty when $\mathcal{I}mp \neq \emptyset$, the proof is immediate using the theorem above. $\square$

Computing a kernel-stable payoff distribution

- There is a transfer scheme converging to an element in the kernel.
- It may require an infinite number of small steps.
- We can consider the ϵ-kernel where the inequality are defined up to an arbitrary small constant ϵ.

R. E. Stearns. **Convergent transfer schemes for n-person games**. *Transactions of the American Mathematical Society*, 1968.

Computing a kernel-stable payoff distribution

Algorithm 1: Transfer scheme converging to a ϵ-Kernel-stable payoff distribution for the CS $\mathcal{S}$

compute-ϵ-Kernel-Stable(N, v, $\mathcal{S}$, ϵ)
repeat
  for *each coalition* $\mathcal{C} \in \mathcal{S}$ **do**
    for *each member* $(i,j) \in \mathcal{C}, i \neq j$ **do** // compute the maximum surplus
      // for two members of a coalition in $\mathcal{S}$
      $s_{ij} \leftarrow \max_{R \subseteq N | (i \in R, j \notin R)} v(R) - x(R)$
  $\delta \leftarrow \max_{(i,j) \in \mathcal{C}^2, \mathcal{C} \in \mathcal{S}} s_{ij} - s_{ji}$;
  $(i^\star, j^\star) \leftarrow \mathbf{arg}\max_{(i,j) \in N^2} (s_{ij} - s_{ji})$;
  if $\left(x_{j^\star} - v(\{j\}) < \frac{\delta}{2}\right)$ **then** // payment should be individually rational
    $d \leftarrow x_{j^\star} - v(\{j^\star\})$;
  else
    $d \leftarrow \frac{\delta}{2}$;
  $x_{i^\star} \leftarrow x_{i^\star} + d$;
  $x_{j^\star} \leftarrow x_{j^\star} - d$;
until $\frac{\delta}{v(S)} \leqslant \epsilon$;

- The complexity for one side-payment is $O(n \cdot 2^n)$.
- Upper bound for the number of iterations for converging to an element of the ϵ-kernel: $n \cdot log_2(\frac{\delta_0}{\epsilon \cdot v(S)})$, where δ_0 is the maximum surplus difference in the initial payoff distribution.
- To derive a polynomial algorithm, the number of coalitions must be bounded. For example, only consider coalitions which size is bounded in $[K_1, K_2]$. The complexity of the truncated algorithm is $O(n^2 \cdot n_{coalitions})$ where $n_{coalitions}$ is the number of coalitions with size in$[K_1, K_2]$, which is a polynomial of order K_2.

• M. Klusch and O. Shehory. **A polynomial kernel-oriented coalition algorithm for rational information agents**. In *Proceedings of the Second International Conference on Multi-Agent Systems*, 1996.

• O. Shehory and S. Kraus. **Feasible formation of coalitions among autonomous agents in non-superadditve environments**. *Computational Intelligence*, 1999.

The Shapley value

Lloyd S. Shapley. **A Value for *n*-person Games**. In *Contributions to the Theory of Games, volume II (Annals of Mathematical Studies)*, 1953.

Definition (marginal contribution)

The **marginal contribution** of agent i for a coalition $\mathcal{C} \subseteq N \setminus \{i\}$ is $mc_i(\mathcal{C}) = v(\mathcal{C} \cup \{i\}) - v(\mathcal{C})$.

$\langle mc_1(\emptyset), mc_2(\{1\}), mc_3(\{1,2\}) \rangle$ is an efficient payoff distribution for any game $(\{1,2,3\}, v)$. This payoff distribution may model a dynamic process in which 1 starts a coalition, is joined by 2, and finally 3 joins the coalition $\{1,2\}$, and where the incoming agent gets its marginal contribution.

An agent's payoff depends on which agents are already in the coalition. This payoff may not be **fair**. To increase fairness,one could take the average marginal contribution over all possible joining orders.

Let σ represent a joining order of the grand coalition N, i.e., σ is a permutation of $\langle 1, \ldots, n \rangle$.
We write $mc(\sigma) \in \mathbb{R}^n$ the payoff vector where agent i obtains $mc_i(\{\sigma(j) \mid j < i\})$. The vector mc is called a **marginal vector**.

Shapley value: version based on marginal contributions

Let (N, v) be a TU game. Let $\Pi(N)$ denote the set of all permutations of the sequence $\langle 1, \ldots, n \rangle$.

$$Sh(N,v) = \frac{\sum_{\sigma \in \Pi(N)} mc(\sigma)}{n!}$$

the Shapley value is a **fair** payoff distribution based on marginal contributions of agents averaged over joining orders of the coalition.

An example

$$N = \{1,2,3\},\ v(\{1\}) = 0,\ v(\{2\}) = 0,\ v(\{3\}) = 0,$$
$$v(\{1,2\}) = 90,\ v(\{1,3\}) = 80,\ v(\{2,3\}) = 70,$$
$$v(\{1,2,3\}) = 120.$$

	1	2	3
$1 \leftarrow 2 \leftarrow 3$	0	90	30
$1 \leftarrow 3 \leftarrow 2$	0	40	80
$2 \leftarrow 1 \leftarrow 3$	90	0	30
$2 \leftarrow 3 \leftarrow 1$	50	0	70
$3 \leftarrow 1 \leftarrow 2$	80	40	0
$3 \leftarrow 2 \leftarrow 1$	50	70	0
total	270	240	210
Shapley value	45	40	35

Let $y = \langle 50, 40, 30 \rangle$

$\mathcal{C}$	$e(\mathcal{C}, x)$	$e(\mathcal{C}, y)$
$\{1\}$	-45	0
$\{2\}$	-40	0
$\{3\}$	-35	0
$\{1,2\}$	5	0
$\{1,3\}$	0	0
$\{2,3\}$	-5	0
$\{1,2,3\}$	120	0

This example shows that the Shapley value may not be in the core, and may not be the nucleolus.

- There are $|\mathcal{C}|!$ permutations in which all members of $\mathcal{C}$ precede i.
- There are $|N \setminus (\mathcal{C} \cup \{i\})|!$ permutations in which the remaining members succede i, i.e. $(|N| - |\mathcal{C}| - 1)!$.

The Shapley value $Sh_i(N,v)$ of the TU game (N,v) for player i can also be written

$$Sh_i(N,v) = \sum_{\mathcal{C} \subseteq N \setminus \{i\}} \frac{|\mathcal{C}|!(|N| - |\mathcal{C}| - 1)!}{|N|!} \left(v(\mathcal{C} \cup \{i\}) - v(\mathcal{C})\right).$$

Using definition, the sum is over $2^{|N|-1}$ instead of $|N|!$.

Some interesting properties

Let (N,v) and (N,u) be TU games and ϕ be a value function.

- **Symmetry or substitution (SYM):** If $\forall (i,j) \in N$, $\forall \mathcal{C} \subseteq N \setminus \{i,j\}$, $v(\mathcal{C} \cup \{i\}) = v(\mathcal{C} \cup \{j\})$ then $\phi_i(N,v) = \phi_j(N,v)$
- **Dummy (DUM):** If $\forall \mathcal{C} \subseteq N \setminus \{i\}$ $v(\mathcal{C}) = v(\mathcal{C} \cup \{i\})$, then $\phi_i(N,v) = 0$.
- **Additivity (ADD):** Let $(N, u+v)$ be a TU game defined by $\forall \mathcal{C} \subseteq N$, $(u+v)(N) = u(N) + v(N)$. $\phi(u+v) = \phi(u) + \phi(v)$.

Theorem

The Shapley value is the unique value function ϕ that satisfies (SYM), (DUM) and (ADD).

Discussion about the axioms

- SYM: it is desirable that two subsitute agents obtain the same value ✔
- DUM: it is desirable that an agent that does not bring anything in the cooperation does not get any value. ✔
- What does the addition of two games mean?
  - + if the payoff is interpreted as an expected payoff, ADD is a desirable property.
  - + for cost-sharing games, the interpretation is intuitive: the cost for a joint service is the sum of the costs of the separate services.
  - – there is no interaction between the two games.
  - – the structure of the game $(N, v+w)$ may induce a behavior that has may be unrelated to the behavior induced by either games (N, v) or (N, w).
- The axioms are independent. If one of the axiom is dropped, it is possible to find a different value function satisfying the remaining two axioms.

Some properties

Note that other axiomatisations are possible.

Theorem

For superadditive games, the Shapley value is an imputation.

Lemma

For convex game, the Shapley value is in the core.