Uncertainty & Decision

von Neumann Morgenstern's Theorem

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von Neumann Morgenstern's Theorem

- A strategy to build an **interval scale**.
- Ask the decision maker her preferences over **risky acts**.
- The outcome of the act cannot be controlled by the decision maker, but the probabilities are **known** (decision under risk).
- preferences over risky acts $\rightarrow$ utility function u
- vNM propose a set of constraints on rational preferences (or axioms).
- If a decision maker follows these axioms, she behaves as if she maximizes expected utility.

- X is the set of outcomes
- Risky acts are lotteries with finite support:

$$L = \left\{ P : X \to [0,1] \;\middle|\; \begin{array}{c} \#\{x \mid P(x) > 0\} < \infty \\ \sum_{x \in X} P(x) = 1 \end{array} \right\}$$

- A **mixing** operation on L is defined as follows: for $A, B \in L$, for a given probability $p \in [0,1]$, $pA + (1-p)B \in L$ is given by

$$(pA + (1-p)B)(x) = pA(x) + (1-p)B(x)$$

"if A and B are lotteries, then so is the prospect of getting A with probability p and B with probability $1-p$.

➩ the decision maker gives her preferences $\succ$ over lotteries (no longer on a set of certain outcomes)

vNM1 (completness) (transitivity) (asymmetric)

$$A \succ B \Rightarrow B \not\succ A$$

$$A \succ B \text{ or } A \sim B \text{ or } B \succ A$$

$$\text{If } A \succ B \text{ and } B \succ C \text{ then } A \succ C$$

The issues raised when we talked about preferences over certain outcomes remain the same.

vNM3 (continuity) For every $A \succ B \succ C$ there exists p and $q \in (0,1)$ such that $pA + (1-p)C \succ B \succ qA + (1-q)C$

$$A \leftrightarrow € \ 10M,\ B \leftrightarrow € \ 9M,\ A \leftrightarrow € \ 0.$$

With continuity axiom, if $A \succ B \succ C$, then there is

p such that
€ $10M$ with prob p and € 0 with prob $1-p \succ$ € 9 for certain.

q such that
€ $9M$ for certain $\succ$ € 10 with prob p and € 0 with prob $1-q$.

vNM4 (independence) $A \succ B$ iff $pA + (1-p)C \succ pB + (1-p)C$

Some kind of independence of irrelevant alternatives: either p or $1-p$ occurs (so you can disregard the other event).

Example:

- lottery A: 1M € for sure
- lottery B: 0 € with probability 0.1 or 5M € with probability 0.9

Suppose you prefer lottery A to lottery B, i.e. $A \succ B$.

Allais paradox can appear as there are **no constraints** on the lottery C.

- $pA + (1-p)C$: 0€ with probability 0.9 or 1M € with 0.1
- $pB + (1-p)C$: 0€ with probability 0.91 or 5M€ with probability 0.09

Now, you cannot guarantee 1M€ for sure, so it may now be worth getting the risk to get 5M€ .

van Neumann Morgenstern's theorem

Theorem (vNM theorem)

The **preference relation** $\succ$ satisfies vNM 1–4 iff there exists a function u that takes a lottery as its argument and returns a real number between 0 and 1 with the following properties:

(1) $A \succ B$ iff $u(A) > u(B)$.

(2) $u(pA + (1-p)B) = pu(A) + (1-p)u(B)$.

(3) for every other function satisfying (1) and (2), there are numbers $c > 0$ and $d \in \mathbb{R}$ such that $u' = c \cdot u + d$.

- From (1), we can see that $A \sim B$ iff $u(A) = u(B)$.
- (2) is the expected utility property: anyone agreeing with the 4 axioms acts in accordance with the principle of maximizing expected utility.
- (1) and (2) are the representation part of the theorem
- (3) is the uniqueness part: all functions satisfying (1) and (2) are all positive linear transformation of each other ⇒ this is an interval scale.

Objections

- axioms are too strong
- No action guidance: to compute the utility, the decision maker should first know her preferences over lotteries. the output is not a preference over acts, it is indeed the input!
- The output is a set of functions that can be used for describing the agent as an expected utility maximiser.
- Agents do not prefer an act *because* its expected utility is higher, but it can only be described as if they were acting from this principle.
- For some agents that are not fully rational
  - detect any inconsistencies in her preferences
  - the expected utility function may help to fill some gaps (preferences over lotteries that haven't been computed)
- Utility without chance: meaning of utility is linked to preference over lotteries? Does utility have relationship with risk?